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MODEL STRUCTURES ON MODULES OVER DING-CHEN RINGS

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Abstract

An n-FC ring is a left and right coherent ring whose left and right self-FP-injective dimension is n. The work of Ding and Chen shows that these rings possess properties which generalize those of n-Gorenstein rings. In this paper we call a (left and right) coherent ring with finite (left and right) self-FP-injective dimension a Ding-Chen ring. In the case of Noetherian rings, these are exactly the Gorenstein rings. We look at classes of modules we call Ding projective, Ding injective and Ding flat which are meant as analogs to Enochs' Gorenstein projective, Gorenstein injective and Gorenstein flat modules. We develop basic properties of these modules. We then show that each of the standard model structures on Mod-R, when R is a Gorenstein ring, generalizes to the Ding-Chen case. We show that when R is a commutative Ding-Chen ring and G is a finite group, the group ring $R[G]$ is a Ding-Chen ring.

1. Introduction

Noncommutative Gorenstein rings were introduced and studied by Y. Iwanaga in [**Iwa79**] and [**Iwa80**]. Later Enochs and Jenda and their coauthors defined and studied the so-called Gorenstein injective, Gorenstein projective and Gorenstein flat modules and developed Gorenstein homological algebra. Hovey [**Hov02**] showed that this theory can be formalized in the language of model categories and showed that when R is a Gorenstein ring, the category of R-modules has two Quillen equivalent model structures, a projective model structure and an injective model structure. Their homotopy categories are what we call the stable module category of the ring R. There is also a third Quillen equivalent model structure called the flat model structure, as was shown in [**GH09**].

The point of this paper is to describe how the homotopy theory on modules over a Gorenstein ring generalizes to a homotopy theory on modules over a so-called n-FC ring, or what we call, a Ding-Chen ring. While an n-Gorenstein ring is a Noetherian ring with self-injective dimension n, an n-FC ring is a coherent ring with self-FP-injective dimension n. These rings were introduced and studied by Ding and Chen in [**DC93**] and [**DC96**] and were seen to have many properties similar to n-Gorenstein

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rings. Just as a ring is called Gorenstein when it is n-Gorenstein for some natural number n, we will call a ring "Ding-Chen" when it is n-FC for some n. (The term FC ring is already taken! In the language of Ding and Chen, these are the 0-FC rings.) By now, the work of Ding and coauthors is sophisticated enough to readily show that each of Hovey's model structures generalizes to modules over a Ding-Chen ring. In fact, Ding and Mao explicitly remark in [**DM07**] that one of these model structures (the one we call the injective model structure) must exist.

We can describe the basic idea rather simply in terms of cotorsion pairs, which are intimately related to model category structures by [**Hov02**]. When R is a Gorenstein ring, we have the class of "trivial" modules $\mathcal{W}$, which are the modules of finite injective dimension. It turns out that a module is trivial if and only if it has finite projective dimension, if and only if has finite flat dimension. Hovey's injective model structure relies on the fact that $(\mathcal{W}, \mathcal{GI})$ is a complete cotorsion pair, where $\mathcal{GI}$ is Enochs' class of Gorenstein injective modules. In this model structure every module is cofibrant and $\mathcal{GI}$ makes up the class of fibrant modules. On the other hand, Hovey's projective model structure relies on the fact that $(\mathcal{GP}, \mathcal{W})$ is a complete cotorsion pair, where $\mathcal{GP}$ is Enochs' class of Gorenstein projective modules. In this dual model structure every module is fibrant, while $\mathcal{GP}$ makes up the class of cofibrant modules. When R is a Ding-Chen ring, the class $\mathcal{W}$ of trivial modules is relaxed to consist of the modules of finite FP-injective dimension. It now turns out that a module is trivial if and only if it has finite flat dimension. Now the injective model structure relies on the fact that $(\mathcal{W}, \mathcal{DI})$ is a complete cotorsion pair, where $\mathcal{DI}$ is the class of Ding injective modules (fibrant modules). On the other hand, the projective model structure relies on the fact that $(\mathcal{DP}, \mathcal{W})$ is a complete cotorsion pair, where $\mathcal{DP}$ is the class of Ding projective modules (cofibrant modules). Furthermore, when R is Noetherian, a Ding-Chen ring is automatically Gorenstein. In this case $\mathcal{DI} = \mathcal{GI}$ and $\mathcal{DP} = \mathcal{GP}$ and our model structures coincide with Hovey's in [**Hov02**].

There is a similar result involving a flat model structure on modules over a Ding-Chen ring. In the process we introduce and look at first properties of Ding injective, Ding projective and Ding flat modules. They are the obvious generalizations of Gorenstien injective, Gorenstein projective and Gorenstein flat modules to the Ding-Chen ring case. However, during the writing of this paper, the author was made aware that Ding injective modules have appeared in [**DM08**] under the name *Gorenstein* FP-*injective modules*, while Ding projective modules have appeared in [**DLM09**] under the name *strongly Gorenstein flat modules*. The author feels that the "Ding" names are quite fitting and illuminate the analogy with the homotopy theory of modules over a Gorenstein ring.

