INVARIANT WEDDERBURN FACTORS

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The Wedderburn principal theorem for a class of algebras states that if an algebra modulo its radical is separable, then it contains a subalgebra with the same structure as the difference algebra. We wish to investigate the problem of when such a subalgebra is invariant under a group of operators on the algebra. The natural setting for this question is that of the extensions of an algebra. In section 2, conditions are given on the groups and algebras considered, which guarantee the existence of such subalgebras. In section 3, a special case of the main theorem for alternative algebras is used to give a proof of the Wedderburn principal theorem for Jordan algebras of characteristic not two. In section 4, a uniqueness theorem is given for a special case: self-adjoint Wedderburn factors of an associative algebra over a field of characteristic zero.

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1. Preliminaries

Let $\mathfrak{A}$ denote a (finite-dimensional) not necessarily associative algebra over a field Φ. The concept of an extension of $\mathfrak{A}$ is found in [4], and we assume familiarity with the discussion given there. The extension $(\mathfrak{B}, \sigma)$ of $\mathfrak{A}$ with kernel $\mathfrak{K}$ may be represented by the diagram

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{B} \xrightarrow{\sigma} \mathfrak{A} \longrightarrow 0.$$

Recall that σ is a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$. If $\mathfrak{K}^2 = \{0\}$, the extension is said to be singular. If $\mathfrak{K}$ is nilpotent, the extension is said to be nilpotent. $\mathfrak{A}$ is segregated in $\mathfrak{B}$ if $\mathfrak{B}$ contains a subalgebra $\mathfrak{A}'$ such that $\mathfrak{B} = \mathfrak{A}' + \mathfrak{K}$, $\mathfrak{A}' \cong \mathfrak{B}/\mathfrak{K}$, and $\mathfrak{A}' \cap \mathfrak{K} = \{0\}$. Such a subalgebra $\mathfrak{A}'$ of $\mathfrak{B}$ will be called a *Wedderburn factor of* $\mathfrak{B}$. $\mathfrak{A}$ is *segregated* if it is segregated in every extension.

We will say $\mathfrak{A}$ is semi-simple if it is the direct sum of simple algebras with nonzero squares, and define the radical of $\mathfrak{A}$ as the minimal ideal $\mathfrak{R}$ such that $\mathfrak{A}/\mathfrak{R}$ is semi-simple. (See [1], [7].) We say $\mathfrak{A}$ is separable if it is semi-simple and remains so under extensions of the base field (or that the centers of its simple components are separable field extensions of the base field).

Before proceeding to introduce group operators, we recall here the Wedderburn principal theorem, which has been proved for several classes of algebras:

If $\mathfrak{A}$ is an algebra with radical $\mathfrak{R}$ such that $\mathfrak{A}/\mathfrak{R}$ is separable, then $\mathfrak{A}$ contains a subalgebra $\mathfrak{S}$ such that $\mathfrak{A} = \mathfrak{S} + \mathfrak{R}$, $\mathfrak{S} \cap \mathfrak{R} = 0$, $\mathfrak{S} \cong \mathfrak{A}/\mathfrak{R}$.

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Since $0 \to \mathfrak{N} \to \mathfrak{A} \xrightarrow{\sigma} \mathfrak{A}/\mathfrak{N} \to 0$ is an extension of $\mathfrak{A}/\mathfrak{N}$, where σ is the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{N}$, $\mathfrak{S}$ is a Wedderburn factor of $\mathfrak{A}$.

All the algebras considered are finite-dimensional over the base field.

2. The main theorem

Let $(\mathfrak{B}, \sigma)$ be an extension of the (not necessarily associative) algebra $\mathfrak{A}$. Let $\mathfrak{G}$ be a group. If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathfrak{G}$-modules, and σ is a $\mathfrak{G}$-homomorphism, we say $(\mathfrak{B}, \sigma)$ is a $\mathfrak{G}$-extension of $\mathfrak{A}$. If $\mathfrak{A}$ is segregated in $(\mathfrak{B}, \sigma)$, such that a subalgebra of $\mathfrak{B}$ isomorphic to $\mathfrak{A}$ can be chosen to be a $\mathfrak{G}$-module, then we say $\mathfrak{A}$ is *$\mathfrak{G}$-segregated in* $(\mathfrak{B}, \sigma)$. $\mathfrak{A}$ is *$\mathfrak{G}$-segregated* if it is $\mathfrak{G}$-segregated in every $\mathfrak{G}$-extension.

