SOME EXAMPLES OF ALMOST HERMITIAN MANIFOLDS

BY

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1. Introduction

Because of their nice topological properties Kähler manifolds have been studied much more extensively than other kinds of almost Hermitian manifolds. In the study of non-Kähler almost Hermitian manifolds it is natural to consider those whose almost complex structure satisfies similar but weaker conditions than those of Kähler manifolds. Calabi and Eckmann [3] have given several examples of complex non-Kähler manifolds which show that the Betti numbers of complex manifolds are not as regularly determined as those of Kähler manifolds. In a different direction it is known [1], [9] that the homogeneous spaces $G_2/A_2 = S^6$, $F_4/A_2 \times A_2$, $E_6/A_2 \times A_2 \times A_2$, $E_7/A_2 \times A_5$, E_8/A_8, $E_8/A_2 \times E_6$, and $E_8/A_4 \times A_4$ have homogeneous almost complex structures which are not complex. Calabi [2] has also proved that orientable six-dimensional hypersurfaces of R^7 have almost complex structures which are usually non-Kählerian.

In §2 we propose several conditions for an almost complex manifold which are slightly weaker than its being parallel (in which case the manifold would be Kähler). The manifolds we discuss include complex and almost Kähler manifolds; also S^6 with the almost complex structure derived from the Cayley numbers falls into a class of manifolds which we call nearly Kählerian. Our definitions were first given by Kotō [12], but we have reformulated some of them in terms of the exterior derivative d and the co-derivative δ.

We also determine the inclusion relations between the classes of manifolds we discuss; these relations formally resemble those in the classification theory of Riemann surfaces. Next comes the question of whether the various inclusions are strict. Two methods of constructing almost Hermitian manifolds for the examples of strict inclusion are given. The first in §3 uses conformal diffeomorphisms. In §4 we give the second method, which may be described as follows: a certain class of seven-dimensional manifolds, which includes parallelizable manifolds, is considered. These manifolds have a "vector cross product tensor" which is defined by means of the Cayley numbers. Then orientable (six-dimensional) hypersurfaces of these seven-dimensional manifolds are almost complex and provide a great many interesting examples. Our work generalizes that of Calabi [2], who considered orientable hypersurfaces of R^7. A particularly interesting consequence of our work is that for any orientable hypersurface of R^7 with its almost complex structure induced from the Cayley numbers, the Kähler form (or fundamental 2-form) is co-closed.

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2. Types of almost Hermitian manifolds

Let M be a C^∞ real differentiable manifold, $\mathfrak{F}(M)$ the ring of real-valued differentiable functions on M, and $\mathfrak{X}(M)$ the module of derivations of $\mathfrak{F}(M)$. Then $\mathfrak{X}(M)$ is a Lie algebra over the real numbers and the elements of $\mathfrak{X}(M)$ are called vector fields. An *almost complex manifold* M is a differentiable manifold equipped with a $(1, 1)$ tensor J (which we may regard as an $\mathfrak{F}(M)$-linear map $J : \mathfrak{X}(M) \to \mathfrak{X}(M)$) which satisfies $J^2 = -I$, where I is the identity. Such a manifold is orientable and even-dimensional. M is *almost Hermitian* provided it is almost complex and has a Riemannian metric $\langle\ ,\ \rangle$ for which $\langle JX, JY\rangle = \langle X, Y\rangle$ for all $X, Y \in \mathfrak{X}(M)$. Any Riemannian almost complex manifold may be made almost Hermitian, and so we shall henceforth deal only with almost Hermitian manifolds. To describe the geometry of an almost Hermitian manifold M, it is useful to consider two special tensors. The first is a 2-form F, called the *Kähler form,* and it is defined for $X, Y \in \mathfrak{X}(M)$ by $F(X, Y) = \langle JX, Y\rangle$. Since it is skew symmetric, it is in fact a differential form. The second, called the *Nijenhuis tensor*, is a $(1, 2)$ tensor S defined by

$$S(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY], \tag{2.1}$$

for $X, Y \in \mathfrak{X}(M)$. It is easy to see that

$$S(X, Y) = -S(Y, X), \qquad S(JX, Y) = S(X, JY) = -JS(X, Y).$$

If we extend the Riemannian connection ∇_X of M to be a derivative on the tensor algebra of M, then we have the following formulas:

$$\nabla_X(J)(Y) = \nabla_X(JY) - J\nabla_X(Y), \tag{2.2}$$

$$\nabla_X(F)(Y, Z) = \langle \nabla_X(J)(Y), Z\rangle. \tag{2.3}$$

It will be necessary to have explicit formulas for the exterior derivative and the co-derivative of F. By standard formulas (cf. [11]) these are computed to be

$$dF(X, Y, Z) = \mathfrak{S}\nabla_X(F)(Y, Z), \tag{2.4}$$

$$\delta F(X) = -\textstyle\sum_{i=1}^{m} \{\nabla_{E_i}(F)(E_i, X) + \nabla_{JE_i}(F)(JE_i, X)\} \tag{2.5}$$

where $\mathfrak{S}$ denotes the cyclic sum over $X, Y, Z \in \mathfrak{X}(M)$ and

$$\{E_1, \cdots, E_m, JE_1, \cdots, JE_m\}$$

is a frame field on an open subset of M.