The layout of the paper is as follows: In Section 2 we give definitions and review concepts which are basic to the rest of the paper. In Section 3 we introduce Ding modules and use techniques of Enochs and coauthors to show that Ding modules have properties analogous to Gorenstein modules. Similar results have appeared in [**DM08**] and [**DLM09**]. In Section 4 we define Ding-Chen rings and show the existence of the injective, projective, and flat model structures. This follows rather easily from the work of Ding and coauthors. In Section 5 we show that when R is a commutative Ding-Chen ring and G is a finite group, the group ring $R[G]$ is a Ding-Chen ring. The model structures on $R[G]$ are relevant for defining Tate cohomology.

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2. Preliminaries

Throughout we assume that all rings have an identity and all modules are unital. Unless stated otherwise, an R-module will be understood to be a *right* R-module. We denote the injective (resp. projective, resp. flat) dimension of an R-module M by $\mathrm{id}(M)$ (resp. $\mathrm{pd}(M)$ resp. $\mathrm{fd}(M)$).

If R is commutative and Noetherian and of finite injective dimension when viewed as a module over itself, then R is called a Gorenstein ring. When R is noncommutative, Iwanaga extended the definition of Gorenstein rings as follows: R is (left and right) Noetherian and both $\mathrm{id}({}_RR)$ and $\mathrm{id}(R_R)$ are finite. In this case Iwanaga showed that $\mathrm{id}({}_RR) = \mathrm{id}(R_R)$, and if this number is n, we say that R is n-Gorenstein. The book [**EJ01**] is a standard reference for Gorenstein rings and modules.

In [**DC93**], and especially [**DC96**], Ding and Chen extended this idea yet further by replacing (left and right) Noetherian with (left and right) coherence and replacing the (left and right) injective dimension of R with the (left and right) FP-injective dimension of R. The FP-injective dimension of a module was introduced by Stenström in [**Sten70**] and will be defined in Section 3. The rings introduced by Ding and Chen (called n-FC rings, or Ding-Chen rings) will be defined in Section 4. Examples of Ding-Chen rings include all Gorenstein rings, and the group rings of Section 5. Any von-Neumann regular ring is also a Ding-Chen ring. In particular, if R is an infinite product of fields, then R is a Ding-Chen ring. Furthermore it follows from Theorem 7.3.1 of [**GL89**] that $R[x_1, x_2, x_3, \dots, x_n]$ is a commutative Ding-Chen ring. Another example of a Ding-Chen ring is the group ring $R[G]$, where R is an FC-ring and G is a locally finite group. See [**Dam79**].

A *cotorsion pair* (of R-modules) is a pair of classes of modules $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}^\perp = \mathcal{B}$ and $\mathcal{A} = {}^\perp\mathcal{B}$. Here $\mathcal{A}^\perp$ is the class of modules M such that $\mathrm{Ext}^1_R(A, M) = 0$ for all $A \in \mathcal{A}$, and similarly ${}^\perp\mathcal{B}$ is the class of modules M such that $\mathrm{Ext}^1_R(M, B) = 0$ for all $B \in \mathcal{B}$. Two simple examples of cotorsion pairs are $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{I})$ where $\mathcal{P}$ is the class of projectives, $\mathcal{I}$ is the class of injectives and $\mathcal{A}$ is the class of all R-modules. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to have *enough projectives* if for any module M there is a short exact sequence $0 \to B \to A \to M \to 0$ where $B \in \mathcal{B}$ and $A \in \mathcal{A}$. We say it has *enough injectives* if it satisfies the dual statement. These two statements are in fact equivalent for the category of R-modules. We say that the cotorsion pair is *complete* if it has enough projectives and injectives. Note that the cotorsion pairs $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{I})$ above are each complete. The book [**EJ01**] is also an excellent reference for cotorsion pairs. The equivalence of the enough injectives and enough projectives, although not difficult, is proved as Proposition 7.1.7 in [**EJ01**].

The most well-known (nontrivial) example of a cotorsion pair is $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F}$ is the class of flat modules and $\mathcal{C}$ are the cotorsion modules. A proof that this is a complete cotorsion theory can be found in [**EJ01**].

Definition 2.1. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *hereditary* if one of the following holds:

1. $\mathcal{A}$ is resolving. That is, $\mathcal{A}$ is closed under taking kernels of epis.
2. $\mathcal{B}$ is coresolving. That is, $\mathcal{B}$ is closed under taking cokernels of monics.
3. $\operatorname{Ext}^i_R(A, B) = 0$ for any R-modules $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and $i \geqslant 1$.