PROPOSITION 1. *Let $\mathfrak{A}$ be a $\mathfrak{G}$-algebra over Φ, where $\mathfrak{G}$ is a finite group whose order is not a multiple of the characteristic of Φ. Let $(\mathfrak{B}, \sigma)$ be a singular $\mathfrak{G}$-extension of $\mathfrak{A}$ in which $\mathfrak{A}$ is segregated. If the elements of $\mathfrak{G}$ induce either automorphisms or anti-automorphisms in $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{A}$ is $\mathfrak{G}$-segregated in $\mathfrak{B}$.*

Proof. Let $\mathfrak{K}$ be the kernel of σ. Let the order of $\mathfrak{G}$ be r. Let ρ be a particular embedding of $\mathfrak{A}$ in $\mathfrak{B}$. We wish to show that $\rho' = (1/r)\sum_{\eta \in G} \eta\rho\eta^{-1}$ is a $\mathfrak{G}$-embedding of $\mathfrak{A}$ in $\mathfrak{B}$.

$\mathfrak{K}$ is a two-sided $\mathfrak{A}$-module under the compositions $a \cdot k = \rho(a)k$, and $k \cdot a = k\rho(a)$ for $a \in \mathfrak{A}$, $k \in \mathfrak{K}$. By $Z'(\mathfrak{A}, \mathfrak{K})$ we denote the derivations of $\mathfrak{A}$ into $\mathfrak{K}$. If $g \in Z'(\mathfrak{A}, \mathfrak{K})$, then it is easy to see that $\rho - g$ is an embedding of $\mathfrak{A}$ in $\mathfrak{B}$. Conversely, if $\rho - g$ is an embedding of $\mathfrak{A}$ in $\mathfrak{B}$, and g has range in $\mathfrak{K}$, then $g \in Z'(\mathfrak{A}, \mathfrak{K})$. Use is made here of the assumption $\mathfrak{K}^2 = \{0\}$.

Let $\eta \in \mathfrak{G}$. Then $\eta\rho\eta^{-1}$ embeds $\mathfrak{A}$ in $\mathfrak{B}$. But $\eta\rho\eta^{-1} = \rho - (\rho - \eta\rho\eta^{-1})$. Hence $\rho - \eta\rho\eta^{-1} \in Z'(\mathfrak{A}, \mathfrak{K})$ for any $\eta \in \mathfrak{G}$. Since $Z'(\mathfrak{A}, \mathfrak{K})$ is linear,

$$(1/r)\sum_{\eta \in G} (\rho - \eta\rho\eta^{-1}) \in Z'(\mathfrak{A}, \mathfrak{K}).$$

Hence

$$\rho - (1/r)\sum_{\eta \in G} \eta\rho\eta^{-1} \in Z'(\mathfrak{A}, \mathfrak{K}),$$

and

$$\rho - (\rho - (1/r)\sum_{\eta \in G} \eta\rho\eta^{-1}) = (1/r)\sum_{\eta \in G} \eta\rho\eta^{-1}$$

embeds $\mathfrak{A}$ in $\mathfrak{B}$.

Finally we note that $\rho' = (1/r)\sum_{\eta \in G} \eta\rho\eta^{-1}$ is a $\mathfrak{G}$-mapping. Let $\tau \in \mathfrak{G}$. Then

$$\tau\rho' = (1/r)\sum_{\eta \in G} \tau\eta\rho\eta^{-1} = (1/r)\sum_{\eta \in G} \tau\eta\rho\eta^{-1}\tau^{-1}\tau$$
$$= (1/r)\sum_{\eta \in G} (\tau\eta)\rho(\tau\eta)^{-1}\tau = \rho'\tau.$$

Hence $\rho'\mathfrak{A}$ is a $\mathfrak{G}$-invariant Wedderburn factor in $\mathfrak{B}$, Q.E.D.

We now wish to extend Proposition 1 from singular extensions to general extensions. The procedure will be to pass first from singular extensions to nilpotent extensions, and thence to the general case. To effect this passage, we will restrict the discussion to a class C of algebras defined by identities, such that the algebras in C satisfy the following three conditions:

(I) If $\mathfrak{A} \in C$, $\mathfrak{N}$ a nonzero nilpotent ideal in $\mathfrak{A}$, then there exists an ideal $\mathfrak{N}_1$ in $\mathfrak{A}$ such that

$$(a)\ \mathfrak{N}_1 \subset \mathfrak{N}, \qquad (b)\ \mathfrak{N}^2 \subseteqq \mathfrak{N}_1 .$$

If $\mathfrak{N}$ is a $\mathfrak{G}$-module, then $\mathfrak{N}_1$ is required to be a $\mathfrak{G}$-module.