THEOREM 2.1. *Let $X, Y, Z \in \mathfrak{X}(M)$. Then*

$$S(X, Y) = -\nabla_X(J)(JY) + \nabla_{JY}(J)(X) - \nabla_{JX}(J)(Y) + \nabla_Y(J)(JX), \tag{2.6}$$

$$2\nabla_X(F)(Y, Z) = dF(X, Y, Z) - dF(X, JY, JZ) - \langle X, S(Y, JZ)\rangle, \tag{2.7}$$

$$(2.8)\quad \begin{aligned} 2\nabla_X(F)(Y, Z) &+ 2\nabla_{JX}(F)(JY, Z) \\ &= dF(X, Y, Z) - dF(X, JY, JZ) + dF(Z, JX, JY) \\ &\quad + dF(Y, JZ, JX), \end{aligned}$$

$$(2.9)\quad \begin{aligned} 2\nabla_X(F)(Y, Z) &- 2\nabla_{JX}(F)(JY, Z) \\ &= \langle S(X, JY), Z\rangle - \langle S(X, Z)\, JY\rangle - \langle S(JY, Z), X\rangle. \end{aligned}$$

Proof. The proof of (2.6) follows from the fact that

$$\nabla_X(Y) - \nabla_Y(X) = [X, Y];$$

(2.7), (2.8) and (2.9) are consequences of (2.6) and the formula

$$(2.10)\qquad \nabla_X(F)(JY, Z) = \nabla_X(F)(Y, JZ).$$

We shall call an almost Hermitian manifold *Kählerian* if $\nabla_X(J) = 0$ for all $X \in \mathfrak{X}(M)$, *almost Kählerian* if $dF = 0$, *nearly Kählerian* if

$$\nabla_X(J)(Y) + \nabla_Y(J)(X) = 0$$

for all $X, Y \in \mathfrak{X}(M)$, *quasi-Kählerian if*

$$\nabla_X(J)(Y) + \nabla_{JX}(J)(JY) = 0$$

for all $X, Y \in \mathfrak{X}(M)$, *semi-Kählerian* if $\delta F = 0$, and *Hermitian* if $S = 0$. Kotō [12] uses the terms H-space for almost Kählerian, K-space for nearly Kählerian, $*0$-space for quasi-Kählerian, and almost semi-Kählerian for semi-Kählerian. An almost complex manifold is complex if and only if $S = 0$.

As a consequence of Theorem 2.1 we get the following corollary, which gives a useful alternate characterization of Hermitian manifolds.

COROLLARY 2.2. $\nabla_X(F)(Y, Z) = \nabla_{JX}(F)(JY, Z)$ *for all* $X, Y, Z \in \mathfrak{X}(M)$ *if and only if M is Hermitian.* $\nabla_X(F)(Y, Z) = -\nabla_{JX}(F)(JY, Z)$ *if and only if M is quasi-Kählerian.*

Let $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$, $\mathcal{QK}$, $\mathcal{SK}$, and $\mathcal{H}$ denote the classes of Kähler, almost Kähler, nearly Kähler, quasi-Kähler, semi-Kähler, and Hermitian manifolds respectively.

THEOREM 2.3. *We have*

$$\mathcal{K}\ \begin{array}{c} \subseteq \mathcal{AK} \subseteq \\ \\ \subseteq \mathcal{NK} \subseteq \end{array}\ \mathcal{QK} \subseteq \mathcal{SK} \quad \text{and} \quad \mathcal{K} \subseteq \mathcal{H}.$$

Furthermore $\mathcal{K} = \mathcal{H} \cap \mathcal{QK} = \mathcal{AK} \cap \mathcal{NK}$.

Proof. That $\mathcal{K} \subseteq \mathcal{AK}$ follows from (2.3) and (2.4), $\mathcal{AK} \subseteq \mathcal{QK}$ from (2.3) and (2.8), $\mathcal{QK} \subseteq \mathcal{SK}$ from (2.3) and (2.5), and $\mathcal{K} \subseteq \mathcal{H}$ from (2.6). It is obvious that $\mathcal{K} \subseteq \mathcal{NK}$, and $\mathcal{NK} \subseteq \mathcal{QK}$ is a consequence of (2.10). Furthermore $\mathcal{K} \subseteq \mathcal{H} \cap \mathcal{QK}$ is obvious and the reverse inclusion follows from (2.9). Finally if $M \in \mathcal{NK}$ we have $dF(X, Y, Z) = 3\nabla_X(F)(Y, Z)$, for $X, Y, Z \in \mathfrak{X}(M)$, and hence $\mathcal{NK} \cap \mathcal{AK} = \mathcal{K}$.

We remark that quasi-Kählerian manifolds have the property that any almost Hermitian submanifold is a minimal variety [8]. Theorem 2.3 is proved by Kotō [12] in a different form.

3. Conformal diffeomorphisms of almost Hermitian manifolds

Let $(M, \langle\ ,\ \rangle)$ and $(M^0, \langle\ ,\ \rangle^0)$ be Riemannian manifolds and $\varphi : M \to M^0$ a diffeomorphism. If $X \in \mathfrak{X}(M)$ let $X^0 \in \mathfrak{X}(M^0)$ be the vector field on M^0 corresponding to X induced by φ. Then φ is called a *conformal diffeomorphism* provided there exists $\sigma \in \mathfrak{F}(M)$ such that

$$\langle X^0, Y^0\rangle^0 \circ \varphi = e^{2\sigma}\langle X, Y\rangle \tag{3.1}$$

for all $X, Y \in \mathfrak{X}(M)$. For $f \in \mathfrak{F}(M)$ define $\operatorname{grad} f \in \mathfrak{X}(M)$ by

$$\langle \operatorname{grad} f, X\rangle = X(f)$$

for all $X \in \mathfrak{X}(M)$. Then we have

LEMMA 3.1. *If $\varphi : M \to M^0$ is a conformal diffeomorphism with*

$$\langle X^0, Y^0\rangle^0 \circ \varphi = e^{2\sigma}\langle X, Y\rangle,$$

then if ∇_X and $\nabla^0_{X^0}$ are the Riemannian connections of M and M^0 respectively

$$\nabla^0_{X^0}(Y^0) = \{\nabla_X(Y) + X(\sigma)Y + Y(\sigma)X - \langle X, Y\rangle \operatorname{grad}\sigma\}^0. \tag{3.2}$$

