Homology, Homotopy and Applications, vol. 8(2), 2006, pp.157–203

A CONSTRUCTION OF QUOTIENT A_∞-CATEGORIES

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(*communicated by Jim Stasheff*)

Abstract

We construct an A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ from a given A_∞-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$. The construction is similar to a particular case of Drinfeld's construction of the quotient of differential graded categories. We use $\mathsf{D}(\mathcal{C}|\mathcal{B})$ to construct an A_∞-functor of K-injective resolutions of a complex, when the ground ring is a field. The conventional derived category is obtained as the 0-th cohomology of the quotient of the differential graded category of complexes over acyclic complexes. This result follows also from Drinfeld's theory of quotients of differential graded categories.

Introduction

In [**Dri04**] Drinfeld reviews and develops Keller's construction of the quotient of differential graded categories [**Kel99**] and gives a new construction of the quotient. This construction consists of two parts. The first part replaces the given pair $\mathcal{B} \subset \mathcal{C}$ of a differential graded category $\mathcal{C}$ and its full subcategory $\mathcal{B}$ with another such pair $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{C}}$, where $\widetilde{\mathcal{C}}$ is homotopically flat over the ground ring $\Bbbk$ (K-flat) [**Dri04**, Section 3.3], and there is a quasi-equivalence $\widetilde{\mathcal{C}} \to \mathcal{C}$ [**Dri04**, Section 2.3]. The first step is not needed, when $\mathcal{C}$ is already homotopically flat, for instance, when $\Bbbk$ is a field. In the second part, a new differential graded category $\mathcal{C}/\mathcal{B}$ is produced from a given pair $\mathcal{B} \subset \mathcal{C}$, by adding to $\mathcal{C}$ new morphisms $\varepsilon_U : U \to U$ of degree -1 for every object U of $\mathcal{B}$, such that $d(\varepsilon_U) = \mathrm{id}_U$.

In the present article we study an A_∞-analogue of the second part of Drinfeld's construction. Namely, to a given pair $\mathcal{B} \subset \mathcal{C}$ of an A_∞-category $\mathcal{C}$ and its full subcategory $\mathcal{B}$, we associate another A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ via a construction related to the bar resolution of $\mathcal{C}$. The A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ plays the role of the quotient of $\mathcal{C}$ over $\mathcal{B}$ in some cases, for instance, when $\Bbbk$ is a field. When $\mathcal{C}$ is a differential graded category, $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is precisely the category $\mathcal{C}/\mathcal{B}$ constructed by Drinfeld [**Dri04**, Section 3.1].

The research of V. L. was supported in part by grant 01.07/132 of State Fund for Fundamental Research of Ukraine.

Received April 13, 2003, revised October 30, 2003, August 7, 2006; published September 30, 2006.

2000 Mathematics Subject Classification: 16E45, 18G10, 18G55, 57T30.

Key words and phrases: A_∞-category, A_∞-functor, A_∞-transformation, K-injective resolution, quotient A_∞-category.

There exists another construction of quotient A_∞-category $\mathsf{q}(\mathcal{C}|\mathcal{B})$, see [**LM04**]. It enjoys some universal property, and is significantly bigger in size than $\mathsf{D}(\mathcal{C}|\mathcal{B})$. However, when $\mathcal{C}$ is unital, the two quotient constructions turn out to be A_∞-equivalent. When $\mathcal{C}$ is strictly unital, so is $\mathsf{D}(\mathcal{C}|\mathcal{B})$, while $\mathsf{q}(\mathcal{C}|\mathcal{B})$ is unital, but not strictly unital.

We apply our construction to the case of practical interest: $\mathcal{C} = \mathsf{C}(\mathcal{A})$ is the differential graded category of complexes in an Abelian category $\mathcal{A}$, and $\mathcal{B} = \mathsf{A}(\mathcal{A})$ is the full subcategory of acyclic complexes. When the full subcategory $\mathcal{I} \subset \mathcal{C}$ of K-injective complexes is big enough (every complex has a right K-injective resolution) and $\Bbbk$ is a field, we obtain an A_∞-functor $i : \mathcal{C} \to \mathcal{I}$, which assigns a K-injective resolution to a complex. Using it we get a new proof of the already known result: $H^0\big(\mathsf{D}(\mathsf{C}(\mathcal{A})|\mathsf{A}(\mathcal{A}))\big)$ is equivalent to the derived category $\mathsf{D}(\mathcal{A})$ of $\mathcal{A}$.

Outline of the article with comments

In the first section, we describe conventions and notations used in the article. In particular, we recall some conventions and useful formulas from [**Lyu03**].

In the second section, we describe a construction of the quotient A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$, departing from an A_∞-category $\mathcal{B}$ fully embedded into an A_∞-category $\mathcal{C}$. The underlying quiver of $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is described in Definition 2.1. Its particular case $\mathsf{D}(\mathcal{C}|\mathcal{C})$ is $s^{-1}T^+s\mathcal{C} = \oplus_{n>0}s^{-1}T^n s\mathcal{C}$, where $s\mathcal{C} = \mathcal{C}[1]$ stands for the suspended quiver $\mathcal{C}$. We introduce two A_∞-category structures for $s^{-1}T^+s\mathcal{C}$. The first, $\underline{\mathcal{C}} = (s^{-1}T^+s\mathcal{C}, \underline{b})$ uses the differential $\underline{b}$, whose components all vanish except $\underline{b}_1 = b : T^+s\mathcal{C} \to T^+s\mathcal{C}$, which is the A_∞-structure of $\mathcal{C}$. The second, $\overline{\mathcal{C}} = (s^{-1}T^+s\mathcal{C}, \overline{b})$ is isomorphic to the first via a coalgebra automorphism $\boldsymbol{\mu} : T(T^+s\mathcal{C}) \to T(T^+s\mathcal{C})$, whose components are multiplications in the tensor algebra $T^+s\mathcal{C}$. The resulting differential $\overline{b} = \boldsymbol{\mu}\underline{b}\boldsymbol{\mu}^{-1} : T(T^+s\mathcal{C}) \to T(T^+s\mathcal{C})$ is described componentwise in Proposition 2.2. The subquiver $\mathsf{D}(\mathcal{C}|\mathcal{B}) \subset \overline{\mathcal{C}}$ turns out to be an A_∞-subcategory (Proposition 2.2). $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ are, in a sense, trivial (they are contractible if $\mathcal{C}$ is unital), but $\mathsf{D}(\mathcal{C}|\mathcal{B})$, in general, is not trivial. If $\mathcal{C}$ is strictly unital, then so are $\overline{\mathcal{C}}$ and $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and their units are identified (Section 2.1). Notice that $\underline{\mathcal{C}}$ is never strictly unital except when $\mathcal{C} = 0$. Nevertheless, $\underline{\mathcal{C}}$ can be unital. When $\mathcal{B}$, $\mathcal{C}$ are differential graded categories, we show in Section 2.2, that $\mathsf{D}(\mathcal{C}|\mathcal{B})$ coincides with the category $\mathcal{C}/\mathcal{B}$ defined by Drinfeld [**Dri04**, Section 3.1].

In the third section, we construct functors between the obtained A_∞-categories. When $\mathcal{B} \subset \mathcal{C}$, $\mathcal{J} \subset \mathcal{I}$ are full A_∞-subcategories, and $i : \mathcal{C} \to \mathcal{I}$ is an A_∞-functor which maps objects of $\mathcal{B}$ into objects of $\mathcal{J}$, we construct a strict A_∞-functor $\underline{i} : \underline{\mathcal{C}} \to \underline{\mathcal{I}}$, whose first component $\underline{i}_1 = i : T^+s\mathcal{C} \to T^+s\mathcal{I}$ is given by i itself. The components of the conjugate A_∞-functor $\overline{\imath} = \boldsymbol{\mu}\underline{i}\boldsymbol{\mu}^{-1} : \overline{\mathcal{C}} \to \overline{\mathcal{I}}$ are described in Proposition 3.1. It turns out that $\overline{\imath}$ restricts to an A_∞-functor $\mathsf{D}(i) = \overline{\imath} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J})$ (Proposition 3.1). If $\mathcal{C}$ is strictly unital, then $\underline{\mathcal{C}}$ is unital (and contractible) and its unit transformation is computed in Section 3.1.

In the fourth section, we construct A_∞-transformations between functors obtained in the third section. When $\mathcal{B} \hookrightarrow \mathcal{C}$ and $\mathcal{J} \hookrightarrow \mathcal{I}$ are full A_∞-subcategories, $f, g : \mathcal{C} \to \mathcal{I}$ are A_∞-functors, which map objects of $\mathcal{B}$ into objects of $\mathcal{J}$, and $r : f \to g : \mathcal{C} \to \mathcal{I}$ is an A_∞-transformation, we construct an A_∞-transformation $\underline{r} : \underline{f} \to \underline{g} : \underline{\mathcal{C}} \to \underline{\mathcal{I}}$, whose only non-trivial components are $\underline{r}_0 = r_0$ and $\underline{r}_1 = r\big|_{T^+s\underline{\mathcal{C}}}$. The components of the conjugate A_∞-transformation $\overline{r} = \boldsymbol{\mu}\underline{r}\boldsymbol{\mu}^{-1} : \overline{f} \to \overline{g} : \overline{\mathcal{C}} \to \overline{\mathcal{J}}$ are computed in Proposition 4.1. It turns out that $\overline{r}$ restricts to an A_∞-transformation $\mathsf{D}(r) : \mathsf{D}(f) \to \mathsf{D}(g) : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{J}|\mathcal{J})$ (Proposition 4.1). Thus, $\underline{\text{-}}$ and $\overline{\text{-}}$ are defined as maps on objects, 1-morphisms and 2-morphisms of the 2-category A_∞ of A_∞-categories. Actually, they are strict 2-functors $A_\infty \to A_\infty$ (Corollaries 4.4 and 4.6). We prove more: they are strict $\mathcal{K}$-2-functors $\mathcal{K}A_\infty \to \mathcal{K}A_\infty$, where the 2-category $\mathcal{K}A_\infty$ is enriched in $\mathcal{K}$ — the category of differential graded $\Bbbk$-modules, whose morphisms are chain maps modulo homotopy (Proposition 4.3, Corollary 4.5). Compatibility of $\overline{\text{-}}$ with the composition of 2-morphisms is expressed via explicit homotopy (23). Components of this homotopy are found in Proposition 4.7. It turns out that this homotopy restricts to subcategories $\mathsf{D}(\text{-}|\text{-})$ (Proposition 4.7). Therefore, D is a $\mathcal{K}$-2-functor from the non-2-unital $\mathcal{K}$-2-category of pairs (A_∞-category, full A_∞-subcategory) to $\mathcal{K}A_\infty$ (Corollary 4.8). It can be viewed also as a 2-functor D from the non-2-unital 2-category of pairs (A_∞-category, full A_∞-subcategory) to A_∞ (Corollary 4.9).

In the fifth section, we consider unital A_∞-categories and prove that some of our A_∞-categories are contractible. If $\mathcal{B}$ is a full subcategory of a unital A_∞-category $\mathcal{C}$, then $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is unital as well, and $\mathsf{D}(\mathbf{i}^\mathcal{C})$ is its unit transformation (Proposition 5.1). In particular, for a unital A_∞-category $\mathcal{C}$, both $\overline{\mathcal{C}}$ and $\underline{\mathcal{C}}$ are unital with the unit transformation $\overline{\mathbf{i}^\mathcal{C}}$ (resp. $\underline{\mathbf{i}^\mathcal{C}}$) (Corollaries 5.2 and 5.3). If $i : \mathcal{C} \to \mathcal{J}$ is a unital A_∞-functor, then $\overline{\imath} : \overline{\mathcal{C}} \to \overline{\mathcal{J}}$, $\underline{i} : \underline{\mathcal{C}} \to \underline{\mathcal{J}}$ and (whenever defined) $\mathsf{D}(i) : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{J}|\mathcal{J})$ are unital as well (Corollaries 5.5 and 5.6). When we restrict $\underline{\text{-}}$, $\overline{\text{-}}$ or D to unital A_∞-categories (and unital A_∞-functors), we get strict 2-functors of (usual 1-2-unital) ($\mathcal{K}$-)2-categories.

In the sixth section, we consider contractible A_∞-categories and A_∞-functors. A unital A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is called *contractible* if many equivalent conditions hold, including contractibility of complexes $(s\mathcal{B}(Xf, Y), b_1)$, $(s\mathcal{B}(Y, Xf), b_1)$ for all $X \in \operatorname{Ob}\mathcal{A}$, $Y \in \operatorname{Ob}\mathcal{B}$ (Propositions 6.1 and 6.3, Definition 6.4). A unital A_∞-category $\mathcal{A}$ is called *contractible* if several equivalent conditions hold, including contractibility of complexes $(s\mathcal{A}(X, Y), b_1)$ for all objects X, Y of $\mathcal{A}$ (Definition 6.4, Proposition 6.7). If $\mathcal{C}$ is a unital A_∞-category, then $\underline{\mathcal{C}}$, $\overline{\mathcal{C}}$ are contractible (Example 6.5). Nevertheless, in general, the subcategories $\mathsf{D}(\mathcal{C}|\mathcal{B}) \subset \overline{\mathcal{C}}$ are not contractible. Contractible A_∞-categories $\mathcal{B}$ may be considered as trivial, because in this case any natural A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ is equivalent to 0 (Corollary 6.8). Moreover, non-empty contractible categories are equivalent to the 1-object-1-morphism A_∞-category $\mathbb{1}$, such that $\operatorname{Ob}\mathbb{1} = \{*\}$ and $\mathbb{1}(*, *) = 0$ (Proposition 6.7, Remark 6.9).

In the seventh section, we consider the case of a contractible full subcategory $\mathcal{F}$ of a unital A_∞-category $\mathcal{E}$. In this case the canonical strict embedding $\mathcal{E} \to \mathsf{D}(\mathcal{E}|\mathcal{F})$ is an equivalence (Proposition 7.4).

In the eighth section, we prepare to construct the K-injective resolution A_∞-functor. This concrete construction is deferred until the next section. In the eighth section we consider an abstract version of it. Given an A_∞-functor $f : \mathcal{B} \to \mathcal{C}$, a map

$g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ and cycles $r_X \in \mathcal{C}^0(Xf, Xg)$, $X \in \operatorname{Ob}\mathcal{B}$, producing certain quasi-isomorphisms, we make g into an A_∞-functor $g : \mathcal{B} \to \mathcal{C}$ and r_X into 0-th component ${}_X r_0 s^{-1}$ of a natural A_∞-transformation $r : f \to g : \mathcal{B} \to \mathcal{C}$ (Proposition 8.1). Next we prove the uniqueness of so constructed g and r. Assuming that the initial data $(g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}, (r_X)_{x \in \operatorname{Ob}\mathcal{B}})$ give rise to two A_∞-functors $g, g' : \mathcal{B} \to \mathcal{C}$ and two natural A_∞-transformations $r : f \to g : \mathcal{B} \to \mathcal{C}$, $r' : f \to g' : \mathcal{B} \to \mathcal{C}$, we construct another natural A_∞-transformation $p : g \to g' : \mathcal{B} \to \mathcal{C}$, such that r' is the composition $(f \xrightarrow{r} g \xrightarrow{p} g')$ in the 2-category A_∞ (Proposition 8.2). Moreover, such p is unique up to an equivalence (Proposition 8.3). If, in addition, $\mathcal{C}$ is unital, then the constructed p is invertible (Corollary 8.4). If f is unital, then the constructed A_∞-functor g is unital as well (Proposition 8.5).

In the ninth section, we consider categories of complexes. Let $\Bbbk$ be a field, let $\mathcal{A}$ be an Abelian $\Bbbk$-linear category, and let $\mathcal{C} = \mathsf{C}(\mathcal{A})$ be the differential graded category of complexes in $\mathcal{A}$. Let $\mathcal{B} = \mathsf{A}(\mathcal{A})$ be its full subcategory of acyclic complexes, $\mathcal{I} = \mathsf{I}(\mathcal{A})$ denotes K-injective complexes, $\mathcal{J} = \mathsf{AI}(\mathcal{A})$ denotes acyclic K-injective complexes. We assume that each complex $X \in \operatorname{Ob}\mathcal{C}$ has a right K-injective resolution $r_X : X \to Xi$ (a quasi-isomorphism with K-injective $Xi \in \operatorname{Ob}\mathcal{I}$). We notice that quasi-isomorphisms from $\mathcal{C}$ become "invertible modulo boundary" in the differential graded category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ (Section 9.1). From the identity functor $f = \mathrm{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$, a map $g : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}$, $X \mapsto Xi$ and quasi-isomorphisms r_X we produce an A_∞-functor $g : \mathcal{C} \to \mathcal{C}$, which factors as $g = \big(\mathcal{C} \xrightarrow{i} \mathcal{I} \xhookrightarrow{e} \mathcal{C}\big)$, into "K-injective resolution" unital A_∞-functor i and the full embedding e (Section 9.2). The unital A_∞-functor $\bar\imath : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J})$ and the faithful differential graded functor $\bar e : \mathsf{D}(\mathcal{I}|\mathcal{J}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$ are A_∞-equivalences quasi-inverse to each other. Due to contractibility of $\mathcal{J}$ the natural embedding $\mathcal{I} \to \mathsf{D}(\mathcal{I}|\mathcal{J})$ is an equivalence. Since $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\mathcal{I}$ are A_∞-equivalent, their 0-th cohomology categories are equivalent as usual $\Bbbk$-linear categories. That is, $H^0(\mathsf{D}(\mathcal{C}|\mathcal{B})) = H^0\big(\mathsf{D}(\mathsf{C}(\mathcal{A})|\mathsf{A}(\mathcal{A}))\big)$ is equivalent to $H^0(\mathcal{I}) = H^0(\mathsf{I}(\mathcal{A}))$ — homotopy category of K-injective complexes, which is equivalent to the derived category $\mathsf{D}(\mathcal{A})$ of $\mathcal{A}$. This result (Section 9.2) motivated our studies. It follows also from Drinfeld's theory of quotients of differential graded categories [**Dri04**]. This agrees with Bondal and Kapranov's proposal to produce triangulated categories as homotopy categories of some differential graded categories [**BK90**].

1. Conventions

We keep the notations and conventions of [**Lyu03**], sometimes without explicitly mentioning them. Some of the conventions are recalled here.

We assume as in [**Lyu03**] that quivers, A_∞-categories, etc. are small with respect to some universe $\mathscr{U}$.

The ground ring $\Bbbk \in \mathscr{U}$ is a unital associative commutative ring.

We use the right operators: the composition of two maps (or morphisms) $f : X \to Y$ and $g : Y \to Z$ is denoted $fg : X \to Z$; a map is written on elements as $f : x \mapsto xf = (x)f$. However, these conventions are not used systematically, and $f(x)$ might be used instead.

If C is a $\mathbb{Z}$-graded $\Bbbk$-module, then $sC = C[1]$ denotes the same $\Bbbk$-module with the grading $(sC)^d = C^{d+1}$, the suspension of C. The shift "identity" map $C \to sC$ of degree -1 is also denoted s. Getzler and Jones demonstrated in [**GJ90**] that the suspension s and the shift map s are useful in the theory of A_∞-algebras. We follow the Koszul sign convention:

$$(x \otimes y)(f \otimes g) = (-)^{yf} xf \otimes yg = (-1)^{\deg y \cdot \deg f} xf \otimes yg.$$

A chain complex is called *contractible* if its identity endomorphism is homotopic to zero.

The category $\mathscr{Q}/S$ of $\mathscr{U}$-small graded $\Bbbk$-linear quivers with fixed set of objects S admits a monoidal structure with the tensor product $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A} \otimes \mathcal{B}$, $(\mathcal{A} \otimes \mathcal{B})(X, Y) = \oplus_{Z \in S} \mathcal{A}(X, Z) \otimes_\Bbbk \mathcal{B}(Z, Y)$. Thus, we have tensor powers $T^n\mathcal{A} = \mathcal{A}^{\otimes n}$ of a given graded $\Bbbk$-quiver $\mathcal{A}$, such that $\operatorname{Ob} T^n\mathcal{A} = \operatorname{Ob}\mathcal{A}$. Explicitly,

$$T^n\mathcal{A}(X, Y) = \bigoplus_{X_1,\dots,X_{n-1} \in \operatorname{Ob}\mathcal{A}} \mathcal{A}(X_0, X_1) \otimes_\Bbbk \mathcal{A}(X_1, X_2) \otimes_\Bbbk \cdots \otimes_\Bbbk \mathcal{A}(X_{n-1}, X_n),$$

where $X_0 = X$ and $X_n = Y$. In particular, $T^0\mathcal{A}$ denotes the unit object $\Bbbk S$, where $\Bbbk S(X, Y) = \Bbbk$ if $X = Y$ and vanishes otherwise.

As in any monoidal category, there is a notion of coassociative coalgebras $(\mathcal{B}, \overline{\Delta} : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B})$ in $\mathscr{Q}/S$ (in general, without counit). For A_∞-category theory, we need only coalgebras $(\mathcal{B}, \overline{\Delta})$ in $\mathscr{Q}/S$ that satisfy an additional requirement: for all $X, Y \in S$

$$\mathcal{B}(X, Y) = \cup_{k=2}^{\infty} \operatorname{Ker}\bigl(\overline{\Delta}^{(k)} : \mathcal{B}(X, Y) \to \mathcal{B}^{\otimes k}(X, Y)\bigr),$$

where $\overline{\Delta}^{(2)} = \overline{\Delta}$, $\overline{\Delta}^{(3)} = \overline{\Delta}(1 \otimes \overline{\Delta}) = \overline{\Delta}(\overline{\Delta} \otimes 1) : \mathcal{B} \to \mathcal{B}^{\otimes 3}$, etc. Such coalgebras are named *cocomplete cocategories* by Keller [**Kel05**]. A counital coassociative coalgebra $(\mathcal{A} = T^0\mathcal{B} \oplus \mathcal{B}, \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \varepsilon : \mathcal{A} \to T^0\mathcal{A})$ in $\mathscr{Q}/S$ is associated with $(\mathcal{B}, \overline{\Delta})$, namely:

$$\begin{aligned}
\Delta\big|_{T^0\mathcal{B}} &= \bigl(T^0\mathcal{B} \xrightarrow{\sim} T^0\mathcal{B} \otimes T^0\mathcal{B} \xhookrightarrow{\mathrm{in}_0 \otimes \mathrm{in}_0} \mathcal{A} \otimes \mathcal{A}\bigr), \\
\Delta\big|_{\mathcal{B}} &= \bigl(\mathcal{B} \xrightarrow{\sim} \mathcal{B} \otimes T^0\mathcal{B} \xhookrightarrow{\mathrm{in}_1 \otimes \mathrm{in}_0} \mathcal{A} \otimes \mathcal{A}\bigr) \\
&\quad + \bigl(\mathcal{B} \xrightarrow{\overline{\Delta}} \mathcal{B} \otimes \mathcal{B} \xhookrightarrow{\mathrm{in}_1 \otimes \mathrm{in}_1} \mathcal{A} \otimes \mathcal{A}\bigr) \\
&\quad + \bigl(\mathcal{B} \xrightarrow{\sim} T^0\mathcal{B} \otimes \mathcal{B} \xhookrightarrow{\mathrm{in}_0 \otimes \mathrm{in}_1} \mathcal{A} \otimes \mathcal{A}\bigr),
\end{aligned}$$

or simply $f\Delta = f \otimes 1 + f\overline{\Delta} + 1 \otimes f$ for $f \in \mathcal{B}(X, Y)$, and $\varepsilon = \mathrm{pr}_0 : \mathcal{A} \to T^0\mathcal{B} = T^0\mathcal{A}$. The triple $(\mathcal{A}, \Delta, \varepsilon)$ is a *cocategory* in the sense of [**Lyu03**]. In the present article, we shall use only one kind of cocategory associated with quivers $\mathcal{C}$, namely, the cocomplete cocategory $\mathcal{B} = T^+\mathcal{C} = \oplus_{n=1}^{\infty} T^n\mathcal{C}$, equipped with the comultiplication $\overline{\Delta} : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$, $(h_1 \otimes h_2 \otimes \cdots \otimes h_n)\overline{\Delta} = \sum_{k=1}^{n-1} h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n$, gives rise to the cocategory $\mathcal{A} = T\mathcal{C} = \oplus_{n=0}^{\infty} T^n\mathcal{C}$, equipped with the cut comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $(h_1 \otimes h_2 \otimes \cdots \otimes h_n)\Delta = \sum_{k=0}^{n} h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n$, and with the counit $\varepsilon = \mathrm{pr}_0 : \mathcal{A} \to T^0\mathcal{C} = T^0\mathcal{A}$.

By definition, cocategory homomorphisms (in particular, A_∞-functors) respect the cut comultiplication Δ, and A_∞-transformations are coderivations with respect to Δ (see e.g. [**Lyu03**]).

We use the following standard equations for a differential b in an A_∞-category

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^k s\mathcal{A} \to s\mathcal{A}. \tag{1}$$

Since b is a differential and a coderivation, it may be called a *codifferential*. Commutation relation $fb = bf$ for an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ expands to the following

$$\sum_{l>0; i_1+\dots+i_l=k} (f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_l}) b_l = \sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t} : T^k s\mathcal{A} \to s\mathcal{B}. \tag{2}$$

Given A_∞-functors $f, g, h : \mathcal{B} \to \mathcal{C}$ and coderivations $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{B} \to \mathcal{C}$ of arbitrary degree we construct a map $\theta : Ts\mathcal{B} \to Ts\mathcal{C}$ as in Section 3 of [**Lyu03**]. We view θ as a bilinear function $(r \otimes p)\theta$ of r, p. Its components $\theta_{kl} = \theta\big|_{T^k s\mathcal{B}} \operatorname{pr}_l : T^k s\mathcal{B} \to T^l s\mathcal{C}$ are given by Formula (3.0.1) of [**Lyu03**]

$$\theta_{kl} = \sum f_{a_1} \otimes \dots \otimes f_{a_\alpha} \otimes r_j \otimes g_{c_1} \otimes \dots \otimes g_{c_\beta} \otimes p_t \otimes h_{e_1} \otimes \dots \otimes h_{e_\gamma}, \tag{3}$$

where the summation is taken over all terms with

$$\alpha + \beta + \gamma + 2 = l, \qquad a_1 + \dots + a_\alpha + j + c_1 + \dots + c_\beta + t + e_1 + \dots + e_\gamma = k.$$

The same formula can be presented as

$$\theta_{kl} = \sum_{\substack{\alpha+\beta+\gamma+2=l \\ a+j+c+t+e=k}} f_{a\alpha} \otimes r_j \otimes g_{c\beta} \otimes p_t \otimes h_{e\gamma}, \tag{4}$$

where $f_{a\alpha} : T^a\mathcal{A} \to T^\alpha\mathcal{B}$ are matrix elements of f and similarly for g, h. By Proposition 3.1 of [**Lyu03**] the map θ satisfies the equation

$$\theta\Delta = \Delta(f \otimes \theta + r \otimes p + \theta \otimes h).$$

Given A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, one constructs an A_∞-category $A_\infty(\mathcal{A}, \mathcal{B})$ of A_∞-functors $\mathcal{A} \to \mathcal{B}$, equipped with a differential B [**Fuk02, Kel05, KS06, KS, LH03**], [**Lyu03**, Section 5].

The category of graded $\Bbbk$-linear quivers admits a symmetric monoidal structure with the tensor product $\mathcal{A} \times \mathcal{B} \mapsto \mathcal{A} \boxtimes \mathcal{B}$, where $\operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B} = \operatorname{Ob} \mathcal{A} \times \operatorname{Ob} \mathcal{B}$ and $(\mathcal{A} \boxtimes \mathcal{B})\big((X, U), (Y, V)\big) = \mathcal{A}(X, Y) \otimes_\Bbbk \mathcal{B}(U, V)$. The same tensor product was denoted $\otimes$ in [**Lyu03**]. Given A_∞-categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, there is a graded cocategory morphism of degree 0

$$M : TsA_\infty(\mathcal{A}, \mathcal{B}) \boxtimes TsA_\infty(\mathcal{B}, \mathcal{C}) \to TsA_\infty(\mathcal{A}, \mathcal{C}),$$

which satisfies equation $(1 \boxtimes B + B \boxtimes 1)M = MB$ [**Lyu03**, Section 6].

2. An A_∞-category

Let $\mathcal{B} \hookrightarrow \mathcal{C}$ be a full A_∞-subcategory. It means that $\operatorname{Ob} \mathcal{B} \subset \operatorname{Ob} \mathcal{C}$, $\mathcal{B}(X, Y) = \mathcal{C}(X, Y)$ for all $X, Y \in \operatorname{Ob} \mathcal{B}$, and the operations for $\mathcal{B}$ coincide with those for $\mathcal{C}$. Let us define another A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$. If $\mathcal{B}$, $\mathcal{C}$ are differential graded categories, then $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is differential graded as well, and it coincides with the category $\mathcal{C}/\mathcal{B}$ defined by Drinfeld in [**Dri04**, Section 3.1].

