Homology, Homotopy and Applications, vol. 16(2), 2014, pp.107–142

L_∞-ALGEBRAS OF LOCAL OBSERVABLES FROM HIGHER PREQUANTUM BUNDLES

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(*communicated by Johannes Huebschmann*)

Abstract

To any manifold equipped with a higher degree closed form, one can associate an L_∞-algebra of local observables that generalizes the Poisson algebra of a symplectic manifold. Here, by means of an explicit homotopy equivalence, we interpret this L_∞-algebra in terms of infinitesimal autoequivalences of higher prequantum bundles. By truncating the connection data on the prequantum bundle, we produce analogues of the (higher) Lie algebras of sections of the Atiyah Lie algebroid and of the Courant Lie 2-algebroid. We also exhibit the L_∞-cocycle that realizes the L_∞-algebra of local observables as a Kirillov–Kostant–Souriau-type L_∞-extension of the Hamiltonian vector fields. When restricted along a Lie algebra action, this yields Heisenberg-like L_∞-algebras such as the string Lie 2-algebra of a semisimple Lie algebra.

1. Introduction

Geometric objects, such as manifolds, orbifolds, or stacks, equipped with a closed differential form play important roles in many areas of current mathematical interest. The archetypical examples are closed 2-forms in (pre-)symplectic geometry. Higher degree closed forms play crucial roles, for example, in covariant quantum field theory, in Hitchin's generalized complex/Riemannian geometry, and in differential cohomology. It is becoming clear that it is advantageous to consider these forms, in one way or another, as higher degree generalizations of symplectic structures.

In all of these applications, there is a particular focus on integral closed forms. This is because such forms correspond to the curvatures of higher geometric structures known as $U(1)$-n-bundles with connection (or $U(1)$-$(n-1)$-bundle gerbes with connection). Here we refer to these as *higher prequantum bundles*, in analogy with the role that $U(1)$-principal connections play in the geometric prequantization of symplectic manifolds [**21, 38**]. (A modern review can be found in [**8**].) In the companion article [**12**] we present general aspects of such higher geometric prequantum structures; here

Received August 7, 2013, revised February 7, 2014; published on July 30, 2014.
2010 Mathematics Subject Classification: 53D50, 53C08, 18G55.
Key words and phrases: Geometric quantization, gerbes, homotopical algebra.
Article available at `http://dx.doi.org/10.4310/HHA.2014.v16.n2.a6`

we work out details of the general theory specialized to the higher differential geometry over smooth manifolds. In particular, we use homotopy Lie theory to study the infinitesimal autoequivalences of higher prequantum bundles covering infinitesimal diffeomorphisms of the base manifold, i.e., the infinitesimal quantomorphisms.

It is well known that every presymplectic manifold induces a Lie algebra of Hamiltonian functions whose bracket is the Poisson bracket given by the closed 2-form. When the manifold is equipped with a prequantum bundle, this Lie algebra is isomorphic to the Lie algebra of infinitesimal autoequivalences of that structure, i.e., those vector fields on the bundle whose flow preserves the underlying bundle and its connection under pullback. These are also called "prequantum operators." More generally, manifolds equipped with higher degree forms also have Hamiltonian vector fields, which form a Lie algebra just as in symplectic geometry. The differential form induces a bilinear skew-symmetric bracket not on functions, but on higher degree differential forms. However, this bracket fails to satisfy the Jacobi identity. The observation made in [**32**] was that, for the case of non-degenerate forms, this failure is controlled by coherent homotopy. Hence, instead of being a problem, the lack of a genuine Lie bracket indicates the presence of a natural, but higher (homotopy-theoretic) structure. More precisely, the higher Poisson bracket gives rise to a strong-homotopy Lie algebra or L_∞-algebra. The construction in [**32**] extends immediately to the case of degenerate forms, and we call the resulting algebra the "L_∞-algebra of local observables." In this paper, we illuminate its conceptual role further.

Summary of results. We identify the higher Kirillov–Kostant–Souriau L_∞-algebra cocycle that classifies the L_∞-algebra of local observables as an extension of the Hamiltonian vector fields (theorem 3.12) and show how this result immediately gives a construction of "higher Heisenberg L_∞-algebras" (section 3.4). As an example, we obtain a direct rederivation (example 3.16) of the $\mathfrak{string}_{\mathfrak{g}}$-Lie 2-algebra as the Heisenberg Lie 2-algebra of a compact simple Lie group G [**4**].

We briefly recall the construction of the higher prequantum automorphism group of a higher prequantum bundle, which is described with more detail in [**12**]. We construct a dg Lie algebra (def. 4.5) that can be thought of as modeling the "infinitesimal elements" of this higher automorphism group in terms of the Čech–Deligne cocycle for the prequantum bundle. (Similar dg Lie models for the "infinitesimal symmetries" of a $U(1)$-bundle gerbe were constructed by Collier [**11**].)

We prove explicitly that our dg Lie algebra of infinitesimal quantomorphisms is equivalent, as an L_∞-algebra, to the L_∞-algebra of local observables of the corresponding pre-n-plectic form (theorem 4.6).

Finally, we show that this construction induces an inclusion of the L_∞-algebra of local observables into higher Courant and higher Atiyah Lie algebras (section 5).

Remark 1.1. All of the constructions and results that we discuss here apply to the general context of pre–n-plectic manifolds, i.e., manifolds equipped with a closed $(n+1)$-form. Nondegeneracy conditions on the differential form do not play a role. Nevertheless, our formalism allows us to restrict to the case of nondegenerate forms, and it may be interesting to do so in specific applications. This is analogous to the well-known fact that nondegeneracy is not needed to prequantize a symplectic manifold. Indeed, one can proceed even further in this case; the full geometric quantization of presymplectic manifolds is a well-defined and interesting endeavour in its own right (e.g. [**9**]).

Motivation and perspective. The L_∞-algebras of local observables as considered here appear naturally in traditional field theory in the guise of higher order local Noether currents. For instance, it is shown in [**3**] how the energy-momentum tensor for the bosonic string arises in the Lie 2-algebra associated to a multiphase space for a $(1+1)$-dimensional field theory. Generally, the classical Hamilton–de Donder–Weyl field equations in multisymplectic field theory characterize the higher dimensional infinitesimal flows in the L_∞-algebra of local observables (Maurer–Cartan elements in the tensor product with a Grassmann algebra); this is discussed in section 1.2.11.3 of [**36**].

In a broader perspective, these L_∞-algebras naturally arise in the context of higher geometric prequantization and in particular in the geometric quantization of loop groups by the orbit method; see, e.g., [**8**, p. 249] and the discussion in [**12**, Sec. 2.6.1]. This was a motivation behind the refinement of multisymplectic geometry to homotopy theory developed in [**32**], leading to a higher Bohr–Sommerfeld-like geometric quantization procedure for manifolds equipped with closed integral 3-forms [**31**, Chap. 7]. These integral 2-plectic stuctures also naturally appear as the geometric quantization of Poisson manifolds via their associated symplectic groupoids (whose multiplicative symplectic form is secretly a 2-plectic simplicial form); see [**5**].

In terms of quantum field theory, higher geometric prequantization concerns the prequantum incarnation of local quantum field theory, in the way envisioned by Freed [**15**], Baez and Dolan [**2**], and more recently formalized by Lurie [**24**]. While Lurie's theorem gives a full characterization of the topological quantum field theories that are local in this sense, it is an open problem to find a refinement of the process of quantization that would "read in" higher geometric prequantum data and produce a local QFT in this sense. The results of the present article, when placed within the larger context of higher prequantum geometry, as discussed more fully in [**12**], are meant to provide some answers to this open question. Indeed, based on these developments, further progress in this direction has been made recently in [**27**]. A survey is given in [**36**, Section 6].

It should be remarked that in the present article we are solely interested in the L_∞-algebra structure on local observables and we are not investigating the existence of compatible associative and commutative algebra structures (up to homotopy) making the higher local observables a Poisson$_\infty$-algebra. This issue will hopefully be investigated elsewhere. It is also worth mentioning that, in parallel to the L_∞-algebras for n-plectic geometry as considered here, there are various other attempts to formulate generalizations of the algebraic structures found in symplectic geometry to multisymplectic geometry [**14, 20, 30**]. These differing proposals are not manifestly equivalent, and it would be interesting to understand the relations between these various proposals at a deeper level.

Acknowledgements.

We thank Johannes Huebschmann, Jim Stasheff, and the referee for valuable comments and suggestions on a first version of this article, Yael Frégier and Marco Zambon for sharing a preliminary draft [**16**] of their work (joint with C.L.R.) that inspired some of the ideas developed here, and Ruggero Bandiera, Christian Blohmann, and Marco Manetti for the invaluably inspiring conversations they had with D.F. on the homotopy fibers of L_∞-morphisms and on higher symplectic geometry.

C.L.R. acknowledges support from the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen. U.S. was supported by the Dutch Research Organization (NWO project 613.000.802).

Notation and conventions.

1.0.1. Notation for Cartan calculus

The Schouten bracket of two decomposable multivector fields $u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n \in \bigwedge^\bullet \mathfrak{X}(X)$ is

$$[u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n] = \sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}[u_i, v_j] \wedge u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n, \quad (1.0.1)$$

where $[u_i, v_j]$ is the usual Lie bracket of vector fields.

Given a form $\alpha \in \Omega^\bullet(X)$, the **interior product** of a decomposable multivector field $v_1 \wedge \cdots \wedge v_n$ with α is defined as $\iota_{v_1\wedge\cdots\wedge v_n}\alpha = \iota_{v_n}\cdots\iota_{v_1}\alpha$, where $\iota_{v_i}\alpha$ is the usual interior product of vector fields and differential forms. The interior product of an arbitrary multivector field is obtained by extending the above formula by $C^\infty(X;\mathbb{R})$-linearity. The **Lie derivative** $\mathcal{L}_v$ of a differential form along a multivector field $v \in \bigwedge^\bullet \mathfrak{X}(X)$ is defined via the graded commutator of d and $\iota(v)$: $\mathcal{L}_v\alpha = d\iota_v\alpha - (-1)^{|v|}\iota_v d\alpha$, where $\iota(v)$ is considered as a degree $-|v|$ operator.

The last identity we will need involving multivector fields is for the graded commutator of the Lie derivative and the interior product. Given $u, v \in \bigwedge^\bullet \mathfrak{X}(X)$, we have the Cartan identity

$$\iota_{[u,v]}\alpha = (-1)^{(|u|-1)|v|}\mathcal{L}_u\iota_v\alpha - \iota_v\mathcal{L}_u\alpha. \quad (1.0.2)$$

1.0.2. Conventions on chain and cochain complexes

We will work mostly with chain complexes and homological degree conventions. The differential of a chain complex $(A_\bullet, d)$ will have degree -1: $\cdots \to A_{n+1} \xrightarrow{d} A_n \xrightarrow{d} A_{n-1} \to \cdots$. The shift functor $A_\bullet \mapsto A[1]_\bullet$ will act by $A[1]_k = A_{k-1}$. In particular, if V is a vector space, seen as a chain complex concentrated in degree 0, $V[n]$ will be the chain complex consisting of V concentrated in degree n. A cochain complex $(A^\bullet, d)$ will have a differential of degree $+1$, $\cdots \to A^{n-1} \xrightarrow{d} A^n \xrightarrow{d} A^{n+1} \to \cdots$, and will be identified with a chain complex (with the same differential) by the rule $A_k = A^{-k}$. In particular, chain complexes concentrated in nonnegative degree will correspond to cochain complexes concentrated in nonpositive degree, and vice versa. On cochain complexes the shift functor $A^\bullet \mapsto A[1]^\bullet$ will act by $A[1]^k = A^{k+1}$.

1.0.3. Conventions and notation for L_∞-algebras

We will assume the reader is familiar with the homotopy theory of dg-Lie and L_∞-algebras. A comprehensive account can be found in [**23**]. We will follow homological degree conventions, as in [**22**], so that the differential of a dg-Lie algebra and of an L_∞-algebra will have degree -1. All examples of L_∞-algebras $\mathfrak{g}$ given here will have their underlying chain complex $\mathfrak{g}_\bullet$ concentrated in nonnegative degree. An L_∞-algebra concentrated in degrees 0 through $(n-1)$ will be called a *Lie n-algebra*.

An L_∞-algebra whose k-ary brackets for $k \geqslant 2$ are trivial, i.e., a plain chain complex, is called an *abelian* L_∞-algebra. If $\mathfrak{h}$ is an abelian L_∞-algebra with underlying chain complex $\mathfrak{h}_\bullet$, then we also write $\mathbf{B}\mathfrak{h}$ for the abelian L_∞-algebra with underlying chain complex $\mathfrak{h}_\bullet[1]$. In particular, for $n \in \mathbb{N}$ we write $\mathbf{B}^n\mathbb{R} = \mathbb{R}[n]$ for the abelian L_∞-algebra whose underlying chain complex is $\mathbb{R}$ concentrated in degree n.

An L_∞-morphism of the form $\mathfrak{g} \to \mathbf{B}A$, for A an abelian L_∞-algebra, will be called an *L_∞-algebra cocycle* on the L_∞-algebra $\mathfrak{g}$ with coefficients in A. For $\mathfrak{g}$ a Lie algebra and $A = \mathbb{R}[n]$, these are just the traditional cocycles used in Lie algebra cohomology. See [**25, 26**] for a discussion of L_∞-algebra extensions in the broader context of principal ∞-bundles.

The (nonfull) inclusion of dg-Lie algebras into L_∞-algebras is a part of an adjunction

$$(\mathcal{R} \dashv i): \; L_\infty\mathrm{Alg} \underset{\mathcal{R}}{\overset{i}{\rightleftarrows}} \mathrm{dgLie}\; ; \tag{1.0.3}$$

see for instance [**23**, Proposition 11.4.5]. We will call $i \circ \mathcal{R}$ the *rectification functor* for L_∞-algebras, and will often leave the (nonfull) embedding i notationally implicit. In particular, for any L_∞-algebra $\mathfrak{g}$ there is a *canonical* L_∞-algebra homomorphism $\mathfrak{g} \xrightarrow{v_\mathfrak{g}} \mathcal{R}(\mathfrak{g})$, namely, the unit of the adjunction, such that every L_∞ morphism $f_\infty : \mathfrak{g} \to A$ to a dg-Lie algebra A uniquely factors as $\mathfrak{g} \xrightarrow{v_\mathfrak{g}} \mathcal{R}(\mathfrak{g}) \xrightarrow{\xi_A \circ \mathcal{R}(f_\infty)} A$, where $\xi_A : \mathcal{R}(A) \to A$ is the dg-Lie algebra morphism in the factorization of the identity of A as $A \xrightarrow{v_A} \mathcal{R}(A) \xrightarrow{\xi_A} A$.

There is a wealth of presentations for the homotopy theory of L_∞-algebras, given by a web of model category structures with Quillen equivalences between them [**28**]. Here we make use of the model structures due to [**17, 18**], from which one can distill the following statement: the category of dg-Lie algebras (over the real numbers) carries a model category structure in which the weak equivalences are the quasi-isomorphisms on the underlying chain complexes, and the fibrations are the degreewise surjections on the underlying chain complexes. Moreover, if we define a morphism $\mathfrak{g} \to \mathfrak{h}$ in $L_\infty\mathrm{Alg}$ to be a weak equivalence iff the underlying morphism of complexes $\mathfrak{g}_\bullet \to \mathfrak{h}_\bullet$ is a quasi-isomorphism, then the adjunction $(\mathcal{R} \dashv i)$ induces an equivalence between the homotopy theories of dg-Lie algebras and L_∞-algebras. In particular, the components of the unit $\mathfrak{g} \xrightarrow{v_\mathfrak{g}} \mathcal{R}(\mathfrak{g})$ and counit $\mathcal{R}(A) \xrightarrow{\xi_A} A$ of this adjunction are weak equivalences.

1.0.4. Conventions on stacks and higher stacks

While this article focuses on homotopy Lie theory, we do mention at some points the corresponding constructions in higher smooth stacks, according to [**12**]. A detailed overview of this formalism is given in [**13**, Section 3.1]. Smooth stacks are taken to be stacks over the category of all smooth manifolds equipped with its standard Grothendieck topology of good open covers. Equivalently but more conveniently, these are stacks over just the subcategory CartSp of Cartesian spaces $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$ (or equivalently, of open n-balls), regarded as smooth manifolds. A higher smooth stack may always be presented as a Kan-complex valued functor on $\mathrm{CartSp}^{\mathrm{op}}$ and the homotopy theory $\mathbf{H}$ of smooth stacks is given by the category of such functors with stalkwise homotopy equivalences of Kan complexes universally turned into actual homotopy equivalences: $\mathbf{H} := L_{\mathrm{lhe}}\,\mathrm{Func}(\mathrm{CartSp}^{\mathrm{op}}, \mathrm{KanCplx})$. In the applications of the present article all examples of such objects are either given by sheaves of chain complexes $A_\bullet$ of abelian groups in nonnegative degrees under the Dold–Kan correspondence $\mathrm{DK} : \mathrm{Ch}_{\geqslant\bullet}(\mathrm{Ab}) \xrightarrow{\simeq} \mathrm{AbGrp}^{\Delta^{\mathrm{op}}} \xrightarrow{\mathrm{forget}} \mathrm{KanCplx}$, or are the Čech nerve $\check{C}(\mathcal{U})$ of an open cover $\mathcal{U} = \{U_i \to X\}_i$ of a smooth manifold X. If $\mathcal{U}$ is a good cover and if $A_\bullet$ is CartSp-acyclic (which it is in all the examples we consider), then the function complex $\mathbf{H}(X, A) \simeq \mathrm{Func}(\check{C}(\mathcal{U}), \mathrm{DK}(A_\bullet))$ is the traditional cocycle complex of Čech hypercohomology of X with coefficients in $A_\bullet$.