Proof. This follows from (3.1) and the formula

$$2\langle\nabla_X(Y), Z\rangle = X\langle Y, Z\rangle - \langle X, [Y, Z]\rangle + Y\langle X, Z\rangle$$
$$- \langle Y, [X, Z]\rangle - Z\langle X, Y\rangle + \langle Z, [X, Y]\rangle.$$

Next we suppose that $\varphi : M \to M^0$ in addition to being a conformal diffeomorphism is also almost complex; that is, we assume M^0 has an almost complex structure $J^0 : \mathfrak{X}(M^0) \to \mathfrak{X}(M^0)$ which satisfies $J^0X^0 = (JX)^0$; then M^0 is almost Hermitian. Let F^0 and S^0 be the Kähler form and the Nijenhuis tensor corresponding to J^0, δ^0 the coderivative of M^0 determined by $\langle\ ,\ \rangle^0$. Also let φ^* be the map induced by φ which takes differential forms on M^0 back to differential forms on M.

PROPOSITION 3.2. *We have the following formulas:*

$$F^0(X^0, Y^0) \circ \varphi = e^{2\sigma}F(X, Y), \tag{3.3}$$

$$\varphi^*(F^0) = e^{2\sigma}F, \tag{3.4}$$

$$\varphi^*(dF^0) = e^{2\sigma}\{2\, d\sigma \wedge F + dF\}, \tag{3.5}$$

$$\nabla^0_{X^0}(J^0)(Y^0) = \{\nabla_X(J)(Y) + JY(\sigma)X - Y(\sigma)JX + \langle JX, Y\rangle \operatorname{grad}\sigma + \langle X, Y\rangle J \operatorname{grad}\sigma\}^0, \tag{3.6}$$

$$\begin{aligned}\nabla^0_{X^0}(F^0)(Y^0, Z^0)\circ\varphi = e^{2\sigma}\{&\nabla_X(F)(Y, Z) + JY(\sigma)\langle X, Z\rangle \\ &- Y(\sigma)F(X, Z) + F(X, Y)Z(\sigma) \\ &- \langle X, Y\rangle JZ(\sigma)\},\end{aligned} \tag{3.7}$$

$$\delta^0F^0(X^0)\circ\varphi = \delta F(X) + (n-2)JX(\sigma) \qquad (n = \dim M = 2m), \tag{3.8}$$

$$S^0(X^0, Y^0) = S(X, Y)^0, \qquad \textit{for}\quad X, Y, Z \in \mathfrak{X}(M). \tag{3.9}$$

Proof. (3.3), (3.4), (3.5), and (3.6) are elementary consequences of (3.1) and (3.2), while (3.9) follows from (2.1). For (3.8) we first observe that if

$$\{E_1, \cdots, E_m, JE_1, \cdots, JE_m\}$$

is a frame field on an open subset of M, then

$$\{(e^{-\sigma}E_1)^0, \cdots, (e^{-\sigma}E_m)^0, (e^{-\sigma}JE_1)^0, \cdots, (e^{-\sigma}JE_m)^0\}$$

is a frame field on an open subset of M^0. Hence by (3.7)

$$\begin{aligned}{}^0F^0(X^0)\circ\varphi &= -e^{-2\sigma}\textstyle\sum_{i=1}^m \{\nabla^0_{E_i^0}(F^0)(E_i^0, X^0) + \nabla^0_{J^0E_i^0}(F^0)(J^0E_i^0, X^0)\}\circ\varphi\,\rangle \\ &= -\textstyle\sum_{i=1}^m\{\nabla_{E_i}(F)(E_i, X) + JE_i(\sigma)\langle E_i, X\rangle - E_i(\sigma)\langle JE_i, X\rangle \\ &\quad - JX(\sigma) + \nabla_{JE_i}(F)(JE_i, X) - E_i(\sigma)\langle JE_i, X\rangle \\ &\quad + JE_i(\sigma)\langle E_i, X\rangle - JX(\sigma)\} \\ &= \delta F(X) + (n-2)JX(\sigma).\end{aligned}$$

THEOREM 3.3. *Let $\varphi : M \to M^0$ be a conformal diffeomorphism between almost Hermitian manifolds. If $M \in \mathcal{H}$ then $M^0 \in \mathcal{H}$. On the other hand suppose* $\dim M \geq 4$ *and φ is not homothetic (i.e., σ is non-constant). Then if M is in one of the classes $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$, $\mathcal{QK}$, or $\mathcal{SK}$, M^0 is never in any of these classes.*

Proof. The first statement follows from (3.9). Next suppose M is in one of the classes $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$, $\mathcal{QK}$, or $\mathcal{SK}$; then by Theorem 2.3, $M \in \mathcal{SK}$. Consequently by (3.8) for all $X^0 \in \mathfrak{X}(M^0)$ we have

$$\delta^0F^0(X^0)\circ\varphi = (n-2)JX(\sigma).$$

Hence for $n = \dim M > 2$, $\delta^0F^0 \not\equiv 0$. Hence $M^0 \notin \mathcal{SK}$ and so M^0 is not in any of the classes $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$, or $\mathcal{QK}$ either.