See [**EJ01**] for a proof that these are equivalent.

Complete hereditary cotorsion pairs are intimately related to abelian model category structures and we refer the reader to Theorem 2.2 of [**Hov02**] for the precise relationship. We end this section by proving a lemma that will be used frequently in the rest of the paper.

Lemma 2.2. *Let $\mathcal{I}$ be the class of injective modules and $\mathcal{P}$ be the class of projective modules. If $\mathcal{W}$ is a coresolving class of objects containing $\mathcal{I}$ and $(\mathcal{W}, \mathcal{W}^\perp)$ is a cotorsion pair, then $\mathcal{W} \cap \mathcal{W}^\perp = \mathcal{I}$. On the other hand, if $\mathcal{W}$ is resolving and contains $\mathcal{P}$ and $({}^\perp\mathcal{W}, \mathcal{W})$ is a cotorsion pair, then ${}^\perp\mathcal{W} \cap \mathcal{W} = \mathcal{P}$.*

Proof. The two statements are dual. We prove the first one. Clearly $\mathcal{I} \subseteq \mathcal{W} \cap \mathcal{W}^\perp$. On the other hand, let $W \in \mathcal{W} \cap \mathcal{W}^\perp$, and write $0 \to W \to I \to C \to 0$ where I is injective. Since $\mathcal{W}$ contains the injectives and is coresolving, C must also be in $\mathcal{W}$. Now $\operatorname{Ext}^1_R(C, W) = 0$, so the sequence must split. Thus, W is a summand of I, which proves that W is injective. □

3. Ding modules

In this section we consider modules we call Ding injective, Ding projective and Ding flat modules. Just as the Gorenstein modules play the role of fibrant and cofibrant objects in the model structures on modules over Gorenstein rings, the Ding modules play the role of fibrant and cofibrant objects over Ding-Chen rings. Ding injective and projective modules just appeared in the literature in [**DM08**] and [**DLM09**] respectively.

Definition 3.1. A right R-module E is a called FP-*injective* if $\operatorname{Ext}^1_R(F, E) = 0$ for all finitely presented modules F. More generally, the FP-*injective dimension* of a right R-module N is defined to be the least integer $n \geqslant 0$ such that $\operatorname{Ext}^{n+1}_R(F, N) = 0$ for all finitely presented right R-modules F. The FP-injective dimension of N is denoted FP-id(N) and equals ∞ if no such n above exists.

These definitions were introduced in [**Sten70**]. There it is shown (Lemma 3.1) that for a (right) coherent ring, the notion of FP-id(N) behaves analogously to id(N). In particular, FP-id$(N) \leqslant n$, if and only if $\operatorname{Ext}^{n+1}_R(F, N) = 0$ for all finitely presented F, if and only if $\operatorname{Ext}^{n+1}_R(R/I, N) = 0$ for all finitely generated (right) ideals of R, if and only if the nth cosyzygy of any FP-injective coresolution of N is FP-injective. Note that the notion of FP-injective coincides with the notion of injective when R is Noetherian.

3.1. Ding injectives

We now introduce and look at basic properties of Ding injective modules. They also appear in [**DM08**] as "Gorenstein FP-injective modules". We refer the reader to [**DM08**] for more results, including results concerning existence of covers and envelopes.

Definition 3.2. We call a right R-module N *Ding injective* if there exists an exact sequence of injective modules

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

with $N = \ker(I^0 \to I^1)$ and which remains exact after applying $\mathrm{Hom}_R(E, -)$ for any FP-injective module E. We denote the class of Ding injective modules by $\mathcal{DI}$.

Remark 3.3. By definition, Ding injective modules are Gorenstein injective. When R is Noetherian the two notions coincide.

Lemma 3.4. *If E is FP-injective and N is Ding injective, then $\mathrm{Ext}^i_R(E, N) = 0$ for each $i \geqslant 1$.*

Proof. In the definition of Ding injective, N has an injective coresolution $0 \to N \to I^0 \to I^1 \to \cdots$ which remains exact after applying $\mathrm{Hom}_R(E, -)$ for any FP-injective module E. So $\mathrm{Ext}^i_R(E, N) = 0$ for each $i \geqslant 1$. □

Proposition 3.5. *A Ding injective module is either injective or has* FP*-injective dimension ∞.*

Proof. Suppose N is Ding injective and $\mathrm{Ext}^{n+1}_R(F, N) = 0$ for all finitely presented F. Let $0 \to N \to I^0 \to I^1 \to \cdots \to I^{n-1} \to E^n$ be exact with each I^i injective and $E^n = \mathrm{cok}\,(I^{n-2} \to I^{n-1})$. From the isomorphism $\mathrm{Ext}^1_R(F, E_n) \cong \mathrm{Ext}^{n+1}_R(F, N)$, it follows that E^n is FP-injective. Now $0 \to N \to I^0 \to I^1 \to \cdots \to I^{n-1} \to E^n \to 0$ represents an element of $\mathrm{Ext}^n_R(E^n, N)$, but this group equals 0 by Lemma 3.4. Therefore the sequence is split exact and so N is a direct sum of I^0. Thus, N must be injective. □