Definition 2.1. Let $T^+s\mathcal{C} = \oplus_{n>0} T^n s\mathcal{C}$ and $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ be the following graded $\Bbbk$-quivers: the class of objects is $\operatorname{Ob} T^+s\mathcal{C} = \operatorname{Ob}\mathcal{E} = \operatorname{Ob}\mathcal{C}$, the modules of morphisms for $X, Y \in \operatorname{Ob}\mathcal{E}$ are

$$T^+s\mathcal{C}(X,Y)$$
$$= \oplus_{C_1,\dots,C_{n-1}\in \operatorname{Ob}\mathcal{C}} s\mathcal{C}(X,C_1)\otimes s\mathcal{C}(C_1,C_2)\otimes\cdots\otimes s\mathcal{C}(C_{n-2},C_{n-1})\otimes s\mathcal{C}(C_{n-1},Y),$$
$$s\mathcal{E}(X,Y)$$
$$= \oplus_{C_1,\dots,C_{n-1}\in \operatorname{Ob}\mathcal{B}} s\mathcal{C}(X,C_1)\otimes s\mathcal{C}(C_1,C_2)\otimes\cdots\otimes s\mathcal{C}(C_{n-2},C_{n-1})\otimes s\mathcal{C}(C_{n-1},Y),$$

where in the second case summation extends over all sequences of objects $(C_1, \dots, C_{n-1})$ of $\mathcal{B}$. To the empty sequence ($n = 1$) corresponds the summand $s\mathcal{C}(X, Y)$.

Let us endow $s^{-1}T^+s\mathcal{C}$ with a structure of A_∞-category, given by $\underline{b} : T(T^+s\mathcal{C}) \to T(T^+s\mathcal{C})$, with the components $\underline{b}_0 = 0$, $\underline{b}_1 = b : T^+s\mathcal{C} \to T^+s\mathcal{C}$, $\underline{b}_k = 0$ for $k > 1$. This A_∞-category is denoted $\underline{\mathcal{C}} = (s^{-1}T^+s\mathcal{C}, \underline{b})$. There is an A_∞-functor $\underline{j} : \mathcal{C} \to (s^{-1}T^+s\mathcal{C}, \underline{b})$, specified by its components $\underline{j}_k : T^k s\mathcal{C} \to T^+s\mathcal{C}$, $k \geqslant 1$, where $\underline{j}_k$ is the canonical embedding of the direct summand. The property $b\underline{j} = \underline{j}\underline{b}$, or

$$\sum_{r+k+t=n} (1^{\otimes r}\otimes b_k \otimes 1^{\otimes t})\underline{j}_{r+1+t} = \underline{j}_n b : T^n s\mathcal{C} \to T^+s\mathcal{C},$$

is clear — this is just the expression of b in terms of its components.

There is a coalgebra automorphism $\boldsymbol{\mu} : TT^+s\mathcal{C} \to TT^+s\mathcal{C}$, specified by its components $\boldsymbol{\mu}_k = \mu^{(k)} : T^kT^+s\mathcal{C} \to T^+s\mathcal{C}$, $k \geqslant 1$, where $\mu : T^+s\mathcal{C}\otimes T^+s\mathcal{C} \to T^+s\mathcal{C}$ is the multiplication in the tensor algebra, $\mu^{(k)} = 0$ for $k \leqslant 0$, $\mu^{(1)} = 1 : T^+s\mathcal{C} \to T^+s\mathcal{C}$, $\mu^{(2)} = \mu$, $\mu^{(3)} = (\mu\otimes 1)\mu : (T^+s\mathcal{C})^{\otimes 3} \to T^+s\mathcal{C}$ and so on. Its inverse is the coalgebra automorphism $\boldsymbol{\mu}^{-1} = \boldsymbol{\mu}^- : TT^+s\mathcal{C} \to TT^+s\mathcal{C}$, specified by its components $\boldsymbol{\mu}^-_k = (-)^{k-1}\mu^{(k)} : T^kT^+s\mathcal{C} \to T^+s\mathcal{C}$. The fact that $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^-$ are inverse to each other is proven as follows:

$$(\boldsymbol{\mu}\boldsymbol{\mu}^-)_n = \sum_{l_1+\dots+l_k=n} (\boldsymbol{\mu}_{l_1}\otimes\cdots\otimes\boldsymbol{\mu}_{l_k})\boldsymbol{\mu}^-_k = \sum_{l_1+\dots+l_k=n} (-)^{k-1}\mu^{(n)}$$
$$= \mu^{(n)} \sum_{k=1}^{n} (-1)^{k-1}\binom{n-1}{k-1} = \mu^{(n)}(1-1)^{n-1},$$

which equals id for $n = 1$ and vanishes for $n > 1$. Similarly, $\boldsymbol{\mu}^-\boldsymbol{\mu} = \mathrm{id}$.

Proposition 2.2. *The conjugate codifferential $\bar{b} = \boldsymbol{\mu}\underline{b}\boldsymbol{\mu}^{-1} : T(T^+s\mathcal{C}) \to T(T^+s\mathcal{C})$ has the following components: $\bar{b}_0 = 0$, $\bar{b}_1 = b$ and for $n \geqslant 2$*

$$\bar{b}_n = \mu^{(n)} \sum_{m;q<k;t<l} 1^{\otimes q}\otimes b_m \otimes 1^{\otimes t} : T^k s\mathcal{C}\otimes (T^+s\mathcal{C})^{\otimes n-2}\otimes T^l s\mathcal{C} \to T^+s\mathcal{C}, \tag{5}$$

$$\bar{b}_n = \mu^{(n)}b - (1\otimes\mu^{(n-1)}b)\mu - (\mu^{(n-1)}b\otimes 1)\mu + (1\otimes\mu^{(n-2)}b\otimes 1)\mu^{(3)} : (T^+s\mathcal{C})^{\otimes n} \to T^+s\mathcal{C}, \tag{6}$$

for all $n \geqslant 0$. The operations $\bar{b}_n$ restrict to maps $s\mathcal{E}^{\otimes n} \to s\mathcal{E}$ via the natural embedding $s\mathcal{E} \subset T^+s\mathcal{C}$ of graded $\Bbbk$-quivers. Hence, $\bar{b}$ turns $\mathcal{E}$ and $\overline{\mathcal{C}} \overset{def}{=} (s^{-1}T^+s\mathcal{C}, \bar{b})$ into an A_∞-category.

Proof. Let us define a $(1,1)$-coderivation $\bar{b}$ of degree 1 by its components (6). Substituting the definition of b via its components, we get formula (5). Clearly, $\boldsymbol{\mu}\underline{b}\boldsymbol{\mu}^{-1}$ is also a $(1,1)$-coderivation of degree 1. Let us show that it coincides with $\bar{b}$, that is, $\bar{b}\boldsymbol{\mu} = \boldsymbol{\mu}\underline{b}$. This equation expands to

$$\sum_{r+k+t=n} (1^{\otimes r} \otimes \bar{b}_k \otimes 1^{\otimes t})\mu^{(r+1+t)} = \mu^{(n)} b = \mu^{(n)} \underline{b}_1,$$

which follows immediately from (5) and from the standard expression of b via its components.

Clearly, $\bar{b}^2 = \boldsymbol{\mu}\underline{b}^2\boldsymbol{\mu}^{-1} = 0$, hence, $\overline{\mathcal{C}} = (s^{-1}T^+ s\mathcal{C}, \bar{b})$ is an A_∞-category. Map (5) is a sum of maps of the form

$$\begin{aligned}
&1^{\otimes q} \otimes b_m \otimes 1^{\otimes t} : s\mathcal{C}(X, C_1) \otimes \cdots \otimes s\mathcal{C}(C_{q-1}, C_q) \otimes s\mathcal{C}(C_q, C_{q+1}) \otimes \cdots \\
&\otimes (T^+ s\mathcal{C})^{\otimes n-2} \otimes \cdots \otimes s\mathcal{C}(D_{l-t-1}, D_{l-t}) \otimes s\mathcal{C}(D_{l-t}, D_{l-t+1}) \otimes \cdots \otimes \\
&s\mathcal{C}(D_{l-1}, Y) \to s\mathcal{C}(X, C_1) \otimes \cdots \otimes s\mathcal{C}(C_{q-1}, C_q) \otimes s\mathcal{C}(C_q, D_{l-t}) \otimes \\
&s\mathcal{C}(D_{l-t}, D_{l-t+1}) \otimes \cdots \otimes s\mathcal{C}(D_{l-1}, Y),
\end{aligned}$$

where $C_0 = X$ for $q = 0$ and $D_l = Y$ for $t = 0$. If the source is contained in $(s\mathcal{E})^{\otimes n}(X, Y)$, then C_i, D_j are in $\operatorname{Ob}\mathcal{B}$ for all $0 < i < k$, $0 < j < l$. Therefore, the target is a direct summand of $s\mathcal{E}(X, Y)$. Thus the maps $\bar{b}_n$ restrict to maps $\bar{b}_n : (s\mathcal{E})^{\otimes n}(X, Y) \to s\mathcal{E}(X, Y)$. The obtained (1,1)-coderivation $\bar{b} : Ts\mathcal{E} \to Ts\mathcal{E}$ also satisfies $\bar{b}^2 = 0$. Thus it makes $\mathcal{E}$ into an A_∞-category. $\square$

In particular, (6) gives

$$\begin{aligned}
\bar{b}_2 &= \mu b - (1 \otimes b + b \otimes 1)\mu, \\
\bar{b}_3 &= \mu^{(3)} b - (1 \otimes \mu b)\mu - (\mu b \otimes 1)\mu + (1 \otimes b \otimes 1)\mu^{(3)}, \\
\bar{b}_4 &= \mu^{(4)} b - (1 \otimes \mu^{(3)} b)\mu - (\mu^{(3)} b \otimes 1)\mu + (1 \otimes \mu b \otimes 1)\mu^{(3)}.
\end{aligned}$$

Remark 2.3. Let $\mathcal{A}$ be an A_∞-category, defined by a codifferential $b : Ts\mathcal{A} \to Ts\mathcal{A}$, let $\mathcal{B}$ be a graded $\Bbbk$-quiver and let $f : Ts\mathcal{A} \to Ts\mathcal{B}$ (resp. $g : Ts\mathcal{B} \to Ts\mathcal{A}$) be an isomorphism of graded cocategories. Then the codifferential $f^{-1}bf$ (resp. gbg^{-1}) is the unique codifferential on $Ts\mathcal{B}$, which turns f (resp. g) into an invertible A_∞-functor between $\mathcal{A}$ and $\mathcal{B}$.

Corollary 2.4. *The coalgebra isomorphism* $\boldsymbol{\mu}^{-1} : \underline{\mathcal{C}} = (s^{-1}T^+ s\mathcal{C}, \underline{b}) \to \overline{\mathcal{C}} = (s^{-1}T^+ s\mathcal{C}, \bar{b})$ *is an* A_∞*-functor. Its composition with* $\underline{j}$ *is a strict* A_∞*-functor* $\bar{\jmath} = \underline{j}\boldsymbol{\mu}^{-1} : \mathcal{C} \to \mathsf{D}(\mathcal{C}|\mathcal{B})$, $X \mapsto X$, *whose components are the direct summand embedding* $\bar{\jmath}_1 : s\mathcal{C}(X, Y) = T^1 s\mathcal{C}(X, Y) \hookrightarrow s\mathcal{E}(X, Y)$ *and* $\bar{\jmath}_n = 0$ *for* $n > 1$.

Indeed,

$$\bar{\jmath}_n = \sum_{l_1+\cdots+l_k=n} (\underline{j}_{l_1} \otimes \cdots \otimes \underline{j}_{l_k})(-1)^{k-1}\mu^{(k)} = \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \underline{j}_n = (1-1)^{n-1} \underline{j}_n,$$

which equals $\underline{j}_1$ for $n = 1$, and vanishes for $n > 1$.

2.1. Strict unitality.

Assume that A_∞-category $\mathcal{C}$ is strictly unital. It means that for each object X of $\mathcal{C}$ there is an element $1_X \in \mathcal{C}^0(X,X)$, such that the map $\mathbf{i}_0^{\mathcal{C}} : \Bbbk \to (s\mathcal{C})^{-1}(X,X)$, $1 \mapsto 1_X s$ of degree -1 satisfies equations $(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 = 1 : s\mathcal{C}(Y,X) \to s\mathcal{C}(Y,X)$ and $(\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 = -1 : s\mathcal{C}(X,Z) \to s\mathcal{C}(X,Z)$ for all $Y, Z \in \operatorname{Ob}\mathcal{C}$, and $(\cdots \otimes 1_X s \otimes \dots)b_n = 0$ if $n \neq 2$. Since $\mathcal{C}$ is strictly unital, its full A_∞-subcategory $\mathcal{B}$ is strictly unital as well.

Let us show that in these assumptions $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ is also strictly unital. We take the same elements $1_X \in \mathcal{C}^0(X,X) \subset \mathcal{E}^0(X,X)$ as strict units of $\mathcal{E}$. We have $1_X s\bar{b}_1 = 1_X sb = 1_X sb_1 = 0$. Explicit formulas give $(\cdots \otimes 1_X s \otimes \dots)\bar{b}_n = 0$ for $n > 2$. The map $\bar{b}_2 : T^k s\mathcal{C}(Y,X) \otimes s\mathcal{C}(X,X) \to T^+ s\mathcal{C}(Y,X)$ is the sum of maps

$$1^{\otimes k-t} \otimes b_{t+1} : s\mathcal{C}(Y,C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-t},C_{k-t+1}) \otimes \cdots \otimes s\mathcal{C}(C_{k-1},X) \otimes s\mathcal{C}(X,X) \\ \to s\mathcal{C}(Y,C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-t},X)$$

over $t > 0$. Therefore, the map $\mathbf{i}_0^{\mathcal{E}} : \Bbbk \to (s\mathcal{E})^{-1}(X,X)$, $1 \mapsto 1_X s$ satisfies equations $(1 \otimes \mathbf{i}_0^{\mathcal{E}})\bar{b}_2 = (1^{\otimes k} \otimes \mathbf{i}_0^{\mathcal{C}})(1^{\otimes k-1} \otimes b_2) = 1^{\otimes k-1} \otimes 1 = 1$. Similarly, for $\bar{b}_2 : s\mathcal{C}(X,X) \otimes T^k s\mathcal{C}(X,Z) \to T^+ s\mathcal{C}(X,Z)$ we have $(\mathbf{i}_0^{\mathcal{E}} \otimes 1)\bar{b}_2 = (\mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes k})(b_2 \otimes 1^{\otimes k-1}) = -1 \otimes 1^{\otimes k-1} = -1$. Therefore, $\mathcal{E}$ and $\overline{\mathcal{C}}$ are strictly unital with the unit $\mathbf{i}^{\mathcal{E}}$.

2.2. Differential graded categories.

If $b_k = 0$ for $k > 2$, then explicit formulae in the case of $\mathcal{E}$ show that we also have $\bar{b}_k = 0$ for $k > 2$. Combining this fact with the above unitality considerations, we see that if $\mathcal{C}$ is a differential graded category, then so is $\mathsf{D}(\mathcal{C}|\mathcal{B})$. The differential graded category $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B}) = \mathcal{C}/\mathcal{B}$ was constructed by Drinfeld [**Dri04**, Section 3.1]. This construction was a starting point of the present article. Let us describe it in detail.

Write down elements of $\mathcal{E}(X,Y)$ as sequences $f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_{n-1}}f_n$, where $f_i \in \mathcal{C}(C_{i-1},C_i)$, $C_0 = X$, $C_n = Y$, and $C_i \in \operatorname{Ob}\mathcal{B}$ for $0 < i < n$. The symbol ε_C for $C \in \operatorname{Ob}\mathcal{B}$ is assigned degree -1. Its differential is set equal to $\varepsilon_C d = 1_C$. The graded Leibniz rule gives

$$\begin{aligned}(f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_{n-1}}f_n)d &\\ = \sum_{q+1+t=n} &(-)^{f_{n-t+1}+\dots+f_n-t} f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_q}(f_{q+1}m_1)\varepsilon_{C_{q+1}}f_{q+2}\dots\varepsilon_{C_{n-1}}f_n \\ + \sum_{q+2+t=n} &(-)^{f_{n-t}+\dots+f_n-t} f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_q}(f_{q+1}\cdot f_{q+2})\varepsilon_{C_{q+2}}f_{q+3}\dots\varepsilon_{C_{n-1}}f_n,\end{aligned}$$

where $f_{q+1}\cdot f_{q+2} = (f_{q+1} \otimes f_{q+2})m_2$ is the composition. Introduce a degree -1 map

$$s : \mathcal{E} \to s\mathcal{E} \subset T^+ s\mathcal{C}, \quad f_1\varepsilon_{C_1}f_2\dots\varepsilon_{C_{n-1}}f_n \mapsto f_1 s \otimes f_2 s \otimes \cdots \otimes f_n s.$$

One can check that $ds = s\bar{b}_1$, where, naturally, $\bar{b}_1 = b = \sum_{q+1+t=n} 1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t} + \sum_{q+2+t=n} 1^{\otimes q} \otimes b_2 \otimes 1^{\otimes t}$.

The composition $\bar{m}_2$ in $\mathcal{E}$ consists of the concatenation and the composition m_2 in $\mathcal{C}$:

$$(f_1\varepsilon_{C_1}\dots f_{n-1}\varepsilon_{C_{n-1}}f_n\otimes g_1\varepsilon_{D_1}g_2\dots\varepsilon_{D_{m-1}}g_m)\bar m_2 \\ = f_1\varepsilon_{C_1}\dots f_{n-1}\varepsilon_{C_{n-1}}(f_n\cdot g_1)\varepsilon_{D_1}g_2\dots\varepsilon_{D_{m-1}}g_m.$$

One can check that $\bar m_2 s = (s\otimes s)\bar b_2$; here $\bar b_2 = 1^{\otimes n-1}\otimes b_2\otimes 1^{\otimes m-1}$.

Specifically, this construction applies to the case of the differential graded category $\mathcal{C} = \mathsf{C}(\mathcal{A})$ of complexes of objects of an Abelian category $\mathcal{A}$. One may take for $\mathcal{B}$ the subcategory of acyclic complexes $\mathcal{B} = \mathsf{A}(\mathcal{A})$.

3. An A_∞-functor

Let $\mathcal{B}\hookrightarrow\mathcal{C}$, $\mathcal{J}\hookrightarrow\mathcal{I}$ be full A_∞-subcategories. Let $i:\mathcal{C}\to\mathcal{I}$ be an A_∞-functor, such that $Xi\in\operatorname{Ob}\mathcal{J}$ for $X\in\operatorname{Ob}\mathcal{B}$. Then it restricts to an A_∞-functor $\mathcal{B}\to\mathcal{J}$, denoted by i'. We are going to construct an extension of this functor to the A_∞-categories $\mathcal{E} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\mathcal{F} = \mathsf{D}(\mathcal{I}|\mathcal{J})$.

Let us begin with a strict A_∞-functor $\underline{i}:\underline{\mathcal{C}}\to\underline{\mathcal{I}}$, given by its components $\underline{i}_1 = i : T^+s\mathcal{C}\to T^+s\mathcal{I}$ and $\underline{i}_k = 0$ for $k>1$. The equation $\underline{i}\,\underline{b} = \underline{b}\,\underline{i}$ reduces to familiar $ib = bi$. Therefore, $\bar\imath \overset{\text{def}}{=} \boldsymbol{\mu}\underline{i}\boldsymbol{\mu}^{-1} : \overline{\mathcal{C}}\to\overline{\mathcal{I}}$ is an A_∞-functor as well.

The following diagram of A_∞-functors commutes

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook] \arrow[d, "i'"'] & \mathcal{C} \arrow[r, "\underline{j}^{\mathcal{C}}"] \arrow[d, "i"] & \underline{\mathcal{C}} \arrow[r, "\boldsymbol{\mu}^{-1}"] \arrow[d, "\underline{i}"] & \overline{\mathcal{C}} \arrow[d, "\bar\imath"] \\ \mathcal{J} \arrow[r, hook] & \mathcal{I} \arrow[r, "\underline{j}^{\mathcal{I}}"] & \underline{\mathcal{I}} \arrow[r, "\boldsymbol{\mu}^{-1}"] & \overline{\mathcal{I}} \end{tikzcd}$$

Indeed, $\underline{j}^{\mathcal{C}}\underline{i} = i\underline{j}^{\mathcal{J}}$ expands to

$$\underline{j}^{\mathcal{C}}_n i = \sum_{l_1+\dots+l_k=n}(i_{l_1}\otimes\dots\otimes i_{l_k})\underline{j}^{\mathcal{J}}_k : T^n s\mathcal{C}\to T^+s\mathcal{I},$$

which expresses i in terms of its components.

Proposition 3.1. *The A_∞-functor $\bar\imath$ has the following components:*

$$\bar\imath_n = \sum_{l_1+\dots+l_k=n}(-)^{k-1}\big(\mu^{(l_1)}\otimes\dots\otimes\mu^{(l_k)}\big)i^{\otimes k}\mu^{(k)} : (T^+s\mathcal{C})^{\otimes n}\to T^+s\mathcal{I}. \tag{7}$$

The restriction of this map to $T^{k_1}s\mathcal{C}\otimes\dots\otimes T^{k_n}s\mathcal{C}$ is

$$\bar\imath_n = \mu^{(n)}\sum_{(l_1,\dots,l_t)\in L(k_1,\dots,k_n)}(i_{l_1}\otimes\dots\otimes i_{l_t}) : T^{k_1}s\mathcal{C}\otimes\dots\otimes T^{k_n}s\mathcal{C}\to T^+s\mathcal{I}, \tag{8}$$

$$L(k_1,\dots,k_n) = \cup_{t>0}\big\{(l_1,\dots,l_t)\in\mathbb{Z}^t_{>0} \mid \forall q,s\in\mathbb{Z}_{>0}, q\leqslant t, s\leqslant n \\ l_1+\dots+l_q = k_1+\dots+k_s \Longleftrightarrow q=t, s=n\big\}.$$

These maps restrict to maps $\bar\imath_n : T^n s\mathsf{D}(\mathcal{C}|\mathcal{B})\to s\mathsf{D}(\mathcal{I}|\mathcal{J})$, which are components of an A_∞-functor $\mathsf{D}(i) = \bar\imath : \mathsf{D}(\mathcal{C}|\mathcal{B})\to\mathsf{D}(\mathcal{I}|\mathcal{J})$. The restriction of $\bar\imath_n$ to $T^n s\mathcal{C} \xhookrightarrow{\bar{\jmath}_1^{\otimes n}} T^n s\mathsf{D}(\mathcal{C}|\mathcal{B})$ equals $T^n s\mathcal{C}\xrightarrow{i_n} s\mathcal{I}\xhookrightarrow{\bar{\jmath}_1} s\mathsf{D}(\mathcal{I}|\mathcal{J})$.

Proof. Let us prove (8). Let $\bar\imath$ denote the cocategory homomorphism $\bar\imath : \overline{\mathcal{C}} \to \overline{\mathcal{I}}$, defined by its components (8). We are going to prove that it satisfies $\bar\imath\boldsymbol{\mu} = \boldsymbol{\mu}\underline{i}$. Indeed, this equation expands to the following

$$\sum_{n_1+\dots+n_a=p} (\bar\imath_{n_1} \otimes \dots \otimes \bar\imath_{n_a})\mu^{(a)} = \mu^{(p)} i : T^{c_1} s\mathcal{C} \otimes \dots \otimes T^{c_p} s\mathcal{C} \to T^{+} s\mathcal{I}, \qquad (9)$$

which has to be proven for all $p \geqslant 1$. The right-hand side is the sum of terms $i_{m_1} \otimes \dots \otimes i_{m_t}$ such that $m_1 + \dots + m_t = c_1 + \dots + c_p$. Consider a set of positive integers

$$N = \{m_1, m_1 + m_2, \dots, m_1 + \dots + m_t\} \cap \{c_1, c_1 + c_2, \dots, c_1 + \dots + c_p\}.$$

It contains $c_1 + \dots + c_p$. Clearly, $i_{m_1} \otimes \dots \otimes i_{m_t}$ will appear in the term $\bar\imath_{n_1} \otimes \dots \otimes \bar\imath_{n_a}$ if and only if $N = \{n_1, n_1 + n_2, \dots, n_1 + \dots + n_a\}$. Since any finite subset $N \subset \mathbb{Z}_{>0}$ has a unique presentation of this form via $n_1, \dots, n_a$, Equation (9) holds.

Let X, Z_j, Y be objects of $\mathcal{C}$ and let C^i_j be objects of $\mathcal{B}$. When $\bar\imath_n$ is applied to the $\Bbbk$-module

$$\begin{aligned} s\mathcal{C}(X, C^1_1) \otimes \dots \otimes s\mathcal{C}(C^1_{k_1-1}, Z_1) \bigotimes s\mathcal{C}(Z_1, C^2_1) \otimes \dots \otimes s\mathcal{C}(C^2_{k_2-1}, Z_2) \bigotimes \dots \\ \bigotimes s\mathcal{C}(Z_{n-2}, C^{n-1}_1) \otimes \dots \otimes s\mathcal{C}(C^{n-1}_{k_{n-1}-1}, Z_{n-1}) \\ \bigotimes s\mathcal{C}(Z_{n-1}, C^n_1) \otimes \dots \otimes s\mathcal{C}(C^n_{k_n-1}, Y), \end{aligned} \qquad (10)$$

the target space for the term $i_{m_1} \otimes \dots \otimes i_{m_t}$ has the form

$$s\mathcal{I}(Xi, C^{\bullet}_{\bullet} i) \otimes \dots \otimes s\mathcal{I}(C^{\bullet}_{\bullet} i, C^{\bullet}_{\bullet} i) \otimes \dots \otimes s\mathcal{I}(C^{\bullet}_{\bullet} i, Yi),$$

where $C^{\bullet}_{\bullet}$ are objects of $\mathcal{B}$ (no Z_j will appear!). Since $Xi, Yi \in \operatorname{Ob}\mathcal{I}$ and $C^{\bullet}_{\bullet} i \in \operatorname{Ob}\mathcal{J}$, the above space is a direct summand of $s\mathsf{D}(\mathcal{I}|\mathcal{J})(Xi, Yi)$. Therefore, the required map $\bar\imath_n : T^n s\mathsf{D}(\mathcal{C}|\mathcal{B}) \to s\mathsf{D}(\mathcal{I}|\mathcal{J})$ is constructed.

The last statement is a particular case of (8). Indeed, if $k_1 = \dots = k_n = 1$, then $L(1, \dots, 1)$ consists of only one sequence (n) of the length $t = 1$.