4. Vector cross products on seven-dimensional manifolds

Let $\bar{M}$ be a seven-dimensional Riemannian manifold. We shall assume the existence on $\bar{M}$ of a *vector cross product* which is a (1, 2) tensor

$$P : \mathfrak{X}(\bar{M}) \times \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M})$$

(we also write $P(A, B) = A \times B$) having the following properties: for $A, B, C \in \mathfrak{X}(\bar{M})$

$$A \times B = -B \times A, \tag{4.1}$$

$$\langle A \times B, C\rangle = \langle A, B \times C\rangle, \tag{4.2}$$

$$(A \times B) \times C + A \times (B \times C) = 2\langle A, C\rangle B - \langle B, C\rangle A - \langle A, B\rangle C. \tag{4.3}$$

Then $\lambda : \mathfrak{X}(\bar{M})^3 \to \mathfrak{F}(\bar{M})$, called the *scalar triple product*, is defined by

$$\lambda(A, B, C) = \langle A \times B, C\rangle;$$

by (4.1) and (4.2) it is a 3-form. Similarly a *vector triple product* is defined by

$$[A\ B\ C] = (A \times B) \times C - \langle A, C\rangle B + \langle B, C\rangle A.$$

For example, any parallelizable manifold has such a tensor. Let

$$\{E_0, \cdots, E_6\}$$

be a frame field defined globally on $\bar{M}$, and consider $\{E_0, \cdots, E_6\}$ as a basis for a seven-dimensional vector space. We can add a vector I to make $\{I, E_0, \cdots, E_6\}$ a basis for the Cayley numbers with I the identity. Then the vector cross product $\times$ is the projection of the Cayley product onto the space spanned by $\{E_0, \cdots, E_6\}$. In fact for A, B in this space we may define $\times$ by

$$AB = -\langle A, B\rangle I + A \times B \tag{4.4}$$

where AB denotes the Cayley product. Explicitly, $\times$ is given by (4.1) and cyclic permutation of the form

$$E_j \times E_{j+1} = E_{j+3} \qquad (j \in Z_1);$$

each of these seven equations is to be permuted cyclically to yield fourteen other equations. We extend $\times$ to all of $\mathfrak{X}(\bar{M})$ by $\mathfrak{F}(\bar{M})$-linearity. Then (4.1), (4.2), and (4.3) follow from (4.4) and the properties of Cayley multiplication [7].

Roughly speaking, P and λ correspond to J and F, except that the former have one more degree of contravariancy; we shall exploit this analogy in Section 5.

PROPOSITION 4.1. *Let $\bar{M}$ be a seven-dimensional manifold with Riemannian connection $\bar{\nabla}$, vector cross product denoted by P or $\times$, and scalar triple product λ. We have the following formulas:*

$$\bar{\nabla}_A(P)(B, C) = \bar{\nabla}_A(B \times C) - \bar{\nabla}_A(B) \times C - B \times \bar{\nabla}_A(C), \tag{4.5}$$

$$\bar{\nabla}_A(P)(B, C) = -\bar{\nabla}_A(P)(C, B), \tag{4.6}$$

$$\langle \bar{\nabla}_A(P)(B, C), D\rangle = \langle B, \bar{\nabla}_A(P)(C, D)\rangle, \tag{4.7}$$

$$\begin{aligned} \bar{\nabla}_A(P)(B \times C, D) &+ \bar{\nabla}_A(P)(B, C \times D) \\ &= -B \times \bar{\nabla}_A(P)(C, D) - \bar{\nabla}_A(P)(B, C) \times D, \end{aligned} \tag{4.8}$$

$$(4.9) \qquad \bar{\nabla}_A(\lambda)(B, C, D) = \langle \bar{\nabla}_A(P)(B, C), D \rangle,$$

$$(4.10) \qquad d\lambda(A, B, C, D) = \langle \mathfrak{S}\bar{\nabla}_A(P)(B, C), D \rangle - \bar{\nabla}_D(\lambda)(A, B, C),$$

$$(4.11) \qquad \begin{aligned} \bar{\delta}\lambda(A, B) &= -\textstyle\sum_{i=0}^{6} \langle \bar{\nabla}_{E_i}(P)(A, B), E_i \rangle \\ &= -\textstyle\sum_{i=0}^{6} \bar{\nabla}_{E_i}(\lambda)(E_i, A, B), \end{aligned}$$

for $A, B, C, D, \in \mathfrak{X}(\bar{M})$ *and* $\{E_0, \cdots, E_6\}$ *any frame field. Here* $\bar{\delta}$ *denotes the coderivative of* $\bar{M}$.

Proof. These formulas are consequences of (4.1), (4.2) and (4.3).

Remarks. For R^7 we may choose $\{E_0, \cdots, E_6\}$ parallel. Then P is parallel, i.e., $\bar{\nabla}_A(P) = 0$, and so

$$(4.12) \qquad \bar{\nabla}_A(B \times C) = \bar{\nabla}_A(B) \times C + B \times \bar{\nabla}_A(C)$$

for all $A, B, C \in \mathfrak{X}(R^7)$.

For three-dimensional manifolds a vector cross product may be defined in a similar fashion via the quaternions. In this case we have (4.1), (4.2), and

$$(A \times B) \times C = -\langle C, A \rangle B + \langle B, A \rangle C$$

which implies (4.3) and that $[A\ B\ C] = 0$. A three-dimensional Riemannian manifold has a vector cross product if and only if it is orientable, in which case we may assume $\langle A \times B, C \rangle = \omega(A, B, C)$ where ω is the volume element. From this it follows that $B \times C = *(B \wedge C)$ and that

$$\nabla_A(B \times C) = \nabla_A(B) \times C + B \times \nabla_A(C)$$

so that the vector cross product is always parallel. Dimensions three and seven are essentially the only ones where a vector cross product can be defined (cf. [5], [16]).

