Ergodic theorems for dynamical semi-groups on operator algebras

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1. Introduction

Recently the theory of quantum dynamical semi-groups have made very interesting progress ([2, 3, 7, 13]). In the present paper we shall show various ergodic theorems (that is, a mean ergodic theorem, an individual ergodic theorem and a local ergodic theorem) for such dynamical semi-groups. As is well known, von Neumann's mean ergodic theorem was generalized to many abstract spaces.

In 1966, I. Kovács and J. Szücs [9] proved an ergodic type theorem which is considered as a non-commutative mean ergodic theorem and has many applications in mathematical physics. The individual ergodic theorem was also investigated by many authors, but the almost everywhere convergence can not be discussed on abstract spaces. However, recently E. C. Lance [12] and Y. G. Sinai and V. V. Anshelvich [17] have shown non-commutative analogues of the Birkhoff's individual ergodic theorem for automorphisms of operator algebras. As is stated in [12], [17], we can discuss "almost everywhere convergence" on von Neumann algebras because of its rich structure as the natural non-commutative generalization of L^∞-algebras over probability measure spaces. There is no notion of underlying measure space in the present situation. However, Egoroff's theorem implies that a sequence of measurable functions converges almost everywhere if and only if the sequence converges uniformly on measurable subsets of measure arbitrary close to 1. The uniform convergence for bounded functions corresponds to the norm convergence in L^∞-algebras and measurable subsets correspond to its characteristic functions. Thus we may restate the almost everywhere convergence by means of the measure, L^∞-norm and the characteristic functions. This restatements fit naturally into our setting. On the other hand, the so-called local ergodic theorem, which was first established by N. Wiener, gives a powerful tool to investigate the asymptotic behavior of dynamical systems as the time parameter $t \to +0$. This result was also extended to the general settings by many authors. We present

a non-commutative analogue of the local ergodic theorem. To our knowledge, this has never been discussed for operator algebras.

2. Preliminaries

We assume familiarity with the basic theory of von Neumann algebras as contained in [4], [16] and [18]. Let M be a von Neumann algebra and let M_* be the predual of M. Since M is the conjugate space of M_*, M_* has the natural embedding in M^* and we shall identify M_* with its image in M^* which is characterized as the space of all ultraweakly continuous linear functionals of M.

A (quantum) dynamical semi-group on M is a one-parameter semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ of normal positive linear maps of M into itself such that (1) $\alpha_0 = I_M$ (the identity map of M) (2) $\alpha_t(1) = 1$ for all $t \geq 0$ (1 denotes the identity element of M) (3) α is ultraweakly continuous (that is, the function $t \to \phi(\alpha_t(A))$ is continuous in t on $[0, \infty)$ for each $\phi \in M_*$ and $A \in M$). Since each α_t is normal positive and $\alpha_t(1) = 1$, it has the preadjoint α_{t*} which means that α_t is the adjoint map of α_{t*}, and the family $\alpha_* \equiv \{\alpha_{t*} ; \ t \geq 0\}$ becomes a weakly (hence strongly) continuous one-parameter semi-group of positive contractions on M_*.

Let Φ be a unital 2-positive linear map. Then $\Phi(A)^*\Phi(A) \leq \Phi(A^*A)$ for all A in M ([1, 6]). We denote by M^α the set of all fixed elements of M with respect to α, that is, $M^\alpha = \{A \in M : \alpha_t(A) = A$ for all $t \geq 0\}$, then M^α is a ultraweakly closed self adjoint linear subspace of M. If each α_t is 2-positive and there exists a faithful family of α-invariant states of M, then M^α is a von Neumann subalgebra of M because of the above inequality.

Now for each element A of M and $f \in L^1(R^+)$, an integral $\int_0^\infty f(t)\, \alpha_t(A)\, dt$ is well-defined in the sense that $\phi\left(\int_0^\infty f(t)\, \alpha_t(A)\, dt\right) = \int_0^\infty \phi(\alpha_t(A)) f(t)\, dt$ for every $\phi \in M_*$. Indeed, $\phi(\alpha_t(A))$ is a bounded continuous function of t for fixed ϕ and A from the ultraweak continuity of the dynamical semi-group and hence $\int_0^\infty \phi(\alpha_t(A)) f(t)\, dt$ is a continuous linear functional on M_*. We denote by $\int_0^T \alpha_t(A)\, dt$ the above integral for the characteristic function f of $[0, T]$ $(0 < T < \infty)$ and by $f * \alpha(A)$ for a general function $f \in L^1(R^+)$. Then it is easy to verify the inequality $\|f * \alpha(A)\| \leq \|A\| \int_0^\infty |f(t)|\, dt$, and we shall use other elementary properties of this integral without any references.

