ON FINITE MODULES OVER A NOETHERIAN DOMAIN[1]

BY

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1. Introduction

An integral domain $\mathfrak{o}^*$ is an *affine ring* over a subdomain $\mathfrak{o}$ in case $\mathfrak{o}$ is a Noetherian domain and $\mathfrak{o}^*$ is finitely generated over $\mathfrak{o}$. The Noetherian domain $\mathfrak{o}$ is said to satisfy the *condition* (F) in case each affine ring $\mathfrak{o}^*$ over $\mathfrak{o}$ is such that the integral closure of $\mathfrak{o}^*$ in its quotient field is a finite $\mathfrak{o}^*$-module. $\mathfrak{o}$ is said to satisfy the *condition* (SF) in case each separably generated affine ring $\mathfrak{o}^*$ over $\mathfrak{o}$ is such that the integral closure of $\mathfrak{o}^*$ in its quotient field is a finite $\mathfrak{o}^*$-module. It is known that a pseudo-geometric integral domain (for example, a field or a complete local (Noetherian) domain) satisfies the condition (F) [1, p. 133], and a regular Noetherian domain satisfies the condition (SF) [3].

The terminology used in this note will be the same as that in [6, pp. 156–160 and 347–352]. Let $\mathfrak{o}$ be a Noetherian domain, let K be a finitely generated extension field of the quotient field of $\mathfrak{o}$, and let M be a finite (finitely generated) $\mathfrak{o}$-module *contained* in K. In [4] it is proven that if $\mathfrak{o}$ is a field, then the integral closure M' of M in K is a finite $\mathfrak{o}$-module, and in [5] it is proven that there exists a non-negative integer k such that $(M^{k+i})' = (M^k)'M^i$ for all $i \geq 0$ (where $M^0 = \mathfrak{o}$). It will be shown in this note that these two theorems can be generalized to the case where $\mathfrak{o}$ is a Noetherian domain which satisfies the condition (F) (or (SF)) and where K is a finitely generated (respectively a finite separable) extension field of the quotient field of $\mathfrak{o}$. It will also be shown that if $\mathfrak{o}$ is furthermore a local domain, then every finitely generated $\mathfrak{o}$-module M contained in K has a minimal reduction N (see Section 3 for the definition), and, without the assumption that $\mathfrak{o}$ is local, that if N is a reduction of M, then $N' = M'$.

The methods used in Section 2 are similar to those used in [4] and [5].

2. Finiteness of the integral closure of a finite $\mathfrak{o}$-module

Throughout this section the following notation will be used. $\mathfrak{o}$ is a Noetherian domain which satisfies the condition (F) (or (SF)), K is a finitely generated (respectively a finite separable) extension field of the quotient field E of $\mathfrak{o}$, and $M = (m_1, \cdots, m_r)$ is a finite $\mathfrak{o}$-module contained in K. Let t be an element which is transcendental over K. Set $M^* = tM$, and regard M^* as an $\mathfrak{o}$-module contained in the field $K(t)$. Let $M^{*\prime}$ be the integral closure of M^* in $K(t)$, and set $R^* = \mathfrak{o}[tm_1, \cdots, tm_r] = \sum_{q=0}^{\infty} M^{*q}$ (where $M^{*0} = \mathfrak{o}$). Let

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F be the quotient field of R^*, let m be a nonzero element in M, and set $F_0 = E(m_1/m, \cdots, m_r/m)$. Let F_0' be the algebraic closure of F_0 in K, and let $R^{*\prime}$ be the integral closure of R^* in $F_0'(t)$. The following remark is a statement of results proved in [6, pp. 351–352].

Remark 2.1. (1) R^* ($R^{*\prime}$) is a graded subdomain of $K[t]$, M^{*q} (respectively $R^{*\prime} \cap t^qF_0'$) being the set of homogeneous elements of degree q ($q \geq 0$). (2) $R^{*\prime}$ is the integral closure of R^* in $K(t)$. (3) $(M^q)' = (1/t)^q(M^{*q})' = (1/t)^q(R^{*\prime} \cap t^qF_0')$. (If A is an $\mathfrak{o}$-module contained in K, then the integral closure of A in K is equal to the integral closure of A in $K(t)$. Hence if A is an $\mathfrak{o}$-module contained in $K(t)$, then A' will consistently be used to denote the integral closure of A in $K(t)$.)

