INEQUALITIES FOR ASYMMETRIC ENTIRE FUNCTIONS[1]

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Let $p_n(z)$ be a polynomial of degree n such that $|p_n(z)| \leqq 1$ in the unit disk $|z| \leqq 1$. The following results are well known.

THEOREM A. *For* $|z| = R > 1$, $|p_n(z)| \leqq R^n$.

THEOREM B. *For* $|z| = 1$, $|p'_n(z)| \leqq n$.

Theorem A is a simple deduction from the maximum principle (see [11], p. 346, or [10], vol. 1, p. 137, problem III 269). Theorem B is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for references see [12], or [2], pp. 206, 231).

When $p_n(z)$ has no zeros in $|z| < 1$, more precise statements can be made:

THEOREM C. *For* $|z| = R > 1$, $|p_n(z)| \leqq \frac{1}{2}(1 + R^n)$.

THEOREM D. *For* $|z| = 1$, $|p'_n(z)| \leqq \frac{1}{2}n$.

Theorem D was conjectured by Erdös and proved by Lax [8]; for another proof see [4]. Theorem C was deduced from Theorem D by Ankeny and Rivlin [1].

Since $p_n(e^{iz})$ is an entire function of exponential type, these theorems suggest generalizations to such functions. Let $f(z)$ be an entire function of exponential type τ, with $|f(x)| \leqq 1$ for real x.

THEOREM A′. *For all* y, $|f(x + iy)| \leqq e^{\tau|y|}$.

THEOREM B′. *For all real* x, $|f'(x)| \leqq \tau$.

Theorem A′ is a simple consequence of the Phragmén-Lindelöf principle (for references see [2], p. 82; see also [11], pp. 346–347). Theorem B′ is Bernstein's generalization of Theorem B (see references on Theorem B).

In this note I obtain theorems for entire functions which generalize Theorems C and D. To see what to expect, note that $p_n(e^{iz})$ is an entire function $f(z)$ of exponential type of a special kind: if $h(\theta)$ is its indicator, we have $h(-\pi/2) = n$, but $h(\pi/2) > -n$ unless $p_n(z) = cz^n$. If $p_n(z)$ has no zeros in $|z| < 1, f(z)$ has no zeros in $y > 0$, and moreover (since $p_n(0) \neq 0$) $h(\pi/2) = 0$.

Let us consider, then, entire functions $f(z)$ of exponential type τ with $|f(x)| \leqq 1$ for real x, $h(\pi/2) = 0$ (hence necessarily $h(-\pi/2) = \tau$), and $f(z) \neq 0$ for $y > 0$.

THEOREM 1. *For* $y < 0$, $|f(x + iy)| \leqq \frac{1}{2}(e^{\tau|y|} + 1)$.

THEOREM 2. *For all real* x, $|f'(x)| \leqq \frac{1}{2}\tau$.

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Theorems 1 and 2 include Theorems C and D, so that we have new proofs of these theorems.

We can vary the form of Theorems 1 and 2 to a certain extent by reducing the asymmetry of the indicator diagram and applying the theorems as they stand to $e^{-i\sigma z}f(z)$ with a suitable σ.

To illustrate Theorems 1 and 2, consider functions of the form

$$f(z) = \int_0^\tau e^{izt}\,d\alpha(t), \qquad \int_0^\tau |\,d\alpha(t)\,| < \infty. \tag{1}$$

If $\alpha(t)$ is not constant in any interval $0 \leqq t \leqq a$, $a > 0$, we have ([2], p. 108) $h(\pi/2) = 0$ and $h(-\pi/2) \leqq \tau$. Theorems 1 and 2 then apply to this $f(z)$ if (in particular) $d\alpha(t) = \varphi(t)\,dt$ and $\varphi(t)$ is positive and decreasing, since [9] $f(z)$ then has all its zeros in the lower half plane. (If we take $x = 0$ in this special case we find the inequality $\int_0^\tau t\varphi(t)\,dt \leqq \frac{1}{2}\tau \int_0^\tau \varphi(t)\,dt$ which is a special case of Chebyshev's inequality ([7], p. 168).) However, it is clear that if all the derivatives of $f(z)$ satisfied the conditions of Theorems 1 and 2, we should obtain a contradiction by repeated applications of Theorem 2. Unless $t = 0$ is an isolated discontinuity of $\alpha(t)$ (as it is when $f(z) = p_n\,(e^{iz})$), all the derivatives of $f(z)$ have the same indicator as $f(z)$; hence not all the derivatives of $f(z)$ can be free of zeros in the upper half plane. Similar reasoning leads to the following more general result.