3. Statement of Main Theorem

In this section we shall state main theorems. The first result is a mean ergodic theorem for dynamical semi-groups. The results for automorphism groups were obtained in [9, 12, 15]. The existence of the conditional expectation in Theorem A was shown in [11] for semi-groups of certain positive maps. However, the present formulation fits naturally into Theorem B.

Assertion (3) of Theorem A is a non-commutative generalization of the classical mean ergodic theorem for positive contraction semi-groups on L^1-spaces.

THEOREM A. (*A non-commutative mean ergodic theorem*) *Let M be a von Neumann algebra and $\alpha=\{\alpha_t\}_{t\geq 0}$ a dynamical semi-group on M. Suppose that there exists a α-invariant faithful normal state ρ on M. Then the followings hold.*

(1) *For each A in M, the strong limit of $\dfrac{1}{T}\int_0^T \alpha_t(A)\,dt$ as $T\to+\infty$ exists, which will be denoted by $\varepsilon(A)$.*

(2) *ε is a faithful normal positive norm one projection of M onto M^α, and $\varepsilon\circ\alpha_t=\alpha_t\circ\varepsilon=\varepsilon$ for all $t\geq 0$, and $\rho\circ\varepsilon=\rho$. If every α_t is 2-positive, then ε is a conditional expectation of M onto the von Neumann subalgebra M^α.*

(3) *For each $\phi\in M_*$, $\dfrac{1}{T}\int_0^T \alpha_{t*}(\phi)\,dt$ converges in the norm of M_* as $T\to+\infty$ and this limit equals to $\varepsilon_*(\phi)$ where ε_* denotes the preadjoint of ε. Moreover, ε_* is a norm one projection onto $M_*^{\alpha*}\equiv\{\phi\in M_* : \alpha_{t*}(\phi)=\phi$ for all $t\geq 0\}$.*

The second result is a non-commutative analogue of the individual ergodic theorem. The result for discrete automorphism groups was proved in [12] and the similar result was obtained in [17] for the group of translations on the algebra of quasi-local observables.

THEOREM B. (*A non-commutative individual ergodic theorem*) *Let M, $\alpha=\{\alpha_t\}_{t\geq 0}$, ρ and ε be as in Theorem A. Then, for each A in M and positive real $\delta>0$, there exists a projection E in M such that $\rho(1-E)<\delta$ and*

$$\lim_{T\to+\infty}\left\|\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt-\varepsilon(A)\right)E\right\|=0\,.$$

Finally the non-commutative local ergodic theorem is formulated in the present situation as follows. The commutative case was proved in [10].

THEOREM C. (*A non-commutative local ergodic theorem*) *Let M, $\alpha=\{\alpha_t\}_{t\geq 0}$ and ρ be as in Theorem A. Then, for each A in M and $\delta>0$,*

there exists a projection E such that $\rho(1-E)<\delta$ and

$$\lim_{T\to+0}\left\|\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt - A\right)E\right\| = 0\,.$$

The proof of these three theorems will be given in the following sections.

4. Proof of Theorem A

The non-commutative mean ergodic theorems for automorphism groups are proved by applications of the classical mean ergodic theorems for Hilbert space operators. We pursue a similar line.

PROOF OF THEOREM A.

Let π_ρ be the Gelfand-Naimark-Segal representation of M associated with ρ on the Hilbert space H_ρ. Since ρ is faithful normal, π_ρ is faithful normal and $\pi_\rho(M)$ is a von Neumann algebra with the cyclic and separating vector ξ_ρ ([4]). Put $K_\rho=$ the norm closure of $\{\pi_\rho(A)\xi_\rho:\ A=A^*$ in $M\}$. By allowing multiplication by real scalars only, we can consider H_ρ as a real Hilbert space with the inner product $\langle\ ,\ \rangle=\mathrm{Re}(\ ,\ )$ where $(\ ,\ )$ is the inner product in the complex Hilbert space H_ρ constructed from ρ and $\mathrm{Re}\ z$ means the real part of a complex number z. We remark that the two inner products $\langle\ ,\ \rangle$ and $(\ ,\ )$ define the same norm in H_ρ. Hence K_ρ becomes a real Hilbert subspace of the real Hilbert space H_ρ. Since each α_t is positive and $\alpha_t(1)=1$, $(\alpha_t(A))^2\leq\alpha_t(A^2)$ for all hermitian A in M ([8]).