2. Higher prequantum geometry over smooth manifolds

We briefly review here the basic notions of higher prequantum geometry over smooth manifolds that we will use throughout the article. First, in 2.1, we recall the notion of pre-n-plectic manifolds and their Hamiltonian vector fields and then in 2.2 their prequantization by Čech–Deligne cocycles.

2.1. n-Plectic manifolds and their Hamiltonian vector fields

In [**3**] the following terminology has been introduced.

Definition 2.1. A *pre–n-plectic manifold* (X, ω) is a smooth manifold X equipped with a closed $(n+1)$-form $\omega \in \Omega^{n+1}_{\mathrm{cl}}(X)$. If the contraction map $\hat{\omega}\colon TX \to \Lambda^n T^*X$ is injective, then ω is called *nondegenerate* or *n-plectic* and (X, ω) is called an *n-plectic manifold*.

Example 2.2. For $n = 1$ an n-plectic manifold is equivalently an ordinary symplectic manifold. A compact connected simple Lie group equipped with its canonical left invariant differential 3-form $\omega := \langle -, [-, -]\rangle$ is a 2-plectic manifold.

Definition 2.3. Let (X, ω) be a pre–n-plectic manifold. If a vector field v and an $(n-1)$-form H are related by $\iota_v\omega + dH = 0$, then we say that v is a Hamiltonian field for H and that H is a Hamiltonian form for v. We denote by $\mathrm{Ham}^{n-1}(X) \subseteq \mathfrak{X}(X) \oplus \Omega^{n-1}(X)$ the subspace of pairs (v, H) such that $\iota_v\omega + dH = 0$. We call this the space of *Hamiltonian pairs*. The image $\mathfrak{X}_{\mathrm{Ham}}(X) \subseteq \mathfrak{X}(X)$ of the projection $\mathrm{Ham}^{n-1}(X) \to \mathfrak{X}(X)$ is called the space of *Hamiltonian vector fields* of (X, ω).

Remark 2.4. Given a pre–n-plectic manifold (X, ω) We have a short exact sequence of vector spaces $0 \to \Omega^{n-1}_{\mathrm{cl}}(X) \to \mathrm{Ham}^{n-1}(X) \to \mathfrak{X}_{\mathrm{Ham}}(X) \to 0$, i.e., closed $(n-1)$-forms are Hamiltonian, with zero Hamiltonian vector field. It is immediate from the definition that Hamilton vector fields preserve the pre–n-plectic form ω, i.e., $\mathcal{L}_v\omega = 0$. Indeed, since ω is closed, we have $\mathcal{L}_v\omega = d\iota_v\omega = -d^2H_v = 0$. Therefore, the integration of a Hamiltonian vector field gives a diffeomorphism of X preserving the pre-n-plectic form: a *Hamiltonian n-plectomorphism*.

Lemma 2.5. *The subspace* $\mathfrak{X}_{\mathrm{Ham}}(X)$ *is a Lie subalgebra of* $\mathfrak{X}(X)$.

Proof. Let v_1 and v_2 be Hamiltonian vector fields, and let H_1, H_2 be their respective Hamiltonian forms. By $\mathcal{L}_{v_1}\omega = 0$ and by the Cartan formulas, we get $\iota_{[v_1,v_2]}\omega = [\mathcal{L}_{v_1}, \iota_{v_2}]\omega = -\mathcal{L}_{v_1} dH_2 = -d\mathcal{L}_{v_1} H_2 = d\iota_{v_1}\iota_{v_2}\omega$; i.e., the vector field $[v_1, v_2]$ is Hamiltonian, with Hamiltonian $\iota_{v_1\wedge v_2}\omega$. □

Remark 2.6. Hamiltonian vector fields on a pre–n-plectic manifold (X, ω) are by definition those vector fields v such that $\iota_v\omega$ is exact. One may relax this condition and consider *symplectic vector fields* instead, i.e., those vector fields v such that $\mathcal{L}_v\omega = 0$, or, equivalently, such that $\iota_v\omega$ is closed. Then the arguments in Remark 2.4 and in Lemma 2.5 show that symplectic vector fields form a Lie subalgebra $\mathfrak{X}_{\mathrm{symp}}(X)$ of $\mathfrak{X}(X)$ and that $\mathfrak{X}_{\mathrm{Ham}}(X) \subseteq \mathfrak{X}_{\mathrm{symp}}(X)$ is a Lie ideal.

2.2. Prequantization of (pre–)n-plectic manifolds

The traditional notion of prequantization of a presymplectic manifold (X, ω) is equivalently a lift of the presymplectic form, regarded as a de Rham 2-cocycle, to a degree 2 cocycle in *ordinary differential cohomology* (see, for instance, [**8**, Section 2.2]). Equivalently, this is a lift of ω to a connection ∇ on a $U(1)$-principal bundle on X with curvature $F_\nabla = \omega$. Accordingly, the prequantization of a pre–n-plectic manifold is naturally defined to be a lift of ω regarded as a degree $(n+1)$ cocycle in de Rham cohomology to a cocycle of degree $(n + 1)$ in ordinary differential cohomology.

Definition 2.7. For X a smooth manifold and $\mathcal{U} = \{U_i \to X\}$ an open cover, we write $(\mathrm{Tot}(\mathcal{U}, \Omega), d_{\mathrm{Tot}})$ for the corresponding Čech–de Rham total complex, i.e., the cochain complex with underlying graded vector space $\mathrm{Tot}^n(\mathcal{U}, \Omega) = \bigoplus_{i+j=n} \check{C}^i(\mathcal{U}, \Omega^j)$ and whose differential is given on elements $\bar\theta = \sum_{i=0}^n \theta^{n-i}$ with $\theta^{n-i} \in \check{C}^i(\mathcal{U}, \Omega^{n-i})$ by $d_{\mathrm{Tot}}\theta^{n-i} = \delta\theta^{n-i} + (-1)^i d\theta^{n-i}$.

Definition 2.8. The cochain complex of sheaves

$$C^\infty(-; U(1)) \xrightarrow{d\log} \Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots\cdots \xrightarrow{d} \Omega^n(-) \xrightarrow{d} \Omega^{n+1}(-) \to \cdots ,$$

with $C^\infty(-; U(1))$ in degree 0, will be called the *Deligne complex* and will be denoted by the symbol $\underline{U}(1)_{\mathrm{Del}}$. Its truncation

$$C^\infty(-; U(1)) \xrightarrow{d\log} \Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots\cdots \xrightarrow{d} \Omega^n(-) \to 0 \to 0 \to \cdots$$

will be denoted by $\underline{U}(1)^{\leqslant n}_{\mathrm{Del}}$.

It follows from the above definition that a degree n Čech–Deligne cocycle in $\underline{U}(1)^{\leqslant n}_{\mathrm{Del}}$ is $\bar A = \sum_{i=0}^n A^{n-i}$, with $A^{n-i} \in \check{C}^i(\mathcal{U}, \Omega^{n-i})$ and $A^0 \in \check{C}^n(\mathcal{U}, \underline{U}(1))$, satisfying

$$\begin{aligned} \delta A^{n-i} &= (-1)^i dA^{n-i-1}, \quad i = 0, \ldots, n-2 \\ \delta A^1 &= (-1)^{n-1} d\log A^0; \qquad \delta A^0 = 1. \end{aligned} \tag{2.2.1}$$

Definition 2.9. The *n-stack of principal $U(1)$-n-bundles (or $(n-1)$-bundle gerbes) with connection* is the n-stack presented via applying the Dold–Kan construction to the presheaf $\underline{U}(1)^{\leqslant n}_{\mathrm{Del}}[n]$, regarded as a presheaf of chain complexes concentrated in nonnegative degree. It will be denoted by the symbol $\mathbf{B}^n U(1)_{\mathrm{conn}}$.

The commutative diagram

$$\begin{array}{ccccccccc} C^\infty(-;U(1)) & \xrightarrow{d\log} & \Omega^1(-) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1}(-) & \xrightarrow{d} & \Omega^n(-) \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle d} \\ 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \Omega^{n+1}(-)_{\mathrm{cl}} \end{array}$$

presents the morphism of stacks $F : \mathbf{B}^n U(1)_{\mathrm{conn}} \to \Omega^{n+1}(-)_{\mathrm{cl}}$ that maps a principal $U(1)$-n-bundle with connection to its *curvature* $(n+1)$-form.

Definition 2.10. Let (X, ω) be a pre–n-plectic manifold. A *prequantization* of (X, ω) is a lift

$$\begin{array}{ccc} & & \mathbf{B}^n U(1)_{\mathrm{conn}} \\ & \nearrow^{\nabla} & \downarrow{\scriptstyle F} \\ X & \xrightarrow{\omega} & \Omega^{n+1}(-)_{\mathrm{cl}}. \end{array}$$

We call the triple (X, ω, ∇) a prequantized pre–n-plectic manifold.

Local data for a prequantization (X, ω, ∇) are conveniently expressed in terms of the Čech–Deligne total complex. Namely, let $\mathcal{U}$ be a good cover of X; then a pre–n-plectic structure on X is the datum of a closed element ω in $\check{C}^0(\mathcal{U}, \underline{U}(1)^{\leqslant n+1}_{\mathrm{Del}})$. Moreover, if (X, ω) admits a prequantization, then the datum of a prequantization is an element A in $\mathrm{Tot}^n(\mathcal{U}, \underline{U}(1)_{\mathrm{Del}})$ such that $d_{\mathrm{Tot}} A = \omega$.

Remark 2.11. It is a well-known fact that (X, ω) admits a prequantization if and only if it is an *integral* presymplectic manifold, i.e., if and only if the closed form ω represents an integral class in de Rham cohomology; see, e.g., [**8**]. Indeed, since the shifted Deligne complex $\underline{U}(1)_{\mathrm{Del}}[n]$ is an acyclic resolution of the cochain complex of sheaves $\flat \mathbf{B}^n U(1)$ consisting of locally constant $U(1)$-valued functions placed in degree $-n$, we see that a pre–n-plectic structure ω is prequantizable if and only if ω defines the trivial class in the degree $n+1$ Čech cohomology of X with coefficients in the discrete abelian group $U(1)$. By the short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$ and by the Čech–de Rham isomorphism $H^n_{\mathrm{dR}}(X, \mathbb{R}) \cong \check{H}^n(X, \mathbb{R})$, this is equivalent to requiring that the de Rham class of ω is an integral class.

3. The L_∞-algebra of local observables and its KKS L_∞-cocycle

To any pre–n-plectic manifold (X, ω) one can associate an L_∞-algebra $L_\infty(X, \omega)$, as defined in [**16, 32**], which we may think of as the higher local observables on (X, ω). This is an L_∞-extension of the Lie algebra of Hamiltonian vector fields on (X, ω) by the $(n-1)$-shifted truncated de Rham complex of X. We briefly recall this construction in 3.1, below.

For (V, ω) an ordinary symplectic vector space, we may regard it as a symplectic manifold that is canonically equipped with a V-action by Hamiltonian vector fields, with V regarded as the abelian Lie algebra of constant (left invariant) vector fields on itself. The evaluation map at zero $\iota_{-\wedge-}\omega|_0 : V \times V \to \mathbb{R}$ of the symplectic form is then a Lie algebra 2-cocycle on V and hence defines an extension of Lie algebras.

This is famous as the *Heisenberg Lie algebra* extension, and $\iota_{-\wedge -}\omega|_0$ is the *Kirillov–Kostant–Souriau cocycle* that classifies it (see example 3.10, below). More generally, for any symplectic manifold, the KKS 2-cocycle classifies the underlying Lie algebra of the Poisson algebra as a central extension of the Hamiltonian vector fields [**21, 38**]. For symplectic vector spaces, the restriction of the KKS 2-cocycle to the constant Hamiltonian vector fields is precisely the above cocycle. We describe in Section 3.2 a further generalization of this to a class of L_∞-algebra $(n+1)$-cocycles on Hamiltonian vector fields over pre–n-plectic manifolds. We call these the *higher Kirillov–Kostant–Souriau L_∞-cocycles*. In Section 3.3 we prove that the L_∞-algebra extension that is classified by the KKS $(n+1)$-cocycle is indeed again the Poisson-bracket L_∞-algebra of local observables.

3.1. The L_∞-algebra of local observables

We recall the construction of the L_∞-algebra of local observables associated to a pre–n-plectic manifold. It is best seen in the light of the following immediate consequence of Cartan's "magic formula" $\mathcal{L}_v = d\iota_v + \iota_v d$.

Lemma 3.1. *Let X be a smooth manifold and let β be an n-form (not necessarily closed) on X. Given k vector fields $v_1, \dots, v_k$ $(k \geqslant 1)$ on X, the following identity holds:*

$$(-1)^k d\iota_{v_1\wedge\cdots\wedge v_k}\beta = \sum_{1\leqslant i<j\leqslant k} (-1)^{i+j}\iota_{[v_i,v_j]\wedge v_1\wedge\cdots\wedge\hat{v}_i\wedge\cdots\wedge\hat{v}_j\wedge\cdots\wedge v_k}\beta + \sum_{i=1}^{k}(-1)^i\iota_{v_1\wedge\cdots\wedge\hat{v}_i\wedge\cdots\wedge v_k}\mathcal{L}_{v_i}\beta + \iota_{v_1\wedge\cdots\wedge v_k}d\beta.$$

A special case of the above appeared as Lemma 3.7 in [**32**]. We thank M. Zambon for pointing out to us this generalization.

Proposition 3.2 ([**32**], Theorem 5.2; [**16**], Theorem 4.7)**.** *Let (X,ω) be a pre–n-plectic manifold. There exists a Lie n-algebra $L_\infty(X,\omega)$ whose underlying chain complex is*

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \mathrm{Ham}^{n-1}(X)\,,$$

with $\mathrm{Ham}^{n-1}(X)$ *in degree 0, and whose multilinear brackets l_i are*

$$l_1(x) = \begin{cases} 0\oplus dx & \text{if } |x| = 1,\\ dx & \text{if } |x|>1,\end{cases} \qquad l_2(x_1,x_2) = \begin{cases} [v_1,v_2] + \iota_{v_1\wedge v_2}\omega & \text{if } |x_1| = |x_2| = 0,\\ 0 & \text{otherwise},\end{cases}$$

and, for $k>2$:

$$l_k(x_1,\dots,x_k) = \begin{cases} -(-1)^{\binom{k+1}{2}}\iota_{v_1\wedge\cdots\wedge v_k}\omega & \text{if } |x_1| = \cdots = |x_k| = 0,\\ 0 & \text{otherwise},\end{cases}$$

where $x = v+\eta^\bullet$ denotes a generic element $(\eta^0,\eta^1,\dots,v+\eta^{n-1})$ in the chain complex.

Definition 3.3. We call the Lie n-algebra $L_\infty(X,\omega)$ defined in the statement of Proposition 3.2 the *L_∞-algebra of local observables* on (X,ω).

Remark 3.4. The projection map of Definition 2.3 uniquely extends to a morphism of L_∞-algebras of the form $L_\infty(X,\omega) \xrightarrow{\pi_L} \mathfrak{X}_{\mathrm{Ham}}(X)$, i.e., local observables of (X,ω) cover Hamiltonian vector fields. Below, in Section 3.2 we turn to the classification of this map by an L_∞-algebra cocycle.

Example 3.5. If $n = 1$ then (X,ω) is a presymplectic manifold, the chain complex underlying $L_\infty(X,\omega)$ is $\mathrm{Ham}^0(X) = \{v + H \in \mathfrak{X}(X) \oplus C^\infty(X;\mathbb{R}) \mid \iota_v\omega + dH = 0\}$, and the Lie bracket is $[v_1 + H_1, v_2 + H_2] = [v_1, v_2] + \iota_{v_1\wedge v_2}\omega$. If, moreover, ω is non-degenerate so that (X,ω) is symplectic, then the projection $v + H \mapsto H$ is a linear isomorphism $\mathrm{Ham}^0(X) \xrightarrow{\simeq} C^\infty(X;\mathbb{R})$. It is easy to see that under this isomorphism $L_\infty(X,\omega)$ is the underlying Lie algebra of the usual Poisson algebra of functions. See also Proposition 2.3.9 in [**8**].

3.2. The Kirillov–Kostant–Souriau L_∞-algebra cocycle

Here we present an L_∞-algebra cocycle on the Lie algebra of Hamiltonian vector fields on a pre–n-plectic manifold, which generalizes the traditional KKS cocycle and the Heisenberg cocycle to higher geometry.