Corollary 3.6. *If the class of Ding injective right R-modules is closed under direct sums, then R is right Noetherian.*

Proof. Suppose the class of Ding injectives is closed under direct sums. Then since the class of FP-injective modules is always closed under direct sums by Corollary 2.4 of [**Sten70**], it follows that a direct sum of injectives must be both Ding injective and FP-injective. But then it must be injective by Proposition 3.5. So R is (right) Noetherian. □

3.2. Ding projectives

We now introduce a kind of dual notion, that of Ding projective module. To be more precise, we will see in Section 4 that the duality holds when R is a Ding-Chen ring. Ding projective modules just appeared in [**DLM09**] as "strongly Gorenstein flat modules". We refer the reader to [**DLM09**] for more results, in particular, for results concerning existence of covers and envelopes.

Definition 3.7. We call a right R-module M *Ding projective* if there exists an exact sequence of projective modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \ker(P^0 \to P^1)$ and which remains exact after applying $\mathrm{Hom}_R(-, F)$ for any flat module F. We denote the class of all Ding projective modules by $\mathcal{DP}$.

Remark 3.8. By definition, Ding projective modules are Gorenstein projective. For a general coherent ring R, it follows from Proposition 10.2.6 of [**EJ01**] that a finitely presented module is Ding projective if and only if it is Gorenstein projective. Also, from Corollary 4.6 the Ding projectives are exactly the Gorenstein projectives when R is a Gorenstein ring.

Lemma 3.9. *If F is flat and M is Ding projective, then $\mathrm{Ext}^i_R(M, F) = 0$ for each $i \geqslant 1$.*

Proposition 3.10. *A Ding projective module is either projective or has flat dimension ∞.*

Proof. Suppose M is Ding projective and $\mathrm{fd}(M) = n < \infty$. Then we can construct an exact sequence $0 \to F_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective and $F_n = \ker(P_{n-1} \to P_{n-2})$ flat. Then this exact sequence is an element of $\mathrm{Ext}^n_R(M, F_n)$ which equals 0 by Lemma 3.9. Therefore the resolution is split exact and so M must be projective. □

Corollary 3.11.

1. *If the Ding projective right R-modules are closed under direct limits, then R is right perfect.*
2. *If R is left coherent and the Ding projective right R-modules are closed under direct products, then R is right perfect.*

Proof. Recall that a ring R is right perfect if and only if every flat right R-module is projective. So by Lazard's theorem it follows that R is right perfect if and only if the class of projective right R-modules is closed under direct limits. Suppose the class of Ding projective right R-modules is closed under direct limits. Then it follows that a direct limit of projectives must be both Ding projective and flat. But then it must be projective by Proposition 3.10. So R is right perfect.

For the second statement, we use the results of S. U. Chase, which are summarized on page 139 and page 147 of [**Lam99**]: First, a ring R is left coherent if and only if direct products of flat right R-modules are flat. Second, a ring R is left coherent and right perfect if and only if direct products of projective right R-modules are projective. Then the proof is similar to the last paragraph. □

3.3. Ding flats

One could introduce the notion of a "Ding flat" module but it turns out that they are nothing more than the Gorenstein flat modules.

Definition 3.12. Call a left R-module M *Ding flat* if there exists an exact sequence of flat modules

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

with $M = \ker(F^0 \to F^1)$ and which remains exact after applying $E \otimes_R -$ for any FP-injective right R-module E.

The following result is a restatement of Ding and Mao's Lemma 2.8 of [**DM08**].

Proposition 3.13. *A left R-module M is Ding flat if and only if it is Gorenstein flat. In this case M^+ is a Ding injective right R-module. The converse holds when R is a right coherent ring.*

We refer the reader to Theorem 5 of [**DC96**] for characterizations of Ding flat (i.e., Gorenstein flat) modules over Ding-Chen rings.

Over a Gorenstein ring R, it is known that Gorenstein projective modules are Gorenstein flat. (See Corollary 10.3.10 of [**EJ01**].) However it is not known whether or not this holds over more general rings R. Affirmative answers have been given for some other types of rings. For example, see Proposition 3.4 of [**HH04**]. Of course, one could ask the analogous question: Are Ding projective modules Ding flat? While this is true when R is a Ding-Chen ring, the following shows that it holds in much more generality.