Put $T_t\pi_\rho(A)\,\xi_\rho=\pi_\rho(\alpha_t(A))\,\xi_\rho$ for hermitian element A in $M(t\geq0)$. Then each T_t is a real linear contraction on the dense real linear subspace $\{\pi_\rho(A)\,\xi_\rho:\ A=A^*$ in $M\}$ of K_ρ by the above inequality for α_t. We denote its extension to K_ρ also by T_t. Then the family $\{T_t;\ t\geq0\}$ becomes a weakly (hence strongly) continuous one-parameter semi-group of real linear contractions on K_ρ. Then the proof will proceed along the following three steps.

Step 1. (The existence of the limit in (1))

Let E be the orthogonal projection onto $\{\xi\in K_\rho;\ T_t\xi=\xi$ for all $t\geq0\}$. Then, from the mean ergodic theorem for the contraction semi-group $\{T_t\}$ (see [5]), $\dfrac{1}{T}\displaystyle\int_0^T T_t\,dt$ converges strongly to E as $T\to+\infty$. Hence for any hermitian A in M and $X\in\pi_\rho(M)'$ (the commutant of $\pi_\rho(M)$), we have

$$\pi_\rho\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt\right)X\xi_\rho = X\pi_\rho\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt\right)\xi_\rho$$

$$= X\frac{1}{T}\int_0^T T_t\,dt\,\pi_\rho(A)\,\xi_\rho\,.$$

Hence $\pi_\rho\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt\right)X\xi_\rho$ converges to $XE\pi_\rho(A)\,\xi_\rho$ in the norm of K_ρ (hence in the norm of H_ρ) as $T\to+\infty$. Hence for any A in M and $X\in\pi_\rho(M)'$, $\pi_\rho\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt\right)X\xi_\rho$ converges to a limit in H_ρ as $T\to+\infty$. Since $\left\{\pi_\rho\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt\right)\,;\,T>0\right\}$ is uniformly bounded and since $\{X\xi_\rho\,;\,X\in\pi_\rho(M)'\}$ is dense in H_ρ by the separation property of ξ_ρ, $\pi_\rho\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt\right)$ converges strongly to a limit as $T\to+\infty$. Since $\pi_\rho(M)$ is strongly closed ([4]), this limit must belong to $\pi_\rho(M)$, that is, there is an uniquely determined element of M, denoted by $\varepsilon(A)$, such that $\pi_\rho\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt\right)\to\pi_\rho(\varepsilon(A))$ strongly as $T\to+\infty$. Since π_ρ is faithful normal, this means that $\frac{1}{T}\int_0^T \alpha_t(A)\,dt$ converges to $\varepsilon(A)$ strongly.

Step 2. (Properties of ε)

From the definition of ε, it is easy to see that ε is an unital norm one positive linear map, and $\rho\circ\varepsilon=\rho$ which implies that ε is faithful.

To see the normality of the map ε it suffices to prove that $0\leq A_r\leq 1$ and $A_r\downarrow_r 0$ implies that the weak convergence of $\pi_\rho(\varepsilon(A_r))$ to 0. Fix $\xi\in H$ and $X\in\pi_\rho(M)'$. Then by considering the real part and the imaginary part of the bounded real linear functional $(\cdot,X^*\xi)$ on K_ρ, we have η_1 and η_2 in K_ρ such that $(\cdot,X^*\xi)=\langle\cdot,\eta_1\rangle+i\langle\cdot,\eta_2\rangle$ on K_ρ. Hence we have

$$\left(\pi_\rho\big(\varepsilon(A_r)\big)X\xi_\rho,\xi\right)$$
$$=\left(E\pi_\rho(A_r)\xi_\rho,X^*\xi\right)$$
$$=\left\langle E\pi_\rho(A_r)\xi_\rho,\eta_1\right\rangle+i\left\langle E\pi_\rho(A_r)\xi_\rho,\eta_2\right\rangle$$
$$=\mathrm{Re}\left(\pi_\rho(A_r)\xi_\rho,E\eta_1\right)+i\,\mathrm{Re}\left(\pi_\rho(A_r)\xi_\rho,E\eta_2\right)$$

From this relation and the fact that $A_r\downarrow_r 0$ implies $\pi_\rho(A_r)\downarrow_r 0$, it follows easily that $\pi_\rho(\varepsilon(A_r))$ converges to 0 weakly.