Since $\underline{i}$ is strict and $\underline{i}_1 = i$, Equation (7) is the expansion of the definition $\bar\imath = \boldsymbol{\mu}\underline{i}\boldsymbol{\mu}^{-1}$. □

For example,

$$\begin{aligned} \bar\imath_1 &= i, \\ \bar\imath_2 &= \mu i - (i \otimes i)\mu, \\ \bar\imath_3 &= \mu^{(3)} i - (i \otimes \mu i)\mu - (\mu i \otimes i)\mu + (i \otimes i \otimes i)\mu^{(3)}, \\ \bar\imath_4 &= \mu^{(4)} i - (i \otimes \mu^{(3)} i)\mu - (\mu i \otimes \mu i)\mu - (\mu^{(3)} i \otimes i)\mu \\ &\quad + (i \otimes i \otimes \mu i)\mu^{(3)} + (i \otimes \mu i \otimes i)\mu^{(3)} + (\mu i \otimes i \otimes i)\mu^{(3)} - (i \otimes i \otimes i \otimes i)\mu^{(4)}. \end{aligned}$$

Corollary 3.2. *We have a commutative diagram of A_∞-functors*

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook] \arrow[d, "i"'] & \mathcal{C} \arrow[r, "\bar{\jmath}^{\mathcal{C}}"] \arrow[d, "i"] & \mathsf{D}(\mathcal{C}|\mathcal{B}) \arrow[d, "\bar\imath"] \\ \mathcal{J} \arrow[r, hook] & \mathcal{I} \arrow[r, "\bar{\jmath}^{\mathcal{I}}"] & \mathsf{D}(\mathcal{I}|\mathcal{J}) \end{tikzcd}$$

When $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ are A_∞-functors, then $\underline{f}\,\underline{g} = \underline{fg} : \underline{\mathcal{A}} \to \underline{\mathcal{C}}$. This implies $\overline{f}\,\overline{g} = \overline{fg} : \overline{\mathcal{A}} \to \overline{\mathcal{C}}$. Assume that $\mathcal{A}' \hookrightarrow \mathcal{A}$, $\mathcal{B}' \hookrightarrow \mathcal{B}$, $\mathcal{C}' \hookrightarrow \mathcal{C}$ are full A_∞-subcategories such that $(\operatorname{Ob}\mathcal{A}')f \subset \operatorname{Ob}\mathcal{B}'$, $(\operatorname{Ob}\mathcal{B}')g \subset \operatorname{Ob}\mathcal{C}'$. Denote $f' = f\big|_{\mathcal{A}'}$, $g' = g\big|_{\mathcal{B}'}$. Since $\mathsf{D}(f)$ and $\mathsf{D}(g)$ are just the restrictions of $\overline{f}$ and $\overline{g}$, we conclude that

$$\mathsf{D}(f)\mathsf{D}(g) = \mathsf{D}(fg) : \mathsf{D}(\mathcal{A}|\mathcal{A}') \to \mathsf{D}(\mathcal{C}|\mathcal{C}'). \tag{11}$$

3.1. Strict unitality.

Assume that the A_∞-category $\mathcal{C}$ is strictly unital. As we know from Section 2.1, $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\overline{\mathcal{C}}$ are strictly unital with the unit transformation $\mathbf{i}^{\overline{\mathcal{C}}}$. Since $\boldsymbol{\mu}^{-1} : \underline{\mathcal{C}} \to \overline{\mathcal{C}}$ is an invertible A_∞-functor, $\underline{\mathcal{C}}$ is unital (see [**Lyu03**, Section 8.12]). Notice that $\underline{\mathcal{C}}$ is never strictly unital except when $\mathcal{C} = 0$, because $\underline{b}_2 = 0$. The transformation $\mathbf{i}^{\underline{\mathcal{C}}} = \boldsymbol{\mu}^{-1}\mathbf{i}^{\overline{\mathcal{C}}}\boldsymbol{\mu} : \mathrm{id}_{\underline{\mathcal{C}}} \to \mathrm{id}_{\underline{\mathcal{C}}} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$, whose components are

$$\begin{aligned}
\mathbf{i}^{\underline{\mathcal{C}}}_0 &= \mathbf{i}^{\overline{\mathcal{C}}}_0, \\
\mathbf{i}^{\underline{\mathcal{C}}}_1 &= (\mathbf{i}^{\overline{\mathcal{C}}}_0 \otimes 1 + 1 \otimes \mathbf{i}^{\overline{\mathcal{C}}}_0)\mu, \\
\mathbf{i}^{\underline{\mathcal{C}}}_2 &= (1 \otimes \mathbf{i}^{\overline{\mathcal{C}}}_0 \otimes 1)\mu^{(3)}, \\
\mathbf{i}^{\underline{\mathcal{C}}}_k &= 0 \qquad \text{for } k > 2,
\end{aligned}$$

is a unit transformation of $\underline{\mathcal{C}}$. Indeed, let us define $\mathbf{i}^{\underline{\mathcal{C}}}$ by the above components and let us prove that $\boldsymbol{\mu}\mathbf{i}^{\underline{\mathcal{C}}} = \mathbf{i}^{\overline{\mathcal{C}}}\boldsymbol{\mu}$. Clearly, $(\boldsymbol{\mu}\mathbf{i}^{\underline{\mathcal{C}}})_0 = \mathbf{i}^{\overline{\mathcal{C}}}_0 = (\mathbf{i}^{\overline{\mathcal{C}}}\boldsymbol{\mu})_0$. For $n > 0$ we have

$$\begin{aligned}
(\boldsymbol{\mu}\mathbf{i}^{\underline{\mathcal{C}}})_n &= \mu^{(n)}\mathbf{i}^{\underline{\mathcal{C}}}_1 + \sum_{k+l=n} (\mu^{(k)} \otimes \mu^{(l)})\mathbf{i}^{\underline{\mathcal{C}}}_2 \\
&= (\mathbf{i}^{\overline{\mathcal{C}}}_0 \otimes 1^{\otimes n})\mu^{(n+1)} + (1^{\otimes n} \otimes \mathbf{i}^{\overline{\mathcal{C}}}_0)\mu^{(n+1)} + \sum_{k,l>0;k+l=n} (1^{\otimes k} \otimes \mathbf{i}^{\overline{\mathcal{C}}}_0 \otimes 1^{\otimes l})\mu^{(n+1)} \\
&= \sum_{k,l\geqslant 0;k+l=n} (1^{\otimes k} \otimes \mathbf{i}^{\overline{\mathcal{C}}}_0 \otimes 1^{\otimes l})\mu^{(n+1)} = (\mathbf{i}^{\overline{\mathcal{C}}}\boldsymbol{\mu})_n.
\end{aligned}$$

4. An A_∞-transformation

Let $\mathcal{B} \hookrightarrow \mathcal{C}$ and $\mathcal{J} \hookrightarrow \mathcal{I}$ be full A_∞-subcategories. Let $f, g : \mathcal{C} \to \mathcal{I}$ be two A_∞-functors such that $(\operatorname{Ob}\mathcal{B})f \subset \operatorname{Ob}\mathcal{J}$, $(\operatorname{Ob}\mathcal{B})g \subset \operatorname{Ob}\mathcal{J}$, and let $r : f \to g : \mathcal{C} \to \mathcal{I}$ be an A_∞-transformation. Denote by $r' : f' \to g' : \mathcal{B} \to \mathcal{J}$ the restriction of r to $\mathcal{B}$. We already have $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ for A_∞-categories $\mathcal{C}$, $\underline{f}$ and $\overline{f}$ for A_∞-functors f. Now let us proceed with A_∞-transformations.

Let us define an A_∞-transformation $\underline{r} : \underline{f} \to \underline{g} : \underline{\mathcal{C}} \to \underline{\mathcal{I}}$ via its components

$$\begin{aligned}
&\underline{r}_0 = r_0\underline{j}_1, \qquad &&\underline{r}_0 = \big[\Bbbk \xrightarrow{r_0} (s\mathcal{I})(Xf, Xg) \xhookrightarrow{\underline{j}_1} (s\underline{\mathcal{I}})(Xf, Xg)\big]; \\
&\underline{r}_1 = r, &&\underline{r}_1 = r\big|_{T^+s\mathcal{C}} : T^+s\mathcal{C} = s\underline{\mathcal{C}} \to T^+s\mathcal{I} = s\underline{\mathcal{I}}; \\
&\underline{r}_k = 0 &&\text{for } k > 1.
\end{aligned} \tag{12}$$

Let us check that $\underline{}$ maps the ω-globular set A_ω [**Lyu03**, Definition 6.4] into itself (so that sources and targets are preserved). It suffices to notice that the correspondence $r \mapsto \underline{r}$ is additive, and if $r = [v, b]$, then $\underline{r} = [\underline{v}, \underline{b}]$. Indeed,

$$
\begin{aligned}
&[\underline{v},\underline{b}]_0 = \underline{v}_0\underline{b}_1 = v_0\underline{j}_1 b = v_0 b_1\underline{j}_1 = r_0\underline{j}_1 = \underline{r}_0,\\
&[\underline{v},\underline{b}]_1 = \underline{v}_1\underline{b}_1 - (-)^v\underline{b}_1\underline{v}_1 = vb - (-)^v bv = r = \underline{r}_1, \\
&[\underline{v},\underline{b}]_k = \underline{v}_k\underline{b}_1 - (-)^v\underline{b}_k\underline{v}_1 = 0 \pm 0 = 0 = \underline{r}_k \qquad \text{for } k > 1.
\end{aligned} \tag{13}
$$

In particular, a natural A_∞-transformation $r : f \to g : \mathcal{C} \to \mathcal{I}$ goes to the natural A_∞-transformation $\underline{r} : \underline{f} \to \underline{g} : \underline{\mathcal{C}} \to \underline{\mathcal{I}}$, and equivalent natural A_∞-transformations r, p go to equivalent $\underline{r}$, $\underline{p}$.

We claim that

$$r\underline{j}^{\mathcal{I}} = \underline{j}^{\mathcal{C}}\underline{r} : f\underline{j} = \underline{j}\ \underline{f} \to g\underline{j} = \underline{j}\ \underline{g} : \mathcal{C} \to \underline{\mathcal{I}}.$$

Indeed, $(r\underline{j})_0 - (\underline{j}\underline{r})_0 = r_0\underline{j}_1 - \underline{r}_0 = 0$, and for $n > 0$

$$
\begin{aligned}
&(r\underline{j})_n - (\underline{j}\underline{r})_n \\
&= \sum_{a_1+\dots+a_l+k+c_1+\dots+c_m=n} (f_{a_1}\otimes\dots\otimes f_{a_l}\otimes r_k\otimes g_{c_1}\otimes\dots\otimes g_{c_m})\underline{j}_{l+1+m} - \underline{j}_n\underline{r}_1 \\
&\qquad\qquad\qquad\qquad\qquad = r\big|_{T^n s\mathcal{C}} - r\big|_{T^n s\mathcal{C}} = 0.
\end{aligned}
$$

We define also the A_∞-transformation conjugate to $\underline{r}$

$$\overline{r} = \boldsymbol{\mu}\underline{r}\boldsymbol{\mu}^{-1} : \overline{f} = \boldsymbol{\mu}\underline{f}\boldsymbol{\mu}^{-1} \to \overline{g} = \boldsymbol{\mu}\underline{g}\boldsymbol{\mu}^{-1} : \overline{\mathcal{C}} \to \overline{\mathcal{I}}$$

(not necessarily natural). Summing up, we have a commutative cylinder

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, hook] \arrow[d, "f'"'] \arrow[d, phantom, shift left=4, "\overset{r'}{\Longrightarrow}"] & \mathcal{C} \arrow[r, "\underline{j}^{\mathcal{C}}"] \arrow[d, "f"'] \arrow[d, phantom, shift left=4, "\overset{r}{\Longrightarrow}"] & \underline{\mathcal{C}} \arrow[r, "\boldsymbol{\mu}^{-1}"] \arrow[d, "\underline{f}"'] \arrow[d, phantom, shift left=4, "\overset{\underline{r}}{\Longrightarrow}"] & \overline{\mathcal{C}} \arrow[d, "\overline{f}"'] \arrow[d, phantom, shift left=4, "\overset{\overline{r}}{\Longrightarrow}"] \\
\mathcal{J} \arrow[r, hook] & \mathcal{I} \arrow[r, "\underline{j}^{\mathcal{I}}"] & \underline{\mathcal{I}} \arrow[r, "\boldsymbol{\mu}^{-1}"] & \overline{\mathcal{I}}
\end{tikzcd}
\tag{14}
$$

The correspondence $\overline{}$ also maps the ω-globular set A_ω into itself. Indeed, if $r = [v, b]$, then

$$\overline{r} = \boldsymbol{\mu}\underline{r}\boldsymbol{\mu}^{-1} = \boldsymbol{\mu}[\underline{v},\underline{b}]\boldsymbol{\mu}^{-1} = [\boldsymbol{\mu}\underline{v}\boldsymbol{\mu}^{-1}, \boldsymbol{\mu}\underline{b}\boldsymbol{\mu}^{-1}] = [\overline{v},\overline{b}].$$

Proposition 4.1. *The A_∞-transformation $\overline{r}$ has the following components*

$$
\begin{aligned}
\overline{r}_n = &\sum_{l_1+\dots+l_t=n}^{0\leqslant q\leqslant t} (-)^t(\mu^{(l_1)}f\otimes\dots\otimes\mu^{(l_q)}f\otimes r_0\underline{j}_1\otimes\mu^{(l_{q+1})}g\otimes\dots\otimes\mu^{(l_t)}g)\mu^{(t+1)} \\
+ &\sum_{l_1+\dots+l_t=n}^{1\leqslant q\leqslant t} (-)^{t-1}(\mu^{(l_1)}f\otimes\dots\otimes\mu^{(l_{q-1})}f\otimes\mu^{(l_q)}r\otimes\mu^{(l_{q+1})}g\otimes\dots\otimes\mu^{(l_t)}g)\mu^{(t)}.
\end{aligned}
\tag{15}
$$

Explicitly, $\overline{r}_0 = \underline{r}_0 = r_0\underline{j}_1$ *and for* $n > 0$ *the restriction of* $\overline{r}_n$ *to* $T^{k_1}s\mathcal{C} \otimes \dots \otimes T^{k_n}s\mathcal{C}$ *is*

$$\overline{r}_n = \mu^{(n)} \sum_{(a_1,\dots,a_\alpha;k;c_1,\dots,c_\beta)\in P(k_1,\dots,k_n)} (f_{a_1} \otimes \dots \otimes f_{a_\alpha} \otimes r_k \otimes g_{c_1} \otimes \dots \\ \otimes g_{c_\beta})\underline{j}_{\alpha+1+\beta} : T^{k_1}s\mathcal{C} \otimes \dots \otimes T^{k_n}s\mathcal{C} \to T^+s\mathcal{I}, \quad (16)$$

$$P(k_1,\dots,k_n) = \sqcup_{\alpha,\beta\geqslant 0}\big\{(l_1,\dots,l_\alpha;l_{\alpha+1};l_{\alpha+2},\dots,l_{\alpha+1+\beta}) \in \mathbb{Z}^\alpha_{>0} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}^\beta_{>0} \mid \\ \forall q \in \mathbb{Z}_{>0}, q \leqslant \alpha+1+\beta \; \forall s \in \mathbb{Z}_{\geqslant 0}, s \leqslant n \\ l_1 + \dots + l_q = k_1 + \dots + k_s \Leftrightarrow q = \alpha+1+\beta, s = n\big\}.$$

These maps restrict to maps $\overline{r}_n : T^n s\mathsf{D}(\mathcal{C}|\mathcal{B}) \to s\mathsf{D}(\mathcal{I}|\mathcal{J})$*, which are components of an* A_∞*-transformation* $\mathsf{D}(r) = \overline{r} : \overline{f} \to \overline{g} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J})$*. The restriction of* $\overline{r}_n$ *to* $T^n s\mathcal{C} \xhookrightarrow{\bar{j}_1^{\otimes n}} T^n s\mathsf{D}(\mathcal{C}|\mathcal{B})$ *equals* $T^n s\mathcal{C} \xrightarrow{r_n} s\mathcal{I} \xhookrightarrow{\bar{j}_1} s\mathsf{D}(\mathcal{I}|\mathcal{J})$*.*

Proof. Similarly to the case of A_∞-functors, discussed in Proposition 3.1, let us define an A_∞-transformation $\overline{r} : \overline{f} \to \overline{g} : \overline{\mathcal{C}} \to \overline{\mathcal{I}}$ by its components (16) and prove that the equation $\overline{r}\boldsymbol{\mu} = \boldsymbol{\mu}\underline{r}$ holds. Clearly, $(\overline{r}\boldsymbol{\mu})_0 = \overline{r}_0 = \underline{r}_0 = (\boldsymbol{\mu}\underline{r})_0$. We have to prove that for $n > 0$

$$\sum_{i_1+\dots+i_t=n} (\overline{f}_{i_1} \otimes \dots \otimes \overline{f}_{i_{q-1}} \otimes \overline{r}_{i_q} \otimes \overline{g}_{i_{q+1}} \otimes \dots \otimes \overline{g}_{i_t})\mu^{(t)} = \mu^{(n)}r : \\ T^{k_1}s\mathcal{C} \otimes \dots \otimes T^{k_n}s\mathcal{C} \to T^+s\mathcal{I}. \quad (17)$$

The right-hand side is the sum of terms $f_{a_1} \otimes \dots \otimes f_{a_\alpha} \otimes r_k \otimes g_{c_1} \otimes \dots \otimes g_{c_\beta}$, such that $a_1 + \dots + a_\alpha + k + c_1 + \dots + c_\beta = k_1 + \dots + k_n$. Denote by $(l_1,\dots,l_{\alpha+1+\beta})$ the sequence $(a_1,\dots,a_\alpha,k,c_1,\dots,c_\beta)$. Consider the subsequence N of the sequence $L = (0, l_1, l_1 + l_2, \dots, l_1 + \dots + l_{\alpha+1+\beta})$ consisting of all elements which belong to the set $\{0, k_1, k_1 + k_2, \dots, k_1 + \dots + k_n\}$. The term $f_{a_1} \otimes \dots \otimes f_{a_\alpha} \otimes r_k \otimes g_{c_1} \otimes \dots \otimes g_{c_\beta}$ will appear as a summand of the term $\overline{f}_{i_1} \otimes \dots \otimes \overline{f}_{i_{q-1}} \otimes \overline{r}_{i_q} \otimes \overline{g}_{i_{q+1}} \otimes \dots \otimes \overline{g}_{i_t}$ if and only if

$$N = (0, i_1, i_1 + i_2, \dots, i_1 + \dots + i_t), \quad (18)$$

$$i_1 + \dots + i_{q-1} \leqslant a_1 + \dots + a_\alpha, \qquad a_1 + \dots + a_\alpha + k \leqslant i_1 + \dots + i_q, \quad (19)$$

$$\forall 1 \leqslant y \leqslant t \qquad i_y = 0 \implies y = q. \quad (20)$$

Let us prove that for a given sequence $(a_1,\dots,a_\alpha;k;c_1,\dots,c_\beta) \in \mathbb{Z}^\alpha_{>0} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}^\beta_{>0}$ there exists exactly one sequence $(i_1,\dots,i_t;q) \in \mathbb{Z}^t_{\geqslant 0} \times \mathbb{Z}_{>0}$ such that conditions (18)–(20) are satisfied. Indeed, the sequence N determines uniquely a sequence $(i_1,\dots,i_t)$ of non-negative integers such that (18) holds. If $k > 0$ or $a_1 + \dots + a_\alpha$ does not belong to N, then all i_y are positive. This implies that the interval $[a_1 + \dots + a_\alpha, a_1 + \dots + a_\alpha + k]$ is contained in a unique interval of the form $[i_1 + \dots + i_{q-1}, i_1 + \dots + i_q]$.

If $k = 0$ and $a_1 + \dots + a_\alpha$ belongs to N, then $a_1 + \dots + a_\alpha = a_1 + \dots + a_\alpha + k$ is repeated in N. Hence, there exists $q > 0$ such that $a_1 + \dots + a_\alpha = i_1 + \dots + i_{q-1}$,

$i_q = 0$. Since M and N may contain no more than one repeated element, i_y is positive for $y \neq q$. Therefore, conditions (18)–(20) are satisfied.

We conclude that (17) holds. Formula (15) is the expansion of the proven property $\overline{r} = \boldsymbol{\mu}\underline{r}\boldsymbol{\mu}^{-1}$.

The target space for map (16) applied to $\Bbbk$-module (10) has the form

$$s\mathcal{I}(Xf, C^\bullet_\bullet f) \otimes \cdots \otimes s\mathcal{I}(C^\bullet_\bullet f, C^\bullet_\bullet g) \otimes \cdots \otimes s\mathcal{I}(C^\bullet_\bullet g, Yg),$$

where $C^\bullet_\bullet$ are objects of $\mathcal{B}$ (and no Z_j will appear). Since objects $C^\bullet_\bullet f$ and $C^\bullet_\bullet g$ belong to $\mathcal{J}$, the above space is a direct summand of $s\mathsf{D}(\mathcal{I}|\mathcal{J})(Xf, Yg)$. Therefore, the required map $\overline{r}_n : T^n s\mathsf{D}(\mathcal{C}|\mathcal{B}) \to s\mathsf{D}(\mathcal{I}|\mathcal{J})$ is constructed.

The last statement is a particular case of (16). Indeed, if $k_1 = \cdots = k_n = 1$, then $P(1, \dots, 1)$ consists of only one element $(;n;)$, that is, $\alpha = \beta = 0$, $i_1 = n \in \mathbb{Z}_{\geqslant 0}$. $\square$

In particular, the correspondence $r \mapsto \mathsf{D}(r) = \overline{r}$ maps natural A_∞-transformations to natural ones, and equivalent $r, p : f \to g : \mathcal{B} \to \mathcal{C}$ are mapped to equivalent

$$\mathsf{D}(r), \mathsf{D}(p) : \overline{f} \to \overline{g} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J}).$$

For example,

$$\begin{aligned}
\overline{r}_1 &= r - (f \otimes r_0 + r_0 \otimes g)\mu, \\
\overline{r}_2 &= \mu r - (f \otimes r + r \otimes g)\mu - \mu(f \otimes r_0 + r_0 \otimes g)\mu \\
&\qquad + (f \otimes f \otimes r_0 + f \otimes r_0 \otimes g + r_0 \otimes g \otimes g)\mu^{(3)}.
\end{aligned}$$

Corollary 4.2. *We have a commutative cylinder*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, hook] \arrow[d, shift right=3, "f'"'] \arrow[d, shift left=3, "g'"] \arrow[d, phantom, "\xRightarrow{r'}"] & \mathcal{C} \arrow[r, "\overline{\jmath}^{\mathcal{C}}"] \arrow[d, shift right=3, "f"'] \arrow[d, shift left=3, "g"] \arrow[d, phantom, "\xRightarrow{r}"] & \mathsf{D}(\mathcal{C}|\mathcal{B}) \arrow[d, shift right=3, "\overline{f}"'] \arrow[d, shift left=3, "\overline{g}"] \arrow[d, phantom, "\xRightarrow{\overline{r}}"] \\
\mathcal{J} \arrow[r, hook] & \mathcal{I} \arrow[r, "\overline{\jmath}^{\mathcal{I}}"] & \mathsf{D}(\mathcal{I}|\mathcal{J})
\end{tikzcd}$$

4.1. $\mathcal{K}$-2-categories and $\mathcal{K}$-2-functors.

Let $\mathcal{K}$ denote the category $\mathsf{K}(\Bbbk\text{-mod}) = H^0(\mathsf{C}(\Bbbk\text{-mod}))$ of differential graded complexes of $\Bbbk$-modules, whose morphisms are chain maps modulo homotopy. A 1-unital, non-2-unital $\mathcal{K}$-2-category $\mathcal{K}A_\infty$ of A_∞-categories is described in [**Lyu03**, Proposition 7.1]. Instead of the complex of 2-morphisms $(A_\infty(\mathcal{A}, \mathcal{B})(f, g), m_1)$, $m_1 = sB_1s^{-1}$, we work with the shifted complex $(sA_\infty(\mathcal{A}, \mathcal{B})(f, g), B_1)$. There is an obvious notion of a strict $\mathcal{K}$-2-functor between such $\mathcal{K}$-2-categories — a map of objects, maps of 1-morphisms and chain maps of 2-morphisms, which preserve all operations. The operations involving 2-morphisms are subject to equations in $\mathcal{K}$, which mean equations between chain maps up to homotopy.

We have applied the "underline" construction $\underline{\text{-}}$ to three kinds of arguments - : A_∞-categories, A_∞-functors and A_∞-transformations. Let us summarize the properties of this construction.

Proposition 4.3. *The following assignment defines a strict $\mathcal{K}$-2-functor $\underline{\text{-}} : \mathcal{K}A_\infty \to \mathcal{K}A_\infty$: an A_∞-category $\mathcal{A}$ is mapped to $\underline{\mathcal{A}}$, an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is mapped to $\underline{f} : \underline{\mathcal{A}} \to \underline{\mathcal{B}}$, and the chain map of complexes of 2-morphisms is*

$$\underline{} : (sA_\infty(\mathcal{A},\mathcal{B})(f,g),B_1) \to (sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{g}),\underline{B}_1), \qquad r \mapsto \underline{r},$$

where $\underline{B}$ denotes the codifferential in $TsA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})$, in particular, $v\underline{B}_1 = [v,\underline{b}]$.

Proof. We have seen in (13) that $\underline{rB_1} = \underline{[r,b]} = [\underline{r},\underline{b}] = \underline{r}\ \underline{B}_1$, thus, $r \mapsto \underline{r}$ is a chain map. The composition of A_∞-functors is preserved, $\underline{fg} = \underline{f}\ \underline{g}$. The right action of a 1-morphism h on a 2-morphism r is preserved, since $\underline{rh} = (r_0h_1\underline{j}_1, rh, 0, 0, \dots) = \underline{r}\ \underline{h}$. The left action of a 1-morphism e on a 2-morphism r is preserved, since $\underline{er} = ({}_e r_0\underline{j}_1, er, 0, 0, \dots) = \underline{e}\ \underline{r}$. The identity A_∞-functor $\mathrm{id}_\mathcal{A}$ is mapped to the identity A_∞-functor $\underline{\mathrm{id}_\mathcal{A}} = \mathrm{id}_{\underline{\mathcal{A}}}$.

It remains to prove that the vertical composition of 2-morphisms

$$m_2 = (s \otimes s)B_2s^{-1} : A_\infty(\mathcal{A},\mathcal{B})(f,g) \otimes A_\infty(\mathcal{A},\mathcal{B})(g,h) \to A_\infty(\mathcal{A},\mathcal{B})(f,h)$$

is preserved, that is, the diagram

$$\begin{tikzcd}
A_\infty(\mathcal{A},\mathcal{B})(f,g) \otimes A_\infty(\mathcal{A},\mathcal{B})(g,h) \arrow[r, "s\underline{}s^{-1}\otimes s\underline{}s^{-1}"] \arrow[d, "m_2"'] & A_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{g}) \otimes A_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{g},\underline{h}) \arrow[d, "\underline{m}_2"] \\
A_\infty(\mathcal{A},\mathcal{B})(f,h) \arrow[r, "s\underline{}s^{-1}"] & A_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{h})
\end{tikzcd}$$

is commutative in $\mathcal{K}$. Here $s\underline{}s^{-1}$ denotes the composition

$$A_\infty(\mathcal{A},\mathcal{B})(f,h) \xrightarrow{s} sA_\infty(\mathcal{A},\mathcal{B})(f,h) \xrightarrow{\underline{}} sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{h}) \xrightarrow{s^{-1}} A_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{h}).$$

Since $\underline{b}_k = 0$ for $k \geqslant 2$, we have $\underline{B}_2 = 0$ due to [**Lyu03**, Equation (5.1.3)], hence, $\underline{m}_2 = 0$. Let us prove that $m_2(s\underline{}s^{-1}) \sim 0$. The homotopy is sought in the form $(s \otimes s)Hs^{-1}$, where

$$H : sA_\infty(\mathcal{A},\mathcal{B})(f,g) \otimes sA_\infty(\mathcal{A},\mathcal{B})(g,h) \to sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{h})$$

is a $\Bbbk$-linear map of degree 0. It has to satisfy the equation

$$B_2\underline{} = H\underline{B}_1 - (1 \otimes B_1 + B_1 \otimes 1)H,$$

that is, for each $r \in sA_\infty(\mathcal{A},\mathcal{B})(f,g)$, $p \in sA_\infty(\mathcal{A},\mathcal{B})(g,h)$

$$\underline{(r \otimes p)B_2} = [(r \otimes p)H, \underline{b}] - [(r \otimes p)(1 \otimes B_1 + B_1 \otimes 1)]H. \tag{21}$$

A candidate for H is chosen similarly to Equation (12). We choose the components of the $(\underline{f},\underline{h})$-coderivation $(r \otimes p)H$ as follows:

$$\begin{aligned}
&[(r \otimes p)H]_0 = (r_0 \otimes p_0)\underline{j}_2, \\
&[(r \otimes p)H]_1 = (r \otimes p)\theta : s\underline{\mathcal{A}}(X,Y) \to s\underline{\mathcal{B}}(Xf,Yh), \\
&[(r \otimes p)H]_k = 0 \qquad\qquad \text{for } k > 1.
\end{aligned}$$

Let us verify Equation (21) for this H. Both sides of (21) are $(\underline{f},\underline{h})$-coderivations. It suffices to check that all their components coincide. The 0-th component of the right-hand side of (21) is

$$\begin{aligned}
&[(r\otimes p)H]_0\underline{b}_1-[(r\otimes[p,b]+(-)^p[r,b]\otimes p)H]_0\\
&=(r_0\otimes p_0)\underline{j}_2b-(r_0\otimes p_0b_1+(-)^pr_0b_1\otimes p_0)\underline{j}_2\\
&=(r_0\otimes p_0)(b-1\otimes b_1-b_1\otimes 1)=(r_0\otimes p_0)b_2\underline{j}_2,
\end{aligned}$$

which equals $[\underline{(r\otimes p)B_2}]_0$. Due to [**Lyu03**, Equation (5.1.2)] the first component of the right-hand side of (21) equals

$$\begin{aligned}
&[(r\otimes p)H]_1\underline{b}_1-(-)^{r+p}\underline{b}_1[(r\otimes p)H]_1-[(r\otimes p)(1\otimes B_1+B_1\otimes 1)H]_1\\
&=(r\otimes p)\theta b-(-)^{r+p}b(r\otimes p)\theta-[(r\otimes p)(1\otimes B_1+B_1\otimes 1)]\theta\\
&=(r\otimes p)B_2,
\end{aligned}$$

which is $[\underline{(r\otimes p)B_2}]_1$. The k-th component of the right-hand side of (21) vanishes for $k>1$, and so does $[\underline{(r\otimes p)B_2}]_k=0$. Therefore, (21) and the proposition are proven. □

Corollary 4.4. *The same assignment $\mathcal{A}\mapsto\underline{\mathcal{A}}$, $f\mapsto\underline{f}$, $r\mapsto\underline{r}$ as in Proposition 4.3 gives a strict 2-functor $\underline{}:A_\infty\to A_\infty$ of non-2-unital 2-categories.*

This is obtained by taking the 0-th cohomology of $\mathcal{K}A_\infty$ in Proposition 4.3.

Similarly, the "overline" construction $\overline{}$, applied to three kinds of arguments, A_∞-categories, A_∞-functors and A_∞-transformations, gives a strict $\mathcal{K}$-2-functor.