5. Certain six-dimensional almost Hermitian manifolds

Let M be a Riemannian manifold imbedded in another Riemannian manifold $\bar{M}$. Let

$$\bar{\mathfrak{X}}(M) = \{X | M \mid X \in \mathfrak{X}(\bar{M})\};$$

then we may write $\bar{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^{\perp}$ where $\mathfrak{X}(M)^{\perp}$ is the collection of vector fields normal to M. Let

$$P_M : \bar{\mathfrak{X}}(M) \to \mathfrak{X}(M)$$

be the natural projection. The *configuration tensor* [8] is an $\mathfrak{F}(M)$-linear map

$$T : \mathfrak{X}(M) \times \bar{\mathfrak{X}}(M) \to \bar{\mathfrak{X}}(M)$$

defined by

$$T_X(Y) = \bar{\nabla}_X(Y) - \nabla_X(Y) \quad \text{for } X, Y \in \mathfrak{X}(M)$$

and

$$T_X(Z) = P_M\bar{\nabla}_X(Z) \quad \text{for } X \in \mathfrak{X}(M), Z \in \mathfrak{X}(M)^\perp.$$

Then $T_X(\mathfrak{X}(M)) \subseteq \mathfrak{X}(M)^\perp$, $T_X(\mathfrak{X}(M)^\perp) \subseteq X(M)$ for $X \in \mathfrak{X}(M)$, $T_X(Y) = T_Y(X)$ for X, $Y \in \mathfrak{X}(M)$, and $\langle T_X(Z), W\rangle = -\langle T_X(W), Z\rangle$ for W, $X \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(M)^\perp$. The configuration tensor is equivalent to the classical second fundamental form, which in our terminology would be the linear transformation $X \to T_X(Z)$ for $X \in \mathfrak{X}(M), Z \in \mathfrak{X}(M)^\perp$. The *mean curvature vector* H of M in $\bar{M}$ is defined by $H = \sum_{i=1}^n T_{E_i}(E_i)$ where $n = \dim M$ and $\{E_1, \cdots, E_n\}$ is a frame field on an open subset of M; this definition is independent of the choice of $\{E_1, \cdots, E_n\}$. M is called a *minimal variety* if $H \equiv 0$. Similarly if $T \subset 0$, M is called *totally geodesic*, and if $T_X(X) = T_Y(Y)$ for all X, Y, $\in \mathfrak{X}(M)$ with $\| X \| = \| Y \|$, M is called *totally umbilic* (cf. [8], [14]).

We now consider a special case. Suppose $\bar{M}$ is a seven-dimensional orientable Riemannian manifold on which there exists a vector cross product denoted by either $P(A, B)$ or $A \times B$, and let M be a (six-dimensional) orientable hypersurface of $\bar{M}$. Then we may choose a unit vector field N globally defined on M, which is a basis for $\mathfrak{X}(M)^\perp$. Since N has constant length, it is easy to see that $T_X(N) = \bar{\nabla}_X(N)$. Define $J : \mathfrak{X}(M) \to \bar{\mathfrak{X}}(M)$ by $JA = N \times A$. On account of (4.1) and (4.2) we have $\langle N, JA\rangle = 0$ so actually $J : \mathfrak{X}(M) \to \mathfrak{X}(M)$. Furthermore from (4.3) it follows that $J^2A = -A$ and from (4.2) and (4.3) that $\langle JA, JB\rangle = \langle A, B\rangle$ for A, $B \in \mathfrak{X}(M)$. From (4.3) we also have the useful formula

$$JA \times B = -J(A \times B) - \langle A, B\rangle N.$$

Let F be the Kähler form of M; by (4.2) we have $F(A, B) = \langle N, A \times B\rangle$ for A, $B \in \mathfrak{X}(M)$. We may choose canonical frames on M so that $E_0 = N$; such frames then have the form $\{N, E_1, JE_1, E_2, JE_2, E_4, JE_4\}$.

THEOREM 5.1. *We have the following formulas for A, B, $C \in \mathfrak{X}(M)$:*

$$(5.1)\quad \begin{aligned} \nabla_A(F)(B, C) &= \langle \bar{\nabla}_A(P)(B, C) - T_A(B \times C), N\rangle \\ &= \bar{\nabla}_A(\lambda)(B, C, N) - \langle T_A(B \times C), N\rangle, \end{aligned}$$

$$(5.2)\quad \begin{aligned} &\nabla_A(F)(B, C) + \nabla_B(F)(A, C) \\ &\quad = \bar{\nabla}_A(\lambda)(B, C, N) + \bar{\nabla}_B(\lambda)(A, C, N) \\ &\quad\quad - \langle T_A(B \times C) + T_B(A \times C), N\rangle, \end{aligned}$$

$$(5.3)\quad \begin{aligned} &\nabla_A(F)(B, C) + \nabla_{JA}(F)(JB, C) \\ &\quad = \bar{\nabla}_A(\lambda)(B, C, N) + \bar{\nabla}_{JA}(\lambda)(JB, C, N) \\ &\quad\quad - \langle (T_A - T_{JA}J)(B \times C), N\rangle, \end{aligned}$$

$$(5.4)\quad \nabla_A(F)(B, C) - \nabla_{JA}(F)(JB, C)$$
$$= \bar{\nabla}_A(\lambda)(B, C, N) - \bar{\nabla}_{JA}(\lambda)(JB, C, N)$$
$$- \langle (T_A + T_{JA}J)(B \times C), N \rangle,$$

$$(5.5)\quad \langle H, N \rangle = \mathfrak{S}_{ijk}\,\{\bar{\nabla}_{E_i}(\lambda)(E_i, E_k, N) - \bar{\nabla}_{JE_i}(\lambda)(JE_i, E_k, N)$$
$$- \nabla_{E_i}(F)(E_j, E_k) + \nabla_{JE_i}(F)(JE_i, E_k)\}$$
$$= \tfrac{1}{2}\mathfrak{S}\{dF(E_i, JE_j, JE_k) + d\lambda(N, E_i, JE_j, JE_k)\}$$
$$- \tfrac{1}{2}\{dF(E_i, E_j, E_k) + d\lambda(N, E_i, E_j, E_k)\},$$

$$(5.6)\quad \delta F(A) = \bar{\delta}\lambda(A, N),$$

for $A, B, C \in \mathfrak{X}(M)$. Here $E_i \times E_j = E_k$.