To prove that the integral closure of M in K is a finite $\mathfrak{o}$-module, it is sufficient by (3) of Remark 2.1 to prove that $R^{*\prime} \cap tF_0'$ is a finite $\mathfrak{o}$-module. Since $K(t)$ is finitely generated over E (and is a finite separable extension of E if K is), $F_0'(t)$ is a finite algebraic (respectively finite separable algebraic) extension field of F. Hence, since $\mathfrak{o}$ satisfies the condition (F) (respectively (SF)), $R^{*\prime}$ is a finite R^*-module.

THEOREM 2.1. *Let M be a finite $\mathfrak{o}$-module contained in K. Then the integral closure M' of M in K is a finite $\mathfrak{o}$-module.*

Proof. Since $R^{*\prime}$ is a finite R^*-module, there exist elements $x_1, \cdots, x_g$ in $R^{*\prime}$ such that $R^{*\prime} = x_1 R^* + \cdots + x_g R^*$. Since $R^{*\prime}$ is a graded subring of $F_0'(t)$ (Remark 2.1), $x_i = f_{i0} + \cdots + f_{ih_i}$, where $f_{ij} \in t^jF_0'$. Since R^* is the graded ring $\sum_{q=0}^{\infty} M^{*q}$, if $y \in R^{*\prime} \cap tF_0'$, then $y = \sum_{i=1}^{g} (m_{i1}^* f_{i0} + m_{i0}^* f_{i1})$, where $m_{ij}^* \in M^{*j}$. Therefore $tM' = R^{*\prime} \cap tF_0'$ is contained in the $\mathfrak{o}$-module generated by $(\mathfrak{o}, M^*)(f_{10}, f_{11}, f_{20}, f_{21}, \cdots, f_{g0}, f_{g1})$. Since M^* is a finite $\mathfrak{o}$-module, and since $\mathfrak{o}$ is Noetherian, tM' is a finite $\mathfrak{o}$-module, hence M' is a finite $\mathfrak{o}$-module, Q.E.D.

COROLLARY 2.1. *If M is a finite $\mathfrak{o}$-module contained in K, then the integral closure in K of M^k ($k \geq 0$) is a finite $\mathfrak{o}$-module.*

Proof. M^k is a finite $\mathfrak{o}$-module, Q.E.D.

Let P denote the prime ideal $tK[t] \cap R^{*\prime}$. Since $R^{*\prime}$ is a graded Noetherian subdomain of $K[t]$, and since an element x in K is in $(M^q)'$ if and only if xt^q is in $M^{*q\prime} \subseteq R^{*\prime}$ (Remark 2.1), $P = (r_1 t^{u_1}, \cdots, r_n t^{u_n})R^{*\prime}$, where $r_i \in (M^{u_i})'$. Therefore an element x in K is in $(M^q)'$ ($q \geq 1$) if and only if xt^q is in P, hence if and only if $xt^q = f_1 r_1 t^{u_1} + \cdots + f_n r_n t^{u_n}$, where $f_i \in R^{*\prime}$. It may clearly be assumed that $f_i = s_i t^{q-u_i}$ if $q \geq u_i$, and that $f_i = 0$ if $q < u_i$. Hence $x \in (M^q)'$ if and only if $x = s_1 r_1 + \cdots + s_n r_n$, where $r_i \in (M^{u_i})'$ and $s_i \in (M^{q-u_i})'$ ($s_i = 0$ if $q < u_i$). Let u be the maximum of the u_i.

LEMMA 2.1. *With the preceding notation if $q \geq ku$ ($k \geq 1$), then*

$$(M^q)' = (\sum (M^{q-k_1u_1-\cdots-k_nu_n})' (M^{u_1})'^{k_1} \cdots (M^{u_n})'^{k_n})$$

($=N_{q,k}$, say) *where the sum is over all non-negative integers* $k_1, \cdots, k_n$ *such that* $k_1 + \cdots + k_n = k$.