Proposition 3.14. *Let R be a left coherent ring. Then any Ding projective right R-module M is Ding flat.*

Proof. Let M be a Ding projective right R-module. By definition there is an exact sequence of projective right R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \ker(P^0 \to P^1)$ and which remains exact after applying $\mathrm{Hom}_R(-, F)$ for any flat right R-module F. Let E be an FP-injective left R-module. Since R is left coherent, E^+ is a flat right R-module by Theorem 2.2 of [**FH72**]. So applying $\mathrm{Hom}_R(-, E^+)$ we get the exact sequence

$$\cdots \to \mathrm{Hom}(P^1, E^+) \to \mathrm{Hom}(P^0, E^+) \to \mathrm{Hom}(P_0, E^+) \to \mathrm{Hom}(P_1, E^+) \to \cdots .$$

But this is naturally isomorphic to

$$\cdots \to (P^1 \otimes_R E)^+ \to (P^0 \otimes_R E)^+ \to (P_0 \otimes_R E)^+ \to (P_1 \otimes_R E)^+ \to \cdots .$$

Therefore $\cdots \to P_1 \otimes_R E \to P_0 \otimes_R E \to P^0 \otimes_R E \to P^1 \otimes_R E \to \cdots$ is exact, proving M is a Ding flat right R-module. □

In considering the above results one is led to the following question which we do not have an answer for: When R is a Ding-Chen ring, are there Gorenstein injective modules that are not Ding injective?

4. Model structures on modules over a Ding-Chen ring

In [**Sten70**], Stenström introduced the notion of an FP-injective module and studied FP-injective modules over coherent rings. FC rings appear to have been introduced by Damiano in [**Dam79**]. These are the "coherent version" of quasi-Frobenius rings where Noetherian is replaced by coherent and self-injective is replaced by self-FP-injective. Finally, just as Gorenstein rings are a natural generalization of quasi-Frobenius rings, Ding and Chen extended FC rings to n-FC rings in [**DC93**] and [**DC96**]. Throughout this section modules are *right* R-modules unless stated otherwise.

Definition 4.1. A ring R is called an n-FC *ring* if it is both left and right coherent and FP-id$({}_RR)$ = FP-id$(R_R) = n$. Often we are not interested in the particular n, and so we say R is a *Ding-Chen ring* if it is an n-FC ring for some $n \geqslant 0$.

We note that Corollary 3.18 of [**DC93**] states that if R is both left and right coherent, and FP-id$({}_RR)$ and FP-id(R_R) are both finite, then FP-id$({}_RR)$ = FP-id(R_R). Ding and Chen go on to prove the following fundamental result, which is the key reason we have a homotopy theory on the category Mod-R when R is a Ding-Chen ring.

Theorem 4.2. *Let R be an n-FC ring and M be a right R-module. Then the following are equivalent:*

1. $\mathrm{fd}(M) < \infty$;
2. $\mathrm{fd}(M) \leqslant n$;
3. $\mathrm{FP}-\mathrm{id}(M) < \infty$;
4. $\mathrm{FP}-\mathrm{id}(M) \leqslant n$.

Due to Theorem 4.2 we can make the following definition.

Definition 4.3. If R is a Ding-Chen ring and M is an R-module, then we say that M is *trivial* if it satisfies one of the equivalent conditions of Theorem 4.2. We denote the class of trivial modules by $\mathcal{W}$.

Note that if a Ding-Chen ring R happens to be left and right Noetherian, then it is automatically Iwanaga-Gorenstein. In this case $M \in \mathcal{W}$ if and only if $\mathrm{id}(M)$ is finite, if and only if $\mathrm{pd}(M)$ is finite, if and only if $\mathrm{fd}(M)$ is finite, and in this case these dimensions all must be less than or equal to n. (See [**Iwa79**] or [**EJ01**] for a proof.) Furthermore, from the remarks in Section 3, the Ding injectives coincide with the Gorenstein injectives and the Ding projectives coincide with the Gorenstein projectives.

Corollary 4.4. *Let R be a Ding-Chen ring. Then the class $\mathcal{W}$ of trivial R-modules is a thick subcategory. This means that $\mathcal{W}$ is closed under retracts and if two out of three terms in a short exact sequence are in $\mathcal{W}$ then so is the third. Furthermore $\mathcal{W}$ is closed under all filtered colimits, in particular transfinite extensions.*

Proof. Use Ding and Chen's Theorem 4.2 above, along with properties of Ext, to argue that $\mathcal{W}$ is a thick subcategory. The filtered colimits argument is exactly as Hovey proves on page 581 of [**Hov02**]. □

It is shown in Theorem 3.4 of [**DM07**] that $(\mathcal{W}, \mathcal{W}^{\perp})$ is a complete cotorsion theory when R is a Ding-Chen ring. It is clear that $\mathcal{W}$ contains $\mathcal{I}$ and is coresolving by Corollary 4.4. So by Lemma 2.2 we have $\mathcal{W} \cap \mathcal{W}^{\perp} = \mathcal{I}$. We see now that $\mathcal{W}^{\perp} = \mathcal{DI}$.