(II) If $\mathfrak{A} \in C$, then the radical of $\mathfrak{A}$ is nilpotent.

(III) If $\mathfrak{A}$, $\mathfrak{B} \in C$, and $\mathfrak{A}$ is a semi-simple ideal of $\mathfrak{B}$, then there is an ideal $\mathfrak{J}$ in $\mathfrak{B}$ such that $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{J}$. If $\mathfrak{A}$ is a $\mathfrak{G}$-module, $\mathfrak{J}$ is required to be a $\mathfrak{G}$-module.

Condition (I) will be used to pass to the nilpotent extensions, and (II), (III) used to pass to the general case. The argument is essentially that given by Hochschild in [4], pages 64–65, and we sketch it here.

THEOREM 1. *Let C be a class of finite-dimensional algebras defined by identities whose members satisfy conditions* (I), (II), *and* (III). *The following discussion refers to algebras in C. Let $\mathfrak{A}$ be a segregated $\mathfrak{G}$-algebra, where $\mathfrak{G}$ is a finite group whose order is not a multiple of the characteristic of the base field Φ. Assume that each element of $\mathfrak{G}$ induces either an automorphism or an anti-automorphism in $\mathfrak{A}$ and its $\mathfrak{G}$-extensions. Then $\mathfrak{A}$ is $\mathfrak{G}$-segregated.*

Proof. $\mathfrak{A}$ is $\mathfrak{G}$-segregated in singular extensions, by Proposition 1.

(a) Let $(\mathfrak{B}, \sigma)$ be a nilpotent $\mathfrak{G}$-extension of $\mathfrak{A}$ with kernel $\mathfrak{K}$. We show $\mathfrak{A}$ is $\mathfrak{G}$-segregated in nilpotent extensions by induction on the dimension of $\mathfrak{K}$. If $\dim \mathfrak{K} = 0$, there is nothing to prove. Suppose $\dim \mathfrak{K} = n$, and the result holds for nilpotent extensions with kernels of dimension less than n.

Let $\mathfrak{K}_1$ be the ideal given by condition (I). σ induces $\bar{\sigma}$ of $\mathfrak{B}/\mathfrak{K}_1$ onto $\mathfrak{A}$ with kernel $\mathfrak{K}/\mathfrak{K}_1$. By (I), $0 \to \mathfrak{K}/\mathfrak{K}_1 \to \mathfrak{B}/\mathfrak{K}_1 \xrightarrow{\bar{\sigma}} \mathfrak{A} \to 0$ is a singular $\mathfrak{G}$-extension of $\mathfrak{A}$. By Proposition 1, $\mathfrak{B}/\mathfrak{K}_1 = \bar{\mathfrak{A}} \oplus \mathfrak{K}/\mathfrak{K}_1$, where $\bar{\mathfrak{A}}$ is a $\mathfrak{G}$-subalgebra of $\mathfrak{B}/\mathfrak{K}_1$. $\bar{\mathfrak{A}}$ has the form $\mathfrak{C}/\mathfrak{K}_1$, where $\mathfrak{C}$ is a $\mathfrak{G}$-subalgebra of $\mathfrak{B}$. Let τ be a $\mathfrak{G}$-homomorphism of $\mathfrak{C}$ onto $\mathfrak{A}$ with kernel $\mathfrak{K}_1$. Then $(\mathfrak{C}, \tau)$ is a $\mathfrak{G}$-extension of $\mathfrak{A}$ with nilpotent kernel $\mathfrak{K}_1$ such that $\dim \mathfrak{K}_1 < \dim \mathfrak{K}$. Hence, there is a $\mathfrak{G}$-subalgebra $\mathfrak{A}_1$ of $\mathfrak{C}$ such that $\mathfrak{C} = \mathfrak{A}_1 \oplus \mathfrak{K}_1$, $\mathfrak{A}_1 \cong \mathfrak{A}$. Then clearly $\mathfrak{B} = \mathfrak{A}_1 \oplus \mathfrak{K}$. Hence $\mathfrak{A}$ is $\mathfrak{G}$-segregated in nilpotent extensions.