Definition 3.6. For X a smooth manifold, denote by $\mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R})$ the abelian Lie $(n+1)$-algebra given by the chain complex $\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(X) \xrightarrow{d} d\Omega^{n-1}(X)$, with $d\Omega^{n-1}(X)$ in degree 0.

Remark 3.7. The complex of Definition 3.6 serves as a resolution of the cocycle complex $\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}_{\mathrm{cl}}(X) \to 0$, for the de Rham cohomology of X up to degree $n-1$ once delooped (i.e., shifted).

Proposition 3.8. *Let (X,ω) be a pre–n-plectic manifold. The multilinear maps*

$$\omega_{[1]} : v \mapsto -\iota_v\omega; \qquad \omega_{[2]} : v_1 \wedge v_2 \mapsto \iota_{v_1\wedge v_2}\omega; \qquad \cdots$$
$$\omega_{[n+1]} : v_1 \wedge v_2 \wedge \cdots v_{n+1} \mapsto -(-1)^{\binom{n+1}{2}}\iota_{v_1\wedge v_2\wedge\cdots\wedge v_{n+1}}\omega$$

define an L_∞-morphism $\omega_{[\bullet]} : \mathfrak{X}_{\mathrm{Ham}}(X) \to \mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R})$, and hence an L_∞-algebra $(n+1)$-cocycle on the Lie algebra of Hamiltonian vector fields, Definition 2.3, with values in the abelian $(n+1)$-algebra of Definition 3.6.

Proof. First, notice that the underlying map on chain complexes is indeed well defined: by definition of Hamiltonian vector fields, if v is Hamiltonian, then there exists an $(n-1)$-form H such that $\iota_v\omega + dH = 0$ and so $\omega_{[\bullet]}$ takes values in $\mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R})$. In general, an L_∞-algebra morphism $f : \mathfrak{g} \to \mathfrak{h}$ from a Lie algebra $\mathfrak{g}$ to an abelian Lie $(n+1)$-algebra $\mathfrak{h}$ is equivalently a collection of linear maps $\{f_k : \wedge^k\mathfrak{g}_\bullet \to \mathfrak{h}_\bullet\}_{k=1}^{n+1}$ with $|f_k| = k-1$ and such that the following holds for all $k \geqslant 1$:

$$d_{\mathfrak{h}} f_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} (-1)^{i+j+1} f_{k-1}([v_i, v_j]_{\mathfrak{g}} \wedge v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_k).$$

Therefore, checking that $\omega_{[\bullet]}$ is an L_∞-morphism reduces to checking the identities

$$d\iota_{v_1\wedge\cdots\wedge v_k}\omega = \sum_{i<j} (-1)^{i+j+k} \iota_{[v_i,v_j]\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_{k+1}}\omega.$$

These are satisfied—since the ω is closed and the v_i are Hamiltonian—by Lemma 3.1. $\square$

Definition 3.9. The degree $(n+1)$ *higher Kirillov–Kostant–Souriau L_∞-cocycle* associated to the pre–n-plectic manifold (X,ω) is the L_∞-morphism $\mathfrak{X}_{\mathrm{Ham}}(X) \xrightarrow{\omega_{[\bullet]}} \mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R})$ given in Proposition 3.8.

If $\rho\colon \mathfrak{g} \to \mathfrak{X}_{\mathrm{Ham}}(X)$ is an L_∞-morphism encoding an action of an L_∞-algebra $\mathfrak{g}$ on (X,ω) by Hamiltonian vector fields, then we call the composite $\rho^*\omega_{[\bullet]}$ the corresponding *Heisenberg L_∞-algebra cocycle*. This terminology is motivated by the following example. Further discussion of this aspect is below in Section 3.4.

Example 3.10. Let V be a vector space equipped with a skew-symmetric multilinear form $\omega : \wedge^{n+1}V \to \mathbb{R}$. Since V is an abelian Lie group, we obtain via left-translation of ω a unique closed invariant form, which we also denote as ω. By identifying V with left-invariant vector fields on V, the Poincare lemma implies that we have a canonical inclusion $j_V : V \hookrightarrow \mathfrak{X}_{\mathrm{Ham}}(V)$ of V regarded as an abelian Lie algebra into the Hamiltonian vector fields on (V,ω) regarded as a pre n-plectic manifold. Since V is contractible as a topological manifold, we have, by Remark 3.7, a quasi-isomorphism $\mathbf{BH}(V; \flat\mathbf{B}^{n-1}\mathbb{R}) \xrightarrow{\simeq} \mathbb{R}[n]$ of abelian L_∞-algebras, given by evaluation at 0. Under this equivalence the restriction of the L_∞-algebra cocycle $\omega_{[\bullet]}$ of Definition 3.9 along j_V is an L_∞-algebra map of the form $j_V^*\omega_{[\bullet]} : V \to \mathbb{R}[n]$ whose single component is the linear map $\iota_{(-)}\omega : \wedge^{n+1}V \to \mathbb{R}$. For $n = 1$ and (V,ω), an ordinary symplectic vector space the map $\iota_{(-)}\omega : V \wedge V \to \mathbb{R}$ is the traditional *Heisenberg cocycle*.

Remark 3.11. The KKS $(n+1)$-cocycle has a natural geometric origin as the Lie differentiation of a morphism of higher smooth groups canonically arising in higher geometric prequantization; see [**12**]. This can be seen as a deeper conceptual justification for Definition 3.9.

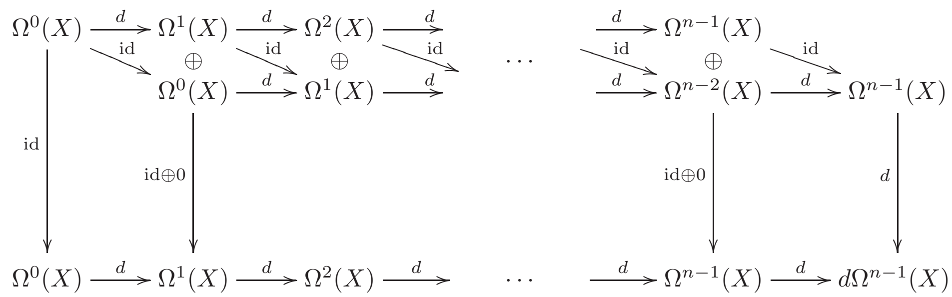
3.3. The Kirillov–Kostant–Souriau L_∞-extension

Using the results presented above, we can now state and prove the main theorem of this section.

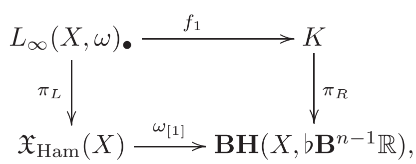
Theorem 3.12. *Given a pre–n-plectic manifold (X,ω), the higher KKS L_∞-cocycle $\omega_{[\bullet]}$ (Definition 3.9) and the projection map $\pi_L\colon L_\infty(X,\omega) \to \mathfrak{X}_{\mathrm{Ham}}(X)$ (Remark 3.4) form a homotopy fiber sequence of L_∞-algebras, i.e., fit into a homotopy pullback diagram of the form*

$$\begin{array}{ccc} L_\infty(X,\omega) & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle \pi_L} & & \big\downarrow \\ \mathfrak{X}_{\mathrm{Ham}}(X) & \xrightarrow{\omega_{[\bullet]}} & \mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R}). \end{array}$$

Proof. By Theorem B.2 it is sufficient to replace the map of chain complexes $0 \to \mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R})$ by any degreewise surjection $K \xrightarrow{\pi_R} \mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R})$ out of an exact chain complex K, such that its pullback along ω_1 is isomorphic to the underlying chain complex of $L_\infty(X,\omega)$, and then to show that the L_∞-structure of $L_\infty(X,\omega)$ sits compatibly in the resulting square diagram. We take K to be the cone of the identity of the chain complex $\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(X)$ with $\Omega^{n-1}(X)$ in degree 0, and take π_R to be the chain map given by the vertical arrows in the following diagram:



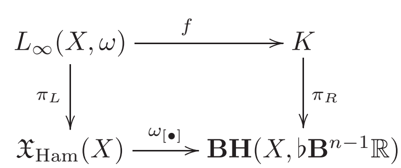
By inspection and comparison with Proposition 3.2, it is easy to see that the fiber product of chain complexes of K and $\mathfrak{X}_{\mathrm{Ham}}(X)$ over $\mathbf{BH}(X, \flat\mathbf{B}^{n-1}\mathbb{R})$ is the chain complex $L_\infty(X,\omega)_\bullet$ that underlies the L_∞-algebra of local observables:



where f_1 is the morphism

$$f_1\colon v + \eta^\bullet \mapsto \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \\ & \eta^0 & \eta^1 & \cdots & \eta^{n-3} & \eta^{n-2} & \eta^{n-1} \end{pmatrix}.$$

As we already observed in Remark 3.4, the chain map underlying π_L uniquely extends to an L_∞-algebra morphism. Therefore to complete the proof, it is sufficient to show that we can lift the horizontal chain map f_1 above to a morphism of L_∞-algebras that makes the diagram



commute. This is easily realized by defining the "Taylor coefficients" of f for $k \geqslant 2$ to be the degree $(k-1)$ maps $f_k\colon \wedge^k L_\infty(X,\omega) \to K$ given by

$$f_k\colon (v_1 + \eta_1^\bullet) \wedge \cdots \wedge (v_k + \eta_k^\bullet) \mapsto \begin{pmatrix} 0 & 0 & 0 & & & \cdots & 0 & 0 & \\ & 0 & 0 & \cdots & 0 & -(-1)^{\binom{k+1}{2}} \iota_{v_1\wedge\cdots\wedge v_k}\omega & 0 & \cdots & 0 \end{pmatrix}.$$

□

3.4. The Heisenberg L_∞-extension

If a Lie algebra $\mathfrak{g}$ acts on an n-plectic manifold by Hamiltonian vector fields, then the KKS L_∞-extension of $\mathfrak{X}_{\mathrm{Ham}}(X)$, discussed in Section 3.3, restricts to an L_∞-extension of $\mathfrak{g}$. This is a generalization of Kostant's construction [**21**] of central extensions of Lie algebras to the context of L_∞-algebras. Perhaps the most famous of these central extensions is the Heisenberg Lie algebra, which is the inspiration behind the following terminology:

Definition 3.13. Let (X,ω) be a pre–n-plectic manifold, and let $\rho : \mathfrak{g} \to \mathfrak{X}_{\mathrm{Ham}}(X)$ be a Lie algebra homomorphism encoding an action of $\mathfrak{g}$ on X by Hamiltonian vector fields. The corresponding *Heisenberg L_∞-algebra extension* $\mathfrak{heis}_\rho(\mathfrak{g})$ of $\mathfrak{g}$ is the extension classified by the composite L_∞-morphism $\omega_{[\bullet]} \circ \rho$, i.e., the homotopy pullback on the left of

$$\begin{array}{ccccc} \mathfrak{heis}_\rho(\mathfrak{g}) & \longrightarrow & L_\infty(X,\omega) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{X}_{\mathrm{Ham}}(X) & \xrightarrow{\omega_{[\bullet]}} & \mathbf{BH}(X, \flat \mathbf{B}^{n-1}\mathbb{R}). \end{array}$$

Remark 3.14. It is natural to call an L_∞-morphism with values in the L_∞-algebra of observables of a pre–n-plectic manifold (X,ω) an "L_∞ comoment map," which generalizes the familiar notion in symplectic geometry. Hence, one could say that an action ρ of a Lie algebra $\mathfrak{g}$ on a pre–n-plectic manifold (X,ω) via Hamiltonian vector fields naturally induces such a comoment map from the Heisenberg L_∞-algebra $\mathfrak{heis}_\rho(\mathfrak{g})$.

Example 3.15. For (V,ω) a symplectic vector space regarded as a symplectic manifold, the translation action of V on itself is via Hamiltonian vector fields (see Example 3.10). If one denotes by $\rho : V \to \mathfrak{X}_{\mathrm{Ham}}(X)$ this action, then the induced Heisenberg L_∞-extension is the traditional Heisenberg Lie algebra.

Example 3.16. Let G be a (connected) compact simple Lie group, regarded as a 2-plectic manifold with its canonical 3-form $\omega := \langle -, [-,-]\rangle$ as in Example 2.2. The infinitesimal generators of the action of G on itself by right translation are the left invariant vector fields $\mathfrak{g}$, which are Hamiltonian. We have $H^1_{\mathrm{dR}}(G) \cong H^1_{\mathrm{CE}}(\mathfrak{g},\mathbb{R}) = 0$, and therefore a weak equivalence: $\mathbf{BH}(G, \flat\mathbf{B}\mathbb{R}) \xrightarrow{\simeq} \mathbb{R}[2]$ given by the evaluation at the identity element of G. The resulting composite cocycle

$$\langle -, [-,-]\rangle : \ \mathfrak{g} \xrightarrow{\rho} \mathfrak{X}_{\mathrm{Ham}}(X) \xrightarrow{\omega_{[\bullet]}} \mathbb{R}[2]$$

is exactly the Lie algebra 3-cocycle that classifies the String Lie-2-algebra. By Theorem B.2 the String Lie 2-algebra is the homotopy fiber of this cocycle, in that we have a homotopy pullback square of L_∞-algebras

$$\begin{array}{ccc} \mathfrak{string}_{\mathfrak{g}} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\langle -,[-,-]\rangle} & \mathbb{R}[2]. \end{array}$$

Hence, the String Lie 2-algebra $\mathfrak{string}_{\mathfrak{g}}$ is the Heisenberg Lie 2-algebra of the 2-plectic manifold $(G, \langle -, [-,-]\rangle)$ with its canonical $\mathfrak{g}$-action ρ, i.e., $\mathfrak{heis}_\rho(\mathfrak{g}) \simeq \mathfrak{string}_{\mathfrak{g}}$. The relationship between $\mathfrak{string}_{\mathfrak{g}}$ and $L_\infty(G,\omega)$ was first explored in [**4**].

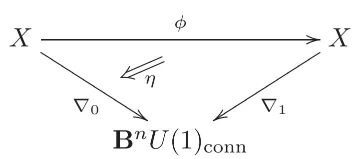
4. The dg-Lie algebra of infinitesimal quantomorphisms

The L_∞-algebra $L_\infty(X,\omega)$ discussed in Section 3 has the nice property that the definition of its brackets generalizes the definition of the traditional Poisson bracket in an elegant way. We now present another L_∞-algebra that looks a little less elegant in components but has a more manifest conceptual interpretation, namely, as the dg Lie algebra of infinitesimal automorphisms of a $U(1)$-n-bundle with connection that cover the diffeomorphisms of the base. A main result of this section is Theorem 4.6, which establishes a weak equivalence between the aforementioned dg Lie algebra and $L_\infty(M,\omega)$.

4.1. Quantomorphism n-groups

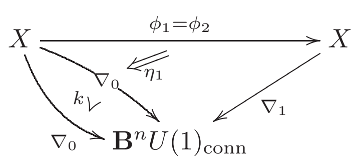
Since, by definition, a prequantization of a pre–n-plectic manifold (X,ω) is a morphism of higher stacks $X \to \mathbf{B}^nU(1)_{\mathrm{conn}}$, a prequantized pre–$n$-plectic manifold is naturally an object in the overcategory (or "slice topos") $\mathbf{H}_{/\mathbf{B}^nU(1)_{\mathrm{conn}}}$. This leads to the following definition.

Definition 4.1. Let $\nabla_0, \nabla_1 : X \to \mathbf{B}^nU(1)_{\mathrm{conn}}$ be two morphisms representing (or "modulating") principal $U(1)$-n-bundles with connection on X. A *1-morphism* $(\phi,\eta)\colon \nabla_0 \to \nabla_1$ in $\mathbf{H}_{/\mathbf{B}^nU(1)_{\mathrm{conn}}}$ is a homotopy commutative diagram of the form



.

A *2-morphism* $(k,h)\colon (\phi_1,\eta_1) \to (\phi_2,\eta_2)$ is only between 1-morphisms such that $\phi_1 = \phi_2$ and is given by a homotopy commutative diagram of the form



,

where one has the (undisplayed) 2-arrow η_2 on the back face of the diagrams and an undisplayed 3-arrow $h : k \circ \eta_1 \to \eta_2$ decorating the bulk of the 3-simplex. Higher morphisms are defined similarly.

Remark 4.2. Since we are dealing with a commutative diagram of morphisms between (higher) stacks, we have the homotopy η appearing here as part of the data of the commutative diagram defining a 1-morphism. In particular, isomorphisms (or better, equivalences) between ∇_0 and ∇_1 will be pairs (ϕ,η) consisting of a diffeomorphism $\phi\colon X \to X$ and a gauge transformation of higher connections $\eta : \phi^*\nabla_1 \xrightarrow{\simeq} \nabla_0$. In particular, for the 1-plectic (i.e., symplectic) case, ∇_0 and ∇_1 correspond to principal $U(1)$-bundles with connection. If X is compact, then the 1-morphisms between them correspond to "strict contactomorphisms" $(P_0,A_0) \to (P_1,A_1)$ between the total spaces of the bundles with their connection 1-forms $A_i \in \Omega^1(P_i;\mathbb{R})$ regarded as "regular" contact forms. If $\nabla = \nabla_0 = \nabla_1$ and ∇ is regarded as the prequantization of its curvature, i.e., the symplectic 2-form ω, then such a contactomorphism is often called a *quantomorphism* in the geometric quantization literature.