We note that in the particular case of when R is an n-FC ring, the authors of [**DM07**] call the modules in $\mathcal{W}^{\perp}$ "n-cotorsion" modules.

Corollary 4.5. *Let R be a Ding-Chen ring and let $\mathcal{W}$ be the class of trivial modules. Then N is Ding injective if and only if $N \in \mathcal{W}^{\perp}$.*

Proof. Suppose N is Ding injective. Let $W \in \mathcal{W}$. We wish to show that $\operatorname{Ext}^1_R(W, N) = 0$. Write a finite FP-injective coresolution $0 \to W \to E^0 \to E^1 \to \cdots \to E^n \to 0$. Then using Lemma 3.4 and dimension shifting we get $\operatorname{Ext}^1_R(W, N) \cong \operatorname{Ext}^{n+1}_R(E^n, N) = 0$.

On the other hand suppose that $N \in \mathcal{W}^{\perp}$. We wish to show that N is Ding injective. First take an injective coresolution of N as below.

$$0 \to N \to I^0 \to I^1 \to I^2 \cdots .$$

Note that since $N \in \mathcal{W}^{\perp}$, and $(\mathcal{W}, \mathcal{W}^{\perp})$ is a hereditary cotorsion pair, the kernel at any spot in the sequence is also in $\mathcal{W}^{\perp}$. Next we use the fact that $(\mathcal{W}, \mathcal{W}^{\perp})$ is a complete cotorsion pair to find a short exact sequence $0 \to K \to I_0 \to N \to 0$ where $I_0 \in \mathcal{W}$ and $K \in \mathcal{W}^{\perp}$. But I_0 must also be in $\mathcal{W}^{\perp}$ since it is an extension of two such modules. As noted before the statement of Corollary 4.5, it follows that I_0 is an injective module. Continuing with the same procedure on K we can build an injective resolution of N as below:

$$\cdots \to I_1 \to I_0 \to N \to 0.$$

Again the kernel at each spot is in $\mathcal{W}^{\perp}$. Pasting this "left" resolution together with the "right" coresolution above we get an exact sequence

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

of injective modules with $N = \ker(I^0 \to I^1)$. This sequence satisfies the definition of N being a Ding injective R-module, since now $\operatorname{Hom}_R(E, -)$ will leave the sequence exact for any FP-injective module E. □

We have a dual result as well. If R is a Ding-Chen ring, then by Theorem 3.8 of [**DM05**], $({}^{\perp}\mathcal{W}, \mathcal{W})$ is a complete cotorsion pair. It is clear that $\mathcal{W}$ contains $\mathcal{P}$ and is a resolving class. So by Lemma 2.2 we have ${}^{\perp}\mathcal{W} \cap \mathcal{W} = \mathcal{P}$. We see now that ${}^{\perp}\mathcal{W} = \mathcal{DP}$.

We note that in the particular case of when R is n-FC, the authors of [**DM05**] call the modules in ${}^{\perp}\mathcal{W}$ "n-FP-projective" modules.

Corollary 4.6. *Let R be a Ding-Chen ring and let $\mathcal{W}$ be the class of trivial modules. Then M is Ding projective if and only if $M \in {}^{\perp}\mathcal{W}$.*

Proof. Suppose M is Ding projective. Let $W \in \mathcal{W}$. We wish to show that $\operatorname{Ext}^1_R(M, W) = 0$. Write a finite flat resolution $0 \to F_n \to \cdots \to F_1 \to F_0 \to W \to 0$. Then by using Lemma 3.9 and dimension shifting we get $\operatorname{Ext}^1_R(M, W) \cong \operatorname{Ext}^{n+1}_R(M, F_n) = 0$.

On the other hand, suppose that $M \in {}^{\perp}\mathcal{W}$. We wish to show that M is Ding projective. First take a projective resolution of M as below.

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 .$$

Note that since $M \in {}^{\perp}\mathcal{W}$, and $({}^{\perp}\mathcal{W}, \mathcal{W})$ is a hereditary cotorsion pair, the kernel at any spot in the sequence is also in ${}^{\perp}\mathcal{W}$. Next we use the fact that $({}^{\perp}\mathcal{W}, \mathcal{W})$ is a complete cotorsion pair to find a short exact sequence $0 \to M \to P^0 \to C \to 0$ where $P^0 \in \mathcal{W}$ and $C \in {}^{\perp}\mathcal{W}$. But P^0 must also be in ${}^{\perp}\mathcal{W}$ since it is an extension of two such modules. It follows that P^0 is a projective module. Continuing with the same procedure on C we can build a projective coresolution of M as below:

$$0 \to M \to P^0 \to P^1 \to P^2 \to \cdots .$$

Again the kernel at each spot is in ${}^{\perp}\mathcal{W}$. Pasting this "right" coresolution together with the "left" resolution above we get an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules which satisfies the definition of M being a Ding projective R-module. □

We now state our results on the existence of three Quillen equivalent homotopy theories on modules over a Ding-Chen ring.