Proof. For (5.1) we have

$$\nabla_A(F)(B, C) = \langle \bar{\nabla}_A(N \times B) - N \times \bar{\nabla}_A(B), C \rangle$$
$$= \langle \bar{\nabla}_A(P)(N, B) + \bar{\nabla}_A(N) \times B, C \rangle$$
$$= \langle \bar{\nabla}_A(P)(B, C), N \rangle + \langle T_A(N) \times B, C \rangle$$
$$= \langle \bar{\nabla}_A(B)(B, C), N \rangle - \langle T_A(B \times C), N \rangle.$$

The proofs of (5.2), (5.3), and (5.4) are obvious consequences of (5.1). For (5.5) we have

$$\langle H, N \rangle = \mathfrak{S}_{ijk}\langle T_{E_i}(E_i) + T_{JE_i}(JE_i), N \rangle$$
$$= \mathfrak{S}\{\langle T_{E_i}(E_j \times E_k), N \rangle - \langle T_{JE_i}(JE_j \times E_k), N \rangle\}$$
$$= \mathfrak{S}\{\bar{\nabla}_{E_i}(\lambda)(E_j, E_k, N) - \bar{\nabla}_{JE_i}(\lambda)(JE_j, E_k, N)$$
$$- \nabla_{E_i}(F)(E_j, E_k) + \nabla_{JE_i}(F)(JE_j, E_k)\}$$

by (5.1). For the second part of (5.5) we compute the Lie derivative $L_N(\lambda)$. We have

$$L_N(\lambda)(E_i, E_j, E_k) = N\lambda(E_i, E_j, E_k) - \lambda([N, E_i], E_j, E_k)$$
$$- \lambda(E_i, [N, E_j], E_k) - \lambda(E_i, E_j, [N, E_k])$$
$$= -\langle \bar{\nabla}_N(E_i) - \bar{\nabla}_{E_i}(N), E_i \rangle - \langle \bar{\nabla}_N(E_j) - \bar{\nabla}_{E_j}(N), E_j \rangle$$
$$- \langle E_k, \bar{\nabla}_N(E_k) - \bar{\nabla}_{E_k}(N) \rangle$$
$$= -\langle T_{E_i}(E_i) + T_{E_j}(E_j) + T_{E_k}(E_k), N \rangle.$$

Similarly

$$L_N(\lambda)(E_i, JE_j, JE_k) = \langle T_{E_i}(E_i) + T_{JE_j}(JE_j) + T_{JE_k}(JE_k), N \rangle.$$

Hence we have

$$(5.7)\quad \mathfrak{S}L_N(\lambda)(E_i, JE_j, JE_k) - L_N(\lambda)(E_i, E_j, E_k) = 2\sum \langle T_{E_i}(E_i) + T_{JE_i}(JE_i), N\rangle = 2\langle H, N\rangle.$$

Let ι_N denote the interior product operator; we have [10, p. 35]

$$(5.8)\qquad L_N\lambda = d\iota_N\lambda + \iota_N d\lambda = dF + \iota_N d\lambda.$$

The second part of (5.5) now follows from (5.7) and (5.8).

For (5.6) we may assume without loss of generality that $A = E_i$. Then

$$\begin{aligned}\delta F(E_i) &= -\nabla_{E_j}(F)(E_j, E_i) - \nabla_{JE_j}(F)(JE_j, E_i) - \nabla_{E_k}(F)(E_k, E_i)\\ &\quad - \nabla_{JE_k}(F)(JE_k, E_i)\\ &= -\bar{\nabla}_{E_j}(\lambda)(E_j, E_i, N) - \bar{\nabla}_{JE_k}(\lambda)(JE_j, E_i, N)\\ &\quad - \bar{\nabla}_{E_k}(\lambda)(E_k, E_i, N) - \bar{\nabla}_{JE_k}(\lambda)(JE_k, E_i, N)\\ &\quad - \langle (T_{E_j} - T_{JE_j}J)(E_j \times E_i), N\rangle - \langle (T_{E_k} - T_{JE_k}J)(E_k \times E_i), N\rangle\\ &= \bar{\delta}\lambda(E_i, N) + \langle T_{E_j}(E_k) - T_{JE_j}(JE_k) - T_{E_k}(E_j) + T_{JE_k}(JE_j), N\rangle\\ &= \bar{\delta}\lambda(E_i, N).\end{aligned}$$

COROLLARY 5.2. *If M is totally geodesic in $\bar{M}$, (5.1) reduces to*

$$(5.9)\qquad \nabla_A(F)(B, C) = \bar{\nabla}_A(\lambda)(B, C, N).$$

M is totally umbilic in $\bar{M}$ if and only if

$$(5.10)\qquad \nabla_A(F)(B, C) + \nabla_B(F)(A, C) = \bar{\nabla}_A(\lambda)(B, C, N) + \bar{\nabla}_B(\lambda)(A, C, N).$$

If M is either Hermitian or almost Kählerian, (5.5) reduces to

$$(5.11)\qquad \begin{aligned}\langle H, N\rangle &= \mathfrak{S}\{\bar{\nabla}_{E_i}(\lambda)(E_j, E_k, N) - \bar{\nabla}_{JE_i}(\lambda)(JE_j, E_k, N)\}\\ &= \tfrac{1}{2}\mathfrak{S}\, d\lambda(N, E_i, JE_j, JE_k) - \tfrac{1}{2}\, d\lambda(N, E_i, E_j, E_k).\end{aligned}$$

Proof. The proofs of (5.9) and (5.11) are obvious, while (5.10) follows from the fact that there is a function $\kappa : M \to R$ such that

$$T_A(B) = \kappa\langle A, B\rangle N \qquad \text{for all } A, B \in \mathfrak{X}(M).$$

In the next section we investigate in detail orientable hypersurfaces of R^7. Other parallelizable seven-dimensional manifolds are, for example, seven-dimensional hyperbolic space, the seven-dimensional sphere, and $U(4)/U(3)$, which with an appropriate invariant metric is diffeomorphic but not isometric to the seven-dimensional sphere. For these manifolds the covariant derivative of λ does not vanish.