(b) Let $(\mathfrak{B}, \sigma)$ be any $\mathfrak{G}$-extension of $\mathfrak{A}$ with kernel $\mathfrak{K}$. Let $\mathfrak{R}$ denote the radical of $\mathfrak{B}$. If $\mathfrak{B} = \mathfrak{R}$, $\mathfrak{K}$ is nilpotent, and the result follows from (a). Hence assume $\mathfrak{R} \neq \mathfrak{B}$, and consider the semi-simple algebra $\mathfrak{B}/\mathfrak{R}$. $(\mathfrak{K} + \mathfrak{R})/\mathfrak{R}$ is a $\mathfrak{G}$-ideal of $\mathfrak{B}/\mathfrak{R}$ and hence is semi-simple. Let $\mathfrak{N} = \mathfrak{K} \cap \mathfrak{R}$. Then $(\mathfrak{K} + \mathfrak{R})/\mathfrak{R} \cong \mathfrak{K}/\mathfrak{N}$. $\mathfrak{N}$ is a nilpotent $\mathfrak{G}$-ideal in $\mathfrak{B}$, $\mathfrak{K}/\mathfrak{N}$ is a semi-simple $\mathfrak{G}$-ideal in $\mathfrak{B}/\mathfrak{N}$, and $\mathfrak{A} \cong \mathfrak{B}/\mathfrak{K} \cong (\mathfrak{B}/\mathfrak{N})/(\mathfrak{K}/\mathfrak{N})$.

By condition (III), there is a $\mathfrak{G}$-ideal $\bar{\mathfrak{D}}$ of $\mathfrak{B}/\mathfrak{N}$ such that $\bar{\mathfrak{D}} \cong \mathfrak{A}$. $\bar{\mathfrak{D}}$ has the form $\mathfrak{D}/\mathfrak{N}$, $\mathfrak{D}$ a $\mathfrak{G}$-ideal in $\mathfrak{B}$. Let τ be a $\mathfrak{G}$-homomorphism of $\mathfrak{D}$ onto

$\mathfrak{A}$ with kernel $\mathfrak{N}$. By (a), there is a $\mathfrak{G}$-subalgebra $\mathfrak{A}_1$ of $\mathfrak{D}$ such that $\mathfrak{D} = \mathfrak{A}_1 \oplus \mathfrak{N}$. Then it is easy to see that $\mathfrak{B} = \mathfrak{A}_1 \oplus \mathfrak{K}$, so that $\mathfrak{A}$ is $\mathfrak{G}$-segregated, Q.E.D.

Now let C be a class of algebras, as above, for which the Wedderburn principal theorem is true. Let $\mathfrak{A}$ be a $\mathfrak{G}$-algebra in C, $\mathfrak{G}$ as in Theorem 1, and let $\mathfrak{N}$ be the radical of $\mathfrak{A}$. Then $0 \rightarrow \mathfrak{N} \rightarrow \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{N} \rightarrow 0$ is a segregated $\mathfrak{G}$-extension of $\mathfrak{A}/\mathfrak{N}$, provided $\mathfrak{A}/\mathfrak{N}$ is separable. Hence $\mathfrak{A}/\mathfrak{N}$ is $\mathfrak{G}$-segregated in $\mathfrak{A}$.

We now give some examples of classes C for which Theorem 1 applies, and for which the Wedderburn principal theorem is known.

1. *Alternative (and hence associative) algebras.* For condition (I), set $\mathfrak{N}_1 = \mathfrak{N}^2$. The radical of an alternative algebra is nilpotent, so (II) holds. If $\mathfrak{A}$ is a semi-simple alternative ideal in $\mathfrak{B}$, then $\mathfrak{A}$ has an identity e, and $\mathfrak{B}_{00}$ in the Peirce decomposition of $\mathfrak{B}$ with respect to e is the desired complementary ideal of $\mathfrak{A}$ in $\mathfrak{B}$, so that (III) holds. The Wedderburn principal theorem for alternative algebras is proved in [10], the result holding for arbitrary characteristic.

2. *Jordan algebras.* For (I), set $\mathfrak{N}_1 = \mathfrak{A}\mathfrak{N}^2 + \mathfrak{N}^2$, see [9]. (II) is also satisfied in this case. As for condition (III), $\mathfrak{B}_0(e)$ in the Peirce decomposition of $\mathfrak{B}$ with respect to e (see [2] or [5]) is the desired complementary ideal for $\mathfrak{A}$. ($\mathfrak{B}_{1/2}(e) = 0$.) In the next section, the Wedderburn principal theorem is proved for Jordan algebras of characteristic different from two.