Corollary 4.5. *The following assignment defines a strict $\mathcal{K}$-2-functor $\overline{}:\mathcal{K}A_\infty\to\mathcal{K}A_\infty$: an A_∞-category $\mathcal{A}$ is mapped to $\overline{\mathcal{A}}$, an A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ is mapped to $\overline{f}:\overline{\mathcal{A}}\to\overline{\mathcal{B}}$, and the chain map of complexes of 2-morphisms is*

$$\overline{}:(sA_\infty(\mathcal{A},\mathcal{B})(f,g),B_1)\to(sA_\infty(\overline{\mathcal{A}},\overline{\mathcal{B}})(\overline{f},\overline{g}),\overline{B}_1),\qquad r\mapsto\overline{r},$$

where $\overline{B}$ denotes the codifferential in $TsA_\infty(\overline{\mathcal{A}},\overline{\mathcal{B}})$, in particular, $v\overline{B}_1=[v,\overline{b}]$. There is an invertible strict $\mathcal{K}$-2-transformation $\boldsymbol{\mu}:\overline{}\to\underline{}$, $\boldsymbol{\mu}_\mathcal{A}:\overline{\mathcal{A}}\to\underline{\mathcal{A}}$.

Proof. Starting with a $\mathcal{K}$-2-functor $\underline{}$, a mapping $\operatorname{Ob}\mathcal{K}A_\infty\to\operatorname{Ob}\mathcal{K}A_\infty$, $\mathcal{A}\mapsto\overline{\mathcal{A}}$, and a family of invertible A_∞-functors $\boldsymbol{\mu}_\mathcal{A}:\overline{\mathcal{A}}\to\underline{\mathcal{A}}$, one may construct another $\mathcal{K}$-2-functor $\overline{}$, which maps an A_∞-category $\mathcal{A}$ to $\overline{\mathcal{A}}$, so that $\boldsymbol{\mu}$ is a strict $\mathcal{K}$-2-transformation. Since $\boldsymbol{\mu}$ is strict and invertible, the values of $\overline{f}$ and $\overline{r}$ are fixed by the requirements $\overline{f}\boldsymbol{\mu}=\boldsymbol{\mu}\underline{f}$, $\overline{r}\boldsymbol{\mu}=\boldsymbol{\mu}\underline{r}$ for each A_∞-functor f and A_∞-transformation r. □

The detailed definition of strict $\mathcal{K}$-2-transformations is left to the interested reader.

Corollary 4.6. *The same assignment $\mathcal{A}\mapsto\overline{\mathcal{A}}$, $f\mapsto\overline{f}$, $r\mapsto\overline{r}$ as in Corollary 4.5 gives a strict 2-functor $\overline{}:A_\infty\to A_\infty$ of non-2-unital categories.*

This is obtained by taking the 0-th cohomology of $\mathcal{K}A_\infty$ in Corollary 4.5.

It is instructive to find the homotopy which forces $\overline{-}$ to preserve the vertical composition of 2-morphisms. Denote $\mathrm{ad}\,\boldsymbol{\mu}$ the maps $sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{g})\to sA_\infty(\overline{\mathcal{A}},\overline{\mathcal{B}})(\overline{f},\overline{g})$, $v\mapsto \boldsymbol{\mu}v\boldsymbol{\mu}^{-1}$. The following diagram commutes modulo homotopy:

$$\begin{tikzcd}
\begin{array}{c} sA_\infty(\mathcal{A},\mathcal{B})(f,g)\otimes \\ \otimes sA_\infty(\mathcal{A},\mathcal{B})(g,h)\end{array} \arrow[rr, "\overline{-}\otimes\overline{-}"] \arrow[dr, "\underline{-}\otimes\underline{-}"'] \arrow[ddd, "B_2"'] & & \begin{array}{c} sA_\infty(\overline{\mathcal{A}},\overline{\mathcal{B}})(\overline{f},\overline{g})\otimes \\ \otimes sA_\infty(\overline{\mathcal{A}},\overline{\mathcal{B}})(\overline{g},\overline{h})\end{array} \arrow[ddd, "\overline{B}_2"] \\
 & \begin{array}{c} sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{g})\otimes \\ \otimes sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{g},\underline{h})\end{array} \arrow[ur, "\mathrm{ad}\,\boldsymbol{\mu}\otimes\mathrm{ad}\,\boldsymbol{\mu}"'] \arrow[d, "0=" ', "\underline{B}_2"] \arrow[u, phantom, "="] \arrow[l, phantom, "\sim"] \arrow[r, phantom, "\sim"] & \\
 & sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{h}) \arrow[dr, "\mathrm{ad}\,\boldsymbol{\mu}"] \arrow[d, phantom, "="] & \\
sA_\infty(\mathcal{A},\mathcal{B})(f,h) \arrow[ur, "\underline{-}"] \arrow[rr, "\overline{-}"'] & & sA_\infty(\overline{\mathcal{A}},\overline{\mathcal{B}})(\overline{f},\overline{h})
\end{tikzcd} \tag{22}$$

The right homotopy commutative square is obtained from [**Lyu03**, Equation (7.1.2)]:

$$\begin{aligned}(\rho\otimes\pi)\underline{B}_2\boldsymbol{\mu}^{-1} &- (\rho\boldsymbol{\mu}^{-1}\otimes\pi\boldsymbol{\mu}^{-1})B_2 \\ &= (\rho\otimes\pi\mid\boldsymbol{\mu}^{-1})M_{20}B_1 - [(\rho\otimes\pi)(1\otimes\underline{B}_1+\underline{B}_1\otimes 1)|\boldsymbol{\mu}^{-1}]M_{20}\end{aligned}$$

for all $\rho\in sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{g})$, $\pi\in sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{g},\underline{h})$. Recall that $\underline{B}_2=0$ and compose with $\boldsymbol{\mu}$ to get

$$\begin{aligned}-&(\boldsymbol{\mu}\rho\boldsymbol{\mu}^{-1}\otimes\boldsymbol{\mu}\pi\boldsymbol{\mu}^{-1})\overline{B}_2 \\ &= [\boldsymbol{\mu}(\rho\otimes\pi\mid\boldsymbol{\mu}^{-1})M_{20}]\overline{B}_1 - \boldsymbol{\mu}[(\rho\otimes\pi)(1\otimes\underline{B}_1+\underline{B}_1\otimes 1)|\boldsymbol{\mu}^{-1}]M_{20}.\end{aligned}$$

In particular, for $\rho=\underline{r}$, $\pi=\underline{p}$ we have

$$\begin{aligned}-&(\overline{r}\otimes\overline{p})\overline{B}_2 \\ &= [\boldsymbol{\mu}(\underline{r}\otimes\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20}]\overline{B}_1 - \boldsymbol{\mu}\{[(r\otimes p)(1\otimes B_1+B_1\otimes 1)](\underline{-}\otimes\underline{-})|\boldsymbol{\mu}^{-1}\}M_{20}.\end{aligned}$$

The left homotopy commutative square, that is, (21) composed with $\mathrm{ad}\,\boldsymbol{\mu}$ gives

$$\overline{(r\otimes p)B_2} = [\boldsymbol{\mu}(r\otimes p)H\boldsymbol{\mu}^{-1}]\overline{B}_1 - \boldsymbol{\mu}[(r\otimes p)(1\otimes B_1+B_1\otimes 1)]H\boldsymbol{\mu}^{-1}.$$

We conclude that the exterior of diagram (22) is commutative modulo homotopy

$$\begin{gathered}R: sA_\infty(\mathcal{A},\mathcal{B})(f,g)\otimes sA_\infty(\mathcal{A},\mathcal{B})(g,h)\to sA_\infty(\overline{\mathcal{A}},\overline{\mathcal{B}})(\overline{f},\overline{h}), \\ (r\otimes p)R = \boldsymbol{\mu}(r\otimes p)H\boldsymbol{\mu}^{-1} + \boldsymbol{\mu}(\underline{r}\otimes\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20},\end{gathered}$$

that is,

$$\overline{(r\otimes p)B_2} - (\overline{r}\otimes\overline{p})\overline{B}_2 = (r\otimes p)R\overline{B}_1 - [(r\otimes p)(1\otimes B_1+B_1\otimes 1)]R. \tag{23}$$

Proposition 4.7. *The $(\overline{f},\overline{h})$-transformation $(r\otimes p)R$ has the following components: $[(r\otimes p)R]_0=0$, and for $n>0$ the restriction of $[(r\otimes p)R]_n$ to $T^{k_1}s\mathcal{A}\otimes\cdots\otimes T^{k_n}s\mathcal{A}$ is*

$$
\begin{aligned}
&[(r\otimes p)R]_n \\
&\quad = \mu^{(n)} \sum_{(\bar a;k;\bar c;t;\bar e)\in Q(k_1,\dots,k_n)} (f_{a_1}\otimes\dots\otimes f_{a_\alpha}\otimes r_k\otimes g_{c_1}\otimes\dots\otimes g_{c_\beta}\otimes p_t\otimes h_{e_1}\otimes\dots \\
&\qquad\qquad \otimes h_{e_\gamma})\underline{j}_{\alpha+\beta+\gamma+2} : T^{k_1}s\mathcal{A}\otimes\dots\otimes T^{k_n}s\mathcal{A}\to T^+s\mathcal{B},
\end{aligned}
\tag{24}
$$

where $(\bar a;k;\bar c;t;\bar e) = (a_1,\dots,a_\alpha;k;c_1,\dots,c_\beta;t;e_1,\dots,e_\gamma)$ *and*

$$
\begin{aligned}
&Q(k_1,\dots,k_n) \\
&\quad = \sqcup_{\alpha,\beta,\gamma\geqslant 0}\big\{(l_1,\dots,l_\alpha;l_{\alpha+1};l_{\alpha+2},\dots,l_{\alpha+\beta+1};l_{\alpha+\beta+2};l_{\alpha+\beta+3},\dots,l_{\alpha+\beta+\gamma+2}) \\
&\in \mathbb{Z}^\alpha_{>0}\times\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}^\beta_{>0}\times\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}^\gamma_{>0} \mid \forall q\in\mathbb{Z}_{>0}, q\leqslant \alpha+\beta+\gamma+2\ \forall s\in\mathbb{Z}_{\geqslant 0}, s\leqslant n \\
&\qquad\qquad l_1+\dots+l_q = k_1+\dots+k_s \Leftrightarrow q=\alpha+\beta+\gamma+2, s=n\big\}.
\end{aligned}
$$

If $\mathcal{A}'\subset\mathcal{A}$, $\mathcal{B}'\subset\mathcal{B}$ *are full* A_∞*-subcategories and* $(\operatorname{Ob}\mathcal{A}')f\subset\operatorname{Ob}\mathcal{B}'$, $(\operatorname{Ob}\mathcal{A}')g\subset\operatorname{Ob}\mathcal{B}'$, $(\operatorname{Ob}\mathcal{A}')h\subset\operatorname{Ob}\mathcal{B}'$, *then* $[(r\otimes p)R]_n$ *restrict to maps*

$$[(r\otimes p)R]_n : T^n s\mathsf{D}(\mathcal{A}|\mathcal{A}')(X,Y)\to s\mathsf{D}(\mathcal{B}|\mathcal{B}')(Xf,Yh),$$

which are components of an A_∞*-transformation*

$$(r\otimes p)R\in sA_\infty\big(\mathsf{D}(\mathcal{A}|\mathcal{A}'),\mathsf{D}(\mathcal{B}|\mathcal{B}')\big)(\overline{f},\overline{h}).$$

Proof. Denote by R' an $(\overline{f},\overline{h})$-coderivation, whose components are $R'_0=0$ and R'_n is given by the right-hand side of (24). We want to prove that $(r\otimes p)R=R'$. This is equivalent to the equation

$$R'\boldsymbol{\mu} = \boldsymbol{\mu}(r\otimes p)H + \boldsymbol{\mu}(\underline{r}\otimes\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20}\boldsymbol{\mu}. \tag{25}$$

Let us transform the last term. Applying the identity $(1\boxtimes M)M\operatorname{pr}_1 = (M\boxtimes 1)M\operatorname{pr}_1$ [**Lyu03**, Proposition 4.1] to an element

$$1\otimes\underline{r}\otimes\underline{p}\otimes 1\in T^0sA_\infty(\overline{\mathcal{A}},\underline{\mathcal{A}})(\boldsymbol{\mu},\boldsymbol{\mu})\otimes T^2sA_\infty(\underline{\mathcal{A}},\underline{\mathcal{B}})(\underline{f},\underline{h})\otimes T^0sA_\infty(\underline{\mathcal{B}},\overline{\mathcal{B}})(\boldsymbol{\mu}^{-1},\boldsymbol{\mu}^{-1})$$

we find from

$$
\begin{aligned}
&(1\otimes\underline{r}\otimes\underline{p}\otimes 1)(1\otimes M)M\operatorname{pr}_1 \\
&\qquad = [1\otimes(\underline{r}\otimes\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20} + 1\otimes\underline{r}\boldsymbol{\mu}^{-1}\otimes\underline{p}\boldsymbol{\mu}^{-1}]M\operatorname{pr}_1 = \boldsymbol{\mu}(\underline{r}\otimes\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20},
\end{aligned}
$$

$$(1\otimes\underline{r}\otimes\underline{p}\otimes 1)(M\otimes 1)M\operatorname{pr}_1 = (\boldsymbol{\mu}\underline{r}\otimes\boldsymbol{\mu}\underline{p}\otimes 1)M\operatorname{pr}_1 = (\boldsymbol{\mu}\underline{r}\otimes\boldsymbol{\mu}\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20},$$

that $\boldsymbol{\mu}(\underline{r}\otimes\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20} = (\boldsymbol{\mu}\underline{r}\otimes\boldsymbol{\mu}\underline{p}\mid\boldsymbol{\mu}^{-1})M_{20}$. Applying the same identity to an element

$$\rho\otimes\pi\otimes 1\otimes 1\in T^2sA_\infty(\overline{\mathcal{A}},\underline{\mathcal{B}})(\boldsymbol{\mu}f,\boldsymbol{\mu}h)\otimes T^0sA_\infty(\underline{\mathcal{B}},\overline{\mathcal{B}})(\lambda,\lambda)\otimes T^0sA_\infty(\overline{\mathcal{B}},\underline{\mathcal{B}})(\boldsymbol{\mu},\boldsymbol{\mu})$$

we find from

$$
\begin{aligned}
(\rho\otimes\pi\otimes 1\otimes 1)(1\otimes M)M\operatorname{pr}_1 &= (\rho\otimes\pi\otimes 1)M\operatorname{pr}_1 = (\rho\otimes\pi\mid\lambda\boldsymbol{\mu})M_{20}, \\
(\rho\otimes\pi\otimes 1\otimes 1)(M\otimes 1)M\operatorname{pr}_1 &= [(\rho\otimes\pi\mid\lambda)M_{20}\otimes 1 + \rho\lambda\otimes\pi\lambda\otimes 1]M\operatorname{pr}_1 \\
&= (\rho\otimes\pi\mid\lambda)M_{20}\boldsymbol{\mu} + (\rho\lambda\otimes\pi\lambda\mid\boldsymbol{\mu})M_{20},
\end{aligned}
$$

that

$$(\rho\otimes\pi \mid \lambda\boldsymbol{\mu})M_{20} = (\rho\otimes\pi \mid \lambda)M_{20}\boldsymbol{\mu} + (\rho\lambda\otimes\pi\lambda \mid \boldsymbol{\mu})M_{20}.$$

When $\lambda = \boldsymbol{\mu}^{-1}$, the left-hand side vanishes. Indeed, for each $k \geqslant 0$

$$[(\rho\otimes\pi \mid \mathrm{id}_{\underline{\mathcal{B}}})M_{20}]_k = \sum_{l\geqslant 2}(\rho\otimes\pi)\theta_{kl}\,\mathrm{id}_l = 0.$$

Hence,

$$(\rho\otimes\pi \mid \boldsymbol{\mu}^{-1})M_{20}\boldsymbol{\mu} = -(\rho\boldsymbol{\mu}^{-1}\otimes\pi\boldsymbol{\mu}^{-1} \mid \boldsymbol{\mu})M_{20}.$$

In particular, for $\rho = \boldsymbol{\mu}\underline{r}$, $\pi = \boldsymbol{\mu}\underline{p}$ we have

$$(\boldsymbol{\mu}\underline{r}\otimes\boldsymbol{\mu}\underline{p} \mid \boldsymbol{\mu}^{-1})M_{20}\boldsymbol{\mu} = -(\overline{r}\otimes\overline{p} \mid \boldsymbol{\mu})M_{20}.$$

Therefore, Equation (25) can be rewritten as follows:

$$\boldsymbol{\mu}(r\otimes p)H = R'\boldsymbol{\mu} + (\overline{r}\otimes\overline{p} \mid \boldsymbol{\mu})M_{20}. \tag{26}$$

Both sides are $(\boldsymbol{\mu}\underline{f},\boldsymbol{\mu}\underline{h})$-coderivations, or $(\overline{f}\boldsymbol{\mu},\overline{h}\boldsymbol{\mu})$-coderivations, which is the same thing. Let us prove that all their components coincide.

The 0-th components coincide, since

$$[(r\otimes p)H]_0 = (r_0\otimes p_0)\underline{j}_2 = (\overline{r}\otimes\overline{p})\theta_{02}\boldsymbol{\mu}_2 = [(\overline{r}\otimes\overline{p} \mid \boldsymbol{\mu})M_{20}]_0.$$

For $n > 0$ we have to verify the following equation for n-th components

$$\mu^{(n)}(r\otimes p)\theta = \sum_{i_1+\dots+i_x=n}\bigl(\overline{f}_{i_1}\otimes\dots\otimes\overline{f}_{i_{q-1}}\otimes R'_{i_q}\otimes\overline{h}_{i_{q+1}}\otimes\dots\otimes\overline{h}_{i_x}\bigr)\mu^{(x)} \\ + \sum_x(\overline{r}\otimes\overline{p})\theta_{nx}\mu^{(x)} : T^{k_1}s\mathcal{A}\otimes\dots\otimes T^{k_n}s\mathcal{A}\to T^{+}s\mathcal{B}.$$

The left-hand side is

$$\sum_{a_1+\dots+a_\alpha+k+c_1+\dots+c_\beta+t+e_1+\dots+e_\gamma=n} f_{a_1}\otimes\dots\otimes f_{a_\alpha}\otimes r_k\otimes g_{c_1}\otimes\dots \\ \otimes g_{c_\beta}\otimes p_t\otimes h_{e_1}\otimes\dots\otimes h_{e_\gamma}. \tag{27}$$

Both sums in the right-hand side consist of some of the above summands. Let us verify that each summand of (27) will occur exactly once either in $\sum(\overline{f}_{i_1}\otimes\dots\otimes\overline{f}_{i_{q-1}}\otimes R'_{i_q}\otimes\overline{h}_{i_{q+1}}\otimes\dots\otimes\overline{h}_{i_x})\mu^{(x)}$, or in $\sum_x(\overline{r}\otimes\overline{p})\theta_{nx}\mu^{(x)}$. Let us rewrite the sequence $(a_1,\dots,a_\alpha;k;c_1,\dots,c_\beta;t;e_1,\dots,e_\gamma)$ as

$$(l_1,\dots,l_\alpha;l_{\alpha+1};l_{\alpha+2},\dots,l_{\alpha+\beta+1};l_{\alpha+\beta+2};l_{\alpha+\beta+3},\dots,l_{\alpha+\beta+\gamma+2}).$$

Consider the subsequence N of the sequence $L = (0,l_1,l_1+l_2,\dots,l_1+\dots+l_{\alpha+\beta+\gamma+2})$ consisting of all elements which belong to the set $\{0,k_1,k_1+k_2,\dots,k_1+\dots+k_n\}$. The term

$$f_{l_1}\otimes\dots\otimes f_{l_\alpha}\otimes r_{l_{\alpha+1}}\otimes g_{l_{\alpha+2}}\otimes\dots\otimes g_{l_{\alpha+\beta+1}}\otimes p_{l_{\alpha+\beta+2}}\otimes h_{l_{\alpha+\beta+3}}\otimes\dots\otimes h_{l_{\alpha+\beta+\gamma+2}} \tag{28}$$

will appear as a summand of

$$(\overline{f}_{i_1}\otimes\dots\otimes\overline{f}_{i_{q-1}}\otimes R'_{i_q}\otimes\overline{h}_{i_{q+1}}\otimes\dots\otimes\overline{h}_{i_x})\mu^{(x)}, \tag{29}$$

if and only if

$$N = (0, i_1, i_1 + i_2, \dots, i_1 + \dots + i_x), \tag{30}$$

$$i_1 + \dots + i_{q-1} \leqslant l_1 + \dots + l_\alpha, \qquad l_1 + \dots + l_{\alpha+\beta+2} \leqslant i_1 + \dots + i_q, \tag{31}$$

$$\forall 1 \leqslant \gamma \leqslant x \qquad i_\gamma = 0 \implies \gamma = q. \tag{32}$$

The term (28) will appear as a summand of

$$(\overline{f}_{i_1} \otimes \dots \otimes \overline{f}_{i_{y-1}} \otimes \overline{r}_{i_y} \otimes \overline{g}_{i_{y+1}} \otimes \dots \otimes \overline{g}_{i_{z-1}} \otimes \overline{p}_{i_z} \otimes \overline{h}_{i_{z+1}} \otimes \dots \otimes \overline{h}_{i_x})\mu^{(x)} \tag{33}$$

(which is a term of $(\overline{r} \otimes \overline{p})\theta_{nx}\mu^{(x)}$) if and only if (30) holds and

$$i_1 + \dots + i_{y-1} \leqslant l_1 + \dots + l_\alpha, \qquad l_1 + \dots + l_{\alpha+1} \leqslant i_1 + \dots + i_y, \tag{34}$$

$$i_1 + \dots + i_{z-1} \leqslant l_1 + \dots + l_{\alpha+\beta+1}, \qquad l_1 + \dots + l_{\alpha+\beta+2} \leqslant i_1 + \dots + i_z, \tag{35}$$

$$y < z \text{ and } \forall\, 1 \leqslant \gamma \leqslant x \qquad i_\gamma = 0 \implies \gamma \in \{y, z\}. \tag{36}$$

A given non-decreasing sequence N determines uniquely a sequence $(i_1, \dots, i_x)$ of non-negative integers such that (30) holds.

If $l_{\alpha+1} > 0$ or $l_1 + \dots + l_\alpha$ does not belong to N, then there exists exactly one element $y = y'$ such that (34) holds. If $l_{\alpha+1} = 0$ and $l_1 + \dots + l_\alpha$ belongs to N, then there are at least 2 such elements. Denote by $y' > 0$ the least of them. Then $l_1 + \dots + l_\alpha = i_1 + \dots + i_{y'-1}$ and $i_{y'} = 0$. If y satisfies both (34) and (36), then $y \leqslant y'$, hence, $y = y'$ is the only solution.

If $l_{\alpha+\beta+2} > 0$ or $l_1 + \dots + l_{\alpha+\beta+1}$ does not belong to N, then there exists exactly one element $z = z'$ such that (35) holds. If $l_{\alpha+\beta+2} = 0$ and $l_1 + \dots + l_{\alpha+\beta+1}$ belongs to N, then there are at least 2 such elements. Denote by z' the biggest of them. Then $l_1 + \dots + l_{\alpha+\beta+1} = i_1 + \dots + i_{z'-1}$ and $i_{z'} = 0$. If z satisfies both (35) and (36), then $z' \leqslant z$, hence, $z = z'$ is the only solution.

Since $[i_1 + \dots + i_{y'-1}, i_1 + \dots + i_{y'}]$ is the leftmost interval with ends in N containing $[l_1 + \dots + l_\alpha, l_1 + \dots + l_{\alpha+1}]$, and the latter lies to the left of $[l_1 + \dots + l_{\alpha+\beta+1}, l_1 + \dots + l_{\alpha+\beta+2}]$, contained in the rightmost interval $[i_1 + \dots + i_{z'-1}, i_1 + \dots + i_{z'}]$, we deduce that $y' \leqslant z'$.

If $y' = z'$, then $y' \leqslant y < z \leqslant z'$ cannot be satisfied, hence, (34)–(36) has no solutions (y, z). On the other hand, for $q = y' = z'$ the interval $[l_1 + \dots + l_\alpha, l_1 + \dots + l_{\alpha+\beta+2}]$ is contained in $[i_1 + \dots + i_{q-1}, i_1 + \dots + i_q]$, that is, (31) holds. Only $l_{\alpha+1}$ and $l_{\alpha+\beta+2}$ might vanish, both are contained in $[l_1 + \dots + l_\alpha, l_1 + \dots + l_{\alpha+\beta+2}]$, hence, i_γ might vanish only for $\gamma = q$, that is, (32) holds. Therefore, $q = y'$ satisfies conditions (31)–(32). This solution is unique, since if (31) is satisfied for $q = q'$, then (34) holds for $y = q'$.

If $y' < z'$, then $y = y'$, $z = z'$ is the only solution of system of conditions (34)–(36). This is proved by examining the four cases which arise from the alternatives in the two paragraphs that follow (36). Let us prove that there are no solutions q of the system of conditions (31)–(32). Suppose q satisfies these conditions, then $y = q$ satisfies (34) and $z = q$ satisfies (36). Therefore, $y' \leqslant q \leqslant z'$, $i_1 + \dots + i_{y'-1} = i_1 + \dots + i_{q-1}$ and $i_1 + \dots + i_q = i_1 + \dots + i_{z'}$. Due to (32) there exists no more than one γ such that $i_\gamma = 0$. Thus, two possibilities exist: either $y' = q < q + 1 = z'$, or $y' = q - 1 < q = z'$. In the first case, (35) and (31) imply

$$i_1 + \cdots + i_q \leqslant l_1 + \cdots + l_{\alpha+\beta+1} \leqslant l_1 + \cdots + l_{\alpha+\beta+2} \leqslant i_1 + \cdots + i_q,$$

hence, $i_1 + \cdots + i_q = l_1 + \cdots + l_{\alpha+\beta+1}$ and $l_{\alpha+\beta+2} = 0$. It follows that $i_{q+1} = 0$, which contradicts (32). In the second case, (31) and (34) imply

$$i_1 + \cdots + i_{q-1} \leqslant l_1 + \cdots + l_\alpha \leqslant l_1 + \cdots + l_{\alpha+1} \leqslant i_1 + \cdots + i_{q-1},$$

hence, $i_1 + \cdots + i_{q-1} = l_1 + \cdots + l_\alpha$ and $l_{\alpha+1} = 0$. It follows that $i_q = 0$. From (31) we deduce that $l_{\alpha+2} + \cdots + l_{\alpha+\beta+2} = 0$, which implies $i_{q+1} = 0$ and this contradicts (32). We conclude that each term (28) either occurs in a unique term (29) or in a unique term (33). Therefore, (26) is proven.

Since (24) is proven, it implies the statement for the transformation $(r \otimes p)R$. $\square$

Denote by $\mathcal{K}A'_\infty$ the non-2-unital $\mathcal{K}$-2-category, whose objects are pairs $(\mathcal{A}, \mathcal{A}')$, consisting of an A_∞-category $\mathcal{A}$ and a full A_∞-subcategory $\mathcal{A}' \subset \mathcal{A}$; 1-morphisms $(\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}')$ are A_∞-functors $f : \mathcal{A} \to \mathcal{B}$ such that $(\operatorname{Ob} \mathcal{A}')f \subset \operatorname{Ob} \mathcal{B}'$;

$$\mathcal{K}A'_\infty\big((\mathcal{A}, \mathcal{A}'), (\mathcal{B}, \mathcal{B}')\big)(f, g) = \big(A_\infty(\mathcal{A}, \mathcal{B})(f, g), m_1\big),$$

and the operations are induced by those of $\mathcal{K}A_\infty$.

Corollary 4.8. *The following assignment defines a strict $\mathcal{K}$-2-functor*

$$\begin{aligned} \mathsf{D} : \mathcal{K}A'_\infty &\longrightarrow \mathcal{K}A_\infty, \\ (\mathcal{A}, \mathcal{A}') &\longmapsto \mathsf{D}(\mathcal{A}|\mathcal{A}'), \\ f : (\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}') &\longmapsto \overline{f} : \mathsf{D}(\mathcal{A}|\mathcal{A}') \to \mathsf{D}(\mathcal{B}|\mathcal{B}'), \\ \big(sA_\infty((\mathcal{A}, \mathcal{A}'), (\mathcal{B}, \mathcal{B}'))(f, g), B_1\big) &\longrightarrow \big(sA_\infty(\mathsf{D}(\mathcal{A}|\mathcal{A}'), \mathsf{D}(\mathcal{B}|\mathcal{B}'))(\overline{f}, \overline{g}), \overline{B}_1\big), \quad r \mapsto \overline{r}. \end{aligned}$$

Proof. Since the coderivation $(r \otimes p)R : Ts\overline{\mathcal{A}} \to Ts\overline{\mathcal{B}}$ restricts to a coderivation $(r \otimes p)R : Ts\mathsf{D}(\mathcal{A}|\mathcal{A}') \to Ts\mathsf{D}(\mathcal{B}|\mathcal{B}')$ by Proposition 4.7, D preserves the vertical composition of 2-morphisms modulo homotopy by (23). $\square$

Corollary 4.9. *Let A'_∞ be a non-2-unital 2-category, whose objects and 1-morphisms are the same as for $\mathcal{K}A'_\infty$, and 2-morphisms are equivalence classes of natural A_∞-transformations:*

$$A'_\infty\big((\mathcal{A}, \mathcal{A}'), (\mathcal{B}, \mathcal{B}')\big)(f, g) = H^0\big(A_\infty(\mathcal{A}, \mathcal{B})(f, g), m_1\big),$$

and the operations are induced by those of A_∞. Then the following assignment defines a strict 2-functor

$$\begin{aligned} \mathsf{D} : A'_\infty &\longrightarrow A_\infty, \\ (\mathcal{A}, \mathcal{A}') &\longmapsto \mathsf{D}(\mathcal{A}|\mathcal{A}'), \\ f : (\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}') &\longmapsto \overline{f} : \mathsf{D}(\mathcal{A}|\mathcal{A}') \to \mathsf{D}(\mathcal{B}|\mathcal{B}'), \\ r : f \to g : (\mathcal{A}, \mathcal{A}') \to (\mathcal{B}, \mathcal{B}') &\longmapsto \overline{r} : \overline{f} \to \overline{g} : \mathsf{D}(\mathcal{A}|\mathcal{A}') \to \mathsf{D}(\mathcal{B}|\mathcal{B}'). \end{aligned}$$

The corollary follows from Corollary 4.8 by taking the 0-th cohomology.