The automorphisms $\mathbf{Aut}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}(\nabla)$ of any object $\nabla \in \mathbf{H}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}$ form an "n-group" (see, for example, [**25**, Section 2.3]). And so, motivated by the terminology used in the above remark, we introduce the following definition.

Definition 4.3. Let $\nabla : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ be a morphism modulating a $U(1)$-n-bundle with connection. The *quantomorphism n-group* of ∇, denoted $\mathbf{QuantMorph}(\nabla)$, is the automorphism n-group $\mathbf{Aut}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}(\nabla)$ equipped with its natural smooth structure.

Remark 4.4. In the above definition we described $\mathbf{QuantMorph}(\nabla)$ as a "smooth n-group." In order to make this precise, we need to say what a smooth family of automorphisms is. This is systematically done by working with smooth families from the very beginning, i.e., by replacing the hom-spaces $\mathbf{H}(X, \mathbf{B}^n U(1)_{\mathrm{conn}})$ by what we call the "concretification" of the internal homs (the higher mapping stacks) $[X, \mathbf{B}^n U(1)_{\mathrm{conn}}]$. See [**12**, Section 2.3.2] for precise discussion of this aspect. The intuition behind this smooth structure—which is all that we need for our purposes here—is that all local bundle data depend smoothly on a parameter varying in the base.

4.2. Infinitesimal quantomorphisms as a strict model for the L_∞-algebra of observables

Since the quantomorphism n-group $\mathbf{QuantMorph}(\nabla)$ is equipped with a smooth structure, it has a notion of "tangent vectors." Roughly speaking, these correspond to maps out of the formal infinitesimal interval, $\mathrm{Spec}\big(\mathbb{R}[\epsilon]/(\epsilon)^2\big) \to \mathbf{QuantMorph}(\nabla)$. So it is not surprising that there is also an abstract notion of "Lie differentiation" in this context that, when applied to the smooth n-group $\mathbf{QuantMorph}(\nabla)$, produces not a Lie algebra, but rather a Lie n-algebra, which will be denoted $\mathrm{Lie}(\mathbf{QuantMorph}(\nabla))$. (See Sec. 3.10.9 and Sec. 4.5.1.2 in [**36**] for more details on Lie differentiation).

The defining equations of $\mathrm{Lie}(\mathbf{QuantMorph}(\nabla))$ can be conceptually described as the infinitesimal versions of the defining equations for the quantomorphism n-group. In particular, a degree zero element in $\mathrm{Lie}(\mathbf{QuantMorph}(\nabla))$ will be an infinitesimal version of a pair $(\phi : X \xrightarrow{\sim} X,\ h : \phi^*\nabla \xrightarrow{\sim} \nabla)$, i.e., a pair (v, b) consisting of a vector field v on X and an "infinitesimal homotopy" b such that $b\colon \mathcal{L}_v \nabla \to 0$, where $\mathcal{L}_v$ is the Lie derivative along v. Degree 1 elements in $\mathrm{Lie}(\mathbf{QuantMorph}(\nabla))$ will be homotopies between the b's, and so on. The notion of taking the Lie derivative of a morphism of higher stacks may give pause, but it has an obvious interpretation if we represent the map $\nabla : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ as a Čech–Deligne cocycle $\bar{A}$ on X (Definition 2.2). In this context, $\mathcal{L}_v \nabla$ corresponds to the usual Lie derivative $\mathcal{L}_v \bar{A}$ for vector fields acting on local differential forms. Moreover, in this context the Dold–Kan correspondence tells us that, for example, an infinitesimal homotopy $b\colon \mathcal{L}_v \nabla \to 0$ is simply an element $\bar{\theta}$ of the total Čech–de Rham complex $\big(\mathrm{Tot}^\bullet(\mathcal{U}, \Omega), d_{\mathrm{Tot}}\big)$ satisfying $\mathcal{L}_v \bar{A} = d_{\mathrm{Tot}} \bar{\theta}$. The above discussion is the intuition behind the following:

Definition/Proposition 4.5. Let X be a smooth manifold and $n \in \mathbb{N}$. If $\bar{A}$ is a Čech–Deligne n-cocycle on X relative to some cover $\mathcal{U}$, then the *dg Lie algebra of infinitesimal quantomorphisms* $\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})$ is the strict Lie n-algebra whose underlying complex is

$$\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})^0 = \{v + \bar{\theta} \in \mathfrak{X}(M) \oplus \mathrm{Tot}^{n-1}(\mathcal{U}, \Omega) \mid \mathcal{L}_v \bar{A} = d_{\mathrm{Tot}} \bar{\theta}\},$$
$$\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})^i = \mathrm{Tot}^{n-1-i}(\mathcal{U}, \Omega) \quad \text{for} \quad 1 \leqslant i \leqslant n-1,$$

with differential

$$\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})^{n-1} \xrightarrow{d_{\mathrm{Tot}}} \mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})^{n-2} \xrightarrow{d_{\mathrm{Tot}}} \cdots \xrightarrow{d_{\mathrm{Tot}}} \mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})^1 \xrightarrow{0 \oplus d_{\mathrm{Tot}}} \mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})^0,$$

and whose graded Lie bracket is the semidirect product bracket for the Lie algebra of vector fields acting on differential forms by Lie derivative:

$$\begin{aligned} &[\![v_1 + \bar{\theta}_1, v_2 + \bar{\theta}_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1} \bar{\theta}_2 - \mathcal{L}_{v_2} \bar{\theta}_1; \\ &[\![v + \bar{\theta}, \bar{\eta}]\!] = - [\![\bar{\eta}, v + \bar{\theta}]\!] = \mathcal{L}_v \bar{\eta}; \qquad [\![\bar{\eta}, \bar{\eta}]\!] = 0. \end{aligned} \tag{4.2.1}$$

The next theorem reveals the relationship between the above dgla of infinitesimal quantomorphisms and the L_∞-algebra of local observables. It is the higher analogue of the well-known fact in traditional prequantization that the underlying Lie algebra of the Poisson algebra on a prequantized symplectic manifold is isomorphic to the Lie algebra of $U(1)$-invariant connection-preserving vector fields on the total space of the prequantum bundle.

Theorem 4.6. *Let (X, ω) be an integral pre–n-plectic manifold (Definition 2.1), $\mathcal{U}$ a good open cover of X, and ∇ a prequantization of (X, ω) (Definition 2.10) presented by a Čech–Deligne cocycle $\bar{A} = \sum_{i=0}^{n} A^{n-i}$ in $\mathrm{Tot}^n(\mathcal{U}, \underline{U}(1)^{\leqslant n}_{\mathrm{Del}})$. There exists an L_∞-quasi-isomorphism $f : L_\infty(X, \omega) \xrightarrow{\simeq} \mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})$ between the L_∞-algebra of local observables (Definition 3.3) and the dgla of infinitesimal quantomorphisms (Definition 4.5), whose linear term is*

$$f_1(x) = \begin{cases} v - H|_{U_\alpha} + \sum_{i=0}^{n} (-1)^i \iota_v A^{n-i} & \forall x = v + H \in \mathrm{Ham}^{n-1}(X) \\ -x|_{U_\alpha} & \forall x \in \Omega^{n-1-i}(X) \quad i \geqslant 1 \end{cases}$$

and whose higher components f_k are explicitly determined by Eq. A.6.

Proof. The linear morphism f_1 is essentially the familiar quasi-isomorphism between the de Rham complex and the total Čech–de Rham complex. Proving that f_1 lifts to an L_∞-morphism and explicitly determining the higher components of this L_∞-morphism is a lengthy but straightforward computation. We report it in Appendix A. □

Remark 4.7. By homological perturbation theory [**19**], one knows that there must exist some L_∞ algebra structure on the chain complex

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \mathrm{Ham}^{n-1}(X)$$

making it an L_∞-algebra quasi-isomorphic to the dgla $\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})$. The remarkable information provided by Theorem 4.6 is that this L_∞ algebra structure is identified with that provided by Proposition 3.2.

Corollary 4.8. *The image of the natural projection* $\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A}) \to \mathfrak{X}(X)$ *is the subspace* $\mathfrak{X}_{\mathrm{Ham}}(X)$ *of Hamiltonian vector fields. That is, the infinitesimal quantomorphisms cover infinitesimal Hamiltonian* n*-plectomorphisms.*

Remark 4.9. Theorem 4.6 implies that $\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})$ is independent, up to equivalence, of the choice of prequantization $\bar{A}$ of ω. It also says that $\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})$ is a "rectification" or "semi-strictification" of the L_∞-algebra $L_\infty(X, \omega)$.

5. Inclusion into Atiyah and Courant L_∞-algebras

If (X, ω) is a prequantized symplectic manifold, and (P, A) is the corresponding prequantum bundle, then there is an embedding, induced by the morphism given in Theorem 4.6, of the Lie algebra of observables on X into the Lie algebra of $U(1)$-invariant vector fields on P. The latter is the Lie algebra of global sections of the Atiyah algebroid of P (see, for example, [**33**, Section 2] and Definition 5.1, below). The integrated analogue of this embedding is a canonical map from the group of quantomorphisms to the group of bisections [**10**, Chap. 15] of the Lie groupoid that integrates the Atiyah algebroid. This groupoid is usually called the gauge groupoid of P, but we prefer to call it the "Atiyah groupoid." Likewise, we call its group of bisections the "Atiyah group." Such a bisection is just an equivariant diffeomorphism of P covering a diffeomorphism of the base X, and hence it "forgets" the connection 1-form A.

In analogy with the above, we now explain how similar embeddings of quantomorphisms naturally arise in the higher case. This provides the motivation for the Lie-theoretic results presented in this section.

5.0.1. Higher Atiyah groups and the Courant n-group

Recall from Section 2.2 that the n-stack $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is presented via the Dold–Kan correspondence by the presheaf of chain complexes

$$C^\infty(-; U(1)) \xrightarrow{d\log} \Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots\cdots \xrightarrow{d} \Omega^n(-)$$

with $\Omega^n(-)$ in degree 0. We can also consider the n-stack $\mathbf{B}\big(\mathbf{B}^{n-1}U(1)_{\mathrm{conn}}\big)$, which is the delooping of the $(n-1)$ stack $\mathbf{B}^{n-1}U(1)_{\mathrm{conn}}$. It is presented by the presheaf

$$C^\infty(-; U(1)) \xrightarrow{d\log} \Omega^1(-) \xrightarrow{d} \Omega^2(-) \xrightarrow{d} \cdots\cdots \xrightarrow{d} \Omega^{n-1}(-) \to 0$$

with $\Omega^{n-1}(-)$ in degree 1. In general, there is more, namely, a commutative diagram

$$\begin{array}{ccccccccc}
C^\infty(-;U(1)) & \xrightarrow{d\log} & \Omega^1(-) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1}(-) & \xrightarrow{d} & \Omega^n(-) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C^\infty(-;U(1)) & \xrightarrow{d\log} & \Omega^1(-) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1}(-) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \cdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
C^\infty(-;U(1)) & \longrightarrow & 0 & \xrightarrow{d} & \cdots & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

corresponding to a sequence of natural forgetful morphisms of stacks

$$\mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}(\mathbf{B}^{n-1}U(1)_{\mathrm{conn}}) \to \mathbf{B}^2(\mathbf{B}^{n-2}U(1)_{\mathrm{conn}}) \to \cdots \to \mathbf{B}^n U(1),$$

where at each step the top differential form data for the connection are forgotten.

If $\nabla : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ is the morphism representing a $U(1)$-n-bundle with connection on a smooth manifold X, then the forgetful morphisms realize X both as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{\mathrm{conn}})$ and as an object over $\mathbf{B}^n U(1)$. Therefore, we have a sequence of automorphism n-groups of ∇

$$\mathbf{Aut}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}(\nabla) \to \mathbf{Aut}_{/\mathbf{B}(\mathbf{B}^{n-1}U(1)_{\mathrm{conn}})}(\nabla) \to \mathbf{Aut}_{/\mathbf{B}^n U(1)}(\nabla)\,.$$

All of these automorphism n-groups have a smooth structure and are "concretified" in the sense of Remark 4.4. We call the n-group $\mathbf{Aut}_{/\mathbf{B}^n U(1)}(\nabla)$ the "Atiyah n-group" of ∇, since for the case $n = 1$, it is the previously mentioned Atiyah group. We call $\mathbf{Aut}_{/\mathbf{B}(\mathbf{B}^{n-1}U(1)_{\mathrm{conn}})}(\nabla)$ the "Courant n-group" of ∇, since for the $n = 2$ case it can be thought of as the object that integrates the Courant Lie 2-algebra. (see Definition 5.6, below). A more detailed discussion of these objects as the bisections of smooth ∞-groupoids appears in [**12**].

Conceptually speaking, the infinitesimal analogue of the above sequence of n-groups is a sequence of Lie n-algebras

$$\mathrm{Lie}\mathbf{QuantMorph}(\nabla) \to \mathrm{Lie}\mathbf{Courant}(\nabla) \to \mathrm{Lie}\mathbf{Atiyah}(\nabla), \tag{5.0.2}$$

where $\mathrm{Lie}\mathbf{QuantMorph}(\nabla)$ is the Lie n-algebra of infinitesimal quantomorphisms described in the beginning of Section 4.2. The elements of $\mathrm{Lie}\mathbf{Atiyah}(\nabla)$ are to be thought of as those infinitesimal autoequivalences that preserve only the underlying $U(1)$-n-bundle of ∇, while $\mathrm{Lie}\mathbf{Courant}(\nabla)$ consists of those infinitesimal autoequivalences that preserve all of the connection data on the n-bundle except the highest degree part.

Recall that we modeled the Lie n-algebra $\mathrm{Lie}\mathbf{QuantMorph}(\nabla)$ by using the dg Lie algebra $\mathrm{dgLie}_{\mathrm{Qu}}(X, \bar{A})$ given in Def./Prop. 4.5. Similarly, we define below dg Lie algebras which can be thought of as models for $\mathrm{Lie}\mathbf{Atiyah}(\nabla)$ and $\mathrm{Lie}\mathbf{Courant}(\nabla)$ for the $n = 1$ and $n = 2$ cases. We consider these particular cases in order to relate our results to the traditional theory of prequantum $U(1)$-bundles and also more recent work on Courant algebroids and $U(1)$-bundle gerbes.

5.1. The $n = 1$ case

Here (X, ω) is an ordinary presymplectic manifold, and the algebra of local observables $L_\infty(X, \omega)$ (Definition 3.3) is the underlying Lie algebra of the Poisson algebra of Hamiltonian functions. A prequantization ∇ is an ordinary $U(1)$-principal bundle with connection over X.

From any closed 2-form ω, one can construct a Lie algebroid over X whose global sections form the following Lie algebra (see, for example, [**33**, Section 2]):

Definition 5.1. Let (X, ω) be a presymplectic manifold. The *Atiyah Lie algebra* $\mathfrak{atiyah}(X, \omega)$ is the vector space $\mathfrak{X}(X) \oplus C^\infty(X; \mathbb{R})$ endowed with the Lie bracket

$$[\![v_1 + c_1, v_2 + c_2]\!]_{\mathfrak{at}} = [v_1, v_2] + \mathcal{L}_{v_1} c_2 - \mathcal{L}_{v_2} c_1 - \omega(v_1, v_2).$$

Obviously, the underlying vector space of the Lie algebra $L_\infty(X,\omega)$ is a subspace of $\mathfrak{atiyah}(X,\omega)$. A straightforward calculation shows that the inclusion

$$L_\infty(X,\omega) \hookrightarrow \mathfrak{atiyah}(X,\omega) \tag{5.1.1}$$

is also a Lie algebra morphism. Just like in our construction of the dgla $\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$ (Def./Prop. 4.5), we now represent the prequantization ∇ by a Čech–Deligne 1-cocycle (2.2) and obtain a model for $\mathrm{Lie}\mathbf{Atiyah}(\nabla)$.

Definition 5.2. If $\bar{A} = A^1 + A^0$ is a Čech–Deligne 1-cocycle on X relative to some cover $\mathcal{U}$, then $\mathrm{Lie}_{\mathrm{At}}(X,\bar{A})$ is the Lie algebra whose underlying vector space is

$$\mathrm{LieAlg}_{\mathrm{At}}(X,\bar{A}) = \{v + \bar{\theta} \in \mathfrak{X}(X) \oplus \check{C}^0(\mathcal{U},\Omega^0) \mid \mathcal{L}_v A^0 = \delta\bar{\theta}\}$$

with Lie bracket $[\![v_1 + \bar{\theta}_1, v_2 + \bar{\theta}_2]\!]_{\mathrm{At}} = [v_1,v_2] + \mathcal{L}_{v_1}\bar{\theta}_2 - \mathcal{L}_{v_2}\bar{\theta}_1$.