THEOREM 3. *If $f(z)$ is an entire function of exponential type τ, such that $h(\pi/2) = 0$ for $f(z)$ and all its derivatives, and $f(z)$ is bounded on the real axis, then every half plane $y > a \geqq 0$ contains zeros of infinitely many derivatives of $f(z)$.*

If $f(z)$ has the form (1) with $d\alpha(t) = \varphi(t)\,dt$ and $\varphi(t)$ positive and increasing, all the zeros of $f(z)$ and its derivatives are in $y \geqq 0$ [9]. If $d\alpha(t) = \varphi(t)\,dt$ and $\varphi(t)$ is an integral, the zeros are always asymptotic to the real axis [5]; Theorem 3 shows, however, that the zeros of the derivatives of $f(z)$ cannot be uniformly asymptotic to the real axis.

The condition $h(\pi/2) = 0$ in Theorem 3 will hold for all the derivatives of $f(z)$ unless 0 is a pole of the Borel transform of $f(z)$.

We deduce Theorem 1 from the following theorem.

THEOREM 4. *If $g(z)$ is an entire function of exponential type τ, if $|\,g(x)\,| \leqq M$ for all real x, and if*

$$|\,g(z)\,| \leqq |\,g(\bar{z})\,|, \qquad y < 0, \tag{2}$$

then

$$|\,g(x + iy)\,| \leqq M \cosh \tau y, \qquad y < 0. \tag{3}$$

This is ostensibly a generalization of a theorem of Duffin and Schaeffer [6], in which $g(z)$ is real on the real axis, so that $|\,g(z)\,| = |\,g(\bar{z})\,|$; but it is actually a corollary of the Duffin-Schaeffer theorem.

To prove Theorem 4, let $\bar{g}(z)$ be the conjugate of $g(z)$, and consider $G(z) = g(z)\bar{g}(z)$, an entire function of exponential type 2τ. We have $|G(x)| \leqq M^2$ for real x; and $G(x)$ is real and non-negative on the real axis. Hence $G(z) - \frac{1}{2}M^2$ is real on the real axis, with absolute value bounded by $\frac{1}{2}M^2$. By the Duffin-Schaeffer theorem,

$$|G(z) - \tfrac{1}{2}M^2| \leqq \tfrac{1}{2}M^2 \cosh 2\tau y,$$

$$|g(z)\bar{g}(z)| \leqq \tfrac{1}{2}M^2 (\cosh 2\tau y + 1) = M^2 \cosh^2 \tau y.$$

Since $|g(z)| \leqq |\bar{g}(z)| = |g(\bar{z})|$ for $y < 0$, the conclusion follows.

The same reasoning shows (as A. C. Schaeffer has pointed out) that, whether or not (2) holds, at least one of $g(x + iy)$, $g(x - iy)$ satisfies (3). (For another proof of this, see [3].)

To prove Theorem 1, put $g(z) = f(z)e^{-\frac{1}{2}i\tau z}$. Then $|g(x)| \leqq 1$ and $g(z)$ is of exponential type $\tau/2$; moreover, the indicator h_g of g satisfies $h_g(-\pi/2) \leqq h_g(\pi/2)$. Since $g(z)$ has no zeros for $y > 0$, by a theorem of B. Levin (see [2], p. 129) we have $|g(z)| \leqq |g(\bar{z})|$ for $y < 0$. Hence, by Theorem 4

$$|f(z)| \leqq e^{\frac{1}{2}\tau|y|} \cosh \tfrac{1}{2}\tau y = \tfrac{1}{2}(e^{\tau|y|} + 1)$$

for $y < 0$.