5. Unitality

Proposition 5.1. *Let $\mathcal{B}$ be a full subcategory of a unital A_∞-category $\mathcal{C}$. Then the A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$ is also unital. If $\mathbf{i}^{\mathcal{C}}$ is a unit transformation of $\mathcal{C}$, then $\mathsf{D}(\mathbf{i}^{\mathcal{C}})$ is a unit transformation of $\mathsf{D}(\mathcal{C}|\mathcal{B})$.*

Proof. The idempotent property $(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}})B_2 \equiv \mathbf{i}^{\mathcal{C}}$ implies by Corollary 4.8 that

$$(\mathsf{D}(\mathbf{i}^{\mathcal{C}}) \otimes \mathsf{D}(\mathbf{i}^{\mathcal{C}}))\overline{B}_2 \equiv \mathsf{D}((\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}})B_2) \equiv \mathsf{D}(\mathbf{i}^{\mathcal{C}}),$$

so $\mathsf{D}(\mathbf{i}^{\mathcal{C}})$ is an idempotent as well. Consider its 0-th component

$${}_X\mathsf{D}(\mathbf{i}^{\mathcal{C}})_0 = \bigl[\Bbbk \xrightarrow{{}_X\mathbf{i}_0^{\mathcal{C}}} s\mathcal{C}(X,X) \hookrightarrow s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,X)\bigr].$$

We have to prove that

$$({}_X\mathsf{D}(\mathbf{i}^{\mathcal{C}})_0 \otimes 1)\bar{b}_2, \quad (1 \otimes {}_Y\mathsf{D}(\mathbf{i}^{\mathcal{C}})_0)\bar{b}_2 : s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y) \to s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y)$$

are homotopy invertible.

Consider the following $\mathbb{Z}_{>0}$-grading of the A_∞-category $\mathsf{D}(\mathcal{C}|\mathcal{B})$

$$G^k = T^k s\mathcal{C} \cap s\mathsf{D}(\mathcal{C}|\mathcal{B}), \qquad k \geqslant 1,$$
$$G^k(X,Y) = \oplus_{C_1,\dots,C_{k-1}\in \operatorname{Ob}\mathcal{B}} s\mathcal{C}(X,C_1) \otimes s\mathcal{C}(C_1,C_2) \otimes \dots \otimes s\mathcal{C}(C_{k-1},Y).$$

Denote also $C_0 = X$, $C_k = Y$. The corresponding increasing filtration

$$0 = \Phi_0 \subset \Phi_1 \subset \dots \subset \Phi_n \subset \Phi_{n+1} \subset \dots \subset s\mathsf{D}(\mathcal{C}|\mathcal{B})$$

is made of $\Phi_n = \oplus_{k=1}^n G^k$. The $\Bbbk$-linear maps

$$\bar{b}_1 = b, \quad ({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)\bar{b}_2, \quad (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})\bar{b}_2 : s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y) \to s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y)$$

preserve the filtration. Consider the $\mathbb{Z}_{>0} \times \mathbb{Z}$-graded quiver, associated with this filtration. The above maps induce on graded components G^k the following maps:

$$d_k = \sum_{\alpha+1+\beta=k} 1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta} : G^k(X,Y) \to G^k(X,Y), \tag{37}$$
$$({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 \otimes 1^{\otimes k-1} : G^k(X,Y) \to G^k(X,Y), \tag{38}$$
$$1^{\otimes k-1} \otimes (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 : G^k(X,Y) \to G^k(X,Y). \tag{39}$$

Let h, h' be homotopies as in

$$(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 = 1 + h_{C_{k-1},Y}b_1 + b_1 h_{C_{k-1},Y} : s\mathcal{C}(C_{k-1},Y) \to s\mathcal{C}(C_{k-1},Y),$$
$$({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 = -1 + h'_{X,C_1}b_1 + b_1 h'_{X,C_1} : s\mathcal{C}(X,C_1) \to s\mathcal{C}(X,C_1).$$

Using them we will present map (39) restricted to $s\mathcal{C}(X,C_1) \otimes \dots \otimes s\mathcal{C}(C_{k-2},C_{k-1}) \otimes s\mathcal{C}(C_{k-1},Y)$ as follows:

$$
\begin{aligned}
1^{\otimes k-1} \otimes (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 &= 1^{\otimes k-1} \otimes (1 + hb_1 + b_1h) \\
&= 1 + (1^{\otimes k-1} \otimes h) \sum_{\alpha+1+\beta=k} 1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta} \\
&\qquad + \Big(\sum_{\alpha+1+\beta=k} 1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta} \Big)(1^{\otimes k-1} \otimes h) \\
&= 1 + (1^{\otimes k-1} \otimes h)d_k + d_k(1^{\otimes k-1} \otimes h).
\end{aligned}
$$

Let us define a $\Bbbk$-linear map $H : s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y) \to s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y)$ of degree -1 as a direct sum of maps

$$
\begin{aligned}
1^{\otimes k-1} \otimes h_{C_{k-1},Y} : s\mathcal{C}(X, C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-2}, C_{k-1}) \otimes s\mathcal{C}(C_{k-1}, Y) \\
\to s\mathcal{C}(X, C_1) \otimes \cdots \otimes s\mathcal{C}(C_{k-2}, C_{k-1}) \otimes s\mathcal{C}(C_{k-1}, Y).
\end{aligned}
$$

Since H preserves the subquivers G^k, it preserves also the filtration Φ_n. Therefore, the chain (with respect to $\bar{b}_1$) map

$$
1 + N \overset{\text{def}}{=} (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})\bar{b}_2 - H\bar{b}_1 - \bar{b}_1H : s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y) \to s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y)
$$

preserves the filtration and the associated map of graded complexes is the identity. Hence, N has a strictly lower triangular matrix with respect to the decomposition $s\mathsf{D}(\mathcal{C}|\mathcal{B}) = \oplus_{k\geqslant 1} G^k$. Therefore, the map $1 + N$ is invertible with an inverse $\sum_{i=0}^{\infty}(-N)^i$. Hence, $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})\bar{b}_2$ is homotopy invertible.

Similarly,

$$
\begin{aligned}
({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 \otimes 1^{\otimes k-1} &= (-1 + h'b_1 + b_1h') \otimes 1^{\otimes k-1} \\
&= -1 + (h' \otimes 1^{\otimes k-1}) \sum_{\alpha+1+\beta=k} 1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta} \\
&\qquad + \Big(\sum_{\alpha+1+\beta=k} 1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta} \Big)(h' \otimes 1^{\otimes k-1}) \\
&= -1 + (h' \otimes 1^{\otimes k-1})d_k + d_k(h' \otimes 1^{\otimes k-1}).
\end{aligned}
$$

Define a map H' as a direct sum of maps

$$
\begin{aligned}
h'_{X,C_1} \otimes 1^{\otimes k-1} : s\mathcal{C}(X, C_1) \otimes s\mathcal{C}(C_1, C_2) \otimes \cdots \otimes s\mathcal{C}(C_{k-1}, Y) \\
\to s\mathcal{C}(X, C_1) \otimes s\mathcal{C}(C_1, C_2) \otimes \cdots \otimes s\mathcal{C}(C_{k-1}, Y).
\end{aligned}
$$

Then the chain map

$$
-1 + N' \overset{\text{def}}{=} ({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)\bar{b}_2 - H'\bar{b}_1 - \bar{b}_1H' : s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y) \to s\mathsf{D}(\mathcal{C}|\mathcal{B})(X,Y)
$$

preserves the filtration and gives -1 on the diagonal. Hence, N' is strictly lower triangular, and $-1 + N'$ is invertible with an inverse $-\sum_{i=0}^{\infty}(N')^i$. Therefore, $({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)\bar{b}_2$ is homotopy invertible. $\square$

Corollary 5.2. *If an A_∞-category $\mathcal{C}$ is unital, then $\overline{\mathcal{C}}$ is unital with a unit transformation $\mathbf{i}^{\overline{\mathcal{C}}}$.*

Indeed, $\overline{\mathcal{C}} = \mathsf{D}(\mathcal{C}|\mathcal{C})$.

Corollary 5.3. *If an A_∞-category $\mathcal{C}$ is unital, then $\underline{\mathcal{C}}$ is unital with a unit transformation $\underline{\mathbf{i}^{\mathcal{C}}}$.*

Proof. The A_∞-functor $\boldsymbol{\mu}^{-1} : \underline{\mathcal{C}} \to \overline{\mathcal{C}}$ is invertible and $\overline{\mathcal{C}}$ is unital. Hence, by [**Lyu03**, Section 8.12] $\underline{\mathcal{C}}$ is unital and $\boldsymbol{\mu}^{-1}\mathbf{i}^{\overline{\mathcal{C}}}\boldsymbol{\mu} = \boldsymbol{\mu}^{-1}\overline{\mathbf{i}^{\mathcal{C}}}\boldsymbol{\mu} = \underline{\mathbf{i}^{\mathcal{C}}}$ is its unit transformation. □

Remark 5.4. Functors $\bar{\jmath}^{\mathcal{C}} : \mathcal{C} \to \mathsf{D}(\mathcal{C}|\mathcal{B})$, $\bar{\jmath}^{\mathcal{C}} : \mathcal{C} \to \overline{\mathcal{C}}$, $\underline{j}^{\mathcal{C}} : \mathcal{C} \to \underline{\mathcal{C}}$ are unital. This follows from Corollary 4.2 for $r = \mathbf{i}^{\mathcal{C}}$ and from commutative diagram (14).

Corollary 5.5. *Let $i : \mathcal{C} \to \mathcal{I}$ be a unital A_∞-functor. Then the A_∞-functors $\bar{\imath} : \overline{\mathcal{C}} \to \overline{\mathcal{I}}$ and $\underline{i} : \underline{\mathcal{C}} \to \underline{\mathcal{I}}$ are unital as well.*

Proof. Since $i\mathbf{i}^{\mathcal{I}} \equiv \mathbf{i}^{\mathcal{C}}i$, we have $\bar{\imath}\overline{\mathbf{i}^{\mathcal{I}}} \equiv \overline{\mathbf{i}^{\mathcal{C}}}\bar{\imath}$ by Corollary 4.6. Therefore, $\bar{\imath}$ is unital by Corollary 5.2. We have also $\underline{i}\,\underline{\mathbf{i}^{\mathcal{I}}} \equiv \underline{\mathbf{i}^{\mathcal{C}}}\,\underline{i}$ by Corollary 4.4. Hence, $\underline{i}$ is unital by Corollary 5.3. □

Corollary 5.6. *Let $i : \mathcal{C} \to \mathcal{I}$ be a unital A_∞-functor, which maps objects of a full A_∞-subcategory $\mathcal{B} \subset \mathcal{C}$ to objects of a full A_∞-subcategory $\mathcal{J} \subset \mathcal{I}$. Then the A_∞-functor $\mathsf{D}(i) : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J})$ is unital as well.*

Proof. Since $i\mathbf{i}^{\mathcal{I}} \equiv \mathbf{i}^{\mathcal{C}}i$, we have $\mathsf{D}(i)\mathsf{D}(\mathbf{i}^{\mathcal{I}}) \equiv \mathsf{D}(\mathbf{i}^{\mathcal{C}})\mathsf{D}(i)$ by Corollary 4.9. Therefore, $\mathsf{D}(i)$ is unital by Proposition 5.1. □

Summing up, when we restrict $\underline{\text{-}}$, $\overline{\text{-}}$ or D to unital A_∞-categories, we get strict 2-functors of (ordinary 1-2-unital) ($\mathcal{K}$-)2-categories. When we restrict $\underline{\text{-}}$, $\overline{\text{-}}$ or D further to unital A_∞-categories and unital A_∞-functors, we also get strict 2-functors of ($\mathcal{K}$-)2-categories.

6. Contractibility

A chain complex C is *contractible* if id_C is null-homotopic. We say that an A_∞-category is contractible if all its complexes of morphisms are contractible. Such A_∞-categories behave like categories with zero morphisms only, although contractibility might not be obvious. An example of this kind is provided by $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$, when $\mathcal{C}$ is unital. In this section, we also collect various notions of contractibility for A_∞-functors. For unital A_∞-functors, all these definitions become equivalent.

Proposition 6.1. *Let $\mathcal{B}$ be a unital A_∞-category. Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor. Then the following conditions are equivalent:*

(C1) *For any $X \in \operatorname{Ob}\mathcal{A}$ and any $V \in \operatorname{Ob}\mathcal{B}$ the complex $(s\mathcal{B}(Xf, V), b_1)$ is contractible;*

(C2) *For any $U \in \operatorname{Ob}\mathcal{B}$ and any $Y \in \mathcal{A}$ the complex $(s\mathcal{B}(U, Yf), b_1)$ is contractible;*

(C3) *For any object X of $\mathcal{A}$ the complex $(s\mathcal{B}(Xf, Xf), b_1)$ is acyclic;*

(C4) *For any object X of $\mathcal{A}$ there is an element ${}_Xv \in (s\mathcal{B})^{-2}(Xf, Xf)$ such that ${}_{Xf}\mathbf{i}_0^{\mathcal{B}} = {}_Xvb_1$;*

(C5) $f\mathbf{i}^{\mathcal{B}} \equiv 0 : f \to f : \mathcal{A} \to \mathcal{B}$.

Proof. Clearly, (C1) $\Longrightarrow$ (C3) $\Longrightarrow$ (C4), (C2) $\Longrightarrow$ (C3) and (C5) $\Longrightarrow$ (C4).

(C4) $\Longrightarrow$ (C1): Consider a $\Bbbk$-linear map $({}_Xv \otimes 1)b_2 : s\mathcal{B}(Xf, V) \to s\mathcal{B}(Xf, V)$ of degree -1. Its commutator with b_1 is

$$({}_Xv \otimes 1)b_2b_1 + b_1({}_Xv \otimes 1)b_2 = -({}_Xvb_1 \otimes 1)b_2 = -({}_{Xf}\mathbf{i}_0^{\mathcal{B}} \otimes 1)b_2 \sim 1 : s\mathcal{B}(Xf, V) \to s\mathcal{B}(Xf, V)$$

by [**Lyu03**, Lemma 7.4]. Therefore, $s\mathcal{B}(Xf, V)$ is contractible.

(C4) $\Longrightarrow$ (C2): Consider a $\Bbbk$-linear map $(1 \otimes {}_Xv)b_2 : s\mathcal{B}(U, Xf) \to s\mathcal{B}(U, Xf)$ of degree -1. Its commutator with b_1 is

$$(1 \otimes {}_Xv)b_2b_1 + b_1(1 \otimes {}_Xv)b_2 = -(1 \otimes {}_Xvb_1)b_2 = -(1 \otimes {}_{Xf}\mathbf{i}_0^{\mathcal{B}})b_2 \sim -1 : s\mathcal{B}(U, Xf) \to s\mathcal{B}(U, Xf)$$

by [**Lyu03**, Lemma 7.4]. Therefore, $s\mathcal{B}(U, Xf)$ is contractible.

(C1) $\Longrightarrow$ (C5): We look for an (f, f)-coderivation v of degree -2 such that $vb - bv = f\mathbf{i}^{\mathcal{C}}$. We choose its 0-th component as ${}_Xv_0 : \Bbbk \to (s\mathcal{B})^{-2}(Xf, Xf)$, $1 \mapsto {}_Xv$, where ${}_Xv$ satisfies condition (C4).

Let n be a positive integer. Assume that $(v_0, v_1, \dots, v_{n-1})$ are already found such that an (f, f)-coderivation $\tilde{v} = (v_0, v_1, \dots, v_{n-1}, 0, 0, \dots)$ of degree -2 satisfies equations $\lambda_m = 0$ for $m < n$, where the (f, f)-coderivation λ of degree -1 is $\lambda = f\mathbf{i}^{\mathcal{C}} - \tilde{v}b + b\tilde{v}$. To make the induction step, we look for a map

$$v_n : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \to s\mathcal{B}(X_0f, X_nf),$$

such that

$$v_nb_1 - \sum_{q+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})v_n = \lambda_n. \tag{40}$$

The identity $\lambda b + b\lambda = 0$ implies

$$\lambda_n d = \lambda_n b_1 + \sum_{q+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\lambda_n = 0,$$

where d is the differential in the complex

$$\operatorname{Hom}\bigl(s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n), s\mathcal{B}(X_0f, X_nf)\bigr) \tag{41}$$

(all complexes are equipped with the differential b_1). Since $s\mathcal{B}(X_0f, X_nf)$ is contractible, so is complex (41). As it is acyclic, there exists v_n such that $v_nd = \lambda_n$, that is, (40) holds. Induction finishes the construction of v. $\square$

Proposition 6.2. *Let $\mathcal{A}$ be a unital A_∞-category. Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor. Then the following conditions are equivalent:*

(C6) For all objects X, Y of $\mathcal{A}$ the chain map $f_1 : (s\mathcal{A}(X, Y), b_1) \to (s\mathcal{B}(Xf, Yf), b_1)$ is homotopic to 0;

(C7) For any object X of $\mathcal{A}$ the chain map $f_1 : (s\mathcal{A}(X, X), b_1) \to (s\mathcal{B}(Xf, Xf), b_1)$ is homotopic to 0;

(C8) For any object X of $\mathcal{A}$ we have

$$H^\bullet(f_1) = 0 : H^\bullet(s\mathcal{A}(X, X), b_1) \to H^\bullet(s\mathcal{B}(Xf, Xf), b_1);$$

(C9) For any object X of $\mathcal{A}$ there is an element ${}_Xw \in (s\mathcal{B})^{-2}(Xf, Xf)$ such that ${}_X\mathbf{i}_0^{\mathcal{A}} f_1 = {}_Xwb_1$.

Proof. Clearly, (C6) $\Longrightarrow$ (C7) $\Longrightarrow$ (C8) $\Longrightarrow$ (C9).

(C9) $\Longrightarrow$ (C6): Since f and b commute, we have

$$(1 \otimes \mathbf{i}_0^{\mathcal{A}})f_2b_1 + (1 \otimes \mathbf{i}_0^{\mathcal{A}})(f_1 \otimes f_1)b_2 = (1 \otimes \mathbf{i}_0^{\mathcal{A}})b_2f_1 + (1 \otimes \mathbf{i}_0^{\mathcal{A}})(1 \otimes b_1 + b_1 \otimes 1)f_2,$$
$$b_1(1 \otimes \mathbf{i}_0^{\mathcal{A}})f_2 + (1 \otimes \mathbf{i}_0^{\mathcal{A}})f_2b_1 - (f_1 \otimes {}_Yw)b_2b_1 - b_1(f_1 \otimes {}_Yw)b_2 = (1 \otimes \mathbf{i}_0^{\mathcal{A}})b_2f_1 \sim f_1$$

by [**Lyu03**, Lemma 7.4]. Therefore, f_1 is homotopic to 0. $\square$

Proposition 6.3. *Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. Let $f : \mathcal{A} \to \mathcal{B}$ be a unital A_∞-functor. Then conditions (C1)–(C9) are equivalent to the following conditions:*

(C10) There is an isomorphism of A_∞-functors $f \simeq \mathcal{O}^f : \mathcal{A} \to \mathcal{B}$, where $\mathcal{O}^f$ is defined as follows: $X\mathcal{O}^f = Xf$, $\mathcal{O}^f_k = 0$ for all $k \geqslant 1$;

(C11) $\mathbf{i}^{\mathcal{A}} f \equiv 0 : f \to f : \mathcal{A} \to \mathcal{B}$.

Proof. Unitality implies that (C5) and (C11) are equivalent, and that (C4) and (C9) are equivalent.

(C5) $\Longrightarrow$ (C10): Consider zero natural A_∞-transformations $0 : f \to \mathcal{O}^f : \mathcal{A} \to \mathcal{B}$ and $0 : \mathcal{O}^f \to f : \mathcal{A} \to \mathcal{B}$. Their composition in one order $0 \cdot 0 = 0 : f \to f : \mathcal{A} \to \mathcal{B}$ is equivalent to $f\mathbf{i}^{\mathcal{B}} = {}_f1s$ by (C5). Their composition in the other order $0 \cdot 0 = 0 : \mathcal{O}^f \to \mathcal{O}^f : \mathcal{A} \to \mathcal{B}$ is equivalent to $\mathcal{O}^f\mathbf{i}^{\mathcal{B}}$. Indeed, there exists an (f, f)-coderivation w of degree -2 such that $f\mathbf{i}^{\mathcal{B}} = wb - bw$. In particular, ${}_X(f\mathbf{i}^{\mathcal{B}})_0 = {}_{Xf}\mathbf{i}_0^{\mathcal{B}} = {}_Xw_0b_1$. Consider the $(\mathcal{O}^f, \mathcal{O}^f)$-coderivation v of degree -2, given by its components $v_0 = w_0$ and $v_k = 0$ for $k > 0$. Then ${}_X(\mathcal{O}^f\mathbf{i}^{\mathcal{B}})_0 = {}_{Xf}\mathbf{i}_0^{\mathcal{B}} = {}_Xv_0b_1$ and

$$(\mathcal{O}^f\mathbf{i}^{\mathcal{B}})_n = 0 = v_nb_1 - \sum_{q+k+t=n} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t})v_{q+1+t} = (vb - bv)_n$$

for $n > 0$. Therefore, $\mathcal{O}^f\mathbf{i}^{\mathcal{B}} = vb - bv$.

(C10) $\Longrightarrow$ (C4): Since f is unital, isomorphic to it A_∞-functor $\mathcal{O}^f$ is unital. Thus,

$$\mathcal{O}^f\mathbf{i}^{\mathcal{B}} \equiv \mathbf{i}^{\mathcal{A}}\mathcal{O}^f = 0 : \mathcal{O}^f \to \mathcal{O}^f : \mathcal{A} \to \mathcal{B}.$$

Therefore, there exists an $(\mathcal{O}^f, \mathcal{O}^f)$-coderivation v of degree -2 such that $\mathcal{O}^f\mathbf{i}^{\mathcal{B}} = vb - bv$. In particular, ${}_{Xf}\mathbf{i}_0^{\mathcal{B}} = {}_X(\mathcal{O}^f\mathbf{i}^{\mathcal{B}})_0 = {}_Xv_0b_1$, hence, (C4) holds. $\square$

Definition 6.4. Let $\mathcal{A}$ be a unital A_∞-category. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is *contractible* if it satisfies equivalent conditions (C6)–(C9) of Proposition 6.2. An A_∞-category $\mathcal{A}$ is *contractible* if complexes $(s\mathcal{A}(X, Y), b_1)$ are contractible for all objects X, Y of $\mathcal{A}$.

A contractible A_∞-category $\mathcal{A}$ is unital. Indeed, ${}_X\mathbf{i}_0^{\mathcal{A}} = 0$ are unit elements of $\mathcal{A}$. The identity A_∞-functor $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ is contractible if and only if $\mathcal{A}$ is contractible. A unital A_∞-functor f is contractible if and only if equivalent conditions (C1)–(C11) hold.

Example 6.5. If $\mathcal{C}$ is a unital A_∞-category, then $\underline{\mathcal{C}}$, $\overline{\mathcal{C}}$ are contractible. Indeed, by Corollaries 5.2 and 5.3 these categories are unital. In particular, for all objects X, Y of $\mathcal{C}$ the chain map

$$0 = (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})\underline{b}_2 : s\underline{\mathcal{C}}(X,Y) \to s\underline{\mathcal{C}}(X,Y)$$

is homotopy invertible. Hence, $(s\underline{\mathcal{C}}(X,Y), \underline{b}_1) = (s\overline{\mathcal{C}}(X,Y), \overline{b}_1)$ is contractible. By Proposition 6.1 (C2) A_∞-categories $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ are contractible.

Example 6.6. Let $\mathcal{B}$ be a full subcategory of a unital A_∞-category $\mathcal{C}$. Then the A_∞-functor $\bar{\jmath}' = \big(\mathcal{B} \hookrightarrow \mathcal{C} \xrightarrow{\bar{\jmath}} \mathsf{D}(\mathcal{C}|\mathcal{B})\big)$ is contractible according to criterion (C4): for any object X of $\mathcal{B}$

$$({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})\bar{b}_1 = ({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})b_2 = {}_X\mathbf{i}_0^{\mathcal{C}} + {}_Xv_0b_1 = {}_X\mathbf{i}_0^{\mathsf{D}(\mathcal{C}|\mathcal{B})} + {}_Xv_0\bar{b}_1.$$

Proposition 6.7. *A unital A_∞-category $\mathcal{A}$ is contractible if and only if the following equivalent conditions hold:*

- *(C0) $\mathcal{A}$ is equivalent in A_∞^u to an A_∞-category $\mathcal{O}$, such that $\mathcal{O}(U,V) = 0$ for all objects U, V of $\mathcal{O}$;*
- *(C1′) For all objects X, Y of $\mathcal{A}$ the complex $(s\mathcal{A}(X,Y), b_1)$ is contractible;*
- *(C2′) For any object X of $\mathcal{A}$ the complex $(s\mathcal{A}(X,X), b_1)$ is contractible;*
- *(C3′) For any object X of $\mathcal{A}$ the complex $(s\mathcal{A}(X,X), b_1)$ is acyclic;*
- *(C4′) For any object X of $\mathcal{A}$ there is an element ${}_Xv \in (s\mathcal{A})^{-2}(X,X)$ such that ${}_X\mathbf{i}_0^{\mathcal{A}} = {}_Xvb_1$;*
- *(C5′) $\mathbf{i}^{\mathcal{A}} \equiv 0 : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$;*
- *(C10′) There is an isomorphism of A_∞-functors $\mathrm{id}_{\mathcal{A}} \simeq \mathcal{O}^{\mathrm{id}} : \mathcal{A} \to \mathcal{A}$, where $X\mathcal{O}^{\mathrm{id}} = X$, $\mathcal{O}^{\mathrm{id}}_k = 0$ for all $k \geqslant 1$.*

Proof. Conditions (C1′)–(C5′), (C10′) are just conditions (C1)–(C10) for $f = \mathrm{id}_{\mathcal{A}}$, hence, they are equivalent to contractibility of $\mathcal{A}$. Notice that any $\mathcal{O}$ as in (C0) is strictly unital.

(C10′) $\Longrightarrow$ (C0): Denote by $\mathcal{O}^{\mathcal{A}}$ the A_∞-category, whose class of objects is $\operatorname{Ob}\mathcal{A}$, and $\mathcal{O}^{\mathcal{A}}(X,Y) = 0$ for all objects $X, Y \in \operatorname{Ob}\mathcal{A} = \operatorname{Ob}\mathcal{O}^{\mathcal{A}}$. Let $\phi : \mathcal{O}^{\mathcal{A}} \to \mathcal{A}$, $\psi : \mathcal{A} \to \mathcal{O}^{\mathcal{A}}$ be the unique A_∞-functors such that $X\phi = X$, $X\psi = X$ for all $X \in \operatorname{Ob}\mathcal{A}$. Then $\phi\psi = \mathrm{id}_{\mathcal{A}}$ and $\psi\phi = \mathcal{O}^{\mathrm{id}_{\mathcal{A}}} \simeq \mathrm{id}_{\mathcal{A}}$ by (C10′).