Theorem 4.7. *Let R be a Ding-Chen ring. Then there are two cofibrantly generated abelian model structures on* $\text{Mod}-R$ *each having $\mathcal{W}$ as the class of trivial objects. In the first model structure, each module is cofibrant while the fibrant objects (resp. trivially fibrant objects) are the Ding injective modules (resp. injective modules). In the second model structure, each module is fibrant while the cofibrant objects (resp. trivially cofibrant objects) are the Ding projective modules (resp. projective modules). We call the first model structure the* injective model structure *and the second the* projective model structure*. When R is Noetherian these model structures coincide with those in [**Hov02**].*

Proof. $\mathcal{DP} \cap \mathcal{W} = \mathcal{P}$ and $\mathcal{DI} \cap \mathcal{W} = \mathcal{I}$, and $(\mathcal{DP}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{DI})$ are complete hereditary cotorsion pairs (each cogenerated by a set). So the result follows from Theorem 2.2 and Corollary 6.8 of [**Hov02**]. □

Remark 4.8. Note that if R is coherent and has weak dimension $n < \infty$, then R is automatically a Ding-Chen ring. However, in this case, $\mathcal{W}$ is the class of all modules. So the model structures in Theorem 4.7 will be trivial. This has an analog in the Noetherian case: When R is Noetherian with global dimension $n < \infty$, then R is automatically a Gorenstein ring. However, the model structures are trivial. So we are interested in examples of Ding-Chen rings with infinite weak dimension.

4.1. The flat model structure

For any ring R, the class $\mathcal{F}$ of flat R-modules form the left side of a complete hereditary cotorsion pair $(\mathcal{F}, \mathcal{C})$. The modules in $\mathcal{C}$ are called cotorsion modules. Similarly, when R is a Ding-Chen ring, the Gorenstein flat modules form the left side of a complete cotorsion theory, denoted $(\mathcal{GF}, \mathcal{GC})$. The modules in $\mathcal{GC}$ are called Gorenstein cotorsion modules.

Lemma 4.9. *When R is a Ding-Chen ring, the Gorenstien flat left R-modules form the left side of a complete hereditary cotorsion pair $(\mathcal{GF}, \mathcal{GC})$.*

Proof. It is easy to show that $\mathcal{GF}$ is resolving by using Theorem 5 of [**DC96**] and arguing with the functors $\text{Tor}_i^R(E, -)$ where E is an arbitrary FP-injective right

R-module and $i \geqslant 1$. A similar Tor argument will show that if $0 \to P \to F \to F/P \to 0$ is a pure exact sequence with $F \in \mathcal{GF}$, then F/P, and consequently P, are also in $\mathcal{GF}$. Now an argument just like Proposition 7.4.3 of [**EJ01**] will show that $(\mathcal{GF}, \mathcal{GC})$ is cogenerated by a set and hence is complete. ☐

Theorem 4.10. *If R is a Ding-Chen ring, then there is a model structure on R-*Mod *in which the cofibrant objects are the Gorenstein flat modules, the fibrant objects are the cotorsion modules, and the trivial objects are the modules of finite* FP*-injective dimension.*

Proof. We know that $(\mathcal{F}, \mathcal{C})$ and $(\mathcal{GF}, \mathcal{GC})$ are each complete hereditary cotorsion pairs. Again, let $\mathcal{W}$ denote the class of modules with finite FP-injective dimension. Using the results of [**Hov02**], the only thing to check is that $\mathcal{GF} \cap \mathcal{W} = \mathcal{F}$ and $\mathcal{C} \cap \mathcal{W} = \mathcal{GC}$. It is clear that $\mathcal{F} \subseteq \mathcal{GF} \cap \mathcal{W}$. Recall that F is a flat left R-module if and only if $F^+ = \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ is injective as a right R-module. Thus, if $F \in \mathcal{W}$, F has finite flat dimension, then F^+ has finite injective dimension. Also, by Theorem 5 of [**DC96**], F is Gorenstein flat if and only if F^+ is Gorenstein injective when R is a Ding-Chen ring. Since we know $\mathcal{GI} \cap \mathcal{W}$ is the class of injectives, we conclude that $F \in \mathcal{GF} \cap \mathcal{W}$ implies F^+ is injective. But this means that F must be flat. So $\mathcal{F} = \mathcal{GF} \cap \mathcal{W}$.