Next, from the definition of ε, we have $\varepsilon\circ\alpha_t=\alpha_t\circ\varepsilon=\varepsilon$ ($t\geq 0$), $\varepsilon\circ\varepsilon=\varepsilon$ and $M^\alpha=\varepsilon(M)$. Thus ε is a normal norm one projection of M onto M^α.

If every α_t is 2-positive, then ε is a conditional expectation of M onto M^α by Tomiyama's Theorem [19], because M^α is a von Neumann subalgebra of M as is stated in section 2.

Step 3. (The assertion (3))

Since ε is ultraweakly continuous, it has a preadjoint ε_* which is a bounded linear operator on M_*. Fix X_1, X_2 in $\pi_\rho(M)'$. Then, by the same reason as in step 2, we have η_1 and η_2 in K_ρ such that $(\cdot, X_1^* X_2 \xi_\rho) = \langle \cdot, \eta_1 \rangle + i\langle \cdot, \eta_2 \rangle$ on K_ρ. Put $\phi(A) = (\pi_\rho(A) X_1 \xi_\rho, X_2 \xi_\rho)$ $(A \in M)$. Then we have

$$\left\| \frac{1}{T} \int_0^T \alpha_{t*}(\phi)\, dt - \varepsilon_*(\phi) \right\|$$

$$= \sup_{\|A\| \leqslant 1, A \in M} \left| A\left(\frac{1}{T} \int_0^T \alpha_{t*}(\phi)\, dt \right) - A\left(\varepsilon_*(\phi) \right) \right|$$

$$\leqq 2 \sup_{\|A\| \leqslant 1, A = A^* \in M} \left| \left(\left(\frac{1}{T} \int_0^T \alpha_t(A)\, dt - \varepsilon(A) \right)(\phi) \right| $$

$$= 2 \sup_{\|A\| \leqslant 1, A = A^* \in M} \left| \left(\pi_\rho\left(\frac{1}{T} \int_0^T \alpha_t(A)\, dt \right) X_1 \xi_\rho, X_2 \xi_\rho \right) \right.$$

$$\left. - \left(\pi_\rho\left(\varepsilon(A) \right) X_1 \xi_\rho, X_2 \xi_\rho \right) \right|$$

$$= 2 \sup_{\|A\| \leqslant 1, A = A^* \in M} \left| \left(\left(\frac{1}{T} \int_0^T T_t\, dt - E \right) \pi_\rho(A)\, \xi_\rho, X_1^* X_2 \xi_\rho \right) \right|$$

$$= 2 \sup_{\|A\| \leqslant 1, A = A^* \in M} \left| \left\langle \left(\frac{1}{T} \int_0^T T_t\, dt - E \right) \pi_\rho(A)\, \xi_\rho, \eta_1 \right\rangle \right.$$

$$\left. + i\left\langle \left(\frac{1}{T} \int_0^T T_t\, dt - E \right) \pi_\rho(A)\, \xi_\rho, \eta_2 \right\rangle \right|$$

$$= 2 \sup_{\|A\| \leqslant 1, A = A^* \in M} \left| \left\langle \pi_\rho(A)\, \xi_\rho, \left(\frac{1}{T} \int_0^T T_t\, dt - E \right)^* \eta_1 \right\rangle \right.$$

$$\left. + i\left\langle \pi_\rho(A)\, \xi_\rho, \left(\frac{1}{T} \int_0^T T_t\, dt - E \right)^* \eta_2 \right\rangle \right|$$

$$\leqq 2 \left\| \left(\frac{1}{T} \int_0^T T_t^*\, dt - E \right) \eta_1 \right\| + 2 \left\| \left(\frac{1}{T} \int_0^T T_t^*\, dt - E \right) \eta_2 \right\|$$

where T_t^* is the adjoint operator of T_t with respect to the inner product $\langle \, , \, \rangle$ in K_ρ.