6. Orientable six-dimensional hypersurfaces of R^7

As we have noted, the vector cross product on R^7 is parallel and so Theorem 5.1 simplifies considerably in this case.

THEOREM 6.1. *Let M be an orientable hypersurface of R^7 with its almost complex structure induced from the Cayley numbers. Then*

(6.1) *$M \in \mathcal{K}$ if and only if M is totally geodesic, i.e., M is a hyperplane of R^7.*

(6.2) *$M \in \mathcal{NK}$ if and only if M is locally isometric to a sphere.*

(6.3) *$M \in \mathcal{QK}$ if and only if $T_A = T_{JA}J$ for all $A \in \mathfrak{X}(M)$.*

(6.4) *$M \in \mathcal{H}$ if and only if $T_A = -T_{JA}J$ for all $A \in \mathfrak{X}(M)$.*

(6.5) *If either $M \in \mathcal{H}$ or $M \in \mathcal{AK}$, then M is a minimal variety of R^7.*

(6.6) *$M \in \mathcal{AK}$ if and only if $T_A = T_{JA}J$ for all $A \in \mathfrak{X}(M)$ and M is a minimal variety of R^7.*

(6.7) $$M \in \mathcal{SK}.$$

Proof. (6.1), (6.3), (6.4), (6.5), and (6.7) follow from Theorem 5.1, and (6.2) follows from Corollary 5.2, since $S^6(r, p)$ is totally umbilic. For (6.6) we observe that if $M \in \mathcal{AK}$, then $M \in \mathcal{QK}$ and so $T_A = T_{JA}J$; this with (6.5) proves the necessity. For the sufficiency we first note that $T_A = T_{JA}J$ for all $A \in \mathfrak{X}(M)$ implies $M \in \mathcal{QK}$. Now for any quasi-Kähler manifold M we have

(6.8) $$dF(JA, B, C) = dF(A, JB, C) = dF(A, B, JC)$$

for all A, P, $C \in \mathfrak{X}(M)$. Hence if M is a minimal variety (5.5) reduces to

(6.9) $$dF(E_i, E_j, E_k) = 0.$$

Furthermore by direct calculation we have

$$dF(JF_i, E_j, E_k) = \langle -T_{E_i}(JE_i) + T_{E_j}(JE_j) + T_{E_k}(JE_k), N\rangle$$

and

$$dF(JE_i, JE_j, JE_k) = \langle T_{E_i}(JE_j) + T_{E_j}(JE_j) + T_{E_k}(JE_k), N\rangle;$$

therefore by (6.8)

(6.10) $$dF(JE_i, E_j, E_k) = dF(JE_i, JE_j, JE_k) = 0.$$

By using (6.8) and linearity any expression $dF(A, B, C)$ can be reduced to either (6.9) or (6.10). Hence $dF = 0$ and $M \in \mathcal{AK}$.

Parts of this theorem are given in a different form by Calabi [2] and Fukami and Ishihara [6].

COROLLARY 6.2. *Let M be an orientable hypersurface of R^7 with its almost* *complex structure induced from the Cayley numbers. Then if either* $M \in \mathcal{AK}$ *or* $M \in \mathcal{K}$, M *cannot be compact.*

Proof. M must be a minimal variety, but there are no compact minimal varieties of R^7 (cf. [13], [15]).

No examples of six-dimensional hypersurfaces of R^7 which are almost Kählerian but not Kählerian are known.

We next investigate some special cases of Theorem 6.1.

THEOREM 6.3. *Let* R^3 *be a three-dimensional linear subspace of* R^7 *closed under the vector cross product, and let* M_1 *be a surface in* R^3. *Define* $M = M_1 \times R^4$, *where* R^4 *is the orthogonal complement of* R^3. *Then* [2] *we may assume that* $\{E_0, \cdots, E_6\}$ *is a frame field on* M *such that* $E_0 = N$ *is normal to* M *and* E_1 *and* E_2 *are tangent to* M_1.

(6.11) $M \in \mathcal{QK}$, $M \notin \mathcal{AK} \cup \mathcal{NK}$ *if and only if* M_1 *is locally isometric to a sphere* $S^2(r, p)$.

(6.12) $M \in \mathcal{H}$, $M \notin \mathcal{K}$ *if and only if* M_1 *is a nonplanar minimal surface of* R^3.

(6.13) $M \in \mathcal{SK}$, $M \notin \mathcal{QK} \cup \mathcal{H}$ *if and only if* M_1 *is neither a minimal surface nor locally isometric to a sphere.*

Proof. If $A \in \mathfrak{X}(M)$ let A_1 denote the component of A tangent to M_1, and let T_1 be the configuration tensor of M_1 in R_3. Then

$$T_A(B) = T_{A_1}(B_1) = T_{1A_1}(B_1)$$

for all $A, B \in \mathfrak{X}(M)$. Moreover M_1 is an almost Hermitian submanifold of M; that is, $JA_1 = (JA)_1$. If $M \in \mathcal{QK}$ then

$$T_A(B) = T_{JA}(JB) = T_{1JA_1}(JB_1)$$

for all $A, B \in \mathfrak{X}(M)$. In particular $T_{E_2}(E_2) = T_{JE_1}(JE_1) = T_{E_1}(E_1)$; this implies that M_1 is totally umbilic, and therefore locally isometric to a sphere. Conversely, if M_1 is locally isometric to a sphere it is totally umbilic and so

$$T_A(A) = T_{1A_1}(A_1) = T_{1JA_1}(JA_1) = T_{JA}(JA).$$

This implies $M \in \mathcal{QK}$.