3. *Lie algebras of characteristic zero.* For these algebras, the term "nilpotent" in condition (II) and the above discussion is to be replaced by "solvable". Then the results will be valid. The square of an ideal is an ideal by the Jacobi identity, so that (I) holds. As for condition (III), set $\mathfrak{J} = (0:\mathfrak{A})$, the annihilator of $\mathfrak{A}$ in $\mathfrak{B}$. The Wedderburn principal theorem (Levi theorem) for these algebras is well-known (e.g. [11]).

COROLLARY. *Theorem* 1 *is valid for C the class of associative, alternative, or Jordan algebras, or Lie algebras over a field of characteristic zero.*

3. The Wedderburn principal theorem for Jordan algebras

In this section, we shall make use of the results of the preceding section to give a proof of the Wedderburn principal theorem for Jordan algebras over fields of characteristic not two. A proof is given for characteristic zero in [9].[2] We will use the reduction given there to the case where the radical $\mathfrak{N}$ of the Jordan algebra $\mathfrak{A}$ has square zero, $\mathfrak{A}$ contains an identity u, the base field Φ is algebraically closed, $\mathfrak{A}/\mathfrak{N}$ is simple. In [9], the five possibilities for the split algebra $\mathfrak{A}/\mathfrak{N}$ are considered separately, and a calculational proof

[2] The proof in [9] is valid for characteristic $\neq 2$. For [6] implies that there are no new split algebras of characteristic $p \neq 2$, and the calculations in [9] involve only characteristic $\neq 2$. (See Bull. Amer. Math. Soc. vol. 61 (1955), p 475.)

involving structure-lifting is given for each case. Here we use this result for split algebras of degree two, and start with $\mathfrak{A}/\mathfrak{N}$ having degree larger than two. This case is taken care of by means of a structure theorem of Jacobson, [5], and Theorem 1 of the last section. So essentially we have reduced the four cases (classes A, B, C, E) to a single case, thereby saving a considerable amount of calculation. As for $\mathfrak{A}/\mathfrak{N}$ a simple Jordan algebra of degree one, this can now be handled using a recent result of Jacobson, [6]. Such algebras are one-dimensional, and hence $\mathfrak{A}/\mathfrak{N}$ is of the form $\Phi\bar{z}$, $\bar{z}$ idempotent. Raising $\bar{z}$ to an idempotent z in $\mathfrak{A}$, we get $\mathfrak{A} = \Phi z \oplus \mathfrak{N}$, since clearly $(\Phi z) \cap \mathfrak{N} = \{0\}$.

Let $\mathfrak{A}/\mathfrak{N}$ be of degree n larger than two, with $\bar{u} = \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_n$. Then it can be shown, ([2], [5]), that $\mathfrak{A}/\mathfrak{N}$ contains elements $\{\bar{u}_{ji}\}$, $j = 2, 3, \cdots, n$, such that

$$\bar{e}_j \bar{u}_{j1} = \bar{e}_1 \bar{u}_{j1} = \tfrac{1}{2}\bar{u}_{j1}, \tag{1}$$

$$\bar{u}_{j1}^2 = 4(\bar{e}_1 + \bar{e}_j) \qquad \text{for } j = 2, 3, \cdots, n. \tag{2}$$

By [9], we may write $u = e_1 + e_2 + \cdots + e_n$, where e_i's are orthogonal idempotents, and $e_i + \mathfrak{N} = \bar{e}_i$. For a fixed j, $\bar{\mathfrak{A}}_{11} + \bar{\mathfrak{A}}_{jj} + \bar{\mathfrak{A}}_{1j}$ is a reduced algebra of degree two, and is a subalgebra of $\mathfrak{A}/\mathfrak{N}$. Let $\mathfrak{A} = \sum_{i \leqq j} \mathfrak{A}_{ij}$ be the Peirce decomposition of $\mathfrak{A}$ with respect to the $\{e_i\}$. Let $\mathfrak{N}_{1j}$ be the radical of $\mathfrak{A}_{e_1+e_j}(1) = \mathfrak{A}_{11} + \mathfrak{A}_{1j} + \mathfrak{A}_{jj}$. Then $(\mathfrak{A}_{11} + \mathfrak{A}_{1j} + \mathfrak{A}_{jj})/\mathfrak{N}_{1j} = \bar{\mathfrak{A}}_{11} + \bar{\mathfrak{A}}_{1j} + \bar{\mathfrak{A}}_{jj}$ (see [9], (4.3)). Since we are assuming the result for such difference algebras of degree two, we can find a subalgebra of $\mathfrak{A}_{11} + \mathfrak{A}_{1j} + \mathfrak{A}_{jj}$ with the same structure as $\bar{\mathfrak{A}}_{11} + \bar{\mathfrak{A}}_{1j} + \bar{\mathfrak{A}}_{jj}$, and which contains e_1 and e_j (see [9], section 6, lemma 6.3). Hence the relations (1), (2) may be "raised" to $\mathfrak{A}$ for a fixed j, and hence for all $j = 2, 3, \cdots, n$.