Since $\{T_t^*\}_{t \geq 0}$ is a weakly (hence strongly) continuous contraction semi-group on K_ρ, by the mean ergodic theorem $\frac{1}{T} \int_0^T T_t^*\, dt$ converges strongly to the projection onto $\{\xi \in K_\rho : T_t^* \xi = \xi \text{ for all } t \geq 0\}$. Since each T_t is a contraction, the subspace coincides with $\{\xi \in K_\rho : T_t \xi = \xi \text{ for all } t \geq 0\}$. Therefore the last term in the above inequality converges to 0 as $T \to +\infty$.

Since ρ is faithful, the linear span of the family $\{(\pi_\rho(\cdot) X_1 \xi_\rho, X_2 \xi_\rho) : X_1, X_2 \in \pi_\rho(M)'\}$ is norm dense in M_* ([16]). Hence for every $\phi \in M_*$, $\frac{1}{T} \int_0^T \alpha_{t*}(\phi)\, dt$

converges to $\varepsilon_*(\phi)$ as $T \to +\infty$ in norm. The remainder of the proof follows immediately from the properties of ε. Thus all the proof are completed.

5. Proof of Theorem B

For the proof of Theorem B we need five lemmas. The first two lemmas are technical ones. Before we state the lemmas, define the functions $f_k(t)$ $(k=1, 2, 3, \cdots)$ as follows:

$$f_k(t) = \begin{cases} \dfrac{2}{\pi}\left(\dfrac{\frac{1}{k}}{\left(\frac{1}{k}\right)^2+t^2} - \dfrac{k}{k^2+t^2}\right) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then $\|f_k\|_1 \leq 2$ and $\displaystyle\int_{-\infty}^{\infty} f_k(t)\, dt = 0$ for every $k \geq 1$.

LEMMA 5. 1. *Let* $\{T_t\}_{t \geq 0}$ *be a strongly continuous one-parameter semi-group of contractions on a (real or complex) Hilbert space* H *such that* $T_0 = I$, *and let* E *be the orthogonal projection onto* $\{\xi \in H; T_t\xi = \xi$ *for all* $t \geq 0\}$. *Let* f_k *be as defined above. Then, for every* $\xi \in H$,

$$\left\| \int_0^{\infty} f_k(t)\, T_t\xi\, dt - (\xi - E\xi) \right\| \longrightarrow 0 \ as\ k \longrightarrow \infty .$$

PROOF. At first, we assume that H is a real Hilbert space and each T_t is a real linear contraction on H. Then we shall consider the complexification H_C of H. More precisely, let H_C denote the cartesian product $H \times H$ in which the algebraic operations and an inner product as a complex space are defined by the relations,

$$\{\xi_1, \xi_2\} + \{\eta_1, \eta_2\} = \{\xi_1+\eta_1, \xi_2+\eta_2\}$$
$$(\alpha+i\beta)\{\xi, \eta\} = \{\alpha\xi-\beta\eta, \alpha\eta+\beta\xi\} .$$
$$\left\langle \{\xi_1, \xi_2\}, \{\eta_1, \eta_2\} \right\rangle$$
$$= (\xi_1, \eta_1)+(\xi_2, \eta_2)+i\{(\xi_2, \eta_1)-(\xi_1, \eta_2)\}$$

where $(,)$ is the inner product of H.

Then H_C becomes a complex Hilbert space. For each T_t, put $\hat{T}_t\{\xi, \eta\} = \{T_t\xi, T_t\eta\}$ for $\{\xi, \eta\} \in H_C$. Then it is readily verified that $\{\hat{T}_t\}_{t \geq 0}$ is a strongly continuous one-parameter semigroup of complex linear contractions on H_C such that $\hat{T}_0 = I_{H_C}$. A bounded operator $\hat{E}$, which is defined by $\hat{E}\{\xi, \eta\} = \{E\xi, E\eta\}$, is the orthogonal projection onto $\{\{\xi, \eta\} \in H_C; \hat{T}_t(\{\xi, \eta\}) = \{\xi, \eta\}$

for all $t \geq 0$}. For any $\xi \in H$, we have

$$\left\| \frac{1}{T} \int_0^T f_k(t) \, \hat{T}_t \{\xi, 0\} \, dt - \left(\{\xi, 0\} - \hat{E} \{\xi, 0\} \right) \right\|$$

$$= \left\| \frac{1}{T} \int_0^T f_k(t) \, T_t \xi dt - (\xi - E\xi) \right\|$$

from the definition of the norm of H_C and easy calculations. Therefore, it is sufficient to prove the lemma for complex linear contractions $\{T_t\}_{t \geq 0}$ on a complex Hilbert space H.