Since $\mathcal{L}_v A^0 = \iota_v d\log A^0$, it is easy to see that an element of $\mathrm{LieAlg}_{\mathrm{At}}(X,\bar{A})$ is the local data corresponding to a $U(1)$-invariant vector field on the total space P of the prequantum bundle, i.e., a global section of the Atiyah algebroid $TP/U(1) \to X$. Moreover, by construction, there is an inclusion of Lie algebras

$$\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A}) \hookrightarrow \mathrm{LieAlg}_{\mathrm{At}}(X,\bar{A}) \tag{5.1.2}$$

exhibiting the infinitesimal quantomorphisms as the Lie subalgebra of vector fields on P that preserve the connection.

The following proposition describes the relationship between $\mathrm{LieAlg}_{\mathrm{At}}(X,\bar{A})$ and $\mathfrak{atiyah}(X,\omega)$, which one can think of as an extension of Theorem 4.6 for the $n = 1$ case.

Proposition 5.3. *There exists a natural Lie algebra isomorphism*

$$\psi\colon \mathfrak{atiyah}(X,\omega) \xrightarrow{\cong} \mathrm{LieAlg}_{\mathrm{At}}(X,\bar{A})$$

such that the following diagram commutes

$$\begin{array}{ccc}
\mathfrak{atiyah}(X,\omega) & \xrightarrow{\psi} & \mathrm{LieAlg}_{\mathrm{At}}(X,\bar{A}) \\
\uparrow & & \uparrow \\
L_\infty(X,\omega) & \xrightarrow{f} & \mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})
\end{array}$$

where $f\colon L_\infty(M,\omega) \xrightarrow{\cong} \mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$ *is the isomorphism of Lie algebras given in Theorem 4.6, and the vertical morphisms are the inclusions* (5.1.1) *and* (5.1.2).

Proof. It follows from Theorem 4.6 that the isomorphism $f\colon L_\infty(M,\omega) \xrightarrow{\cong} \mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$ is $f(v+c) = v - c|_{U_\alpha} + \iota_v A^1$. Hence, we define $\psi\colon \mathfrak{atiyah}(X,\omega) \to \mathrm{LieAlg}_{\mathrm{At}}(X,\bar{A})$ to be $\psi(v+c) = v - c|_{U_\alpha} + \iota_v A^1$. Note that if $\mathcal{L}_v A^0 = \delta\bar{\theta}$, then $\delta(\bar{\theta} + \iota_v A^1) = 0$. Hence ψ is an isomorphism of vector spaces. The fact that ψ preserves the Lie brackets follows from the equalities $\mathcal{L}_{v_1}\iota_{v_2}A^1 - \mathcal{L}_{v_2}\iota_{v_1}A^1 = \iota_{[v_1,v_2]}A^1 + \iota_{v_2}\iota_{v_1}dA^1 = \iota_{[v_1,v_2]}A^1 + \iota_{v_1\wedge v_2}\omega$. ∎

Remark 5.4. Note that the isomorphism ψ in the above proposition uses the connection A to lift horizontally a vector field on M to a vector field on P in the standard way.

5.2. The $n = 2$ case

Here (X, ω) is a pre–2-plectic manifold. A prequantization ∇ of (X, ω) is a $U(1)$-bundle gerbe (or principal $U(1)$ 2-bundle) over X equipped with a 2-connection.

In addition to the Lie 2-algebra of local observables $L_\infty(X, \omega)$, there are two other Lie 2-algebras one can build directly from any closed 3-form ω. It seems that the first of these has not appeared previously in the literature, while the second one originates in Roytenberg and Weinstein's work on Courant algebroids [**35**]. (The 2-term truncation we use here is due to subsequent work by Roytenberg [**34**].)

Definition/Proposition 5.5. Let ω be a pre-2-plectic structure on X. The *Atiyah Lie 2-algebra* $\mathfrak{atiyah}(\omega)$ is the graded vector space

$$\mathfrak{atiyah}(X, \omega)^0 = \mathfrak{X}(X); \qquad \mathfrak{atiyah}(X, \omega)^1 = \Omega^0(X);$$

endowed with the brackets

$$[\![\eta]\!]_1^{\mathfrak{a}} = 0; \qquad [\![v_1, v_2]\!]_2^{\mathfrak{a}} = [v_1, v_2]; \qquad [\![v, \eta]\!]_2^{\mathfrak{a}} = \mathcal{L}_v \eta; \qquad [\![v_1, v_2, v_3]\!]_3^{\mathfrak{a}} = -\iota_{v_1 \wedge v_2 \wedge v_3} \omega$$

(with all other brackets zero by degree reasons).

Definition/Proposition 5.6. Let ω be a pre-2-plectic structure on X. The *Courant Lie 2-algebra* $\mathfrak{courant}(\omega)$ is the graded vector space

$$\mathfrak{courant}(X, \omega)^0 = \mathfrak{X}(X) \oplus \Omega^1(X); \qquad \mathfrak{courant}(X, \omega)^1 = \Omega^0(X);$$

endowed with the brackets

$$[\![\eta]\!]_1^{\mathfrak{c}} = d\eta; \qquad [\![v + \theta, \eta]\!]_2^{\mathfrak{c}} = \tfrac{1}{2} \iota_v d\eta$$

$$[\![v_1 + \theta_1, v_2 + \theta_2]\!]_2^{\mathfrak{c}} = [v_1, v_2] + \mathcal{L}_{v_1} \theta_2 - \mathcal{L}_{v_2} \theta_1 - \tfrac{1}{2} d\big(\iota_{v_1} \theta_2 - \iota_{v_2} \theta_1\big) - \iota_{v_1 \wedge v_2} \omega$$

$$[\![v_1 + \theta_1, v_2 + \theta_2, v_3 + \theta_3]\!]_3^{\mathfrak{c}} = -\tfrac{1}{6} \Big(\langle [\![v_1 + \theta_1, v_2 + \theta_2]\!]_2^{\mathfrak{c}}, v_3 + \theta_3 \rangle + \text{cyc. perm.} \Big)$$

where $\langle\, ,\, \rangle$ is the natural symmetric pairing between sections of $T^*X \oplus TX$, i.e., $\langle v_1 + \theta_1, v_2 + \theta_2 \rangle := \iota_{v_1} \theta_2 + \iota_{v_2} \theta_1$ (and with all other brackets zero by degree reasons).

The relationship between these Lie 2-algebras is given by the next proposition.

Proposition 5.7. *There exists a natural sequence of L_∞ morphisms*

$$L_\infty(X, \omega) \xrightarrow{\phi} \mathfrak{courant}(X, \omega) \xrightarrow{\psi} \mathfrak{atiyah}(X, \omega),$$

where the nontrivial components of the morphism ϕ are

$$\phi_1(v + \theta) = v + \theta; \qquad \phi_1(\eta) = \eta; \qquad \phi_2(v_1 + \theta_1, v_2 + \theta_2) = -\tfrac{1}{2} \left(\iota_{v_1} \theta_2 - \iota_{v_2} \theta_1 \right)$$

and the nontrivial components of the morphism ψ are

$$\psi_1(v + \theta) = v; \qquad \psi_1(\eta) = \eta; \qquad \psi_2(v_1 + \theta_1, v_2 + \theta_2) = -\tfrac{1}{2} \left(\iota_{v_1} \theta_2 - \iota_{v_2} \theta_1 \right).$$

Proof. The fact that ϕ is an L_∞-morphism is the content of Theorem 7.1 in [**33**]. To show ψ is a L_∞ morphism, we first perform several straightforward computations using the Cartan calculus in order to obtain the following equalities:

$$\psi_2(d\eta, v+\theta) = \psi_1\big([\![\eta, v+\theta]\!]^{\mathfrak{c}}\big) - [\![\psi_1(\eta), \psi_1(v+\theta)]\!]_2^{\mathfrak{a}};$$
$$[\![v_1+\theta_1, v_2+\theta_2, v_3+\theta_3]\!]_3^{\mathfrak{c}} = -\tfrac{1}{4}\big(\iota_{v_3}\mathcal{L}_{v_1}\theta_2 - \iota_{v_3}\mathcal{L}_{v_2}\theta_1 + \text{cyc. perm.}\big) + \tfrac{1}{2}\iota_{v_1\wedge v_2\wedge v_3}\omega;$$
$$\psi_2\big([\![v_1+\theta_1, v_2+\theta_2]\!]_2^{\mathfrak{c}}, v_3+\theta_3\big) + \text{cyc. perm.} = -\tfrac{1}{4}\big(\iota_{v_3}\mathcal{L}_{v_1}\theta_2 - \iota_{v_3}\mathcal{L}_{v_2}\theta_1 + \text{cyc. perm.}\big) - \big(\iota_{v_1\wedge v_2}d\theta_3 + \text{cyc. perm.}\big) - \tfrac{3}{2}\iota_{v_1\wedge v_2\wedge v_3}\omega;$$
$$[\![\psi_1(v_1+\theta_1), \psi_2(v_2+\theta_2, v_3+\theta_3)]\!]_2^{\mathfrak{a}} + \text{cyc. perm.} = -\tfrac{1}{2}\big(\iota_{v_3}\mathcal{L}_{v_1}\theta_2 - \iota_{v_3}\mathcal{L}_{v_2}\theta_1 + \text{cyc. perm.}\big) - \big(\iota_{v_1\wedge v_2}d\theta_3 + \text{cyc. perm.}\big).$$

We then use the above to verify that the equalities given in [**1**, Definition 34] are satisfied. □

If (X,ω) is prequantized, then we represent the prequantum 2-bundle $\nabla : X \to \mathbf{B}^2U(1)_{\text{conn}}$ with a Čech–Deligne 2-cocycle and obtain dg Lie algebras that we think of as modeling the previously discussed L_∞-algebras Lie**Atiyah**(∇) and Lie**Courant**(∇). In what follows, $\Omega^{\leqslant 0}$ and $\Omega^{\leqslant 1}$ denote the cochain complexes of sheaves

$$\Omega^0(-) \to 0 \to 0 \to 0 \to \cdots; \qquad \Omega^0(-) \xrightarrow{d} \Omega^1(-) \to 0 \to 0 \to \cdots,$$

respectively, with both having $\Omega^0(-)$ in degree 0.

Definition/Proposition 5.8. If $\bar{A} = A^2 + A^1 + A^0$ is a Čech–Deligne 2-cocycle on X relative to some cover $\mathcal{U}$, then we denote by $\text{dgLie}_{\text{At}}(X,\bar{A})$ and $\text{dgLie}_{\text{Cou}}(X,\bar{A})$ the dg Lie algebras whose underlying complexes are

$$\text{dgLie}_{\text{At}}(X,\bar{A})^0 = \{v+\bar{\theta} \in \mathfrak{X}(X) \oplus \text{Tot}^1(\mathcal{U},\Omega^{\leqslant 0}) \mid \mathcal{L}_v A^0 = d_{\text{Tot}}\bar{\theta}\}$$
$$\text{dgLie}_{\text{At}}(X,\bar{A})^1 = \text{Tot}^0(\mathcal{U},\Omega^{\leqslant 0})$$

and

$$\text{dgLie}_{\text{Cou}}(X,\bar{A})^0 = \{v+\bar{\theta} \in \mathfrak{X}(X) \oplus \text{Tot}^1(\mathcal{U},\Omega^{\leqslant 1}) \mid \mathcal{L}_v(A^1 + A^0) = d_{\text{Tot}}\bar{\theta}\}$$
$$\text{dgLie}_{\text{Cou}}(X,\bar{A})^1 = \text{Tot}^0(\mathcal{U},\Omega^{\leqslant 1}),$$

both equipped with the differential $0 \oplus d_{\text{Tot}}$, and whose graded Lie brackets are (for both cases)

$$\begin{aligned} &[\![v_1+\bar{\theta}_1, v_2+\bar{\theta}_2]\!] = [v_1,v_2] + \mathcal{L}_{v_1}\bar{\theta}_2 - \mathcal{L}_{v_2}\bar{\theta}_1 \\ &[\![v+\bar{\theta}, \bar{\eta}]\!] = -[\![\bar{\eta}, v+\bar{\theta}]\!] = \mathcal{L}_v\bar{\eta}; \qquad [\![\bar{\eta},\bar{\eta}]\!] = 0. \end{aligned} \tag{5.2.1}$$

The dg Lie algebra $\text{dgLie}_{\text{At}}(X,\bar{A})$ was constructed by Collier [**11**, Definition 6.11, Theorem 8.18], and he rigorously proved that its degree 0 elements correspond to infinitesimal autoequivalences of the $U(1)$ 2-bundle represented by the Čech 2-cocycle A^0. He also constructed $\text{dgLie}_{\text{Cou}}(X,\bar{A})$ and proved that its degree 0 elements are the infinitesimal autoequivalences the $U(1)$ 2-bundle equipped with a connective structure represented by the truncated Čech–Deligne 2-cocycle $A^1 + A^0$ [**11**, Definition 10.38, Proposition 10.48]. There is an obvious map of dg Lie algebras $\text{dgLie}_{\text{Cou}}(X,\bar{A}) \xrightarrow{p} \text{dgLie}_{\text{At}}(X,\bar{A})$, which in degree 0 forgets the $\check{C}^0(\mathcal{U},\Omega^1)$ component. It is also clear that the dg Lie algebra $\text{dgLie}_{\text{Qu}}(X,\bar{A})$ of infinitesimal quantomorphisms (Def./Prop. 4.5) embeds into $\text{dgLie}_{\text{Cou}}(X,\bar{A})$. Hence, the next result follows automatically by construction.

Proposition 5.9. *There is a natural sequence of dg Lie algebras*

$$\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A}) \xrightarrow{i} \mathrm{dgLie}_{\mathrm{Cou}}(X,\bar{A}) \xrightarrow{p} \mathrm{dgLie}_{\mathrm{At}}(X,\bar{A})$$

that we interpret as modeling the sequence (5.0.2).

In [**11**, Theorem 12.50], Collier constructed a weak equivalence of Lie 2-algebras between a local Čech description of the Courant Lie 2-algebra (5.6) and the dg Lie algebra $\mathrm{dgLie}_{\mathrm{Cou}}(X,\bar{A})$. We conclude with the following proposition which strengthens this result by incorporating our Theorem 4.6 and Proposition 5.7. It can also be viewed as the higher analogue of Proposition 5.3.

Proposition 5.10. *If (X,ω) is a prequantized pre-2-plectic manifold, then there exist natural weak equivalences of Lie 2-algebras $f^{\mathfrak{a}}\colon \mathfrak{atiyah}(X,\omega) \xrightarrow{\sim} \mathrm{dgLie}_{\mathrm{At}}(X,\bar{A})$ and $f^{\mathfrak{c}}\colon \mathfrak{courant}(X,\omega) \xrightarrow{\sim} \mathrm{dgLie}_{\mathrm{Cou}}(X,\bar{A})$ such that the following diagram of L_∞-algebras (strictly) commutes:*

$$\begin{array}{ccc}
\mathfrak{atiyah}(X,\omega) & \xrightarrow{f^{\mathfrak{a}}} & \mathrm{dgLie}_{\mathrm{At}}(X,\bar{A}) \\
\uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle p} \\
\mathfrak{courant}(X,\omega) & \xrightarrow{f^{\mathfrak{c}}} & \mathrm{dgLie}_{\mathrm{Cou}}(X,\bar{A}) \\
\uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle i} \\
L_\infty(X,\omega) & \xrightarrow{f} & \mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})
\end{array}$$

where $f\colon L_\infty(M,\omega) \xrightarrow{\sim} \mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$ is the weak equivalence given in Theorem 4.6, and the vertical morphisms are those given in Proposition 5.7 and Proposition 5.9.

Proof. In terms of the notation above, Proposition A.1 and equations A.6 imply that the weak equivalence $f\colon L_\infty(M,\omega) \xrightarrow{\sim} \mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$ has nontrivial components

$$f_1(v+\theta) = v - \theta + \iota_v(A^2 - A^1); \qquad f_1(\eta) = -\eta;$$
$$f_2(v_1+\theta_1, v_2+\theta_2) = \iota_{v_1}\theta_2 - \iota_{v_2}\theta_1 + \iota_{v_1\wedge v_2}A^2.$$

(Above we have suppressed the restriction of global forms on X to open sets $U_\alpha \in \mathcal{U}$.) Hence, we define the non-trivial components of $f^{\mathfrak{c}}$ to be

$$f_1^{\mathfrak{c}}(v+\theta) = v - \theta + \iota_v(A^2 - A^1); \qquad f_1^{\mathfrak{c}}(\eta) = -\eta;$$
$$f_2^{\mathfrak{c}}(v_1+\theta_1, v_2+\theta_2) = \tfrac{1}{2}\big(\iota_{v_1}\theta_2 - \iota_{v_2}\theta_1\big) + \iota_{v_1\wedge v_2}A^2.$$

Similarly, we define $f^{\mathfrak{a}}$ by

$$f_1^{\mathfrak{a}}(v) = v - \iota_v A^1; \qquad f_1^{\mathfrak{a}}(\eta) = -\eta; \qquad f_2^{\mathfrak{a}}(v_1, v_2) = \iota_{v_1\wedge v_2}A^2.$$

Note that if $v+\bar{\theta}$ is a degree 0 element of $\mathrm{dgLie}_{\mathrm{Cou}}(X,\bar{A})$, then $d_{\mathrm{Tot}}(\bar{\theta} - \iota_v(A^2 - A^1)) = 0$. Similarly, if $v+\bar{\theta}$ is a degree 0 element of $\mathrm{dgLie}_{\mathrm{At}}(X,\bar{A})$, then $d_{\mathrm{Tot}}(\bar{\theta} + \iota_v A^1) = 0$. It then follows from the Poincaré lemma that both $f_1^{\mathfrak{c}}$ and $f_1^{\mathfrak{a}}$ are quasi-isomorphisms of chain complexes.