Now referring to Theorem 9.1 of [5] and the remark on page 36 there, we conclude that $\mathfrak{A} = H(\mathfrak{D}_n)$, the Jordan algebra of all $n \times n$ hermitian matrices over an alternative algebra $\mathfrak{D}$ with involution $d \to d^*$. Let $\mathfrak{R}$ be the radical of $\mathfrak{D}$. Using the results of section 2 for $\mathfrak{G}$ the group of order two consisting of the identity and involution of $\mathfrak{D}$, we may write $\mathfrak{D} = \mathfrak{S} + \mathfrak{R}$, where $\mathfrak{S}$ is a self-adjoint separable subalgebra of $\mathfrak{D}$.

Now let $S(\mathfrak{D}_n)$, $R(\mathfrak{D}_n)$ be the subalgebra of $\mathfrak{A}$ consisting of those matrices all of whose entries lie in $\mathfrak{S}$ and $\mathfrak{R}$ respectively. Then it is not hard to see that $R(\mathfrak{D}_n) = \mathfrak{N}$, the radical of $\mathfrak{A}$, and that $\mathfrak{A} = S(\mathfrak{D}_n) + R(\mathfrak{D}_n)$ is a Wedderburn decomposition of $\mathfrak{A}$. (Use is made of Theorem 7.1 of [5].) This completes the proof of the Wedderburn principal theorem for Jordan algebras whose characteristic is different from two.

4. Orthogonal conjugacy and the uniqueness theorem for self-adjoint Wedderburn factors of an associative algebra

We assume familiarity with the Malcev theorem for Wedderburn factors of an associative algebra, [8], which states that if the algebra modulo its radical is separable, then any two such factors are conjugate by an element

$1 - z$, where z is in the radical. Here we will show that any two self-adjoint factors in an associative algebra $\mathfrak{A}$ with involution are conjugate by an orthogonal element. Since use is made of the exponential of a derivation, our result is limited to the case of characteristic zero.

We assume familiarity with the notions of exponential of a nilpotent derivation, and the adjoint mapping of $\mathfrak{A}$ into its Lie algebra of derivations. In particular, if z is in the radical $\mathfrak{N}$ of $\mathfrak{A}$, then $\exp z$ is regular (in $\mathfrak{A}_1$, the algebra obtained from $\mathfrak{A}$ by adjunction of an identity, if necessary), and $\exp(\mathrm{Ad}\ z)$ is conjugation by $\exp z$.

We extend the involution $a \to a^*$ of $\mathfrak{A}$ to $\mathfrak{A}_1$ by setting $(\alpha 1)^* = \alpha 1$ for $\alpha \in \Phi$. An element a of $\mathfrak{A}_1$ is *skew* if $a^* = -a$, *self-adjoint* if $a^* = a$, and *orthogonal* if $aa^* = 1 = a^*a$.

Let z be a skew element in $\mathfrak{N}$. Then $\exp z$ is clearly an orthogonal element of $\mathfrak{A}_1$, and $\mathrm{Ad}(\exp z)$ is conjugation by an orthogonal element.

If $\mathfrak{S}$ is a self-adjoint (i.e. $\mathfrak{S}^* = \mathfrak{S}$) Wedderburn factor of $\mathfrak{A}$, and a is orthogonal in $\mathfrak{A}_1$, then $\mathfrak{S}' = a^*\mathfrak{S}a$ is another self-adjoint Wedderburn factor in $\mathfrak{A}$. It is the converse that we wish to prove. Hence we make the following definitions:

DEFINITION. An automorphism of $\mathfrak{A}$ which is given by conjugation by an orthogonal element of $\mathfrak{A}_1$ is called an *orthogonal conjugacy* of A. Two subalgebras $\mathfrak{S}$ and $\mathfrak{T}$ of $\mathfrak{A}$ are *orthogonally conjugate* if there is an orthogonal conjugacy of $\mathfrak{A}$ carrying $\mathfrak{S}$ onto $\mathfrak{T}$.

The relationship of orthogonal conjugacy is an equivalence relation among the subalgebras of $\mathfrak{A}$ since the orthogonal conjugacies form a group.