To prove Theorem 2, consider the same function $g(z)$. By another theorem of Levin (see [2], p. 226, 11.7.5), the function $g'(z) - (\alpha + i\beta)g(z)$ also has no zeros for $y > 0$ if $\beta \geqq 0$. That is, if $y > 0$ and $\beta \geqq 0$,

$$f'(z) - (\tfrac{1}{2}i\tau + \alpha + i\beta)f(z) \neq 0. \tag{4}$$

Since $|f(x)| \leqq 1$ for real x and $h(\pi/2) \leqq 0$, we have $|f(x + iy)| \leqq 1$ for $y \geqq 0$. Thus if λ is any complex number of modulus greater than 1, $f(z) - \lambda$ satisfies the same hypotheses as $f(z)$. Hence we also have, for $y > 0$ and $\beta \geqq 0$, and all λ with $|\lambda| > 1$,

$$f'(z) - \{f(z) - \lambda\}(\tfrac{1}{2}i\tau + \alpha + i\beta) \neq 0. \tag{5}$$

We now show that (4) and (5), with $|f(z)| \leqq 1$, imply $|f'(z)| \leqq \frac{1}{2}\tau$; since this is true for all $y > 0$ it is also true for $y = 0$.

To simplify the notation, put $if'(z) = w$, $f(z) = \zeta$, $\frac{1}{2}\tau - i\alpha + \beta = a + ib$, with $a \geqq \frac{1}{2}\tau$. Then (4) and (5) become

$$w - \zeta(a + ib) \neq 0, \tag{6}$$

$$w - (\zeta + \lambda)(a + ib) \neq 0, \tag{7}$$

where $|\zeta| \leqq 1$, and the inequalities hold for all λ with $|\lambda| > 1$, all $a \geqq \frac{1}{2}\tau$, and all real b. There is no loss in generality from taking $\frac{1}{2}\tau = 1$. If $\zeta = 0$, (7) with $a = 1$ and $b = 0$ says that $w \neq \lambda$ and so $|w| \leqq 1$. If $\zeta \neq 0$, we may assume that ζ is real and positive (otherwise consider $we^{-i\theta}$ instead of w). Then let $\zeta = \sin\psi$, $0 < \psi \leqq \pi/2$.

The points $w = u + iv$ with $|w| > 1$ may be divided into three sets:

(i) The set of points with $|v| \leqq \cos\psi$ and $u \leqq 0$;

(ii) The set of points with $|v| \leqq \cos\psi$ and $u > 0$;
(iii) The set of points with $|v| > \cos\psi$.

We proceed to show that each of these sets is excluded by (6) or (7). (The reasoning is most easily followed on a figure.)

Set (i) is a subset of the set (iv) of points with $|w| > 1$ and $u \leqq 0$. In set (iv), $|w - \zeta| \geqq |w| > 1$, and so, for any w in (iv), $w = \zeta + (w - \zeta) = \zeta + \lambda$, $|\lambda| > 1$, contradicting (7) with $a = 1$, $b = 0$.

Set (ii) is a subset of the set (v) of points with $|w| > 1$ and $u > \zeta$, since $u^2 > \zeta^2$ if $u^2 + v^2 > 1$ and $v^2 \leqq 1 - \zeta^2$. In set (v), $\Re(w/\zeta) > 1$ and this contradicts (6).

For w in set (iii), consider (for definiteness) the case when $v > \cos\psi$. If $\zeta < 1$, take $\lambda = (1 + \varepsilon)i\cos\psi - \zeta$, with $\varepsilon > 0$; then $|\lambda| > 1$ and

$$\Re\left(\frac{w}{\zeta + \lambda}\right) = \frac{v \sec\psi}{1 + \varepsilon} > 1$$

provided ε is small enough, contradicting (7). If $\zeta = 1$, take $\lambda = -1 + i\varepsilon$, and then

$$\Re\left(\frac{w}{\zeta + \lambda}\right) = v/\varepsilon > 1$$

if ε is small enough, again contradicting (7).

We see that (6) and (7) actually restrict $f'(z)$ to a proper subset of the unit disk.

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