It follows immediately from the definitions (Proposition 5.7 and Proposition 5.9) that $f_1^{\mathfrak{c}} \circ \phi_1 = i \circ f_1$ and $f_1^{\mathfrak{a}} \circ \psi_1 = p \circ f_1^{\mathfrak{c}}$. Simple calculations show that the following equations hold:

$$\begin{aligned}(f^{\mathfrak{c}}\circ\phi)_2(v_1+\theta_1,v_2+\theta_2) &:= f_1^{\mathfrak{c}}\phi_2(v_1+\theta_1,v_2+\theta_2)+f_2^{\mathfrak{c}}(\phi_1(v_1+\theta_1),\phi_1(v_2+\theta_2))\\ &= i\circ f_2(v_1+\theta_1,v_2+\theta_2),\\ (f^{\mathfrak{a}}\circ\psi)_2(v_1+\theta_1,v_2+\theta_2) &:= f_1^{\mathfrak{a}}\psi_2(v_1+\theta_1,v_2+\theta_2)+f_2^{\mathfrak{a}}(\psi_1(v_1+\theta_1),\psi_1(v_2+\theta_2))\\ &= p\circ f_2^{\mathfrak{c}}(v_1+\theta_1,v_2+\theta_2).\end{aligned}$$

Hence, the above diagram commutes. Next, using the identities from Section 1.0.1 and the cocycle equation for $\bar{A}$, we obtain the following equalities:

$$\begin{aligned}&[\![f_1^{\mathfrak{c}}(v_1+\theta_1),f_1^{\mathfrak{c}}(v_2+\theta_2)]\!]^{\mathrm{Cou}} = [v_1,v_2]-\mathcal{L}_{v_1}\theta_2+\mathcal{L}_{v_2}\theta_1+\iota_{[v_1,v_2]}(A^2-A^1)\\ &\qquad\qquad+\iota_{v_1\wedge v_2}\omega-d_{\mathrm{Tot}}(\iota_{v_1\wedge v_2}A^2)\\ &f_2^{\mathfrak{c}}\big([\![v_1+\theta_1,v_2+\theta_2]\!]_2^{\mathfrak{c}},v_3+\theta_3\big)+\text{cyc. perm.} = \tfrac{1}{4}\big(\iota_{v_3}\mathcal{L}_{v_1}\theta_2-\iota_{v_3}\mathcal{L}_{v_2}\theta_1+\text{cyc. perm.}\big)\\ &\qquad\qquad+\big(\iota_{v_1\wedge v_2}d\theta_3+\iota_{[v_1,v_2]\wedge v_3}A^2+\text{cyc. perm.}\big)+\tfrac{3}{2}\iota_{v_1\wedge v_2\wedge v_3}\omega\\ &[\![f_1^{\mathfrak{c}}(v_1+\theta_1),f_2^{\mathfrak{c}}(v_2+\theta_2,v_3+\theta_3)]\!]_2^{\mathrm{Cou}}+\text{cyc. perm.} = \tfrac{1}{2}\big(\iota_{v_3}\mathcal{L}_{v_1}\theta_2-\iota_{v_3}\mathcal{L}_{v_2}\theta_1+\text{cyc. perm.}\big)\\ &\qquad\qquad+\big(\iota_{v_1\wedge v_2}d\theta_3+2\iota_{[v_1,v_2]\wedge v_3}A^2+\iota_{v_2\wedge v_3}\mathcal{L}_{v_1}A^2+\text{cyc. perm.}\big).\end{aligned}$$

And similarly for $f^{\mathfrak{a}}$:

$$\begin{aligned}[\![f_1^{\mathfrak{a}}(v_1),f_1^{\mathfrak{a}}(v_2)]\!]^{\mathrm{At}} &= [v_1,v_2]-\iota_{[v_1,v_2]}A^1-d_{\mathrm{Tot}}(\iota_{v_1\wedge v_2}A^2)\\ f_2^{\mathfrak{a}}\big([\![v_1,v_2]\!]_2^{\mathfrak{a}},v_3\big)+\text{cyc. perm.} &= \iota_{[v_1,v_2]\wedge v_3}A^2+\text{cyc. perm.}\\ [\![f_1^{\mathfrak{a}}(v_1),f_2^{\mathfrak{a}}(v_2,v_3)]\!]^{\mathrm{At}}+\text{cyc. perm.} &= \big(2\iota_{[v_1,v_2]\wedge v_3}+\iota_{v_2\wedge v_3}\mathcal{L}_{v_1}\big)A^2+\text{cyc. perm..}\end{aligned}$$

Using these in conjunction with Lemma 3.1, it is easy to verify that $f^{\mathfrak{c}}$ and $f^{\mathfrak{a}}$ are L_∞ morphisms (see, e.g., [**1**, Definition 34]). □

Appendix A. An explicit weak equivalence between $L_\infty(X,\omega)$ and $\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$

In this section, we prove Theorem 4.6. Namely, given a pre–n-plectic manifold (X,ω) and a prequantization presented by a Čech–Deligne n-cocycle $\bar{A}$ with respect to a cover $\mathcal{U}=\{U_\alpha\}$ of X, we shall construct an L_∞-quasi-isomorphism $L_\infty(X,\omega)\xrightarrow{\sim}\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$. We use the following conventions to help simplify calculations:

- We denote by $\mathrm{res}\colon\Omega^\bullet(X)\to\check{C}^0(\mathcal{U},\Omega^\bullet)$ the restriction map $\mathrm{res}(\theta)_\alpha=\theta|_{U_\alpha}\in\Omega^\bullet(U_\alpha)$. For $\bar{A}=\sum_{i=0}^n A^{n-i}$ a Čech–Deligne cocycle, we define for all $m\geqslant 1$
$$\bar{A}(m):=\sum_{i=0}^{n}(-1)^{mi}A^{n-i}. \tag{A.2}$$
- (L,l_1) denotes the underlying complex of the Lie n-algebra $L_\infty(X,\omega)$ introduced in Definition 3.3:
$$\begin{aligned}L_0&=\{v+H\in\mathfrak{X}_{\mathrm{Ham}}(X)\oplus\Omega^{n-1}(X)\mid dH=-\iota_v\omega\}\\ L_i&=\Omega^{n-1-i}(X)\quad i\geqslant 1\end{aligned}$$
with differential
$$l_1\theta=\begin{cases}0+d\theta, & |\theta|=1\\ d\theta, & |\theta|>1.\end{cases}$$

The higher k-ary brackets of $L_\infty(X,\omega)$ are denoted by $l_2,\ldots,l_{n+1}$.

- (L', l'_1) denotes the underlying complex of the dgla $\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$ introduced in Definition 4.5:

$$L'_0 = \{v+\bar{\theta} \in \mathfrak{X}(X) \oplus \mathrm{Tot}^{n-1}(\mathcal{U},\Omega) \mid \mathcal{L}_v \bar{A} = d_{\mathrm{Tot}}\bar{\theta}\}$$
$$L'_i = \mathrm{Tot}^{n-1-i}(\mathcal{U},\Omega) \quad i \geqslant 1,$$

with differential

$$l'_1\bar{\theta} = \begin{cases} 0 + d_{\mathrm{Tot}}\bar{\theta}, & |\bar{\theta}| = 1 \\ d_{\mathrm{Tot}}\bar{\theta}, & |\bar{\theta}| > 1. \end{cases}$$

The Lie bracket on $\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$ is denoted by $l'_2 = [\![\cdot,\cdot]\!]$.

- Elements of arbitrary degree in L (resp. L') will be denoted as $x_1, x_2, \ldots$ (resp. $\bar{x}_1, \bar{x}_2, \ldots$), where

$$x_i := v_i + \theta_i \quad (\text{resp. } \bar{x}_i := v_i + \bar{\theta}_i). \tag{A.3}$$

It is understood that we set $v_i = 0$ if $|x_i| > 0$ (resp. $|\bar{x}_i| > 0$). So, for example, for any $x_1,\ldots,x_k \in L$ and any $\bar{x}_1, \bar{x}_2 \in L'$ the following equalities hold:

$$l_2(x_1,x_2) = [v_1,v_2] + \iota_{v_1\wedge v_2}\omega; \qquad l_{k\geqslant 3}(x_1,\ldots,x_k) = -(-1)^{\binom{k+1}{2}}\iota_{v_1\wedge\cdots\wedge v_k}\omega,$$
$$[\![\bar{x}_1,\bar{x}_2]\!] = [v_1,v_2] + \mathcal{L}_{v_1}\bar{\theta}_2 - \mathcal{L}_{v_2}\bar{\theta}_1.$$

- For all $m \geqslant 2$ we define a map $S_m \colon L^{\otimes m} \to L$, where

$$S_m(x_1,\ldots,x_m) = \sum_{i=1}^{m} (-1)^i \iota_{v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge v_m}\theta_i. \tag{A.4}$$

It is clear from our above notation that $S_m(x_1,\ldots,x_m) = 0$ if two or more arguments have degree > 0. Note that S_m is a graded skew-symmetric map of degree $m-1$ and $S_{m>n} = 0$.

- For all $m \geqslant 1$ we define the linear maps $f_m \colon L^{\otimes m} \to L'$:

$$f_1(x) = v - \mathrm{res}(\theta) + \iota_v \bar{A}(1), \tag{A.5}$$

$$f_{2\leqslant m\leqslant n}(x_1,\ldots,x_m) = -(-1)^{\binom{m+1}{2}}\left(\mathrm{res}\circ S_m(x_1,\ldots,x_m) + \iota_{v_1\wedge\cdots\wedge v_m}\bar{A}(m)\right), \tag{A.6}$$

and $f_{m>n} = 0$. Note that each f_m is graded skew-symmetric with $|f_m| = m-1$. Below, we will often suppress the restriction map in the definitions. These are the structure maps we will use to construct an L_∞ quasi-morphism.

- Finally, we define the following auxiliary linear maps $I^m_{(1)},\ I^m_{(2)},\ I^m_{(3)} \colon L^{\otimes m} \to L'$, for all $m \geqslant 1$, where $I_2^1 = I_3^{m<3} = 0$ and

$$\begin{aligned}
I_{(1)}^m(x_1,\dots,x_m) &= \sum_{\sigma\in \mathrm{Sh}(1,m-1)} \chi(\sigma)(-1)^m f_m(l_1(x_{\sigma(1)}),\dots,x_{\sigma(m)}),\\
I_{(2)}^{m\geqslant 2}(x_1,\dots,x_m) &= -\sum_{\sigma\in \mathrm{Sh}(2,m-2)} \chi(\sigma) f_{m-1}(l_2(x_{\sigma(1)},x_{\sigma(2)}),\dots,x_{\sigma(m)}),\\
I_{(3)}^{m\geqslant 3}(x_1,\dots,x_m) &= \sum_{\substack{k=3\dots m\\ \sigma\in \mathrm{Sh}(k,m-k)}} \chi(\sigma)(-1)^{k(m-k)+1} f_{m+1-k}(l_k(x_{\sigma(1)},\dots,x_{\sigma(k)}),\dots,x_{\sigma(m)}).
\end{aligned} \tag{A.7}$$

Above, $\chi(\sigma) = (-1)^\sigma \epsilon(\sigma)$, where $\epsilon(\sigma)$ is the Koszul sign of the permutation. We also define for all $m \geqslant 1$ maps $J^m\colon L^{\otimes m} \to L'$, where $J^1 = 0$ and

$$J^{m\geqslant 2}(x_1,\dots,x_m) = \sum_{\substack{s+t=m\\ \tau\in \mathrm{Sh}(s,m-s)\\ \tau(1)<\tau(s+1)}} \chi(\tau)(-1)^{s-1}(-1)^{(t-1)\sum_{p=1}^{s}|x_{\tau(p)}|} \left[\!\left[f_s(x_{\tau(1)},\dots,x_{\tau(s)}), f_t(x_{\tau(s+1)},\dots,x_{\tau(m)})\right]\!\right]. \tag{A.8}$$

Proposition A.1. *The linear map* (A.5) $f_1\colon L \to L'$ *is a quasi-isomorphism of chain complexes.*

Proof. It is clear from the definition that f_1 is a chain map. Since $\bar{A}$ is a Čech–Deligne cocycle, and since the interior product ι_v commutes with the Čech differential, we have $d_{\mathrm{Tot}}\iota_v\bar{A}(1) = d\iota_v A^n + \sum_{i=1}^n \mathcal{L}_v A^{n-i}$. This implies that $v+\bar\theta \in L'_0$ if and only if $d_{\mathrm{Tot}}\big(\bar\theta - \iota_v\bar{A}(1)\big) = \mathrm{res}(\iota_v\omega)$. Let $\tilde{L}$ be the complex whose underlying graded vector space is

$$\tilde{L}_0 = \{v+\bar\theta \in \mathfrak{X}(X)\oplus \mathrm{Tot}^{n-1}(\mathcal{U},\Omega) \mid d_{\mathrm{Tot}}\bar\theta = \mathrm{res}(\iota_v\omega)\}; \qquad \tilde{L}_i = L'_i \quad i>0,$$

and whose differential is $\tilde{l}_1 = l'_1$, the same differential as on L'. The chain map f_1 then is equal to the composition: $L \xrightarrow{r} \tilde{L} \xrightarrow{\phi} L'$, where, using notation (A.3), $r(x) = v - \mathrm{res}(\theta)$, and $\phi(\bar{x}) = v + \bar\theta + \iota_v\bar{A}(1)$. Note that ϕ is a isomorphism of complexes.

Next, let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover $\mathcal{U} = \{U_\alpha\}$. Define a map $K\colon \check{C}^i(\mathcal{U},\Omega^j) \to \check{C}^{i-1}(\mathcal{U},\Omega^j)$ to be $(K\theta)_{\alpha_0,\dots,\alpha_{i-1}} = \sum_\alpha \rho_\alpha\theta_{\alpha,\alpha_0,\dots,\alpha_{i-1}}$, and let $D''\colon \check{C}^i(\mathcal{U},\Omega^j) \to \check{C}^i(\mathcal{U},\Omega^{j+1})$ be the "signed" de Rham differential $D''\theta = (-1)^i d\theta$. Then (see [**6**, Proposition 9.5]) there exists a chain map $\jmath\colon\ \mathrm{Tot}^\bullet(\mathcal{U},\Omega) \to \Omega^\bullet(X)$ such that

$$\jmath\circ \mathrm{res} = \mathrm{id}_{\Omega^\bullet(X)}, \quad \mathrm{id}_{\mathrm{Tot}^\bullet(\mathcal{U},\Omega)} - \mathrm{res}\circ\jmath = d_{\mathrm{Tot}}H + Hd_{\mathrm{Tot}}, \tag{A.9}$$

where $H\colon\ \mathrm{Tot}^\bullet(\mathcal{U},\Omega) \to \mathrm{Tot}^\bullet(\mathcal{U},\Omega)$ is the chain homotopy given as follows: if $\bar\theta = \sum_{i=0}^m \theta^{m-i}$, with $\theta^{m-i} \in \check{C}^i(\mathcal{U},\Omega^{m-i})$, then $H(\bar\theta) = \sum_{i=0}^{m-1}(H\bar\theta)_i$, where

$$(H\bar\theta)_i = \sum_{j=i+1}^m K\circ \underbrace{(-D''K)\circ(-D''K)\circ\cdots\circ(-D''K)}_{j-(i+1)}\theta^{m-j} \in \check{C}^i(\mathcal{U},\Omega^{m-1-i}). \tag{A.10}$$

Hence, the restriction map res is a quasi-isomorphism between the de Rham and Čech–de Rham complexes, and $\jmath$ is its homotopy inverse.

Let $\tilde{\jmath}\colon \tilde{L} \to L$ to be the chain map $\tilde{\jmath}(\bar{x}) = v - \jmath(\bar{\theta})$. Note that $\tilde{\jmath}$ is well defined on degree 0 elements since $d_{\mathrm{Tot}}\bar{\theta} = \mathrm{res}(\iota_v\omega)$. Let $\tilde{H}\colon \tilde{L} \to \tilde{L}$ be the degree 1 map $\tilde{H}(\bar{x}) = H(\bar{\theta})$. We now show that $\tilde{H}$ is a chain homotopy i.e., $\mathrm{id}_{\tilde{L}} - r \circ \tilde{\jmath} = \tilde{l}_1 \tilde{H} + \tilde{H}\tilde{l_1}$. Since (A.9) holds, it follows that we just need to check this on degree 0 elements. Since we have the equality $d_{\mathrm{Tot}}\bar{\theta} = \mathrm{res}(\iota_v\omega) \in \check{C}^0(\mathcal{U}, \Omega^n)$ for all $v + \bar{\theta} \in \tilde{L}_0$, it follows from the definition of H (A.10) that $H\big(d_{\mathrm{Tot}}\bar{\eta}\big) = 0$. So (A.9) implies that the above identity holds for degree 0 as well. Therefore, r is a quasi-isomorphism, and hence f_1 is a quasi-isomorphism. □

Technical lemmas

In the remainder of the appendix we show that the maps $f_{2\leqslant m\leqslant n}\colon L^{\otimes m} \to L'$ given by equation (A.6) lift the map $f_1\colon L \to L'$ to an L_∞-morphism between $L_\infty(X,\omega)$ and $\mathrm{dgLie}_{\mathrm{Qu}}(X,\bar{A})$. Proposition A.1 implies that this lift will be an L_∞-quasi-isomorphism. We present here several small computational results necessary for the proof.