Proof. The case $k = 1$ is true by the preceding paragraph. Hence let k^* be greater than one and assume that the statement holds for values of $k < k^*$. Now $(M^q)' \supseteq N_{q,k^*}$, since $(AB)' \supseteq A'B'$ and $(A + B)' \supseteq A' + B'$ holds for all $\mathfrak{o}$-modules A and B [6, pp. 348–349]. Hence let $x \in (M^q)'$. Then

$$x = s_1 r_1 + \cdots + s_n r_n,$$

where $s_i \in (M^{q-u_i})'$. Since $q - u_i \geq q - u \geq (k^* - 1)u$, by induction each $s_i \in N_{q-u_i,k^*-1}$. Since $r_i \in (M^{u_i})'$, $x \in N_{q,k^*}$, Q.E.D.

COROLLARY 2.2 *If* $q > ku$, *then* $(M^{*q})'$ *is contained in the ideal* P^{k+1}.

Proof. If $xt^q \in (M^{*q})'$, then $xt^q \in P$. Therefore

$$xt^q = (s_1 r_1 + \cdots + s_n r_n)t^q,$$

where $s_i \in (M^{q-u_i})'$, hence $s_i t^{q-u_i} \in (M^{*q-u_i})'$. Since

$$q - u_i \geq q - u > (k - 1)u,$$

the conclusion follows by induction on k, Q.E.D.

LEMMA 2.2. *If* M *is a finite* $\mathfrak{o}$*-module contained in* K, *then there exists a positive integer* q *(depending on* M*) such that* $(M')^{q+i} = (M')^q M^i$, *for all* $i \geq 0$.

Proof. If $x \in M'$, then x satisfies an equation of the form

$$X^g + m_1 X^{g-1} + \cdots + m_g = 0,$$

where $m_i \in M^i$, hence

$$x^g \in M(\textstyle\sum_{i=1}^{g} M^{i-1}(M')^{g-i}) \subseteq M(M')^{g-1}.$$

Let $M' = (x_1, \cdots, x_k)$ (Theorem 2.1), let g_i be an integer such that $x_i^{g_i} \in M(M')^{g_i-1}$, and let g be the maximum of the g_i. Since $(M')^r$ is generated by the power products of degree r of the x_i, if $q = k(g - 1)$, then each power product of degree $q + 1$ is in $M(M')^q$. Hence $(M')^{q+1} = (M')^q M$, and the conclusion follows immediately, Q.E.D.

THEOREM 2.2. *If* M *is a finite* $\mathfrak{o}$*-module contained in* K, *then there exists a positive integer* q *(depending on* M*) such that* $(M^{q+i})' = (M^q)'M^i$, *for all* $i \geq 0$.

Proof. Given M, the ring $R^{*\prime}$ can be constructed, hence a basis

$$(r_1 t^{u_1}, \cdots, r_n t^{u_n})$$

of $tK[t] \cap R^{*\prime}$ can be found. By Lemma 2.2 let g_j $(j = 1, \cdots, n)$ be positive integers such that $(M^{u_j})'^{g_j+i} = (M^{u_j})'^{g_j}(M^{u_j})^i$. Let g be the maximum of the g_j, and let k be a positive integer such that if $k_1, \cdots, k_n$ are positive integers which sum to k, then $k_j \geq g + 1$ for at least one j. Let $q = ku - 1$, and let $i \geq 1$. By Lemma 2.1, $(M')^{q+i} = N_{q+i,k}$, and each summand of $N_{q+i,k}$ has a factor $(M^{u_1})'^{k_1} \cdots (M^{u_n})'^{k_n}$. Since $k_j \geq g + 1 \geq g_j + 1$ for some j, each summand of $N_{q+i,k}$ has M as a factor (since $u_j \geq 1$). Therefore $(M^{q+i})' = AM$, where A is $N_{q+i,k}$ with M factored out of each summand. By the properties of the integral closure of a sum of products, A is clearly contained in $(M^{q+i-1})'$. Hence

$$(M^{q+i})' = MA \subseteq M(M^{q+i-1})' \subseteq (M^{q+i})',$$

so $(M^{q+i})' = M(M^{q+i-1})'$ $(i \geq 1)$. Therefore by induction on $i \geq 0$, $(M^{q+i})' = (M^q)'M^i$, Q.E.D.