Let $\{U_t\}_{-\infty < t < \infty}$ be a unitary dilation of the complex linear contraction semi-group $\{T_t\}_{t \geq 0}$, that is, $\{U_t\}_{-\infty < t < \infty}$ is a strongly continuous one-parameter group of unitary operators on a Hilbert space K containing H as a closed subspace such that $U_t = P_H T_t|_H$ for $t \geq 0$, where P_H is the orthogonal projection onto H ([14]). Then by the Stone's Theorem we have the spectral representation

$$U_t = \int_{-\infty}^{\infty} e^{it\lambda} dE_\lambda \qquad (-\infty < t < \infty).$$

Then for each $\xi \in H$,

$$\int_0^{\infty} f_k(t) \, T_t \xi dt$$

$$= P_H \left(\int_0^{\infty} \int_{-\infty}^{\infty} f_k(t) \, e^{it\lambda} dE_\lambda \xi dt \right)$$

$$= P_H \left(\int_{-\infty}^{\infty} \int_0^{\infty} f_k(t) \, e^{it\lambda} dt dE_\lambda \xi \right).$$

Put $g_k(\lambda) = \int_0^{\infty} f_k(t) \, e^{it\lambda} dt$. Then each $g_k(\lambda)$ is continuous in λ and $\{g_k\}_{k=1}^{\infty}$ is uniformly bounded. Moreover, a simple calculation implies that

$$g_k(\lambda) = \frac{2}{\pi} \left(\int_0^{\infty} \frac{e^{i\frac{1}{k}s\lambda}}{1+s^2} ds - \int_0^{\infty} \frac{e^{iks\lambda}}{1+s^2} ds \right),$$

and hence by the dominated convergence Theorem and a property of the Fourier transform, we have $\lim_{k \to \infty} g_k(\lambda) = 1$ for any non-zero λ, and $g_k(0) = 0$ for all k. Thus it follows that

$$\lim_{k \to \infty} \int_0^{\infty} f_k(t) \, T_t \xi dt = P_H \left(\xi - \int_{-\infty}^{\infty} \chi_0(\lambda) \, dE_\lambda \xi \right) = P_H \left(\xi - (E_0 - E_{0-0}) \, \xi \right)$$

for $\xi \in H$ where χ_0 is the characteristic function of $\{0\}$. Since $E_0 - E_{0-0}$ is the projection onto the space of all fixed points of $\{U_t\}$, the relation $P_H U_t|_H = T_t (t \geq 0)$ implies $P_H (E_0 - E_{0-0})|_H = E$. Thus we have the desired conclusion.

The next lemma is a continuous version of Lemma 4. 1 in [12].

LEMMA 5. 2.   *If f is an $L^1(R)$-function, then*

$$\lim_{T\to+\infty}\int_{-\infty}^{\infty}\left|\frac{1}{T}\int_{0}^{T}f(s-t)\,dt\right|ds = 0 .$$

PROOF.   Consider the one-parameter contraction semi-group $\{V_t\}_{t\geq 0}$ of translations on $L^1(R)$, that is, $V_t h(s)=h(s-t)$ for $h\in L^1(R)$. It is known (see [5]) that for this semi-group the mean ergodic theorem holds in $L^1(R)$:
$\frac{1}{T}\int_{0}^{T}V_t dt$ converges strongly to 0 as $T\to+\infty$. The assertion of the lemma follows from the relation

$$\int_{-\infty}^{\infty}\left|\frac{1}{T}\int_{0}^{T}f(s-t)\,dt\right|ds = \left\|\left(\frac{1}{T}\int_{0}^{T}V_t dt\right)f\right\|_1 .$$

Now we shall show an uniform ergodic lemma which is one of the key steps.