(C0) $\Longrightarrow$ (C3′): There is a 2-functor $H^\bullet : A_\infty^u \to \mathcal{C}at$ such that $\operatorname{Ob} H^\bullet(\mathcal{A}) = \operatorname{Ob}\mathcal{A}$, $H^\bullet(\mathcal{A})(X,Y) = H^\bullet(\mathcal{A}(X,Y), b_1)$. If $\mathcal{A}$ is equivalent to $\mathcal{O}$ in A_∞^u, then $H^\bullet(\mathcal{A})$ is equivalent to $H^\bullet(\mathcal{O})$ in $\mathcal{C}at$. Clearly, $H^\bullet(\mathcal{O})(U,V) = 0$ for all objects U, V of $\mathcal{O}$. Hence, $H^\bullet(\mathcal{A})(X,Y) = 0$ for all objects X, Y of $\mathcal{A}$. □

Corollary 6.8. *If the A_∞-category $\mathcal{B}$ is contractible, then for any A_∞-category $\mathcal{A}$, any natural A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ is equivalent to 0.*

Remark 6.9. Any A_∞-category $\mathcal{O}$, such that $\mathcal{O}(U,V) = 0$ for all objects U, V of $\mathcal{O}$ with non-empty $\operatorname{Ob}\mathcal{O}$ is equivalent to 1-object-1-morphism A_∞-category $\mathbb{1}$, such that $\operatorname{Ob}\mathbb{1} = \{*\}$ and $\mathbb{1}(*,*) = 0$. Indeed, choose an object $Z \in \operatorname{Ob}\mathcal{O}$. Consider A_∞-functors $\phi : \mathcal{O} \to \mathbb{1}$, $U \mapsto *$ and $\psi : \mathbb{1} \to \mathcal{O}$, $* \mapsto Z$. We have $\psi\phi = \mathrm{id}_{\mathbb{1}}$ and $\phi\psi$ is isomorphic to $\mathrm{id}_{\mathcal{O}}$ via inverse 2-morphisms $0 : \phi\psi \to \mathrm{id}_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}$ and $0 : \mathrm{id}_{\mathcal{O}} \to \phi\psi : \mathcal{O} \to \mathcal{O}$.

Remark 6.10. Let $\mathcal{C}$ be strictly unital. Then $\mathsf{D}(\mathcal{C}|\mathcal{B})$ has a strict unit $\mathbf{i}^{\mathsf{D}(\mathcal{C}|\mathcal{B})}$ described in Section 2.1. On the other hand, $\mathsf{D}(\mathbf{i}^{\mathcal{C}})$ is a unit of $\mathsf{D}(\mathcal{C}|\mathcal{B})$ as well. Hence, $\mathbf{i}^{\mathsf{D}(\mathcal{C}|\mathcal{B})} \equiv \mathsf{D}(\mathbf{i}^{\mathcal{C}})$ by [**Lyu03**, Corollary 7.10].

7. The case of a contractible subcategory

Taking the quotient $\mathsf{D}(\mathcal{E}|\mathcal{F})$ can be interpreted as contracting the full A_∞-subcategory $\mathcal{F} \subset \mathcal{E}$. If $\mathcal{F}$ were already contractible, one would expect that no further contracting is required. And, in fact, if $\mathcal{E}$ is unital, we shall prove below that $\mathsf{D}(\mathcal{E}|\mathcal{F})$ is equivalent to $\mathcal{E}$. In the proof we shall construct inductively a new A_∞-structure on $\mathcal{E}$. So first of all, we consider direct limits of A_∞-structures on a given graded $\Bbbk$-linear quiver.

Lemma 7.1. *Let $\mathcal{B}$ be a graded $\Bbbk$-quiver. Let $\mathcal{A}_k$, $k \geqslant 1$, be a sequence of A_∞-categories, whose underlying graded $\Bbbk$-quiver is $\mathcal{B}$. Let $\mathcal{A}_1 \xrightarrow{f^1} \mathcal{A}_2 \xrightarrow{f^2} \cdots$ be a sequence of A_∞-functors, such that $f_1^k = \mathrm{id}_{\mathcal{A}_k}$ for all k, and let N_i, $i \geqslant 2$ be an increasing sequence of positive integers, such that $f_i^k = 0$ for $k \geqslant N_i$. Then there exists a direct (2-)limit $\mathcal{A} = \varinjlim_{f^i} \mathcal{A}_i$ of this diagram, and the structure A_∞-functors $p^k : \mathcal{A}_k \to \mathcal{A}$ are invertible.*

Proof. If $g : \mathcal{D} \to \mathcal{C}$ is an A_∞-functor, such that $g_1 = \mathrm{id}_{\mathcal{D}}$ and $g_i = 0$ for $i = 2, \dots, k$, then for any such i there exists a commutative diagram

$$\begin{tikzcd} s\mathcal{D}^{\otimes i} \arrow[r, "\mathrm{id}_{s\mathcal{D}^{\otimes i}}"] \arrow[d, "b_i"'] & s\mathcal{C}^{\otimes i} \arrow[d, "b_i"] \\ s\mathcal{D} \arrow[r, "\mathrm{id}_{s\mathcal{D}}"] & s\mathcal{C} \end{tikzcd}$$

which allows us to identify the A_∞-operations b_i, $i = 1, \dots, k$ on $\mathcal{D}$ and $\mathcal{C}$.

Due to this remark, we take $\mathcal{B}$ as the underlying $\Bbbk$-quiver of $\mathcal{A}$, we set $b_1 : s\mathcal{A} \to s\mathcal{A}$ to be $b_1 : s\mathcal{A}_1 \to s\mathcal{A}_1$ and we set $b_i : s\mathcal{A}^{\otimes i} \to s\mathcal{A}$ equal $b_i : s\mathcal{A}_{N_i}^{\otimes i} \to s\mathcal{A}_{N_i}$. This equips $\mathcal{A}$ with an A_∞-structure. We define p^k setting its i-th component equal to $p_i^k = (f^k f^{k+1} \dots f^l)_i$ for $l = \max(k, N_i)$.

Given an A_∞-category $\mathcal{C}$ and A_∞-functors $\pi^k : \mathcal{A}_k \to \mathcal{C}$, $k = 1, \dots,$ such that $\pi^k = f^k \pi^{k+1}$, then there exists the unique A_∞-functor $\pi : \mathcal{A} \to \mathcal{C}$, such that $\pi^k = p^k \pi$, defined by $\pi_i = \pi_i^{N(i)}$, $i \geqslant 1$. It shows that the constructed $\mathcal{A}$ is a direct limit of the diagram $(\mathcal{A}_i, f^i, i \geqslant 1)$. □

Lemma 7.2. *Let $\mathcal{E}$ be an A_∞-category and let $\mathcal{F}$ be its full A_∞-subcategory such that the complex of $\Bbbk$-modules $(s\mathcal{E}(X,Y), b_1)$ is contractible provided at least one of X, Y belongs to $\operatorname{Ob}\mathcal{F}$. Denote by $D_n(\mathcal{E}|\mathcal{F})$, $n = 2, 3, \dots$ the $\Bbbk$-submodule in $(s\mathcal{E})^{\otimes n}$, which is a sum of $s\mathcal{E}(X_0, X_1) \otimes s\mathcal{E}(X_1, X_2) \otimes \dots \otimes s\mathcal{E}(X_{n-2}, X_{n-1}) \otimes s\mathcal{E}(X_{n-1}, X_n)$, such that at least for one $i = 0, \dots, n$ object X_i belongs to $\operatorname{Ob}\mathcal{F}$. Then there exists an invertible A_∞-functor $g : \mathcal{E} \to \mathcal{E}_{\mathcal{F}}$, such that A_∞-category $(\mathcal{E}_{\mathcal{F}}, b')$ coincides with $\mathcal{E}$ as a graded differential (with respect to b_1) $\Bbbk$-quiver (that is, $g_1 = \mathrm{id}_{\mathcal{E}}$) and $(D_n(\mathcal{E}|\mathcal{F}))b'_n = 0$ for any $n > 1$.*

Proof. We construct a chain of A_∞-isomorphisms $f^i : \mathcal{E}_i \to \mathcal{E}_{i+1}$, $i = 1, \dots$, $\mathcal{E}_1 = \mathcal{E}$ as in Lemma 7.1 and then we set $\mathcal{E}_{\mathcal{F}} = \varinjlim_{f^i} \mathcal{E}_i$ and $g = p^1$. For the constructed $\mathcal{E}_j$ and f^j, $j = 1, \dots$ the following will hold: 1) $b_k|_{D_k(\mathcal{E}|\mathcal{F})} = 0$ holds in $\mathcal{E}_j$ for $k = 2, \dots, j$; 2) $f^j_i = 0$ for $i \neq 1, j+1$.

Given an A_∞-category $\mathcal{E}_k$ and a $\Bbbk$-quiver morphism $f^k = (\mathrm{id}_\mathcal{E}, 0, \dots, 0, f^k_{k+1}, 0, \dots) : Ts\mathcal{E}_k \to s\mathcal{E}$ of degree 0, they define (following Remark 2.3) a unique A_∞-category structure $\mathcal{E}_{k+1}$ on the graded $\Bbbk$-quiver $\mathcal{E}$ such that f^k turns into an A_∞-functor $f^k : \mathcal{E}_k \to \mathcal{E}_{k+1}$. Assume $f^j : \mathcal{E}_j \to \mathcal{E}_{j+1}$, $j = 1, \dots, k-1$ are constructed (the reasoning is valid for $k = 1$ too).

Let us fix, for any sequence $X_0, X_1, \dots, X_n$ as in the hypothesis of the lemma, an index $l(X_0, \dots, X_n)$ such that $X_{l(X_0,\dots,X_n)}$ belongs to $\operatorname{Ob}\mathcal{F}$. Any choice of $f^k_{k+1} : T^{k+1}s\mathcal{E}_k \to s\mathcal{E}$ determines by Remark 2.3 an A_∞-category $\mathcal{E}_{k+1}$ with the operations $b_p = b_p^{\mathcal{E}_{k+1}}$. Notice that the conditions $b_2^{\mathcal{E}_{k+1}}|_{D_2(\mathcal{E}|\mathcal{F})} = 0, \dots, b_k^{\mathcal{E}_{k+1}}|_{D_k(\mathcal{E}|\mathcal{F})} = 0$ hold automatically: in view of $f^k_1 = \mathrm{id}_\mathcal{E}$ in $\mathcal{E}_k$ and $f^k_j = 0$, $1 < j \leqslant i \leqslant k$ the i-th condition (2) shows, that $b_1, \dots, b_k$ in $\mathcal{E}_k$ and $\mathcal{E}_{k+1}$ coincide. The $(k+1)$-th condition (1) for $\mathcal{E}_k$ on $D_{k+1}(\mathcal{E}|\mathcal{F})$ turns into

$$\sum_{r+1+t=k+1} (1^{\otimes r} \otimes b_1 \otimes 1^{\otimes t}) b_{k+1} + b_{k+1} b_1 = 0$$

(for any other summand in the sum $\sum_{r+n+t=k+1} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t}$, either the first or the second factor vanishes by induction). On the other hand, by the induction assumptions, the $(k+1)$-th condition (2) turns on $D_{k+1}(\mathcal{E}|\mathcal{F})$ into

$$(f^k_1)^{\otimes(k+1)} b_{k+1} + f^k_{k+1} b_1 = \sum_{r+1+t=k+1} (1^{\otimes r} \otimes b_1 \otimes 1^{\otimes t}) f^k_{k+1} + b_{k+1} f^k_1 : \\ sD_{k+1}(\mathcal{E}|\mathcal{F}) \to s\mathcal{E}_{k+1}. \tag{42}$$

We consider the condition "$b_{k+1}|_{D_{k+1}(\mathcal{E}|\mathcal{F})} = 0$ holds in the A_∞-category $\mathcal{E}_{k+1}$" as an equation with respect to f^k_{k+1}. Denote $l = \min\{l(X_0, \dots, X_{k+1}), k\}$. Choose a contracting homotopy, i.e. $\Bbbk$-module morphism $h : s\mathcal{E}(X_l, X_{l+1}) \to s\mathcal{E}(X_l, X_{l+1})$ of degree -1 such that $hb_1 + b_1 h = \mathrm{id}_{s\mathcal{E}(X_l, X_{l+1})}$. We define f^k_{k+1} on $D_{k+1}(\mathcal{E}|\mathcal{F})(X_0, \dots, X_{k+1})$ as

$$f^k_{k+1} = -(1^{\otimes l} \otimes h \otimes 1^{\otimes(k-l)}) b_{k+1}.$$

On $s\mathcal{E}(X_0, X_1) \otimes \dots \otimes s\mathcal{E}(X_{n-1}, X_n)$ such that all $X_i \notin \operatorname{Ob}\mathcal{F}$, we set $f^k_{k+1} = 0$.

Compare Equation (42) restricted to $D_{k+1}(\mathcal{E}|\mathcal{F})(X_0, \dots, X_{k+1})$ with the following computation

$$
\begin{aligned}
f_{k+1}^k b_1 &= -(1^{\otimes l} \otimes h \otimes 1^{\otimes (k-l)}) b_{k+1} b_1 \\
&= (1^{\otimes l} \otimes h \otimes 1^{\otimes (k-l)}) \sum_{r+1+t=k+1} (1^{\otimes r} \otimes b_1 \otimes 1^{\otimes t}) b_{k+1} \\
&= \sum_{r+1+t=k+1} (1^{\otimes r} \otimes b_1 \otimes 1^{\otimes t})(-(1^{\otimes l} \otimes h \otimes 1^{\otimes (k-l)})) b_{k+1} \\
&\quad + (1^{\otimes l} \otimes b_1 h \otimes 1^{\otimes (k-l)}) b_{k+1} + (1^{\otimes l} \otimes h b_1 \otimes 1^{\otimes (k-l)}) b_{k+1} \\
&= \sum_{r+1+t=k+1} (1^{\otimes r} \otimes b_1 \otimes 1^{\otimes t}) f_{k+1}^k + b_{k+1}.
\end{aligned}
$$

We deduce that in $\mathcal{E}_{k+1}$ the restriction of b_{k+1} to $D_{k+1}(\mathcal{E}|\mathcal{F})$ vanishes. Now the lemma follows from the definition of the limit morphism $g : \mathcal{E} \to \varinjlim_{f^i} \mathcal{E}_i$. □

Lemma 7.3. *Let $\mathcal{F} \subset \mathcal{E}_{\mathcal{F}}$ be a full A_∞-subcategory such that $b_n\big|_{D_n(\mathcal{E}_{\mathcal{F}}|\mathcal{F})}$ vanishes for all $n \geqslant 2$. Then the canonical strict embedding $\bar{\jmath} : \mathcal{E}_{\mathcal{F}} \to \mathsf{D}(\mathcal{E}_{\mathcal{F}}|\mathcal{F})$ admits a splitting strict A_∞-functor $\pi : \mathsf{D}(\mathcal{E}_{\mathcal{F}}|\mathcal{F}) \to \mathcal{E}_{\mathcal{F}}$, that is, $\bar{\jmath}\pi = \mathrm{id}_{\mathcal{E}_{\mathcal{F}}}$. Its first component is the projection*

$$\pi_1 = \big(s\mathsf{D}(\mathcal{E}_{\mathcal{F}}|\mathcal{F}) \hookrightarrow T^+ s\mathcal{E}_{\mathcal{F}} \xrightarrow{\mathrm{pr}_1} s\mathcal{E}_{\mathcal{F}}\big).$$

Proof. Denote $\mathcal{A} = \mathsf{D}(\mathcal{E}_{\mathcal{F}}|\mathcal{F}) \cap (s^{-1}T^{\geqslant 2} s\mathcal{E}_{\mathcal{F}}) = \operatorname{Ker}\pi_1$. Then $\mathsf{D}(\mathcal{E}_{\mathcal{F}}|\mathcal{F}) = \mathcal{E}_{\mathcal{F}} \oplus \mathcal{A}$ as a graded $\Bbbk$-quiver and $s\mathcal{A} \subset \oplus_{n\geqslant 2} D_n(\mathcal{E}|\mathcal{F})$. Let us check that π is an A_∞-functor, that is, $\pi_1^{\otimes k} b_k = \bar{b}_k \pi_1$ for all $k \geqslant 1$.

Notice that $\mathcal{A}(X,Y) = \oplus^{n\geqslant 2}_{X_1,\dots,X_{n-1}\in \operatorname{Ob}\mathcal{F}} s\mathcal{E}(X,X_1) \otimes \dots \otimes s\mathcal{E}(X_{n-1},Y)$, and each substring of such a tensor product of length $k \geqslant 2$ is in $D_k(\mathcal{E}|\mathcal{F})$. The restriction to $s\mathcal{E}_{\mathcal{F}}^{\otimes k}$ of the equation $\pi_1^{\otimes k} b_k = \bar{b}_k \pi_1$ follows from (5) or Corollary 2.4. Both sides, restricted to $s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_k$ where $\mathcal{X}_j$ is $\mathcal{E}_{\mathcal{F}}$ or $\mathcal{A}$, vanish if at least one $\mathcal{A}$ is present. Indeed, $\bar{b}_k$ vanishes in that case if $k > 1$ due to (5). For $k = 1$, $(s\mathcal{A})\bar{b}_1 = (s\mathcal{A})b \subset s\mathcal{A}$ as equation $\bar{b}_1\big|_{\mathcal{A}} = b\big|_{\mathcal{A}} = \sum_{q+1+t=n} 1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t} : s\mathcal{A} \to s\mathcal{A}$ shows. Other claims are clear. □

Proposition 7.4. *Let $\mathcal{F}$ be a contractible full subcategory of a unital A_∞-category $\mathcal{E}$. Then there exists a quasi-inverse to the canonical strict embedding $\bar{\jmath}^{\mathcal{E}} : \mathcal{E} \to \mathsf{D}(\mathcal{E}|\mathcal{F})$ unital A_∞-functor $\pi^{\mathcal{E}} : \mathsf{D}(\mathcal{E}|\mathcal{F}) \to \mathcal{E}$ such that $\bar{\jmath}^{\mathcal{E}}\pi^{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$. In particular, $\mathsf{D}(\mathcal{E}|\mathcal{F})$ is equivalent to $\mathcal{E}$.*

Proof. First of all, we prove the statements for the full embedding $\mathcal{F} \subset \mathcal{E}_{\mathcal{F}}$ constructed in Lemma 7.2. Since $\mathcal{E}$ is unital and $g : \mathcal{E} \to \mathcal{E}_{\mathcal{F}}$ is invertible, the A_∞-category $\mathcal{E}_{\mathcal{F}}$ is unital by [**Lyu03**, Section 8.12]. Let us prove that the A_∞-functor $\pi^{\mathcal{E}_{\mathcal{F}}} = \pi : \mathsf{D}(\mathcal{E}_{\mathcal{F}}|\mathcal{F}) \to \mathcal{E}_{\mathcal{F}}$ from Lemma 7.3 is unital, that is, $\pi \mathbf{i}^{\mathcal{E}_{\mathcal{F}}} \equiv \mathsf{D}(\mathbf{i}^{\mathcal{E}_{\mathcal{F}}})\pi$.

We look for a 3-morphism

$$v : \pi\mathbf{i}^{\mathcal{E}_{\mathcal{F}}} \to \mathsf{D}(\mathbf{i}^{\mathcal{E}_{\mathcal{F}}})\pi : \pi \to \pi : \mathsf{D}(\mathcal{E}_{\mathcal{F}}|\mathcal{F}) \to \mathcal{E}_{\mathcal{F}}.$$

Since $(\pi\mathbf{i}^{\mathcal{E}_{\mathcal{F}}})_0 = \mathbf{i}_0^{\mathcal{E}_{\mathcal{F}}} = (\mathsf{D}(\mathbf{i}^{\mathcal{E}_{\mathcal{F}}})\pi)_0$, we may take $v_0 = 0$. Let us proceed by induction. Assume that we have already found components $(v_0, v_1, \dots, v_{n-1})$ of v such that v_m vanishes on $(s\mathcal{E}_{\mathcal{F}})^{\otimes m}$ for all $m < n$. Define a (π,π)-transformation

$\tilde{v} = (v_0, v_1, \dots, v_{n-1}, 0, 0, \dots)$ by these components. Denote by λ the (π,π)-transformation $\pi \mathbf{i}^{\mathcal{E}_\mathcal{F}} - \mathsf{D}(\mathbf{i}^{\mathcal{E}_\mathcal{F}})\pi - \tilde{v}b + \bar{b}\tilde{v}$. Our assumption is that $\lambda_m = 0$ for $m < n$. Clearly, $\lambda b + \bar{b}\lambda = 0$. This implies

$$0 = (\lambda b + \bar{b}\lambda)_n = \lambda_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes \bar{b}_1 \otimes 1^{\otimes t})\lambda_n,$$

that is, $\lambda_n \in \mathrm{Hom}^{-1}\big((s\mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}))^{\otimes n}, s\mathcal{E}_\mathcal{F}\big)$ is a cocycle. We wish to prove that it is a coboundary of an element $v_n \in \mathrm{Hom}^{-2}\big((s\mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}))^{\otimes n}, s\mathcal{E}_\mathcal{F}\big)$, that is,

$$\lambda_n = v_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes \bar{b}_1 \otimes 1^{\otimes t}) v_n = v_n d.$$

We have $(s\mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}))^{\otimes n} = \oplus_{\mathcal{X}_i \in \{\mathcal{E}_\mathcal{F}, \mathcal{A}\}} s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n$. If at least one $\mathcal{A}$ is present in $s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n$, then this summand is contractible, hence, $\mathrm{Hom}(s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n, s\mathcal{E}_\mathcal{F})$ is contractible. Therefore, there are such $v'_n \in \mathrm{Hom}^{-2}(s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n, s\mathcal{E}_\mathcal{F})$ that $v'_n d = \lambda_n$ on $s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n$. It remains to look at the case $s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n = (s\mathcal{E}_\mathcal{F})^{\otimes n}$. By restriction to this submodule, we have

$$(\pi \mathbf{i}^{\mathcal{E}_\mathcal{F}})_n = \mathbf{i}_n^{\mathcal{E}_\mathcal{F}} = (\mathsf{D}(\mathbf{i}^{\mathcal{E}_\mathcal{F}})\pi)_n : (s\mathcal{E}_\mathcal{F})^{\otimes n} \to s\mathcal{E}_\mathcal{F},$$
$$(\tilde{v}b - \bar{b}\tilde{v})_n = 0 : (s\mathcal{E}_\mathcal{F})^{\otimes n} \to s\mathcal{E}_\mathcal{F}.$$

Therefore, λ_n vanishes on $(s\mathcal{E}_\mathcal{F})^{\otimes n}$, and $v_n\big|_{(s\mathcal{E}_\mathcal{F})^{\otimes n}} = 0$ satisfies the equation and the induction assumptions.

Define the following natural A_∞-transformations

$$r : \mathrm{id} \to \pi\jmath : \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}) \to \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}),$$
$$p : \pi\jmath \to \mathrm{id} : \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}) \to \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F})$$

via its components, restricted to $s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n \subset (s\mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}))^{\otimes n}$, namely, $r_k = \mathbf{i}_k^{\mathcal{E}_\mathcal{F}}$ and $p_k = \mathbf{i}_k^{\mathcal{E}_\mathcal{F}}$ if $s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n = (s\mathcal{E}_\mathcal{F})^{\otimes n}$, and $r_k = 0$, $p_k = 0$ otherwise. We have to check the equation $r\bar{b} + \bar{b}r = 0$. Its restriction to $(s\mathcal{E}_\mathcal{F})^{\otimes n}$ holds because on this submodule $\bar{b}$ can be replaced with b and r with $\mathbf{i}$. If $\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$ contains $\mathcal{A}$, then all terms in the following sums vanish on $s\mathcal{X}_1 \otimes \dots \otimes s\mathcal{X}_n$, hence,

$$\sum_{q+k+t=n} (1^{\otimes q} \otimes r_k \otimes (\pi\jmath)_1^{\otimes t})\bar{b}_{q+1+t} + \sum_{q+k+t=n} (1^{\otimes q} \otimes \bar{b}_k \otimes 1^{\otimes t}) r_{q+1+t} = 0.$$

In the same way, we prove that $p\bar{b} + \bar{b}p = 0$. Thus, r and p are, indeed, natural A_∞-transformations.

Let us prove now that r and p are inverse to each other 2-morphisms, that is,

$$(r \otimes p)B_2 \equiv \mathsf{D}(\mathbf{i}^{\mathcal{E}_\mathcal{F}}) : \mathrm{id} \to \mathrm{id} : \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}) \to \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}),$$
$$(p \otimes r)B_2 \equiv \pi\jmath\mathsf{D}(\mathbf{i}^{\mathcal{E}_\mathcal{F}}) : \pi\jmath \to \pi\jmath : \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}) \to \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}).$$

We look for a 3-morphism

$$v : (r \otimes p)B_2 \to \mathsf{D}(\mathbf{i}^{\mathcal{E}_\mathcal{F}}) : \mathrm{id} \to \mathrm{id} : \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}) \to \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}).$$

Let us look at the restriction of the equation

$$(r \otimes p)B_2 - \mathsf{D}(\mathbf{i}^{\mathcal{E}_\mathcal{F}}) = v\bar{b} - \bar{b}v \tag{43}$$

to $(s\mathcal{E}_\mathcal{F})^{\otimes n}$. First of all, $(\pi\bar{\jmath})_1 = 1$ on $s\mathcal{E}_\mathcal{F}$. Summands $\bar{b}_k$, contained in B_2, are applied to elements of $(s\mathcal{E}_\mathcal{F})^{\otimes k}$ only. Hence, they can be replaced with b_k. Therefore, $B_2^{\mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F})}$ is replaced with $B_2^{\mathcal{E}_\mathcal{F}}$ on $(s\mathcal{E}_\mathcal{F})^{\otimes n}$. The problem of finding a 3-morphism

$$w : (\mathbf{i}^{\mathcal{E}_\mathcal{F}} \otimes \mathbf{i}^{\mathcal{E}_\mathcal{F}})B_2 \to \mathbf{i}^{\mathcal{E}_\mathcal{F}} : \mathrm{id} \to \mathrm{id} : \mathcal{E}_\mathcal{F} \to \mathcal{E}_\mathcal{F}$$

is solvable. We set the restriction of v_n to $(s\mathcal{E}_\mathcal{F})^{\otimes n}$ equal $v_n = w_n : (s\mathcal{E}_\mathcal{F})^{\otimes n} \to s\mathcal{E}_\mathcal{F}$ and it solves (43) on this submodule. The restriction of Equation (43) to $s\mathcal{X}_1 \otimes \cdots \otimes s\mathcal{X}_n$ that contains factor $\mathcal{A}$, can be solved by induction due to contractibility of $s\mathcal{X}_1 \otimes \cdots \otimes s\mathcal{X}_n$ as above. Thus v is constructed.

Similarly, a 3-morphism

$$u : (p \otimes r)B_2 \to \pi\bar{\jmath}\mathsf{D}(\mathbf{i}^{\mathcal{E}_\mathcal{F}}) : \pi\bar{\jmath} \to \pi\bar{\jmath} : \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}) \to \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F})$$

is constructed.

The property $\bar{\jmath}\pi = \mathrm{id}_{\mathcal{E}_\mathcal{F}}$ is proved in Lemma 7.3.

Now we turn to the general case. The invertible A_∞-functor $g : \mathcal{E} \to \mathcal{E}_\mathcal{F}$, constructed in Lemma 7.2 is the identity on objects. Denoting $\pi^{\mathcal{E}_\mathcal{F}} = \pi$ as above, $g' = g\big|_\mathcal{F} : \mathcal{F} \to \mathcal{F}$, and $\pi^\mathcal{E} = \bar{g}\pi^{\mathcal{E}_\mathcal{F}}g^{-1}$, we get a diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d, "g'"'] & \mathcal{E} \arrow[r, shift left, "\bar{\jmath}^\mathcal{E}"] \arrow[d, "g"] & \mathsf{D}(\mathcal{E}|\mathcal{F}) \arrow[l, shift left, "\pi^\mathcal{E}"] \arrow[d, "\bar{g}"] \\
\mathcal{F} \arrow[r, hook] & \mathcal{E}_\mathcal{F} \arrow[r, shift left, "\bar{\jmath}^{\mathcal{E}_\mathcal{F}}"] & \mathsf{D}(\mathcal{E}_\mathcal{F}|\mathcal{F}) \arrow[l, shift left, "\pi^{\mathcal{E}_\mathcal{F}}"]
\end{tikzcd}$$

All the required properties of $\pi^\mathcal{E}$ follow immediately from those of $\pi^{\mathcal{E}_\mathcal{F}}$. □

7.1. Reducing a full contractible subcategory to 0.

Let $\mathcal{F}$ be a full contractible subcategory of a unital A_∞-category $\mathcal{E}$. Let us consider another A_∞-category $\mathcal{E}/_p\mathcal{F}$, whose class of objects is $\operatorname{Ob}\mathcal{E}$. Here $/_p$ stands for the plain quotient. The morphisms are $\mathcal{E}/_p\mathcal{F}(X,Y) = \mathcal{E}(X,Y)$, if $X, Y \in \operatorname{Ob}\mathcal{E} - \operatorname{Ob}\mathcal{F}$ and $\mathcal{E}/_p\mathcal{F}(X,Y) = 0$ otherwise. The component of the differential for $\mathcal{E}/_p\mathcal{F}$

$$b_n : s\mathcal{E}/_p\mathcal{F}(X_0,X_1) \otimes \cdots \otimes s\mathcal{E}/_p\mathcal{F}(X_{n-1},X_n) \to s\mathcal{E}/_p\mathcal{F}(X_0,X_n)$$

equals b_n for $\mathcal{E}$ if $X_0, \dots, X_n \notin \operatorname{Ob}\mathcal{F}$, and vanishes otherwise.