Lemma A.2. *For all $m \geqslant 2$ and $x_1,\ldots,x_m \in L$, we have*

$$I^{m\geqslant 2}_{(1)}(x_1,\ldots,x_m) = -(-1)^{\binom{m+1}{2}}(-1)^m \sum_{i=1}^{m} (-1)^i \iota_{v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge v_m}(l_1\theta_i). \tag{A.11}$$

Proof. Equations (A.6) and (A.7) imply that

$$I^{m\geqslant 2}_{(1)}(x_1,\ldots,x_m) = -(-1)^{\binom{m+1}{2}}(-1)^m \sum_{i=1}^{m} (-1)^{i-1}\epsilon(\sigma(i)) S_m(l_1x_i, x_1,\ldots,\widehat{x}_i,\ldots,x_m). \tag{A.12}$$

The vector field associated to $l_1x_i = l_1(v_i+\theta_i)$ is zero; hence

$$S_m(l_1x_i, x_1,\ldots,\widehat{x}_i,\ldots,x_m) = -\iota_{v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge v_m}(l_1\theta_i),$$

and furthermore, any non-zero terms contributing to the sum (A.12) necessarily have $\epsilon(\sigma(i)) = 1$. □

Lemma A.3. *If $x_1, x_2 \in L$, then $I^2_{(2)}(x_1,x_2) = -[v_1,v_2] + \iota_{v_1\wedge v_2}\omega - \iota_{[v_1,v_2]}\bar{A}(1)$, and for all $m > 2$ and $x_1,\ldots,x_m \in L$, the following equality holds:*

$$\begin{aligned} I^{m>2}_{(2)}(x_1,\ldots,x_m) = & -(-1)^{\binom{m}{2}} \left(\binom{m}{2}\iota_{v_1\wedge\cdots\wedge v_m}\omega + \sum_{i<k}(-1)^{i+k}\iota_{[v_i,v_k]\wedge v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_k\cdots\wedge v_m}\bar{A}(m-1)\right) \\ & +(-1)^{\binom{m}{2}}\Big(\sum_{i<k<j} - \sum_{i<j<k} + \sum_{j<i<k}\Big)(-1)^{i+k+j}\iota_{[v_i,v_k]\wedge v_1\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\widehat{v}_k\cdots\wedge v_m}\theta_j. \end{aligned} \tag{A.13}$$

Proof. The $m=2$ case follows immediately from the definitions. For $m>2$, recall the definition of $I^m_{(2)}$ (A.7) and note the following equality of summations:

$$-\sum_{\sigma\in \mathrm{Sh}(2,m-2)} \chi(\sigma) = \sum_{1\leqslant i<k\leqslant m} (-1)^{i+k}\epsilon(i,k). \tag{A.14}$$

A summand contributing to $I^m_{(2)}$ is of the form

$$f_{m-1}(l_2(x_i,x_k),x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_k,\ldots x_m) = -(-1)^{\binom{m}{2}}\Big(S_{m-1}(l_2(x_i,x_k),x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_k,\ldots x_m) + \iota_{[v_i,v_k]\wedge v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_k\cdots\wedge v_m}\bar{A}(m-1)\Big). \tag{A.15}$$

The second term on the right-hand side above vanishes if $|x_i|>0$ for any i; hence taking the summation (A.14) of all such terms gives

$$\sum_{1\leqslant i<k\leqslant m}(-1)^{i+k}\iota_{[v_i,v_k]\wedge v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_k\cdots\wedge v_m}\bar{A}(m-1). \tag{A.16}$$

Using (A.4), we rewrite the first term on the right-hand side of (A.15) as

$$S_{m-1}(l_2(x_i,x_k),x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_k,\ldots x_m) = (-1)^{i+k}\iota_{v_1\wedge\cdots\wedge v_m}\omega + \Big(-\sum_{j=1}^{i}+\sum_{j=i+1}^{k-1}-\sum_{j=k+1}^{m}\Big)(-1)^j\iota_{[v_i,v_k]\wedge v_1\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\widehat{v}_k\cdots\wedge v_m}\theta_j. \tag{A.17}$$

The first term on the right-hand side of (A.17) vanishes if $|x_i|>0$ for any i. The second term vanishes if more than one x_i has degree >0. Hence, the summation (A.14) of the terms (A.17) is

$$\binom{m}{2}\iota_{v_1\wedge\cdots\wedge v_m}\omega + \Big(-\sum_{i<k<j}+\sum_{i<j<k}-\sum_{j<i<k}\Big)(-1)^{i+k+j}\iota_{[v_i,v_k]\wedge v_1\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\widehat{v}_k\cdots\wedge v_m}\theta_j. \tag{A.18}$$

Combining the above with (A.16) completes the proof. ☐

Lemma A.4. *For all $m\geqslant 3$ and $x_1,\ldots,x_m\in L$, the following equality holds:*

$$I^{m\geqslant 3}_{(3)}(x_1,\ldots,x_m) = (-1)^{\binom{m+1}{2}}(-1)^m\left(\binom{m}{2}-m+1\right)\iota_{v_1\wedge\cdots\wedge v_m}\omega. \tag{A.19}$$

Proof. Let $\sigma\in\mathrm{Sh}(k,m-k)$. We have the following equalities:

$$f_1(l_m(x_{\sigma(1)},\ldots,x_{\sigma(m)})) = (-1)^{\binom{m+1}{2}}\iota_{v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(m)}}\omega, \tag{A.20}$$

and, for all $k<m$,

$$\begin{aligned} f_{m+1-k}(l_k(x_{\sigma(1)},\ldots,x_{\sigma(k)}),\ldots,x_{\sigma(m)}) &= -(-1)^{\binom{m-k+2}{2}}S_{m+1-k}(l_k(x_{\sigma(1)},\ldots,x_{\sigma(k)}),\ldots,x_{\sigma(m)})\\ &= -(-1)^{\binom{m-k+2}{2}}(-1)^1\iota_{v_{\sigma(k+1)}\wedge\cdots\wedge v_{\sigma(m)}}l_k(x_{\sigma(1)},\ldots,x_{\sigma(k)})\\ &= -(-1)^{\binom{k+1}{2}}(-1)^{\binom{m-k+2}{2}}\iota_{v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(m)}}\omega. \end{aligned} \tag{A.21}$$

The second-to-last equality above follows from the fact that $|l_k|>0$ for $k\geqslant 3$. Combining (A.20) and (A.21) with the definition of $I^m_{(3)}$ (A.7) gives

$$I^{m\geqslant 3}_{(3)}(x_1,\ldots,x_m) = \sum_{k=3}^{m}\sum_{\sigma\in\mathrm{Sh}(k,m-k)}\chi(\sigma)(-1)^{k(m-k)}(-1)^{\binom{k+1}{2}}(-1)^{\binom{m-k+2}{2}}\iota_{v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(m)}}\omega. \tag{A.22}$$

The sum on the right-hand side above vanishes if, for any i, $|x_i| > 0$. Nonzero summands above have $\chi(\sigma) = (-1)^\sigma$, and since ω is skew-symmetric, reordering the vector fields will cancel this sign. The number of unshuffles appearing in the summation is $\binom{m}{k}$; therefore, summing over σ gives

$$I_{(3)}^{m\geqslant 3}(x_1,\ldots,x_m) = \sum_{k=3}^{m}(-1)^{k(m-k)}(-1)^{\binom{k+1}{2}}(-1)^{\binom{m-k+2}{2}}\binom{m}{k}\iota_{v_1\wedge\cdots\wedge v_m}\omega. \quad \text{(A.23)}$$

It's easy to see that $(-1)^{k(m-k)}(-1)^{\binom{k+1}{2}}(-1)^{\binom{m-k+2}{2}} = -(-1)^{\binom{m+1}{2}}(-1)^m(-1)^k$. Substituting the above sign into (A.23) and using the fact that $\sum_{k=0}^m \binom{m}{k}(-1)^k = 0$ gives the equality (A.19). □

Lemma A.5. *For all $m \geqslant 2$ and $x_1,\ldots,x_m \in L$, the following equality holds:*

$$J^{m\geqslant 2}(x_1,\ldots,x_m) = \sum_{i=1}^{m}(-1)^{i-1}\left[\!\left[f_1(x_i), f_{m-1}(x_1,\ldots,\widehat{x}_i,\ldots,x_m)\right]\!\right]. \quad \text{(A.24)}$$

Proof. Recalling the definition of J^m (A.8), it is easy to see that $J^2(x_1,x_2) = [\![f_1(x_1), f_1(x_2)]\!]$. For the $m > 2$ case, it follows from the definition of the bracket (A) that

$$\begin{aligned} J^{m\geqslant 3}(x_1,\ldots,x_m) &= [\![f_1(x_1), f_{m-1}(x_2,\ldots,x_m)]\!] \\ &\quad + \sum_{i\geqslant 2}\chi(\tau(i))(-1)^m\,[\![f_{m-1}(x_1,x_2,\ldots,\widehat{x}_i,\ldots,x_m), f_1(x_i)]\!]. \end{aligned} \quad \text{(A.25)}$$

Above, $x_i = x_{\tau(m)}$, so $\chi(\tau(i)) = (-1)^{m-i}\epsilon(\tau(i)) = (-1)^{m-i}(-1)^{|x_i|\sum_{j>i}|x_j|}$. It follows from the antisymmetry of the bracket and the definition of the structure maps that the summation on the right-hand side of (A.25) is

$$\sum_{i\geqslant 2}(-1)^{i-1}\,[\![f_1(x_i), f_{m-1}(x_1,x_2,\ldots,\widehat{x}_i,\ldots,x_m)]\!]. \quad \text{(A.26)}$$

Hence, the equality (A.24) holds. □

Lemma A.6. *For all $m \geqslant 3$ and $x_1,\ldots,x_m \in L$ the following equality holds:*

$$\begin{aligned} -(-1)^{\binom{m}{2}}J^{m\geqslant 3}(x_1,\ldots,x_m) &= 2\Big(\sum_{i<k<j} - \sum_{i<j<k} + \sum_{j<i<k}\Big)(-1)^{i+j+k}\iota_{[v_i,v_k]\wedge v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\widehat{v}_k\cdots\wedge v_m}\theta_j \\ &\quad + \Big(\sum_{i<j} - \sum_{j<i}\Big)(-1)^{i+j}\iota_{v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\wedge v_m}\mathcal{L}_{v_i}\theta_j - 2\sum_{i<j}(-1)^{i+j}\iota_{[v_i,v_j]\wedge v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\wedge v_m}\bar{A}(m-1) \\ &\quad - \sum_i (-1)^i\iota_{v_1\wedge\cdots\widehat{v}_i\cdots\wedge v_m}\mathcal{L}_{v_m}\bar{A}(m-1). \end{aligned} \quad \text{(A.27)}$$

Proof. Lemma A.5 and the definitions of the bracket $[\![\cdot,\cdot]\!]$ and f_{m-1} imply that

$$-(-1)^{\binom{m}{2}}J^{m>2}(x_1,\ldots,x_m) = \sum_i(-1)^{i-1}\big(\mathcal{L}_{v_i}S_{m-1}(x_1,\ldots,\widehat{x}_i,\ldots,x_m) + \mathcal{L}_{v_i}\iota_{v_1\wedge\cdots\widehat{v}_i\cdots\wedge v_m}\bar{A}(m-1)\big). \quad \text{(A.28)}$$

The definition of S_{m-1} (A.4) implies that the first summation on the right-hand side of (A.28) is

$$\sum_i (-1)^{i-1} \mathcal{L}_{v_i} S_{m-1}(x_1, \ldots, \widehat{x}_i, \ldots, x_m) = \sum_{i<j} (-1)^{i+j+1} \Big(\mathcal{L}_{v_j} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \theta_i - \mathcal{L}_{v_i} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \theta_j \Big). \tag{A.29}$$

The commutator (1.0.2) implies that

$$\iota_{[v_j, v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m]} = \mathcal{L}_{v_j} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} - \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \mathcal{L}_{v_j}.$$

This and the definition of the Schouten bracket (1.0.1) give the following equalities:

$$\sum_{i<j} (-1)^{i+j+1} \mathcal{L}_{v_j} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \theta_i = \Big(-\sum_{i<j<k} + 2 \sum_{j<i<k} \Big) (-1)^{i+j+k} \iota_{[v_i, v_k] \wedge v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \widehat{v}_k \cdots \wedge v_m} \theta_j + \sum_{j<i} (-1)^{i+j+1} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \mathcal{L}_{v_i} \theta_j, \tag{A.30}$$

$$\sum_{i<j} (-1)^{i+j+1} \mathcal{L}_{v_i} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \theta_j = \Big(-2 \sum_{i<k<j} + \sum_{i<j<k} \Big) (-1)^{i+j+k} \iota_{[v_i, v_k] \wedge v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \widehat{v}_k \cdots \wedge v_m} \theta_j + \sum_{i<j} (-1)^{i+j+1} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \mathcal{L}_{v_i} \theta_j. \tag{A.31}$$

As for the second summation on the right-hand side of (A.28), note that the identity (1.0.2) for the commutator gives

$$\sum_i (-1)^{i-1} \mathcal{L}_{v_i} \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \wedge v_m} \bar{A}(m-1) = \sum_i (-1)^{i-1} \big(\iota_{[v_i, v_1 \wedge \cdots \widehat{v}_i \cdots \wedge v_m]} + \iota_{v_1 \wedge \cdots \widehat{v}_i \cdots \wedge v_m} \mathcal{L}_{v_i} \big) \bar{A}(m-1). \tag{A.32}$$

The definition of the Schouten bracket implies that

$$\sum_i (-1)^{i-1} \iota_{[v_i, v_1 \wedge \cdots \widehat{v}_i \cdots \wedge v_m]} \bar{A}(m-1) = -2 \sum_{i<j} (-1)^{i+j} \iota_{[v_i, v_j] \wedge v_1 \wedge \cdots \widehat{v}_i \cdots \widehat{v}_j \cdots \wedge v_m} \bar{A}(m-1).$$

The above equality, along with (A.32), (A.30), and (A.31), gives the desired expression for $J^{m \geqslant 3}$. ☐

Lemma A.7. *For all $m \geqslant 2$ and $x_1, \ldots, x_m \in L$, the following equality holds:*

$$l_1' f_m(x_1, \ldots, x_m) = -(-1)^{\binom{m+1}{2}} \big(dS_m(x_1, \ldots, x_m) + (-1)^m \iota_{v_1 \wedge \cdots \wedge v_m} \omega + \mathcal{L}_{v_1 \wedge \cdots \wedge v_m} \bar{A}(m-1) \big).$$

Proof. The definitions of f_m and l_1' imply that

$$l_1' f_m(x_1, \ldots, x_m) = -(-1)^{\binom{m+1}{2}} \big(dS_m(x_1, \ldots, x_m) + d_{\mathrm{Tot}} \iota_{v_1 \wedge \cdots \wedge v_m} \bar{A}(m) \big).$$

The Čech differential commutes with interior product. Hence,

$$d_{\mathrm{Tot}} \iota_{v_1 \wedge \cdots \wedge v_m} \bar{A}(m) = \iota_{v_1 \wedge \cdots \wedge v_m} \delta \bar{A}(m) + d \iota_{v_1 \wedge \cdots \wedge v_m} A^n + \sum_{i=1}^n (-1)^{mi+i} d \iota_{v_1 \wedge \cdots \wedge v_m} A^{n-i}.$$

Since $\bar{A}$ is a Čech–Deligne n-cocycle,

$$\iota_{v_1\wedge\cdots\wedge v_m}\delta\bar{A}(m) = -(-1)^m \iota_{v_1\wedge\cdots\wedge v_m}\sum_{i=1}^{n}(-1)^{(m-1)i} dA^{n-i}.$$

Hence, Cartan's formula $\mathcal{L}_{v_1\wedge\cdots\wedge v_m} = d\iota_{v_1\wedge\cdots\wedge v_m} - (-1)^m \iota_{v_1\wedge\cdots\wedge v_m} d$ implies that

$$\begin{aligned} d_{\mathrm{Tot}}\iota_{v_1\wedge\cdots\wedge v_m}\bar{A}(m) &= d\iota_{v_1\wedge\cdots v_m}A^n + \sum_{i=1}^{n}(-1)^{(m-1)i}\mathcal{L}_{v_1\wedge\cdots\wedge v_m}A^{n-i} \\ &= (-1)^m \iota_{v_1\wedge\cdots\wedge v_m}\omega + \sum_{i=0}^{n}(-1)^{(m-1)i}\mathcal{L}_{v_1\wedge\cdots\wedge v_m}A^{n-i}. \end{aligned}$$