THEOREM. *Let $\mathfrak{A}$ be an associative algebra over a base field of characteristic zero. Let $a \to a^*$ be an involution in $\mathfrak{A}$. Let $\mathfrak{N}$ be the radical of $\mathfrak{A}$, ($\mathfrak{A}/\mathfrak{N}$ separable). Let $\mathfrak{S}$ be a self-adjoint separable subalgebra of $\mathfrak{A}$, and let $\mathfrak{A} = \mathfrak{T} + \mathfrak{N}$ be a Wedderburn decomposition of $\mathfrak{A}$ such that $\mathfrak{T}$ is self-adjoint. Then $\mathfrak{S}$ is orthogonally conjugate to a subalgebra of $\mathfrak{T}$, and the conjugacy is given by* $\exp(\mathrm{Ad}\ z)$, *z a skew element of $\mathfrak{N}$.*

Proof. First we note that, by using the Baker-Hausdorff formula (see [3]), it is not hard to show that mappings $\exp(\mathrm{Ad}\ z)$, z skew in $\mathfrak{N}$, are closed under multiplication.

Let $G_1 = \exp(\mathrm{Ad}\ 0) = I$. Then trivially $\mathfrak{S}^{G_1} \subseteqq \mathfrak{T} + \mathfrak{N} = \mathfrak{A}$. Now suppose we have found $G_i = \exp(\mathrm{Ad}\ z_i)$, z_i skew in $\mathfrak{N}$, $i = 1, 2, \cdots, k$, such that $\mathfrak{S}_k = \mathfrak{S}^{G_1 G_2 \cdots G_k} \subseteqq \mathfrak{T} + \mathfrak{N}^k$. Then $\mathfrak{S}_k$ is a self-adjoint separable subalgebra of $\mathfrak{A}$. We wish to construct $G_{k+1} = \exp(\mathrm{Ad}\ z_{k+1})$, z_{k+1} skew in $\mathfrak{N}$, such that $\mathfrak{S}_k^{G_{k+1}} \subseteqq \mathfrak{T} + \mathfrak{N}^{k+1}$. Since $\mathfrak{N}^r = \{0\}$, $G_1 G_2 \cdots G_r$ will be the desired orthogonal conjugacy of $\mathfrak{S}$ into $\mathfrak{T}$.

If $s \in \mathfrak{S}$, let $s = t(s) + n(s)$, $t(s) \in \mathfrak{T}$, $n(s) \in \mathfrak{N}$. By the induction hypothesis, we take $s \in \mathfrak{S}_k$, $n(s) \in \mathfrak{N}^k$. If $s_1, s_2 \in \mathfrak{S}_k$, then

$$t(s_1 s_2) = t(s_1)t(s_2), \tag{1}$$

$$n(s_1s_2) = n(s_1)t(s_2) + t(s_1)n(s_2) + n(s_1)n(s_2), \tag{2}$$

$$t(s^*) = t(s)^*, \qquad n(s^*) = n(s)^*. \tag{3}$$

The last equation results from $\mathfrak{T}$, $\mathfrak{N}^k$, $\mathfrak{S}_k$ being self-adjoint.

Consider $\mathfrak{N}^k/\mathfrak{N}^{k+1}$. This has square zero, and may be considered as a two-sided $\mathfrak{S}_k$-bimodule by means of the compositions:

$$s\cdot\bar{z} = \overline{t(s)z}$$

$$\bar{z}\cdot s = \overline{zt(s)} \qquad \text{for} \quad s \in \mathfrak{S}_k, \quad z \in \mathfrak{N}^k, \quad \bar{z} = z + \mathfrak{N}^{k+1} \in \mathfrak{N}^k/\mathfrak{N}^{k+1}.$$

By (2), $\overline{n(s_1s_2)} = \overline{n(s_1)}\cdot s_2 + s_1\cdot\overline{n(s_2)}$. Hence $s \to \overline{n(s)}$ is a derivation of $\mathfrak{S}_k$ into $\mathfrak{N}^k/\mathfrak{N}^{k+1}$. Since $\mathfrak{S}_k$ is separable, $H'(\mathfrak{S}_k, \mathfrak{N}^k/\mathfrak{N}^{k+1}) = 0$ (see [4]), and there is a $z \in \mathfrak{N}^k$ such that

$$\overline{n(s)} = \delta\bar{z}(s) \equiv s\cdot\bar{z} - \bar{z}\cdot s, \qquad \text{for all } s \in \mathfrak{S}_k. \tag{4}$$