There is a strict embedding $e : \mathcal{E}/_p\mathcal{F} \to \mathcal{E}$, which is the identity on objects, $e_1 = \mathrm{id} : s\mathcal{E}/_p\mathcal{F}(X,Y) \to s\mathcal{E}(X,Y)$ if $X, Y \notin \operatorname{Ob}\mathcal{F}$ and vanishes otherwise.The identity $e_1^{\otimes n}b_n^\mathcal{E} = b_n^{\mathcal{E}/_p\mathcal{F}}e_1$ is obvious.

If $\mathcal{E}$ is strictly unital with the strict unit $\mathbf{i}^\mathcal{E}$, then $\mathcal{E}/_p\mathcal{F}$ is strictly unital with the strict unit $\mathbf{i}^{\mathcal{E}/_p\mathcal{F}}$, defined as follows. Its 0-th component is ${}_X\mathbf{i}_0^{\mathcal{E}/_p\mathcal{F}} = {}_X\mathbf{i}_0^\mathcal{E}$ if $X \notin \operatorname{Ob}\mathcal{F}$, and vanishes otherwise.

Let us consider the general case of a unital $\mathcal{E}$. Each complex $(s\mathcal{E}(X,Y), b_1)$ is contractible if X or Y is an object of $\mathcal{F}$ due to Proposition 6.1 (C1), (C2). Therefore, $e_1 : s\mathcal{E}/_p\mathcal{F}(X,Y) \to s\mathcal{E}(X,Y)$ is homotopy invertible for all pairs X, Y of objects of $\mathcal{E}$. Consider the following data: identity map $h = \mathrm{id} : \operatorname{Ob}\mathcal{E} \to \operatorname{Ob}\mathcal{E}/_p\mathcal{F}$ and $\Bbbk$-linear maps ${}_Xr_0 = {}_Xp_0 = {}_X\mathbf{i}_0^\mathcal{E} : \Bbbk \to (s\mathcal{E})^{-1}(X,X)$. Clearly, ${}_X\mathbf{i}_0^\mathcal{E}b_1 = 0$ and

$({}_X\mathbf{i}_0^{\mathcal{E}} \otimes {}_X\mathbf{i}_0^{\mathcal{E}})b_2 - {}_X\mathbf{i}_0^{\mathcal{E}} \in \operatorname{Im} b_1$. Therefore, the hypotheses of Theorem 8.8 of [**Lyu03**] are satisfied. By this theorem we conclude that $\mathcal{E}/_p\mathcal{F}$ is unital and $e : \mathcal{E}/_p\mathcal{F} \to \mathcal{E}$ is a unital A_∞-equivalence.

8. An A_∞-functor related by an A_∞-transformation to a given A_∞-functor

Given an A_∞-functor f and the 0-th component r_0 of a natural A_∞-transformation $r : f \to g$, we construct the A_∞-functor g and extend r_0 to the whole A_∞-transformation r. We do it under additional assumptions on r_0 which are satisfied, for instance, when r_0 is invertible. In the next section we apply this construction to the case of the quasi-isomorphisms r_0.

8.1. Assumptions.

Let $\mathcal{B}$, $\mathcal{C}$ be A_∞-categories, let $f : \mathcal{B} \to \mathcal{C}$ be an A_∞-functor and let $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be a map. Assume that for each object $X \in \operatorname{Ob}\mathcal{B}$, there is an element $r_X \in \mathcal{C}^0(Xf, Xg)$ such that $r_X s b_1 = 0$. For any object $Y \in \operatorname{Ob}\mathcal{B}$, this element determines a chain map

$$(r_X s \otimes 1)b_2 : s\mathcal{C}(Xg, Yg) \to s\mathcal{C}(Xf, Yg), \qquad p \mapsto (-)^p (r_X s \otimes p)b_2.$$

Finally, we assume that for any chain complex of $\Bbbk$-modules of the form $N = s\mathcal{B}(X_0, X_1) \otimes_\Bbbk s\mathcal{B}(X_1, X_2) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{B}(X_{n-1}, X_n)$, $n \geqslant 0$, the following chain map

$$u = \operatorname{Hom}(N, (r_X s \otimes 1)b_2) : \operatorname{Hom}_\Bbbk^\bullet(N, s\mathcal{C}(Xg, Yg)) \to \operatorname{Hom}_\Bbbk^\bullet(N, s\mathcal{C}(Xf, Yg)) \quad (44)$$

is a quasi-isomorphism. For $n = 0$, we have $N = \Bbbk$ and the 0-th condition means that $(r_X s \otimes 1)b_2$ is a quasi-isomorphism.

Proposition 8.1. *Under the above assumptions, the map $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ extends to an A_∞-functor $g : \mathcal{B} \to \mathcal{C}$. There exists a natural A_∞-transformation $r : f \to g : \mathcal{B} \to \mathcal{C}$ such that its 0-th component is $r_0 : \Bbbk \to s\mathcal{C}(Xf, Xg)$, $1 \mapsto r_X s$.*

All statements in this section (existence of the A_∞-functor g and the natural A_∞-transformation r, their uniqueness in a certain sense, unitality of g and invertibility of r) are proved in a similar fashion, using acyclicity of the cone of the quasi-isomorphism u.

Proof. The components $g_0 = 0$ and r_0 are already known. Let us build the remaining components by induction. Assume that we have already found components g_m, r_m of the sought for g, r for $m < n$, such that the equations

$$gb\operatorname{pr}_1 = bg\operatorname{pr}_1 : s\mathcal{B}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 g, X_m g),$$
$$(rb + br)\operatorname{pr}_1 = 0 : s\mathcal{B}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 f, X_m g)$$

are satisfied for all $m < n$. Under these assumptions, we will find such g_n, r_n that the above equations are satisfied for $m = n$. Let us write down these equations explicitly. The terms which contain unknown maps g_n, r_n are singled out on the left-hand side. The right-hand side consists of already known terms:

$$\begin{aligned}-g_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) g_n = &\sum_{l>1; i_1+\dots+i_l=n} (g_{i_1} \otimes g_{i_2} \otimes \dots \otimes g_{i_l}) b_l \\ &- \sum_{k>1; q+k+t=n} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) g_{q+1+t}, \qquad (45)\end{aligned}$$

$$\begin{aligned}r_n b_1 &+ \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) r_n + (r_0 \otimes g_n) b_2 \\ &= - \sum_{\substack{k<n; (q,k,t)\neq(0,0,1) \\ i_1+\dots+i_q+k+j_1+\dots+j_t=n}} (f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_k \otimes g_{j_1} \otimes \dots \otimes g_{j_t}) b_{q+1+t} \\ &\qquad - \sum_{\substack{k>1 \\ q+k+t=n}} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) r_{q+1+t}. \qquad (46)\end{aligned}$$

Let us prove that there exist $\Bbbk$-linear maps $g_n : s\mathcal{B}(X_0, X_1) \otimes_\Bbbk \dots \otimes_\Bbbk s\mathcal{B}(X_{n-1}, X_n) \to s\mathcal{B}(X_0 g, X_n g)$, $r_n : s\mathcal{B}(X_0, X_1) \otimes_\Bbbk \dots \otimes_\Bbbk s\mathcal{B}(X_{n-1}, X_n) \to s\mathcal{B}(X_0 f, X_n g)$ which solve the above equations.

Since the map $u = \operatorname{Hom}(N, (r_{X_0} s \otimes 1) b_2)$ from (44) is a quasi-isomorphism, $\operatorname{Cone}(u)$ is acyclic. As a differential graded $\Bbbk$-module

$$\begin{aligned}\operatorname{Cone}(u) &= \operatorname{Hom}^\bullet_\Bbbk(N, s\mathcal{C}(X_0 f, X_n g)) \oplus \operatorname{Hom}^\bullet_\Bbbk(N, s\mathcal{C}(X_0 g, X_n g))[1], \\ (v, p)d &= (vd + pu, -pd),\end{aligned}$$

where $N = s\mathcal{B}(X_0, X_1) \otimes_\Bbbk \dots \otimes_\Bbbk s\mathcal{B}(X_{n-1}, X_n)$. Denote by $\lambda_n \in \operatorname{Hom}^1_\Bbbk(N, s\mathcal{C}(X_0 g, X_n g))$ the right-hand side of (45) and by $\nu_n \in \operatorname{Hom}^0_\Bbbk(N, s\mathcal{C}(X_0 f, X_n g))$ the right-hand side of (46). Equations (45) and (46) mean that $(r_n, g_n)d = (\nu_n, \lambda_n) \in \operatorname{Cone}^0(u)$. Since $\operatorname{Cone}(u)$ is acyclic, such a pair $(r_n, g_n) \in \operatorname{Cone}^{-1}(u)$ exists if and only if $(\nu_n, \lambda_n) \in \operatorname{Cone}^0(u)$ is a cycle, that is, equations $-\lambda_n d = 0$, $\nu_n d + \lambda_n u = 0$ are satisfied. Let us verify them now.

Introduce a cocategory homomorphism $\tilde{g} : Ts\mathcal{B} \to Ts\mathcal{C}$ by its components $(g_1, \dots, g_{n-1}, 0, 0, \dots)$ (these are already known). The map $\lambda = \tilde{g}b - b\tilde{g}$ is a $(\tilde{g}, \tilde{g})$-coderivation. Its components $(\tilde{g}b - b\tilde{g})_k$ vanish for $0 \leqslant k \leqslant n-1$, and

$$\begin{aligned}&(\tilde{g}b - b\tilde{g})_n \\ &= \sum_{l>1; i_1+\dots+i_l=n} (g_{i_1} \otimes g_{i_2} \otimes \dots \otimes g_{i_l}) b_l - \sum_{k>1; q+k+t=n} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) g_{q+1+t} = \lambda_n\end{aligned}$$

is the right-hand side of (45). This coderivation commutes with b, since $(\tilde{g}b - b\tilde{g})b + b(\tilde{g}b - b\tilde{g}) = 0$. Applying this identity to $T^n s\mathcal{B}$ and composing it with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$, we get an identity

$$(\tilde{g}b - b\tilde{g})_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) (\tilde{g}b - b\tilde{g})_n = 0,$$

which means precisely that $\lambda_n d = 0$.

Introduce a $(f,\tilde{g})$-coderivation $\tilde{r}: Ts\mathcal{B} \to Ts\mathcal{C}$ by its components $(r_0, r_1, \dots, r_{n-1}, 0, 0, \dots)$ (these are already known). The commutator $\tilde{r}b + b\tilde{r}$ has the following property:

$$(\tilde{r}b + b\tilde{r})\Delta = \Delta\big[f \otimes (\tilde{r}b + b\tilde{r}) + (\tilde{r}b + b\tilde{r}) \otimes \tilde{g} + \tilde{r} \otimes (\tilde{g}b - b\tilde{g})\big].$$

Let us construct a map $\theta = [\tilde{r} \otimes (\tilde{g}b - b\tilde{g})]\theta : Ts\mathcal{B} \to Ts\mathcal{C}$ for the data $f \xrightarrow{\tilde{r}} \tilde{g} \xrightarrow{\tilde{g}b - b\tilde{g}} \tilde{g} : Ts\mathcal{B} \to Ts\mathcal{C}$ as in Section 3 of [**Lyu03**] (see also Section 1). Its components $\theta_{kl} = \theta\big|_{T^k s\mathcal{B}} \mathrm{pr}_l : T^k s\mathcal{B} \to T^l s\mathcal{C}$ are given by Formula (4)

$$\theta_{kl} = \sum_{\substack{\alpha+\beta+\gamma+2=l \\ a+j+c+t+e=k}} f_{a\alpha} \otimes \tilde{r}_j \otimes \tilde{g}_{c\beta} \otimes (\tilde{g}b - b\tilde{g})_t \otimes \tilde{g}_{e\gamma}.$$

By Proposition 3.1 of [**Lyu03**] the map θ satisfies the equation

$$\theta\Delta = \Delta\big[f \otimes \theta + \theta \otimes \tilde{g} + \tilde{r} \otimes (\tilde{g}b - b\tilde{g})\big].$$

Therefore, $\nu = -\tilde{r}b - b\tilde{r} + [\tilde{r} \otimes (\tilde{g}b - b\tilde{g})]\theta : Ts\mathcal{B} \to Ts\mathcal{C}$ is a $(f,\tilde{g})$-coderivation. Since $\theta_{k1} = 0$ for all k, the components ν_k vanish for $k < n$, and

$$\nu_n = - \sum_{\substack{k<n;(q,k,t)\neq(0,0,1) \\ i_1+\dots+i_q+k+j_1+\dots+j_t=n}} (f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_k \otimes g_{j_1} \otimes \dots \otimes g_{j_t}) b_{q+1+t} \\ - \sum_{k>1;q+k+t=n} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) r_{q+1+t}$$

which is the right-hand side of (46). We have an obvious identity

$$\nu b - b\nu = (-\tilde{r}b - b\tilde{r} + \theta)b - b(-\tilde{r}b - b\tilde{r} + \theta) = \theta b - b\theta.$$

Applying this identity to $T^n s\mathcal{B}$ and composing it with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$, we get an identity

$$\nu_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\nu_n = \big[r_0 \otimes (\tilde{g}b - b\tilde{g})_n\big] b_2,$$

since $[\tilde{r} \otimes (\tilde{g}b - b\tilde{g})]\theta_{nl}$ vanishes for $l \neq 2$, and equals $r_0 \otimes (\tilde{g}b - b\tilde{g})_n$ for $l = 2$. The above equation means precisely that $\nu_n d = -\lambda_n u$. Indeed, $(\tilde{g}b - b\tilde{g})_n (r_{X_0} \otimes 1) = -r_0 \otimes (\tilde{g}b - b\tilde{g})_n$. Thus, proposition is proved by induction. □

8.2. Transformations between the constructed A_∞-functors.

Let $\mathcal{B}$, $\mathcal{C}$ be A_∞-categories, let $f : \mathcal{B} \to \mathcal{C}$ be an A_∞-functor, let $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be a map, and assume that for each object $X \in \operatorname{Ob}\mathcal{B}$ there is a map $r_0 : \Bbbk \to (s\mathcal{C})^{-1}(Xf, Xg)$ such that $r_0 b_1 = 0$. Let the assumptions of Section 8.1 hold. Let $g, g' : \mathcal{B} \to \mathcal{C}$ be two A_∞-functors, whose underlying map is the given $g : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$. Let $r : f \to g : \mathcal{B} \to \mathcal{C}$, $r' : f \to g' : \mathcal{B} \to \mathcal{C}$ be natural A_∞-transformations, whose 0-th component $r_0 = r'_0$ is the given map $r_0 : \Bbbk \to (s\mathcal{C})^{-1}(Xf, Xg)$.

Proposition 8.2. *Under the above assumptions, there exists a natural A_∞-transformation $p : g \to g' : \mathcal{B} \to \mathcal{C}$ such that $r' = (f \xrightarrow{r} g \xrightarrow{p} g')$ in the 2-category A_∞.*

Proof. Let us construct a (g, g')-coderivation $p : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -1 and a (f, g')-coderivation $v : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -2 such that

$$pb + bp = 0,$$
$$(r \otimes p)B_2 - r' = [v, b],$$

that is, $p : g \to g' : \mathcal{B} \to \mathcal{C}$ is a 2-morphism and $v : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}$ is a 3-morphism. Let us build the components of p and v by induction. We have $p_k = 0$ and $v_k = 0$ for $k < 0$. Given non-negative n, assume that we have already found components p_m, v_m of the sought for p, v for $m < n$, such that equations

$$(pb + bp)\,\mathrm{pr}_1 = 0 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 g, X_m g),$$

$$\{(r \otimes p)B_2 - r' - [v, b]\}\,\mathrm{pr}_1 = 0 : \\ s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 f, X_m g)$$

are satisfied for all $m < n$. Under these assumptions we will find such p_n, v_n that the above equations are satisfied for $m = n$. Notice that for $m = n = 0$ the source complexes reduce to $\Bbbk$. Let us write down these equations explicitly. The terms which contain unknown maps p_n, v_n are singled out on the left-hand side. The right-hand side consists of already known terms:

$$- p_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) p_n = \sum_{q+k+t=n}^{k>1} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) p_{q+1+t} \\ + \sum_{i_1+\cdots+i_q+k+j_1+\cdots+j_t=n}^{k<n} (g_{i_1} \otimes \cdots \otimes g_{i_q} \otimes p_k \otimes g'_{j_1} \otimes \cdots \otimes g'_{j_t}) b_{q+1+t}, \quad (47)$$

$$v_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) v_n - (r_0 \otimes p_n) b_2 \\ = \sum_{a_1+\cdots+a_\alpha+j+c_1+\cdots+c_\beta+k+e_1+\cdots+e_\gamma=n}^{k<n} (f_{a_1} \otimes \cdots \otimes f_{a_\alpha} \otimes r_j \otimes g_{c_1} \otimes \dots \\ \otimes g_{c_\beta} \otimes p_k \otimes g'_{e_1} \otimes \cdots \otimes g'_{e_\gamma}) b_{\alpha+\beta+\gamma+2} \\ - r'_n - \sum_{i_1+\cdots+i_q+k+j_1+\cdots+j_t=n}^{k<n} (f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes v_k \otimes g'_{j_1} \otimes \cdots \otimes g'_{j_t}) b_{q+1+t} \\ + \sum_{q+k+t=n}^{k>1} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) v_{q+1+t}. \quad (48)$$

The components of $(r \otimes p)B_2$ are computed by Formula (5.1.3) of [**Lyu03**]:

$$[(r \otimes p)B_2]_n = \sum_l (r \otimes p)\theta_{nl} b_l.$$

Denote by $\lambda_n \in \mathrm{Hom}^0_\Bbbk(N, s\mathcal{C}(X_0 g, X_n g))$ the right-hand side of (47) and by $\nu_n \in \mathrm{Hom}^{-1}_\Bbbk(N, s\mathcal{C}(X_0 f, X_n g))$ the right-hand side of (48), where $N = s\mathcal{B}(X_0, X_1) \otimes_\Bbbk$

$\cdots \otimes_{\Bbbk} s\mathcal{B}(X_{n-1}, X_n)$. In particular, $N = \Bbbk$ for $n = 0$. Equations (47) and (48) mean that $(v_n, p_n)d = (\nu_n, \lambda_n) \in \operatorname{Cone}^{-1}(u)$. Since $\operatorname{Cone}(u)$ is acyclic, such a pair $(v_n, p_n) \in \operatorname{Cone}^{-2}(u)$ exists if and only if $(\nu_n, \lambda_n) \in \operatorname{Cone}^{-1}(u)$ is a cycle, that is, equations $-\lambda_n d = 0$, $\nu_n d + \lambda_n u = 0$ are satisfied. Let us verify them now.

Introduce a (g, g')-coderivation $\tilde{p} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -1 by its components $(p_0, p_1, \dots, p_{n-1}, 0, 0, \dots)$. The commutator $\lambda = \tilde{p}b + b\tilde{p}$ is also a (g, g')-coderivation (of degree 0). Its components λ_m vanish for $m < n$. The component $\lambda_n = (\tilde{p}b + b\tilde{p})_n$ is the right-hand side of (47). Consider the identity

$$(\tilde{p}b + b\tilde{p})b - b(\tilde{p}b + b\tilde{p}) = 0.$$

Applying this identity to $T^n s\mathcal{B}$ and composing it with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$, we get an identity

$$\lambda_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\lambda_n = 0,$$

that is, $\lambda_n d = 0$.

Introduce a (f, g')-coderivation $\tilde{v} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -2 by its components $(v_0, v_1, \dots, v_{n-1}, 0, 0, \dots)$. All summands of the map $\nu = (r \otimes \tilde{p})B_2 - r' - [\tilde{v}, b]$ are (f, g')-coderivations of degree -1. Hence, the same holds for ν. The components ν_m vanish for $m < n$. The component ν_n is the right-hand side of (48). Consider the commutator

$$\begin{aligned}[\nu, b] = \nu B_1 = (r \otimes \tilde{p})B_2 B_1 - r' B_1 - \tilde{v} B_1 B_1 &= -(r \otimes \tilde{p})(1 \otimes B_1 + B_1 \otimes 1)B_2 \\ &= -(r \otimes \tilde{p}B_1)B_2 = -(r \otimes \lambda)B_2.\end{aligned}$$

Applying this identity to $T^n s\mathcal{B}$ and composing it with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$ we get an identity

$$\nu_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\nu_n = -(r_0 \otimes \lambda_n)b_2,$$

that is, $\nu_n d = -\lambda_n u$. Thus, the proposition is proved by induction. $\square$

Proposition 8.3 (Uniqueness of the transformations)**.** *Let assumptions of Sections 8.1 and 8.2 hold. The natural A_∞-transformation $p : g \to g' : \mathcal{B} \to \mathcal{C}$, such that $r' = (f \xrightarrow{r} g \xrightarrow{p} g')$ in the 2-category A_∞, is unique up to an equivalence.*

Proof. Assume that we have two such 2-morphisms $p, q : g \to g' : \mathcal{B} \to \mathcal{C}$ and two 3-morphisms $v : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}$ and $w : (r \otimes q)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}$. We are looking for a 3-morphism $x : p \to q : g \to g' : \mathcal{B} \to \mathcal{C}$ and the following 4-morphism, whose source depends on x. Assuming that $p - q = xB_1$ we deduce that

$$-(r \otimes x)B_2 B_1 = (r \otimes xB_1)B_2 = (r \otimes p)B_2 - (r \otimes q)B_2.$$

Since $(r \otimes q)B_2 - r' = wB_1$, we find out that $(r \otimes p)B_2 - r' = [w - (r \otimes x)B_2]B_1$. Thus, we have two 3-morphisms with the common source and target $v, w - (r \otimes x)B_2 : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C}$. We are looking for a 4-morphism

$$z : w - (r \otimes x)B_2 \to v : (r \otimes p)B_2 \to r' : f \to g' : \mathcal{B} \to \mathcal{C},$$

as well as for x. In other terms, we have to find coderivations x of degree -2 and z of degree -3 such that the following equations hold:

$$\begin{aligned} p - q &= xb - bx, \\ w - (r \otimes x)B_2 - v &= zb + bz. \end{aligned}$$

Let us build the components of x and z by induction. We have $x_k = 0$ and $z_k = 0$ for $k < 0$. Given non-negative n, assume that we have already found components x_m, z_m of the sought x, z for $m < n$, such that equations

$$(p-q)\,\mathrm{pr}_1 = (xb - bx)\,\mathrm{pr}_1 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 g, X_m g),$$

$$\begin{aligned} &[w - (r \otimes x)B_2 - v]\,\mathrm{pr}_1 = (zb + bz)\,\mathrm{pr}_1 : \\ &\qquad\qquad s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 f, X_m g) \end{aligned}$$

are satisfied for all $m < n$. Under these assumptions, we will find such x_n, z_n that the above equations are satisfied for $m = n$. Notice that for $m = n = 0$ the source complexes reduce to $\Bbbk$. Let us write down these equations explicitly. The terms which contain unknown maps x_n, z_n are singled out on the left-hand side. The right-hand side consists of already known terms:

$$\begin{aligned} &- x_n b_1 + \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta}) x_n = q_n - p_n \\ &\quad + \sum_{i_1+\cdots+i_\alpha+k+j_1+\cdots+j_\beta=n}^{k<n} (g_{i_1} \otimes \cdots \otimes g_{i_\alpha} \otimes x_k \otimes g'_{j_1} \otimes \cdots \otimes g'_{j_\beta}) b_{\alpha+1+\beta} \\ &\qquad - \sum_{\alpha+k+\beta=n}^{k>1} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) x_{\alpha+1+\beta}, \end{aligned} \tag{49}$$

$$\begin{aligned} &z_n b_1 + \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta}) z_n + (r_0 \otimes x_n) b_2 \\ &= - \sum_{a_1+\cdots+a_\alpha+j+c_1+\cdots+c_\beta+k+e_1+\cdots+e_\gamma=n}^{k<n} (f_{a_1} \otimes \cdots \otimes f_{a_\alpha} \otimes r_j \otimes g_{c_1} \otimes \dots \\ &\qquad\qquad \otimes g_{c_\beta} \otimes x_k \otimes g'_{e_1} \otimes \cdots \otimes g'_{e_\gamma}) b_{\alpha+\beta+\gamma+2} \\ &+ w_n - v_n - \sum_{i_1+\cdots+i_\alpha+k+j_1+\cdots+j_\beta=n}^{k<n} (f_{i_1} \otimes \cdots \otimes f_{i_\alpha} \otimes z_k \otimes g'_{j_1} \otimes \cdots \otimes g'_{j_\beta}) b_{\alpha+1+\beta} \\ &\qquad - \sum_{\alpha+k+\beta=n}^{k>1} (1^{\otimes\alpha} \otimes b_k \otimes 1^{\otimes\beta}) z_{\alpha+1+\beta}. \end{aligned} \tag{50}$$

Denote by $\lambda_n \in \mathrm{Hom}_\Bbbk^{-1}(N, s\mathcal{C}(X_0 g, X_n g))$ the right-hand side of (49) and by $\nu_n \in \mathrm{Hom}_\Bbbk^{-2}(N, s\mathcal{C}(X_0 f, X_n g))$ the right-hand side of (50), where $N = s\mathcal{B}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{B}(X_{n-1}, X_n)$. In particular, $N = \Bbbk$ for $n = 0$. Equations (49) and (50) mean that $(z_n, x_n)d = (\nu_n, \lambda_n) \in \mathrm{Cone}^{-2}(u)$. Since $\mathrm{Cone}(u)$ is acyclic, such a pair

$(z_n, x_n) \in \mathrm{Cone}^{-3}(u)$ exists if and only if $(\nu_n, \lambda_n) \in \mathrm{Cone}^{-2}(u)$ is a cycle, that is, equations $-\lambda_n d = 0$, $\nu_n d + \lambda_n u = 0$ are satisfied. Let us verify them now.

Introduce a (g, g')-coderivation $\tilde{x} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -2 by its components $(x_0, x_1, \dots, x_{n-1}, 0, 0, \dots)$. The commutator $\tilde{x}b - b\tilde{x}$ is also a (g, g')-coderivation (of degree -1). Hence, the map $\lambda = -p + q + \tilde{x}b - b\tilde{x}$ is also a (g, g')-coderivation of degree -1. Its components λ_m vanish for $m < n$. The component λ_n is the right-hand side of (49). Consider the identity

$$\lambda B_1 = -pB_1 + qB_1 + \tilde{x}B_1B_1 = 0.$$

Applying this identity to $T^n s\mathcal{B}$ and composing it with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$ we get an identity

$$\lambda_n b_1 + \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\lambda_n = 0,$$

that is, $\lambda_n d = 0$.

Introduce a (f, g')-coderivation $\tilde{z} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -3 by its components $(z_0, z_1, \dots, z_{n-1}, 0, 0, \dots)$. All summands of the map $\nu = w - (r \otimes \tilde{x})B_2 - v - [\tilde{z}, b]$ are (f, g')-coderivations of degree -2. Hence, the same holds for ν. The components ν_m vanish for $m < n$. The component ν_n is the right-hand side of (50). Consider the commutator

$$\begin{aligned}
[\nu, b] = \nu B_1 &= wB_1 - (r \otimes \tilde{x})B_2B_1 - vB_1 - \tilde{z}B_1B_1 \\
&= (r \otimes q)B_2 - r' + (r \otimes \tilde{x}B_1)B_2 - (r \otimes p)B_2 + r' \\
&= [r \otimes (q - p + \tilde{x}B_1)]B_2 = (r \otimes \lambda)B_2.
\end{aligned}$$

Applying this identity to $T^n s\mathcal{B}$ and composing it with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$ we get an identity

$$\nu_n b_1 - \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\nu_n = (r_0 \otimes \lambda_n)b_2,$$

that is, $\nu_n d = -\lambda_n u$. Thus, the proposition is proved by induction. □

Corollary 8.4. *In the assumptions of Proposition 8.3, let $\mathcal{C}$ be unital. Then the constructed 2-morphism $p : g \to g' : \mathcal{B} \to \mathcal{C}$ is invertible in A_∞.*

Proof. Exchanging the pairs (g, r) and (g', r'), we see that there is a 2-morphism $t : g' \to g : \mathcal{B} \to \mathcal{C}$, such that $r = (f \xrightarrow{r'} g' \xrightarrow{t} g)$. Therefore, $r = (f \xrightarrow{r} g \xrightarrow{p \cdot t} g)$. Since $\mathcal{C}$ is unital, there is a unit 2-endomorphism $1_g s = g\mathbf{i}^{\mathcal{C}} : g \to g : \mathcal{B} \to \mathcal{C}$. It satisfies the equation $r = (f \xrightarrow{r} g \xrightarrow{1_g s} g)$. The uniqueness proved in Proposition 8.3 implies that $p \cdot t = 1_g s$. Similarly, $t \cdot p = 1_{g'} s$. □

Proposition 8.5 (Unitality of A_∞-functors)**.** *Let the assumptions of Section 8.1 hold. If A_∞-categories $\mathcal{B}$, $\mathcal{C}$ are unital and A_∞-functor $f : \mathcal{B} \to \mathcal{C}$ is unital, then the A_∞-functor $g : \mathcal{B} \to \mathcal{C}$ constructed in Proposition 8.1 is unital as well.*

Proof. We are given a 2-morphism $r : f \to g : \mathcal{B} \to \mathcal{C}$ and a 3-morphism $v : f\mathbf{i}^{\mathcal{C}} \to \mathbf{i}^{\mathcal{B}} f : f \to f : \mathcal{B} \to \mathcal{C}$. We are looking for a 3-morphism $w : g\mathbf{i}^{\mathcal{C}} \to \mathbf{i}^{\mathcal{B}} g : g \to g : \mathcal{B} \to \mathcal{C}$ and a 4-morphism x, whose target depends on w. Let us describe x now for the above w.