The result then follows from the definition of $\bar{A}(m-1)$ (A.2). □

Proof of Theorem 4.6

To prove that the maps $f_k\colon L^{\otimes k} \to L'$ give an L_∞-morphism [**22**, Definition 5.2], we must verify that $\forall m \geqslant 1$

$$\begin{aligned} &l'_1 f_m(x_1,\dots,x_m) + \sum_{j+k=m+1}\ \sum_{\sigma\in \mathrm{Sh}(k,m-k)} \chi(\sigma)(-1)^{k(j-1)+1} f_j(l_k(x_{\sigma(1)},\dots,x_{\sigma(k)}), x_{\sigma(k+1)},\dots,x_{\sigma(m)}) \\ &\quad + \sum_{s+t=m}\ \sum_{\substack{\tau\in \mathrm{Sh}(s,m-s)\\ \tau(1)<\tau(s+1)}} \chi(\tau)(-1)^{s-1}(-1)^{(t-1)\sum_{p=1}^{s}|x_{\tau(p)}|} \left[\!\left[f_s(x_{\tau(1)},\dots,x_{\tau(s)}), f_t(x_{\tau(s+1)},\dots,x_{\tau(m)}) \right]\!\right] = 0, \end{aligned}$$

or, in our notation:

$$\big(l'_1 f_m + I^m_{(1)} + I^m_{(2)} + I^m_{(3)} + J^m\big)(x_1,\dots,x_m) = 0 \quad \forall m \geqslant 1. \tag{A.33}$$

For $m=1$, (A.33) holds, since f_1 is a chain map. For $m=2$, we have $I^2_{(3)} = 0$ by definition, and it follows from Lemmas A.2 and A.3 that

$$I^2_{(1)}(x_1,x_2) + I^2_{(2)}(x_1,x_2) = -[v_1,v_2] - \iota_{v_2}l_1\theta_1 + \iota_{v_1}l_1\theta_2 + \iota_{v_1\wedge v_2}\omega - \iota_{[v_1,v_2]}\bar{A}(1).$$

From Lemma A.5 we have

$$J^2(x_1,x_2) = [v_1,v_2] - \iota_{v_1}d\theta_2 + \iota_{v_2}d\theta_1 - dS_2(x_1,x_2) + \iota_{[v_1,v_2]}\bar{A}(1) - \mathcal{L}_{v_1\wedge v_2}\bar{A}(1).$$

Hence, the above equalities, along with Lemma A.7, imply that the left-hand side of (A.33) is $\iota_{v_1}(l_1 - d)\theta_2 + \iota_{v_2}(d - l_1)\theta_1 + 2\iota_{v_1\wedge v_2}\omega$. If $|x_1| = |x_2| = 0$, then $l_1 = 0$ and the θ_i are Hamiltonian, i.e., $-\iota_{v_1}d\theta_2 = \iota_{v_2}d\theta_1 = -\iota_{v_1\wedge v_2}\omega$. If $|x_i| > 0$, then $v_i = 0$ and $l_1\theta_i = d\theta_i$. Therefore, in either case, (A.33) holds.

For the $m \geqslant 3$ case, note that Lemma 3.1 combined with Cartan's formula for the Lie derivative implies that for any $x_1,\dots,x_m \in L$,

$$\begin{aligned} (-1)^m\mathcal{L}_{v_1\wedge\cdots\wedge v_m}\bar{A}(m-1) &= \sum_{i<j}(-1)^{i+j}\iota_{[v_i,v_j]\wedge v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\wedge v_m}\bar{A}(m-1) \\ &\quad + \sum_i (-1)^i \iota_{v_1\wedge\cdots\widehat{v}_i\cdots\wedge v_m}\mathcal{L}_{v_m}\bar{A}(m-1) \end{aligned}$$

and

$$(-1)^{m-1}\sum_{j=1}^m(-1)^j\mathcal{L}_{v_1\wedge\cdots\widehat{v}_j\cdots\wedge v_m}\theta_j=\Big(\sum_{i<k<j}-\sum_{i<j<k}+\sum_{j<i<k}\Big)(-1)^{i+j+k}\iota_{[v_i,v_k]\wedge v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\widehat{v}_k\cdots\wedge v_m}\theta_j$$
$$+\Big(\sum_{i<j}-\sum_{j<i}\Big)(-1)^{i+j}\iota_{v_1\wedge\cdots\widehat{v}_i\cdots\widehat{v}_j\cdots\wedge v_m}\mathcal{L}_{v_i}\theta_j. \tag{A.34}$$

Combining the above equalities with Lemmas A.3, A.4, and A.6 gives

$$(I^m_{(2)}+I^m_{(3)}+J^m)(x_1,\dots,x_m)=-(-1)^{\binom{m}{2}}\Big((-1)^{m-1}\sum_{j=1}^m(-1)^j\mathcal{L}_{v_1\wedge\cdots\widehat{v}_j\cdots\wedge v_m}\theta_j$$
$$-(-1)^m\mathcal{L}_{v_1\wedge\cdots\wedge v_m}\bar{A}(m-1)+(m-1)\iota_{v_1\wedge\cdots\wedge v_m}\omega\Big). \tag{A.35}$$

Cartan's formula also implies that

$$\sum_{j=1}^m(-1)^j\mathcal{L}_{v_1\wedge\cdots\widehat{v}_j\cdots\wedge v_m}\theta_j=dS_m(x_1,\dots,x_m)-(-1)^{m-1}\sum_{j=1}^m(-1)^j\iota_{v_1\wedge\cdots\widehat{v}_j\cdots\wedge v_m}d\theta_j.$$

Using this, along with Eq. (A.35) and the results of Lemmas A.2 and A.7, we conclude that the left-hand side of (A.33) is

$$-(-1)^{\binom{m}{2}}\Big(\sum_{i=1}^m(-1)^i\iota_{v_1\wedge\cdots\wedge\widehat{v}_i\wedge\cdots\wedge v_m}(l_1\theta_i)-\sum_{j=1}^m(-1)^j\iota_{v_1\wedge\cdots\widehat{v}_j\cdots\wedge v_m}d\theta_j+m\iota_{v_1\wedge\cdots\wedge v_m}\omega\Big). \tag{A.36}$$

If all x_i are degree 0, then $l_1=0$, and all θ_i are Hamiltonian, which implies that $\sum_{j=1}^m(-1)^j\iota_{v_1\wedge\cdots\widehat{v}_j\cdots\wedge v_m}d\theta_j=m\iota_{v_1\wedge\cdots\wedge v_m}\omega$. If $|x_k|>0$ for some x_k, then $v_k=0$ and $l_1\theta_k=d\theta_k$. Hence, in either case, (A.33) holds. This completes the proof of the theorem. □

Appendix B. A recognition principle for homotopy fibers of L_∞-morphisms

In this section we provide a proof of the recognition principle for homotopy fibers of L_∞-algebra morphisms that has been used in Section 3. The proof is based on the following two facts recalled in the Introduction. First, every L_∞-morphism $f_\infty:\mathfrak{g}\to A$ to a dg Lie algebra A uniquely factors as $\mathfrak{g}\xrightarrow{v_\mathfrak{g}}\mathcal{R}(\mathfrak{g})\xrightarrow{\xi_A\circ\mathcal{R}(f_\infty)}A$, where $\xi_A:\mathcal{R}(A)\to A$ is the dg-Lie algebra morphism in the factorization of the identity of A as $A\xrightarrow{v_A}\mathcal{R}(A)\xrightarrow{\xi_A}A$. Second, the adjunction $(\mathcal{R}\dashv i)$ induces an equivalence between the homotopy theories of dg-Lie algebras and L_∞-algebras and so, if $f_\infty:\mathfrak{g}\to\mathfrak{h}$ is an L_∞-morphism between two L_∞-algebras, then an L_∞-algebra $\mathfrak{k}$ presents the homotopy fiber of f_∞ if $\mathfrak{k}$ is L_∞-quasi-isomorphic to the homotopy fiber of $\mathcal{R}(f_\infty):\mathcal{R}(\mathfrak{g})\to\mathcal{R}(\mathfrak{h})$ in the category of dglas.

Lemma B.1. *Let $\mathfrak{g}$ be an L_∞-algebra, A a dgla, and $f_\infty:\mathfrak{g}\to A$ an L_∞ morphism. Let $p_A:B\to A$ be a fibration in the category of dglas, with $H_\bullet(B)=0$. The fiber* *product*

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{g}) \times_A B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_{\mathcal{R}(\mathfrak{g})}}\downarrow & & \downarrow{\scriptstyle p_A} \\ \mathcal{R}(\mathfrak{g}) & \xrightarrow{\xi_A \circ \mathcal{R}(f_\infty)} & A \end{array}$$

is a dgla model for the homotopy fiber of f_∞.

Proof. Consider the commutative diagram of dglas

$$\begin{array}{ccccc} \mathcal{R}(\mathfrak{g}) \times_A B & \xrightarrow{(\mathcal{R}(f_\infty), \mathrm{id}_B)} & \mathcal{R}(A) \times_A B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_{\mathcal{R}(\mathfrak{g})}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{R}(A)}} & & \downarrow{\scriptstyle p_A} \\ \mathcal{R}(\mathfrak{g}) & \xrightarrow{\mathcal{R}(f_\infty)} & \mathcal{R}(A) & \xrightarrow{\xi_A} & A, \end{array}$$

where the rightmost diagram and the outer diagram are pullbacks. By the pasting law, also the leftmost diagram is a pullback. Since p_A is a fibration and ξ_A is a weak equivalence, the map $\pi_{\mathcal{R}(A)}$ is a fibration and the map π_B is a weak equivalence. It follows that $\pi_{\mathcal{R}(A)}$ is a fibrant replacement of $0 \to \mathcal{R}(A)$. Hence, $\mathcal{R}(\mathfrak{g}) \times_A B$ is a model for the homotopy fiber of $\mathcal{R}(f_\infty)$ in the category of dglas. □

Theorem B.2. *Let* $\mathfrak{g}$ *be an* L_∞*-algebra,* A *a dgla, and* $f_\infty : \mathfrak{g} \to A$ *an* L_∞ *morphism. Let* $p_A : B \to A$ *be a fibration in the category of dglas, with* $H_\bullet(B) = 0$. *Assume we have a commutative diagram of* L_∞*-algebras*

$$\begin{array}{ccc} (\mathfrak{g} \times_A B, Q) & \xrightarrow{\pi_{B,\infty}} & B \\ {\scriptstyle \pi_{\mathfrak{g},\infty}}\downarrow & & \downarrow{\scriptstyle p_A} \\ \mathfrak{g} & \xrightarrow{f_\infty} & A \end{array}$$

for a suitable L_∞*-structure* Q *on the fiber product of chain complexes* $\mathfrak{g} \times_A B$ *of* p_A *with the linear component of* f_∞*, with* $\pi_{\mathfrak{g},\infty}$ *and* $\pi_{B,\infty}$ L_∞*-morphisms lifting the linear projections* $\pi_{\mathfrak{g}}$ *and* π_B. *Then* $(\mathfrak{g} \times_A B, Q)$ *is a model for the homotopy fiber of* f_∞.

Proof. Applying the rectification functor to the diagram of L_∞-morphisms above, we get a commutative diagram of dglas

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{g} \times_A B, Q) & \xrightarrow{\mathcal{R}(\pi_{B,\infty})} & \mathcal{R}(B) \\ {\scriptstyle \mathcal{R}(\pi_{\mathfrak{g},\infty})}\downarrow & & \downarrow{\scriptstyle \mathcal{R}(p_A)} \\ \mathcal{R}(\mathfrak{g}) & \xrightarrow{\mathcal{R}(f_\infty)} & \mathcal{R}(A). \end{array}$$

Using the counit of the adjunction $(\mathcal{R} \dashv i)$, we can extend this to a commutative diagram of dglas

$$\begin{array}{ccccc} \mathcal{R}(\mathfrak{g} \times_A B, Q) & \xrightarrow{\mathcal{R}(\pi_{B,\infty})} & \mathcal{R}(B) & \xrightarrow{\xi_B} & B \\ {\scriptstyle \mathcal{R}(\pi_{\mathfrak{g},\infty})}\downarrow & & \downarrow{\scriptstyle \mathcal{R}(p_A)} & & \downarrow{\scriptstyle p_A} \\ \mathcal{R}(\mathfrak{g}) & \xrightarrow{\mathcal{R}(f_\infty)} & \mathcal{R}(A) & \xrightarrow{\xi_A} & A. \end{array}$$

By the universal property of the pullback of dglas, the outer rectangle is equivalent to the datum of a morphism of dglas $\psi : \mathcal{R}(\mathfrak{g} \times_A B, Q) \to \mathcal{R}(\mathfrak{g}) \times_A B$, where $\mathcal{R}(\mathfrak{g}) \times_A B$ is the pullback of dglas

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{g}) \times_A B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_{\mathcal{R}(\mathfrak{g})}}\downarrow & & \downarrow{\scriptstyle p_A} \\ \mathcal{R}(\mathfrak{g}) & \xrightarrow{\xi_A \circ \mathcal{R}(f_\infty)} & A. \end{array}$$

The morphism ψ will satisfy $\pi_B \circ \psi = \xi_B \circ \mathcal{R}(\pi_{B,\infty})$ and $\pi_{\mathcal{R}(\mathfrak{g})} \circ \psi = \mathcal{R}(\pi_{\mathfrak{g},\infty})$. By Lemma B.1, the dgla $\mathcal{R}(\mathfrak{g}) \times_A B$ is a dgla model for the homotopy fiber of f_∞. Then to conclude we only need to show that ψ is a quasi-isomorphism. This is equivalent to proving that the L_∞-morphism η given by the composition

$$\eta : (\mathfrak{g} \times_A B, Q) \xrightarrow{v_{(\mathfrak{g}\times_A B,Q)}} \mathcal{R}(\mathfrak{g} \times_A B, Q) \xrightarrow{\psi} \mathcal{R}(\mathfrak{g}) \times_A B$$

is a quasi-isomorphism. The linear part η_1 of η is determined by its compositions with the linear projections to $\mathcal{R}(\mathfrak{g})$ and to B. We have $\pi_B \circ \eta_1 = (\pi_B \circ \eta)_1 = (\pi_B \circ \psi \circ v_{(\mathfrak{g}\times_A B,Q)})_1 = (\xi_B \circ \mathcal{R}(\pi_{B,\infty}) \circ v_{(\mathfrak{g}\times_A B,Q)})_1 = (\pi_{B,\infty})_1 = \pi_B$ and, similarly, $\pi_{\mathcal{R}(\mathfrak{g})} \circ \eta_1 = (\pi_{\mathcal{R}(\mathfrak{g})} \circ \eta)_1 = (\pi_{\mathcal{R}(\mathfrak{g})} \circ \psi \circ v_{(\mathfrak{g}\times_A B,Q)})_1 = (\mathcal{R}(\pi_{\mathfrak{g},\infty}) \circ v_{(\mathfrak{g}\times_A B,Q)})_1 = (v_\mathfrak{g} \circ \pi_{\mathfrak{g},\infty})_1 = (v_\mathfrak{g})_1 \circ \pi_\mathfrak{g}$. This means that the map of chain complexes $\eta_1 : \mathfrak{g} \times_A B \to \mathcal{R}(\mathfrak{g}) \times_A B$ is given by $\eta_1 = ((v_\mathfrak{g})_1, \mathrm{id}_B)$. Now consider the commutative diagram

$$\begin{array}{ccccc} \mathfrak{g} \times_A B & \xrightarrow{((v_\mathfrak{g})_1, \mathrm{id}_B)} & \mathcal{R}(\mathfrak{g}) \times_A B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_\mathfrak{g}}\downarrow & & {\scriptstyle \pi_{\mathcal{R}(\mathfrak{g})}}\downarrow & & \downarrow{\scriptstyle p_A} \\ \mathfrak{g} & \xrightarrow{(v_\mathfrak{g})_1} & \mathcal{R}(\mathfrak{g}) & \xrightarrow{\xi_A \circ \mathcal{R}(f_\infty)} & A. \end{array}$$

The rightmost subdiagram is a pullback by definition, while the total diagram is

$$\begin{array}{ccc} \mathfrak{g} \times_A B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_\mathfrak{g}}\downarrow & & \downarrow{\scriptstyle p_A} \\ \mathfrak{g} & \xrightarrow{(f_\infty)_1} & A \end{array}$$

since $\xi_A \circ \mathcal{R}(f_\infty) \circ (v_\mathfrak{g})_1 = (\xi_A \circ \mathcal{R}(f_\infty) \circ v_\mathfrak{g})_1 = (f_\infty)_1$, and so it is a pullback by hypothesis. Then, by the pasting law, also the leftmost subdiagram is a pullback. The map $\pi_{\mathcal{R}(\mathfrak{g})}$ is a fibration, since p_A is fibration, and all chain complexes are fibrant. Hence, since $(v_\mathfrak{g})_1$ is a quasi-isomorphism, its pullback $\eta_1 = ((v_\mathfrak{g})_1, \mathrm{id}_B)$ is also a quasi-isomorphism. □

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