COROLLARY 2.3. *Let q be such that $(M^{q+i})' = (M^q)'M^i$ for all $i \geq 0$, and set $N = (M^q)'$. Then $(N^j)' = N^j$ for all $j \geq 1$.*

Proof. $(M^{q+q})' = (M^q)'M^q \subseteq (M^q)'(M^q)' \subseteq (M^qM^q)' = (M^{2q})'$, hence $(M^{2q})' = (M^q)'^2$, and by induction $(M^{qn})' = (M^q)'^n$. Therefore $N^n = (M^{qn})' = ((M^q)'^n)' = (N^n)'$, Q.E.D.

3. Reductions of $\mathfrak{o}$-modules

In [2] a reduction of an ideal A in a Noetherian ring Q was defined to be an ideal B in Q such that $B \subseteq A$ and $BA^n = A^{n+1}$ for some $n \geq 1$. B was defined to be a minimal reduction of A in case B is a reduction of A and B is minimal with this property. It was proved in [2] that if Q is a local ring, then every ideal A in Q has a minimal reduction B, and that if A contains an element which is not a zero-divisor in Q, then the integral closure A' of A in Q is equal to the integral closure B' of B in Q. Further B is a reduction of $B' = A'$, and if B is a reduction of C, then $C \subseteq B'$. In this section it will be proved that, with the same domain $\mathfrak{o}$ and field K of Section 2 these results can be extended to a finite $\mathfrak{o}$-module M contained in K. However, to prove the existence of a minimal reduction of M it was necessary to assume that $\mathfrak{o}$ is a local domain.

Let $\mathfrak{o}$ and K be as in Section 2, and let M be a finite $\mathfrak{o}$-module contained in K. An $\mathfrak{o}$-module $N \subseteq M$ is a *reduction* of M in case $M^{n+1} = NM^n$, for some $n > 0$. N is a *minimal reduction* of M in case if L is a reduction of M which is contained in N, then $L = N$. It should be noted that if N is a reduction of M, then N is a finite $\mathfrak{o}$-module (since $\mathfrak{o}$ is Noetherian), and that M is a reduction of M. Further, if N is a reduction of M, then N is a reduction of every $\mathfrak{o}$-module L such that $N \subseteq L \subseteq M$. (That L is also a reduction of M follows immediately from the next paragraph.) Finally, if N is a reduction of M, say $M^{n+1} = NM^n$, then $M^{n+i} = N^iM^n$, for all $i \geq 0$.

By Lemma 2.2, M is a reduction of M' (where A' denotes the integral closure in K of an $\mathfrak{o}$-module A). Also, if N is a reduction of M, then $NM^n = MM^n$, for some n, therefore $(NM^n)' = (MM^n)'$. Hence, since M^n is a finite $\mathfrak{o}$-module, $N' = M'$ [6, p. 348]. Further $N' = M'$ is the largest $\mathfrak{o}$-module for which N is a reduction, for if N is a reduction of L, then $N \subseteq L \subseteq L' = N'$. In summary, THEOREM 3.1. *If M is a finite* $\mathfrak{o}$*-module contained in K, then M is a reduction of M'. If N is a reduction of M, then $N' = M'$, N is a reduction of L, where $N \subseteq L \subseteq N'$, and L is a reduction of $N' = L'$.*

If $\mathfrak{o}$ is a local domain (which does not necessarily have the property (F) (or (SF))), then the proof given in [2] that every ideal has a minimal reduction carries over with minor changes to proof of

THEOREM 3.2. *Let* $\mathfrak{o}$ *be a local domain, and let K be an extension field of the quotient field of* $\mathfrak{o}$*. If M is a finite* $\mathfrak{o}$*-module contained in K, then there exists a minimal reduction of M.*

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