Furthermore $M \notin \mathcal{AK}$ since if H denotes the mean curvature vector of M we have $H = T_{E_1}(E_1) + T_{E_2}(E_2) = 2T_{E_1}(E_1) \neq 0$. Also $M \notin \mathcal{NK}$, since

$$T_{E_1}(E_5 \times E_6) + T_{E_5}(E_1 \times E_6) = T_{E_1}(E_1) - T_{E_5}(E_5) = T_{E_1}(E_1) \neq 0.$$

This proves (6.11); (6.12) and (6.13) can be proved in a similar fashion.

7. The strictness of the inclusions

The following theorem together with Theorem 2.3 determines all possible inclusion relations among the various classes of almost Hermitian manifolds defined in §2, and shows that they are all strict.

THEOREM 7.1. *We have*

$$\mathcal{K} \begin{array}{c} < \mathcal{AK} < \\ \\ < \mathcal{NK} < \end{array} \mathcal{AK} \cup \mathcal{NK} < \mathcal{QK} < \mathcal{QK} \cup (\mathcal{SK} \cap \mathcal{H}) < \mathcal{SK}$$

and

$$\mathcal{K} < \mathcal{SK} \cap \mathcal{H} \begin{array}{c} < \mathcal{SK} < \\ \\ < \mathcal{H} < \end{array} \mathcal{SK} \cup \mathcal{H} < \mathcal{AH}.$$

Here $\mathcal{AH}$ stands for the class of all almost Hermitian manifolds.

Proof. $\mathcal{K} < \mathcal{NK}$. Let S^6 be imbedded in R^7 with its almost complex structure determined from the Cayley numbers. Then $S^6 \in \mathcal{NK}$ by Corollary 5.2, but $S^6 \notin \mathcal{K}$.

$\mathcal{K} < \mathcal{AK}$. Dombrowski [4] has shown that the tangent bundle of any Riemannian manifold whose curvature does not vanish has an almost complex structure which is almost Kählerian but not Kählerian.

$\mathcal{AK} < \mathcal{AK} \cup \mathcal{NK}$. Since $\mathcal{K} = \mathcal{AK} \cap \mathcal{NK}$ we have

$$\mathcal{AK} \cup \mathcal{NK} \backslash \mathcal{AK} = \mathcal{NK} \backslash \mathcal{K} \neq \emptyset.$$

$\mathcal{NK} < \mathcal{AK} \cup \mathcal{NK}$. The proof is similar to that of $\mathcal{AK} < \mathcal{AK} \cup \mathcal{NK}$.

$\mathcal{AK} \cup \mathcal{NK} < \mathcal{QK}$. By (6.11) $S^2 \times R^4 \in \mathcal{QK}$ but $S^2 \times R^4 \notin \mathcal{AK} \cup \mathcal{NK}$ where the almost complex structure is induced from the Cayley numbers.

$\mathcal{QK} \cup (\mathcal{SK} \cap \mathcal{H}) < \mathcal{SK}$. Let M_1 be a surface in R^3 which is neither minimal nor locally isometric to a sphere, and let $M = M_1 \times R^4 \subseteq R^7$. With the almost complex structure induced from the Cayley numbers $M \in \mathcal{SK}$ but $M \notin \mathcal{QK} \cup \mathcal{H}$.

$\mathcal{QK} < \mathcal{QK} \cup (\mathcal{SK} \cap \mathcal{H})$. From the relation $\mathcal{QK} \cap \mathcal{H} = \mathcal{K}$ we have

$$\mathcal{QK} \cup (\mathcal{SK} \cap \mathcal{H}) \backslash \mathcal{QK} = \mathcal{SK} \cap \mathcal{H} \backslash \mathcal{K} \neq \emptyset.$$

$\mathcal{K} < \mathcal{SK} \cap \mathcal{H}$. Let M_1 be a minimal surface in R^3. Then by (6.12), $M_1 \times R^4 \in \mathcal{SK} \cap \mathcal{H}$ but $M_1 \times R^4 \notin \mathcal{K}$, where the almost complex structure is induced from the Cayley numbers.

$\mathcal{SK} \cap \mathcal{H} < \mathcal{SK}$. This is a consequence of the strict inclusion

$$\mathcal{QK} \cup (\mathcal{SK} \cap \mathcal{H}) < \mathcal{SK}.$$

$\mathcal{SK} \cap \mathcal{H} < \mathcal{H}$. Let M be conformally flat, but not flat, and even-dimensional. For example, we may take M to be hyperbolic space, or a sphere minus a point. With the almost complex structure induced via the conformal diffeomorphism of M into Euclidean space $M \in \mathcal{H}$ but $M \notin \mathcal{SK}$ by Theorem 3.3.

$\mathcal{SK} < \mathcal{SK} \cup \mathcal{H}$. We have $\mathcal{SK} \cup \mathcal{H} \backslash \mathcal{SK} = \mathcal{H} \backslash \mathcal{SK} \cap \mathcal{H} \neq \emptyset$.

$\mathcal{H} < \mathcal{SK} \cup \mathcal{H}$. Similarly $\mathcal{SK} \cup \mathcal{H} \backslash \mathcal{H} = \mathcal{SK} \backslash \mathcal{SK} \cap \mathcal{H} \neq \emptyset$.

$\mathcal{SK} \cup \mathcal{H} < \mathcal{AH}$. Let M be conformally equivalent to a manifold $M^0 \in \mathcal{SK}$ for which $M^0 \notin \mathcal{H}$. For example, we may take $M^0 = S^6 \backslash \{p\}$ and $M = R^6$

with the stereographic projection as the conformal diffeomorphism. Then $M \in \mathcal{AH}$ but $M \notin \mathcal{SK} \cup \mathcal{H}$.

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