Now $\mathfrak{A}/\mathfrak{N}^{k+1}$ has an involution, induced by that of $\mathfrak{A}$, and

$$\begin{aligned} ((\delta\bar{z})(s))^* &= \overline{n(s)}^* \text{ by (4)} \\ &= \overline{n(s)^*} \\ &= \overline{n(s^*)} \text{ by (3)} \\ &= (\delta\bar{z})(s^*) \text{ by (4)}. \end{aligned}$$

Hence

$$((\delta\bar{z})(s))^* = (\delta\bar{z})(s^*) \qquad \text{for } s \in \mathfrak{S}_k. \tag{5}$$

Next we show that

$$\delta(\bar{z}^*) = -(\delta\bar{z}). \tag{6}$$

$$\begin{aligned} (\delta\bar{z}^*)(s) &= s\cdot\bar{z}^* - \bar{z}^*\cdot s \\ &= \overline{t(s)z^* - z^*t(s)} \\ &= \overline{t(s^*)^*z^* - z^*t(s^*)^*} \text{ by (3)} \\ &= \overline{(zt(s^*) - t(s^*)z)^*} \\ &= -\,(\delta\bar{z}(s^*))^* \\ &= -\,\delta\bar{z}(s) \text{ by (5)}. \end{aligned}$$

This proves (6). Set $z'_{k+1} = \frac{1}{2}(z - z^*)$. Then z'_{k+1} is skew in $\mathfrak{N}$.

$$\bar{z}'_{k+1} = \tfrac{1}{2}(\bar{z} - \bar{z}^*),$$

so that

$$\begin{aligned} \delta(\bar{z}'_{k+1}) &= \tfrac{1}{2}(\delta\bar{z} - \delta\bar{z}^*) \\ &= \tfrac{1}{2}(\delta\bar{z} + \delta\bar{z}) \text{ by (6)} \\ &= \delta\bar{z}. \end{aligned}$$

Hence, by (4), we get

$$\overline{n(s)} = (\delta\bar{z}'_{k+1})(s) \qquad \text{for } s \in \mathfrak{S}_k\,. \tag{7}$$

Let $G_{k+1} = \exp(\mathrm{Ad}(-z'_{k+1}))$. If $s \in \mathfrak{S}_k$, then

$$s^{G_{k+1}} = s(1 + \mathrm{Ad}(-z'_{k+1}) + \cdots),$$

where the omitted terms involve two or more multiplications by $z'_{k+1} \in \mathfrak{N}^k$, so that they yield elements in $\mathfrak{N}^{k+1}$. Hence

$$\begin{aligned} s^{G_{k+1}} &\equiv (t(s) + n(s))\,(1 + \mathrm{Ad}(-z'_{k+1})) \pmod{\mathfrak{N}^{k+1}} \\ &\equiv t(s) + n(s) - t(s)z'_{k+1} + z'_{k+1}\,t(s) \pmod{\mathfrak{N}^{k+1}} \\ &\equiv t(s) \pmod{\mathfrak{N}^{k+1}} \quad \text{by (7).} \end{aligned}$$

Hence $\mathfrak{S}_k^{G_{k+1}} \subseteq \mathfrak{T} + \mathfrak{N}^{k+1}$, and if we now put $z_{k+1} = -z'_{k+1}$, then z_{k+1} is skew in $\mathfrak{N}$, and $G_{k+1} = \exp(\mathrm{Ad}\, z_{k+1})$ is an orthogonal conjugacy of the desired form.

This completes the proof of the above theorem. The proof also works for $\mathfrak{N}^2 = \{0\}$ and arbitrary characteristic not two. This theorem has the usual two corollaries:

COROLLARY. *Any two self-adjoint Wedderburn factors of an associative algebra $\mathfrak{A}$ over a field Φ of characteristic zero are orthogonally conjugate by a mapping* $\exp(\mathrm{Ad}\, z)$, *z a skew element in the radical of $\mathfrak{A}$.*

COROLLARY. *Let $\mathfrak{S}$ be a separable self-adjoint subalgebra of the associative algebra $\mathfrak{A}$ over a field of characteristic zero. Then $\mathfrak{S}$ may be embedded in a self-adjoint Wedderburn factor of $\mathfrak{A}$.*

In connection with the general uniqueness problem of two $\mathfrak{G}$-invariant Wedderburn factors, where $\mathfrak{G}$ is a group of operators, one might conjecture that any two such factors are automorphic in the algebra by means of an automorphism which commutes with the operators of $\mathfrak{G}$. The orthogonal conjugacy type of mapping can easily be seen to commute with the group of order two consisting of the identity mapping and involution of the algebra.

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