Now by Proposition 3.14 it follows that $\mathcal{DP} \subseteq \mathcal{GF}$, and of course $\mathcal{F} \subseteq \mathcal{GF}$. So taking the right half of the associated cotorsion theories gives us $\mathcal{GC} \subseteq \mathcal{C} \cap \mathcal{W}$. Now suppose $X \in \mathcal{C} \cap \mathcal{W}$. Since $(\mathcal{GF}, \mathcal{GC})$ is complete we can find a short exact sequence $0 \to X \to C \to F \to 0$ where $C \in \mathcal{GC}$ and $F \in \mathcal{GF}$. Since both X and C are in $\mathcal{W}$, so is F. By the last paragraph, we see that $F \in \mathcal{F}$, which makes $0 \to X \to C \to F \to 0$ split and so X is a summand of C. Therefore $X \in \mathcal{GC}$. So $\mathcal{C} \cap \mathcal{W} = \mathcal{GC}$. ☐

5. Applications to group rings

As we commented in Section 4 we are interested in examples of Ding-Chen rings with infinite weak dimension. A good example comes from considering group rings. These model structures are relevant for defining Tate cohomology.

Proposition 5.1. *Let R be a commutative Ding-Chen ring. Then the group ring $R[G]$ is a Ding-Chen ring for any finite group G.*

Proof. First lets recall some things about the group ring $R[G]$. The ring inclusion $R \hookrightarrow R[G]$ induces, by restricting scalars, a functor $U\colon$ Mod-$R[G] \to R$-Mod. The functor $- \otimes_R R[G]\colon R$-Mod $\to$ Mod-$R[G]$ is left adjoint to U. On the other hand, $\mathrm{Hom}_R(R[G], -)$ is right adjoint to U. (Here the right $R[G]$-module structure on $\mathrm{Hom}_R(R[G], M)$ is define by $(fr)(s) = f(rs)$.) Since G is finite we have $\mathrm{Hom}_R(R[G], M) \cong M \otimes_R R[G]$ as right $R[G]$-modules. Similarly, we have a forgetful functor $V\colon R[G]$-Mod $\to R$-Mod with left adjoint functor

$$R[G] \otimes_R - \cong \mathrm{Hom}_R(R[G], -).$$

(This time the left $R[G]$-module structure on $\mathrm{Hom}_R(R[G], M)$ is define by $(rf)(s) = f(sr)$.)

First we show that $R[G]$ is right coherent. For any indexing set I, the product $\prod_{i\in I} R$ is a flat R-module by Chase's theorem (Theorem 4.47 of [**Lam99**]). Since $-\otimes_R R[G]$ is left adjoint to an exact functor it preserves flat modules. Therefore $(\prod_{i\in I} R)\otimes_R R[G]$ is a flat right $R[G]$-module. But since G is finite, $-\otimes_R R[G]$ is also a right adjoint, so $(\prod_{i\in I} R)\otimes_R R[G] \cong \prod_{i\in I}(R\otimes_R R[G]) \cong \prod_{i\in I} R[G]$ is a flat right $R[G]$-module. It follows, again from Chase's theorem, that $R[G]$ is left coherent. Arguing on the other side with $R[G]\otimes_R -$ shows that $R[G]$ is right coherent as well.

Note that since G is finite, U preserves finitely generated modules. Since U is exact it also preserves finitely presented modules. Since $\mathrm{Hom}_R(R[G],-)\colon R\text{-Mod} \to \text{Mod-}R[G]$ is also exact, one can prove

$$\mathrm{Ext}^n_R(U(M),N) \cong \mathrm{Ext}^n_{R[G]}(M,\mathrm{Hom}_R(R[G],N)).$$

From these two observations, it follows immediately that $\mathrm{Hom}_R(R[G],-)$ preserves FP-injective modules. Now since $R[G]$ is coherent and $\mathrm{Hom}_R(R[G],-)$ preserves FP-injective modules, it follows from Lemma 3.1 of [**Sten70**] that $\mathrm{Hom}_R(R[G],-)$ preserves modules of finite FP-injective dimension. Thus $R[G] \cong \mathrm{Hom}_R(R[G],R)$ has finite right self-FP-injective dimension whenever R is a commutative Ding-Chen ring. Arguing on the other side with V and $\mathrm{Hom}_R(R[G],-)\colon R\text{-Mod} \to R[G]\text{-Mod}$ we see that $R[G]$ had finite left self-FP-injective dimension whenever R is a commutative Ding-Chen ring. □

Damiano points out in [**Dam79**] the following result of Colby. A group ring $R[G]$ is an FC ring if and only if R is an FC ring and G is a locally finite group. This raises the following question: Let R be a commutative ring. Is the group ring $R[G]$ a Ding-Chen ring if and only if R is a Ding-Chen ring and G is a locally finite group?

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