We have the following 3-morphisms

$$\begin{aligned}
(r\otimes \mathbf{i}^{\mathcal{C}})M_{11} :&(f\mathbf{i}^{\mathcal{C}}\otimes r)B_2 \to (r\otimes g\mathbf{i}^{\mathcal{C}})B_2 : f\to g:\mathcal{B}\to\mathcal{C},\\
(v\otimes r)B_2 :&(f\mathbf{i}^{\mathcal{C}}\otimes r)B_2 \to (\mathbf{i}^{\mathcal{B}}f\otimes r)B_2 : f\to g:\mathcal{B}\to\mathcal{C},\\
(r\otimes w)B_2 :&(r\otimes \mathbf{i}^{\mathcal{B}}g)B_2 \to (r\otimes g\mathbf{i}^{\mathcal{C}})B_2 : f\to g:\mathcal{B}\to\mathcal{C},\\
(\mathbf{i}^{\mathcal{B}}\otimes r)M_{11} :&(r\otimes \mathbf{i}^{\mathcal{B}}g)B_2 \to (\mathbf{i}^{\mathcal{B}}f\otimes r)B_2 : f\to g:\mathcal{B}\to\mathcal{C}.
\end{aligned}$$

Indeed, for the cocategory homomorphism $M : TsA_\infty \boxtimes TsA_\infty \to TsA_\infty$ we have an equation $MB = (1\boxtimes B + B\boxtimes 1)M$, see Section 6 of [**Lyu03**]. It implies, in particular, that

$$\begin{gathered}
(r\otimes \mathbf{i}^{\mathcal{C}})M_{11}B_1 - (f\mathbf{i}^{\mathcal{C}}\otimes r)B_2 + (r\otimes g\mathbf{i}^{\mathcal{C}})B_2 = (r\otimes \mathbf{i}^{\mathcal{C}})(1\otimes B_1 + B_1\otimes 1)M_{11} = 0,\\
(v\otimes r)B_2B_1 = (vB_1\otimes r)B_2 = (f\mathbf{i}^{\mathcal{C}}\otimes r)B_2 - (\mathbf{i}^{\mathcal{B}}f\otimes r)B_2,\\
(r\otimes w)B_2B_1 = -(r\otimes wB_1)B_2 = (r\otimes \mathbf{i}^{\mathcal{B}}g)B_2 - (r\otimes g\mathbf{i}^{\mathcal{C}})B_2,\\
(\mathbf{i}^{\mathcal{B}}\otimes r)M_{11}B_1 - (r\otimes \mathbf{i}^{\mathcal{B}}g)B_2 + (\mathbf{i}^{\mathcal{B}}f\otimes r)B_2 = (\mathbf{i}^{\mathcal{B}}\otimes r)(1\otimes B_1 + B_1\otimes 1)M_{11} = 0.
\end{gathered}$$

Linear combinations of the above maps form 3-morphisms with the same source and target

$$\begin{aligned}
(r\otimes \mathbf{i}^{\mathcal{C}})M_{11} - (v\otimes r)B_2 &: (\mathbf{i}^{\mathcal{B}}f\otimes r)B_2 \to (r\otimes g\mathbf{i}^{\mathcal{C}})B_2 : f\to g:\mathcal{B}\to\mathcal{C},\\
(r\otimes w)B_2 - (\mathbf{i}^{\mathcal{B}}\otimes r)M_{11} &: (\mathbf{i}^{\mathcal{B}}f\otimes r)B_2 \to (r\otimes g\mathbf{i}^{\mathcal{C}})B_2 : f\to g:\mathcal{B}\to\mathcal{C}.
\end{aligned}$$

We are looking for a 4-morphism between the above 3-morphisms

$$\begin{aligned}
x : (r\otimes \mathbf{i}^{\mathcal{C}})M_{11} - (v\otimes r)B_2 \to (r\otimes w)B_2 - (\mathbf{i}^{\mathcal{B}}\otimes r)M_{11}&\\
: (\mathbf{i}^{\mathcal{B}}f\otimes r)B_2 \to (r\otimes g\mathbf{i}^{\mathcal{C}})B_2 : f\to g:\mathcal{B}\to\mathcal{C}&,
\end{aligned}$$

as well as for w.

In other words, we have to find a (g, g)-coderivation w of degree -2 and an (f, g)-coderivation x of degree -3 such that the following equations hold:

$$\begin{aligned}
-wb + bw &= \mathbf{i}^{\mathcal{B}}g - g\mathbf{i}^{\mathcal{C}},\\
xb + bx &= (r\otimes \mathbf{i}^{\mathcal{C}})M_{11} - (v\otimes r)B_2 - (r\otimes w)B_2 + (\mathbf{i}^{\mathcal{B}}\otimes r)M_{11}.
\end{aligned}$$

Let us construct the components of w and x by induction. We have $w_k = 0$ and $x_k = 0$ for $k < 0$. Given non-negative n, assume that we have already found components w_m, x_m of the sought for x, z for $m < n$, such that the above equations restricted to $T^m s\mathcal{B}$ are satisfied for all $m < n$. Under these assumptions, we will find such w_n, x_n for $m = n$. Let us write down these equations explicitly. The terms which contain unknown maps w_n, x_n are singled out on the left-hand side. The right-hand side consists of already known terms:

$$
\begin{aligned}
&- w_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) w_n \\
&\qquad = \sum_{i_1+\dots+i_q+k+j_1+\dots+j_t=n}^{k<n} (g_{i_1} \otimes \dots \otimes g_{i_q} \otimes w_k \otimes g_{j_1} \otimes \dots \otimes g_{j_t}) b_{q+1+t} \\
&\quad - \sum_{q+k+t=n}^{k>1} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) w_{q+1+t} + \sum_{q+k+t=n} (1^{\otimes q} \otimes \mathbf{i}^{\mathcal{B}}_k \otimes 1^{\otimes t}) g_{q+1} - \mathbf{i}^{\mathcal{C}}_n, \qquad (51)
\end{aligned}
$$

$$
\begin{aligned}
&x_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t}) x_n + (r_0 \otimes w_n) b_2 \\
&\quad = - \sum_{i_1+\dots+i_q+k+j_1+\dots+j_t=n}^{k<n} (f_{i_1} \otimes \dots \otimes f_{i_q} \otimes x_k \otimes g_{j_1} \otimes \dots \otimes g_{j_t}) b_{q+1+t} \\
&\qquad\qquad - \sum_{q+k+t=n}^{k>1} (1^{\otimes q} \otimes b_k \otimes 1^{\otimes t}) x_{q+1+t} \\
&- \sum_{a_1+\dots+a_\alpha+j+c_1+\dots+c_\beta+k+e_1+\dots+e_\gamma=n}^{k<n} (f_{a_1} \otimes \dots \otimes f_{a_\alpha} \otimes v_j \otimes f_{c_1} \otimes \dots \\
&\qquad\qquad \otimes f_{c_\beta} \otimes r_k \otimes g_{e_1} \otimes \dots \otimes g_{e_\gamma}) b_{\alpha+\beta+\gamma+2} \\
&- \sum_{a_1+\dots+a_\alpha+j+c_1+\dots+c_\beta+k+e_1+\dots+e_\gamma=n}^{k<n} (f_{a_1} \otimes \dots \otimes f_{a_\alpha} \otimes r_j \otimes g_{c_1} \otimes \dots \\
&\qquad\qquad \otimes g_{c_\beta} \otimes w_k \otimes g_{e_1} \otimes \dots \otimes g_{e_\gamma}) b_{\alpha+\beta+\gamma+2} \\
&\quad + \sum_{i_1+\dots+i_q+k+j_1+\dots+j_t=n} (f_{i_1} \otimes \dots \otimes f_{i_q} \otimes r_k \otimes g_{j_1} \otimes \dots \otimes g_{j_t}) \mathbf{i}^{\mathcal{C}}_{q+1+t} \\
&\qquad\qquad + \sum_{q+k+t=n} (1^{\otimes q} \otimes \mathbf{i}^{\mathcal{B}}_k \otimes 1^{\otimes t}) r_{q+1+t}. \qquad (52)
\end{aligned}
$$

Denote by $\lambda_n \in \operatorname{Hom}^{-1}_{\Bbbk}(N, s\mathcal{C}(X_0 g, X_n g))$ the right-hand side of (51) and by $\nu_n \in \operatorname{Hom}^{-2}_{\Bbbk}(N, s\mathcal{C}(X_0 f, X_n g))$ the right-hand side of (52), where $N = s\mathcal{B}(X_0, X_1) \otimes_{\Bbbk} \dots \otimes_{\Bbbk} s\mathcal{B}(X_{n-1}, X_n)$. Equations (51) and (52) mean that $(x_n, w_n)d = (\nu_n, \lambda_n) \in \operatorname{Cone}^{-2}(u)$. Since $\operatorname{Cone}(u)$ is acyclic, such a pair $(x_n, w_n) \in \operatorname{Cone}^{-3}(u)$ exists if and only if $(\nu_n, \lambda_n) \in \operatorname{Cone}^{-2}(u)$ is a cycle, that is, equations $-\lambda_n d = 0$, $\nu_n d + \lambda_n u = 0$ are satisfied. Let us verify them now.

Introduce a (g, g)-coderivation $\tilde{w} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -2 by its components $(w_0, w_1, \dots, w_{n-1}, 0, 0, \dots)$. Hence, the map $\lambda = \tilde{w}b - b\tilde{w} + \mathbf{i}^{\mathcal{B}} g - g\mathbf{i}^{\mathcal{C}}$ is also a (g, g)-coderivation of degree -1. Its components λ_m vanish for $m < n$. The component λ_n is the right-hand side of (51). Consider the identity

$$\lambda B_1 = \tilde{w} B_1 B_1 + (\mathbf{i}^{\mathcal{B}} g) B_1 - (g \mathbf{i}^{\mathcal{C}}) B_1 = 0.$$

Applying this identity to $T^n s\mathcal{B}$ and composing it with $\operatorname{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$, we get an identity

$$\lambda_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\lambda_n = 0,$$

that is, $\lambda_n d = 0$.

Introduce a (f,g)-coderivation $\tilde{x} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree -3 by its components $(x_0, x_1, \dots, x_{n-1}, 0, 0, \dots)$. All summands of the map

$$\nu = -\tilde{x}B_1 + (r \otimes \mathbf{i}^{\mathcal{C}})M_{11} - (v \otimes r)B_2 - (r \otimes \tilde{w})B_2 + (\mathbf{i}^{\mathcal{B}} \otimes r)M_{11}$$

are (f,g)-coderivations of degree -2. Hence, the same holds for ν. The components ν_m vanish for $m < n$. The component ν_n is the right-hand side of (52). Consider its differential

$$\begin{aligned}
\nu B_1 &= -\tilde{x}B_1B_1 + (r \otimes \mathbf{i}^{\mathcal{C}})M_{11}B_1 - (v \otimes r)B_2B_1 - (r \otimes \tilde{w})B_2B_1 + (\mathbf{i}^{\mathcal{B}} \otimes r)M_{11}B_1 \\
&= (f\mathbf{i}^{\mathcal{C}} \otimes r)B_2 - (r \otimes g\mathbf{i}^{\mathcal{C}})B_2 - (vB_1 \otimes r)B_2 + (r \otimes \tilde{w}B_1)B_2 + (r \otimes \mathbf{i}^{\mathcal{B}}g)B_2 \\
&\qquad - (\mathbf{i}^{\mathcal{B}}f \otimes r)B_2 \\
&= [r \otimes (\tilde{w}B_1 - g\mathbf{i}^{\mathcal{C}} + \mathbf{i}^{\mathcal{B}}g)]B_2 = (r \otimes \lambda)B_2.
\end{aligned}$$

Applying identity $\nu B_1 = (r \otimes \lambda)B_2$ to $T^n s\mathcal{B}$ and composing it with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$ we get an identity

$$\nu_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\nu_n = (r_0 \otimes \lambda_n)b_2,$$

that is, $\nu_n d = -\lambda_n u$. Thus, the proposition is proved by induction. □

8.3. Invertible transformations.

Let $\mathcal{B}$, $\mathcal{C}$ be unital A_∞-categories, and let $f, g : \mathrm{Ob}\,\mathcal{B} \to \mathrm{Ob}\,\mathcal{C}$ be maps. Assume that for each object X of $\mathcal{B}$ there are $\Bbbk$-linear maps

$$\begin{aligned}
{}_Xr_0 &: \Bbbk \to (s\mathcal{C})^{-1}(Xf, Xg), \qquad & {}_Xp_0 &: \Bbbk \to (s\mathcal{C})^{-1}(Xg, Xf), \\
{}_Xw_0 &: \Bbbk \to (s\mathcal{C})^{-2}(Xf, Xf), \qquad & {}_Xv_0 &: \Bbbk \to (s\mathcal{C})^{-2}(Xg, Xg),
\end{aligned}$$

such that

$$\begin{gathered}
{}_Xr_0b_1 = 0, \qquad {}_Xp_0b_1 = 0, \\
({}_Xr_0 \otimes {}_Xp_0)b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{C}} = {}_Xw_0b_1, \\
({}_Xp_0 \otimes {}_Xr_0)b_2 - {}_{Xg}\mathbf{i}_0^{\mathcal{C}} = {}_Xv_0b_1.
\end{gathered} \tag{53}$$

Proposition 8.6. *Let the assumptions of Section 8.3 hold and, moreover, let $f : \mathcal{B} \to \mathcal{C}$ be a unital A_∞-functor. Then the map g extends to a unital A_∞-functor $g : \mathcal{B} \to \mathcal{C}$ and the given r_0, p_0 extend to natural A_∞-transformations $r : f \to g : \mathcal{B} \to \mathcal{C}$, $p : g \to f : \mathcal{B} \to \mathcal{C}$, inverse to each other.*

Proof. Propositions 8.1 and 8.5 imply the existence and unitality of g. Indeed, since $(r_0 \otimes 1)b_2$ is a homotopy invertible chain map, the map $u = \mathrm{Hom}(N, (r_0 \otimes 1)b_2)$ is also homotopy invertible, hence a quasi-isomorphism. Existence of $r : f \to g : \mathcal{B} \to \mathcal{C}$ is shown in Proposition 8.1. Existence of $p : g \to f : \mathcal{B} \to \mathcal{C}$, inverse to r is proven in [**Lyu03**, Proposition 7.15]. □

9. Derived categories

Let $\mathcal{A}$ be a $\mathscr{U}$-small Abelian $\Bbbk$-linear category, and let $\mathcal{C} = \mathsf{C}(\mathcal{A})$ or $\mathcal{C} = \mathsf{C}^+(\mathcal{A})$ be the differential graded category of complexes (resp. bounded below complexes) of objects of $\mathcal{A}$. Denote by $\mathcal{B} = \mathsf{A}(\mathcal{A})$ its full subcategory of acyclic complexes. Let $\mathcal{D} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ be the constructed $\mathscr{U}$-small differential graded category. We observe first that quasi-isomorphisms in $\mathcal{C}$ become (homotopy) invertible elements in $\mathcal{D}$.

Assuming that the ground ring $\Bbbk$ is a field, we turn to the procedure of finding a K-injective resolution (if they exist) into a unital A_∞-functor. Under these assumptions, we also show that $\mathcal{D} = \mathsf{D}(\mathcal{C}|\mathcal{B})$ is A_∞-equivalent to $\mathcal{I} \subset \mathcal{C}$ — the full subcategory of K-injective complexes. Hence, $H^0(\mathcal{D})$ is equivalent to the derived category $\mathsf{D}(\mathcal{A})$.

9.1. Invertibility of quasi-isomorphisms.

Assume that X, Y are objects of $\mathcal{C}$ and $q : X \to Y$ is a quasi-isomorphism. In particular, $q \in \mathcal{C}^0(X,Y)$, $qm_1 = 0$. Let us prove that $r = qs\bar{\jmath}_1 \in (s\mathcal{D})^{-1}(X,Y)$ is invertible in the sense of Section 8.3, that is, there are elements $p \in (s\mathcal{D})^{-1}(Y,X)$, $w \in (s\mathcal{D})^{-2}(X,X)$, $v \in (s\mathcal{D})^{-2}(Y,Y)$, such that

$$\begin{aligned} r\bar{b}_1 = 0, \qquad p\bar{b}_1 &= 0, \\ (r \otimes p)\bar{b}_2 - 1_X s &= w\bar{b}_1, \\ (p \otimes r)\bar{b}_2 - 1_Y s &= v\bar{b}_1. \end{aligned} \tag{54}$$

Indeed, denote $C = \operatorname{Cone}(q) = (Y \oplus X[1], d^C)$, where $(y,x)d^C = (yd^Y + xq, -xd^X)$ for $y \in Y^l$, $x \in X^{l+1}$. Since q is a quasi-isomorphism, C is acyclic. There is a standard exact sequence of complexes $0 \to Y \xrightarrow{n} C \xrightarrow{k} X[1] \to 0$ with the chain maps n, k, $yn = (y,0)$, $(0,x)k = x$. From now on we denote by n, k also the corresponding elements $n \in \mathcal{C}^0(Y,C)$, $k \in \mathcal{C}^1(C,X)$. Define p as $p = ns \otimes ks \in (s\mathcal{C})^{-1}(Y,C) \otimes (s\mathcal{C})^0(C,X) \subset (s\mathcal{D})^{-1}(Y,X)$. Then

$$p\bar{b}_1 = pb = (ns \otimes ks)(1 \otimes b_1 + b_1 \otimes 1 + b_2) = -(n \otimes k)m_2 s = -(nk)s = 0.$$

Denote by $h \in \mathcal{C}^{-1}(X,C)$ the following $\Bbbk$-linear embedding $X \to C$, $X^l \to C^{l-1} = Y^{l-1} \oplus X^l$, $x \mapsto (0,x)$. Define w as $w = hs \otimes ks \in (s\mathcal{C})^{-2}(X,C) \otimes (s\mathcal{C})^0(C,X) \subset (s\mathcal{D})^{-2}(X,X)$. Then

$$\begin{aligned} w\bar{b}_1 = wb &= (hs \otimes ks)(1 \otimes b_1 + b_1 \otimes 1 + b_2) = hm_1 s \otimes ks - (hk)s \\ &= (qn)s \otimes ks - 1_X s = (qs \otimes ns)b_2 \otimes ks - 1_X s \\ &= [qs \overset{2}{\otimes} (ns \overset{1}{\otimes} ks)]\bar{b}_2 - 1_X s = (r \otimes p)\bar{b}_2 - 1_X s. \end{aligned}$$

Indeed, $qn = hm_1 = hd + dh : X \to C$ as explicit computation shows:

$$xqn = (xq, 0) = (xq, -xd^X) + (0, xd^X) = (0,x)d^C + (0, xd) = x(hd + dh).$$

Denote by $z \in \mathcal{C}^0(C,Y)$ the following $\Bbbk$-linear projection $z : C \to Y$, $(y,x) \mapsto y$. Define v as $v = -ns \otimes zs \in (s\mathcal{C})^{-1}(Y,C) \otimes (s\mathcal{C})^{-1}(C,Y) \subset (s\mathcal{D})^{-2}(Y,Y)$. Then

$$\begin{aligned} v\bar{b}_1 = vb &= -(ns \otimes zs)(1 \otimes b_1 + b_1 \otimes 1 + b_2) = -ns \otimes zm_1 s - (nz)s \\ &= ns \otimes (kq)s - 1_Y s = ns \otimes (ks \otimes qs)b_2 - 1_Y s \\ &= [(ns \overset{1}{\otimes} ks) \overset{2}{\otimes} qs]\bar{b}_2 - 1_Y s = (p \otimes r)\bar{b}_2 - 1_Y s. \end{aligned}$$

Indeed, $-kq = zm_1 = zd - dz : C \to Y$ as explicit computation shows:

$$-(y,x)kq = -xq = yd - yd - xq = (y,x)zd - (y,x)d^C z = (y,x)(zd - dz).$$

Thus, Equations (54) hold true.

9.2. K-injective complexes.

A complex $A \in \operatorname{Ob} \mathcal{C}$ is K-injective if and only if for every quasi-isomorphism $t : X \to Y \in \mathcal{C}$, the chain map $\mathcal{C}(t, A) : \mathcal{C}(Y, A) \to \mathcal{C}(X, A)$ is a quasi-isomorphism [**Spa88**, Proposition 1.5]. Assume that each complex $X \in \mathcal{C}$ has a right K-injective resolution $r_X : X \to Xi$, that is, r_X is a quasi-isomorphism and $Xi \in \operatorname{Ob} \mathcal{C}$ is K-injective. Moreover, if X is K-injective, we assume that $Xi = X$ and $r_X = 1_X$. By definition, $\mathcal{C}(r_X, A) : s\mathcal{C}(Xi, A) \to s\mathcal{C}(X, A)$, $fs \mapsto (r_X f)s$ is a quasi-isomorphism. The assumption is satisfied, when $\mathcal{A}$ has enough injectives and $\mathcal{C} = \mathsf{C}^+(\mathcal{A})$, or when $\mathcal{A} = R$-mod,[1] or when $\mathcal{O}$ is a sheaf of rings on a topological space, and $\mathcal{A}$ is the category of sheaves of left $\mathcal{O}$-modules, see [**Spa88**].

Assume now that $\Bbbk$ is a field. Then for any chain complex of $\Bbbk$-modules of the form $N = s\mathcal{C}(X_0, X_1) \otimes_\Bbbk s\mathcal{C}(X_1, X_2) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{C}(X_{n-1}, X_n)$, $n \geqslant 0$, $X_i \in \operatorname{Ob} \mathcal{C}$, for any quasi-isomorphism $r_X : X \to Y$ and for any K-injective $A \in \mathcal{C}$, the following chain map

$$u = \operatorname{Hom}(N, \mathcal{C}(r_X, A)) : \operatorname{Hom}^\bullet_\Bbbk(N, s\mathcal{C}(Y, A)) \to \operatorname{Hom}^\bullet_\Bbbk(N, s\mathcal{C}(X, A)),$$

is a quasi-isomorphism (any $\Bbbk$-module complex is K-projective). Therefore, we may apply the results of Section 8 to the differential graded category $\mathcal{C} = \mathsf{C}$ or C^+, and its full subcategories $\mathcal{B} = \mathsf{A}(\mathcal{A})$ (resp. $\mathcal{I} = \mathsf{I}(\mathcal{A})$, $\mathcal{J} = \mathsf{AI}(\mathcal{A})$) of acyclic (resp. K-injective, acyclic K-injective) complexes. Denote by $e : \mathcal{I} \hookrightarrow \mathcal{C}$ the full embedding. Starting with the identity functor $f = \mathrm{id}_\mathcal{C}$, we get the existence of $g = ie$ simultaneously with the existence of a unital A_∞-functor $i : \mathcal{C} \to \mathcal{I}$ — the "K-injective resolution functor" — and a natural A_∞-transformation $r : \mathrm{id} \to ie : \mathcal{C} \to \mathcal{C}$ (Propositions 8.1 and 8.5). i and r are unique in the sense of Propositions 8.2, 8.3 and Corollary 8.4. Moreover, while solving Equations (45)–(46) we will choose the solutions

$$\begin{aligned} i_n = g_n = \mathrm{id}_n &: s\mathcal{C}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{C}(X_0, X_n), \\ r_n = \mathbf{i}^\mathcal{C}_n &: s\mathcal{C}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{C}(X_0, X_n), \end{aligned}$$

if $X_0, \dots, X_n$ are K-injective (recall that $X_0 i = X_0$, $X_n i = X_n$).

Extending e, i to A_∞-functors between the constructed categories, we get a unital strict A_∞-embedding (actually, a faithful differential graded functor) $\bar{e} : \mathsf{D}(\mathcal{I}|\mathcal{J}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$, which is injective on objects, and a unital A_∞-functor $\bar{\imath} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{I}|\mathcal{J})$.

---

[1] If $R \in {}'\mathscr{U} \in \mathscr{U}$, where ${}'\mathscr{U}$ is a smaller universe, then $\mathcal{A} = R$-mod is a $\mathscr{U}$-small ${}'\mathscr{U}$-category. Is it possible to replace $\mathcal{B} = \mathsf{A}(\mathcal{A})$ with some ${}'\mathscr{U}$-small category $\mathcal{B}' \subset \mathcal{B}$ to get a ${}'\mathscr{U}$-category $\mathcal{D} = \mathsf{D}(\mathcal{C}|\mathcal{B}')$ A_∞-equivalent to $\mathscr{U}$-small $\mathcal{D} = \mathsf{D}(\mathcal{C}|\mathcal{B})$?

Let us prove that these A_∞-functors are quasi-inverse to each other. First of all, $ei = \mathrm{id}_{\mathcal{J}}$ implies $\overline{e}\,\overline{\imath} = \mathrm{id}_{\mathsf{D}(\mathcal{J}|\mathcal{J})}$. Secondly, there is a natural A_∞-transformation $\overline{r} : \mathrm{id} \to \overline{\imath}\,\overline{e} : \mathsf{D}(\mathcal{C}|\mathcal{B}) \to \mathsf{D}(\mathcal{C}|\mathcal{B})$. Let us prove that it is invertible.

The 0-th component is

$$_X\overline{r}_0 = \big[\Bbbk \xrightarrow{_Xr_0} (s\mathcal{C})^{-1}(X, Xi) \xhookrightarrow{\bar{\jmath}_1} (s\mathsf{D}(\mathcal{C}|\mathcal{B}))^{-1}(X, Xi)\big].$$

We have proved in Section 9.1 that since r_X is a quasi-isomorphism, the above element is invertible modulo boundary in the sense of Section 8.3: there exist p_0, v_0, w_0 such that Equations (53) hold. We conclude by Proposition 8.6 that $\overline{r}$ is invertible, hence, $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\mathsf{D}(\mathcal{J}|\mathcal{J})$ are equivalent.

Each acyclic K-injective complex X is contractible. Indeed, $\mathsf{K}(\mathcal{A})(X, X) \simeq \mathsf{D}(\mathcal{A})(X, X) = 0$ by [**Spa88**, Proposition 1.5]. Hence, $\mathcal{J}$ is a contractible subcategory of $\mathcal{J}$. Thus, $\bar{\jmath} : \mathcal{J} \to \mathsf{D}(\mathcal{J}|\mathcal{J})$ is an equivalence. We deduce that $\mathsf{D}(\mathcal{C}|\mathcal{B})$ and $\mathcal{J}$ are equivalent in A^u_∞. Taking H^0 we get equivalent categories $H^0(\mathsf{D}(\mathcal{C}|\mathcal{B}))$ and $H^0(\mathcal{J})$. The latter is a full subcategory of $\mathsf{K}(\mathcal{A})$, whose objects are K-injective complexes. It is equivalent to the derived category $\mathsf{D}(\mathcal{A})$ (e.g. by [**KS90**, Proposition 1.6.5]). Hence, $H^0(\mathsf{D}(\mathcal{C}|\mathcal{B}))$ is equivalent to the derived category $\mathsf{D}(\mathcal{A})$. This result follows also from Drinfeld's theory [**Dri04**]. It motivated our study of A_∞-categories.

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive $\Bbbk$-linear functor between Abelian categories. The standard recipe [**Spa88**] of producing its right derived functor can be formulated in terms of the K-injective resolution A_∞-functor i as follows. Apply H^0 to the A_∞-functor

$$\mathsf{D}(\mathsf{C}(\mathcal{A})|\mathsf{A}(\mathcal{A})) \xrightarrow{\overline{\imath}} \mathsf{D}(\mathsf{I}(\mathcal{A})|\mathsf{AI}(\mathcal{A})) \xrightarrow{\mathsf{D}(F)} \mathsf{D}(\mathsf{C}(\mathcal{B})|\mathsf{A}(\mathcal{B}))$$

(when $F(\operatorname{Ob}\mathsf{AI}(\mathcal{A})) \subset \operatorname{Ob}\mathsf{A}(\mathcal{B})$). Some work is required to identify the obtained functor [**Lyu03**, Section 8.13]

$$H^0(\overline{\imath}\mathsf{D}(F)) : \mathsf{D}(\mathcal{A}) \simeq H^0\big(\mathsf{D}(\mathsf{C}(\mathcal{A})|\mathsf{A}(\mathcal{A}))\big) \to H^0\big(\mathsf{D}(\mathsf{C}(\mathcal{B})|\mathsf{A}(\mathcal{B}))\big) \simeq \mathsf{D}(\mathcal{B})$$

with RF; however, we shall not consider this topic here.

Acknowledgements

We are grateful to B. L. Tsygan for drawing our attention to the subject. We thank V. G. Drinfeld for the possibility to become acquainted with his work on quotients of differential graded categories before it was completed. We are grateful to all the participants of the A_∞-category seminar at the Institute of Mathematics, Kyiv, especially to Yu. Bespalov and O. Manzyuk. We thank the referee for useful suggestions, which allowed us to improve the exposition.

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