LEMMA 5. 3.   *Let M, $\alpha=\{\alpha_t\}_{t\geq 0}$, ρ and ε be as in Theorem B, and f_k as in Lemma 5.1. For each $A\in M$, define $A_k=f_k*\alpha(A)+\varepsilon(A)$ $(k\geq 1)$. Then $\|A_k\|\leq 3\|A\|$ and A_k converges to A in the strong*-topology as $k\to\infty$, and $\lim_{T\to+\infty}\left\|\frac{1}{T}\int_{0}^{T}\alpha_t(A_k)\,dt-\varepsilon(A_k)\right\|=0$ $(k\geq 1)$.*

PROOF.   The first inequality follows easily. We shall show the last assertion.

$$\left\|\frac{1}{T}\int_{0}^{T}\alpha_s(A_k)\,ds-\varepsilon(A_k)\right\|$$

$$=\left\|\frac{1}{T}\int_{0}^{T}\int_{0}^{\infty}f_k(t)\,\alpha_{t+s}(A)\,dtds-\varepsilon\left(f_k*\alpha(A)\right)\right\|$$

$$=\left\|\frac{1}{T}\int_{0}^{T}\int_{s}^{\infty}f_k(h-s)\,\alpha_h(A)\,dhds-\varepsilon(A)\int_{0}^{\infty}f_k(t)\,dt\right\|$$

$$=\left\|\frac{1}{T}\int_{0}^{T}\int_{0}^{\infty}f_k(h-s)\,\alpha_h(A)\,dhds\right\|$$

$$=\left\|\int_{0}^{\infty}\left(\frac{1}{T}\int_{0}^{T}f_k(h-s)\,ds\right)\alpha_h(A)\,dh\right\|$$

$$\leq\|A\|\int_{0}^{\infty}\left|\frac{1}{T}\int_{0}^{T}f_k(h-s)\,ds\right|dh .$$

Then, from Lemma 5. 2 it follows that $\lim_{T\to+\infty}\left\|\frac{1}{T}\int_{0}^{T}\alpha_t(A_k)\,dt-\varepsilon(A_k)\right\|=0$.
Next, let $(H_\rho,\pi_\rho,\xi_\rho)$, K_ρ and $\{T_t\}_{t\geq 0}$, E be as in the proof of Theorem A. If A is hermitian, then A_k is hermitian and we have

$$\pi_\rho(A_k)\,\xi_\rho = \int_0^\infty f_k(t)\,T_t\pi_\rho(A)\,\xi_\rho\,dt + E\pi_\rho(A)\,\xi_\rho, \quad \text{so} \quad \pi_\rho(A_k)\,\xi_\rho \longrightarrow \pi_\rho(A)\,\xi_\rho$$

$(k\to\infty)$ by Lemma 5.1. Hence, from the same arguments as in the proof of step (1) of Theorem A, $A_k\to A$ strongly as $k\to\infty$.

For two hermitian elements B, C in M, we have

$$(B+iC)_k = f_k * \alpha(B+iC) + \varepsilon(B+iC)$$

$$= f_k * \alpha(B) + \varepsilon(B) + i\big(f_k * \alpha(C) + \varepsilon(C)\big)$$

$$= B_k + iC_k \,.$$

Hence, for any A in M, $A_k\to A$ in the strong*-topology as $k\to\infty$. This completes the proof.

Now we need the following crucial lemma which is essentially due to E. C. Lance [12] and is considered as a non-commutative maximal ergodic theorem.

LEMMA 5.4. (*E. C. Lance*) *Let M be a von Neumann algebra and Φ be a normal positive linear map on M such that $\Phi(1)=1$, and let ρ be a Φ-invariant faithful normal state of M. Let B be a positive element with norm less than 1. Then there exists a positive element C in M such that $\|C\|\leqq 2$, $\rho(C)\leqq 4(\rho(B))^{\frac{1}{2}}$ and $\dfrac{1}{n}(B+\Phi(B)+\cdots+\Phi^{(n-1)}(B))\leqq C$ for every integer n.*

The proof of this lemma follows from the careful inspection of Theorem 2.1, Lemma 5.1 and 5.2 in [12].

The following lemma is easily obtained from the above lemma.

LEMMA 5.5. *Let M be a von Neumann algebra and $\alpha = \{\alpha_t\}_{t\geqq 0}$ a dynamical semi-group on M. Suppose that there exists a α-invariant faithful normal state ρ of M. Let B be a positive element in the unit ball of M. Then there exists a positive element C in M such that $0\leqq C\leqq 2$, $\rho(C) \leqq 4(\rho(B))^{\frac{1}{2}}$ and $\dfrac{1}{T}\int_0^T \alpha_t(B)\,dt \leqq C + \dfrac{1}{T}1$ for all $T\geqq 1$.*

PROOF. Put $A=\displaystyle\int_0^1 \alpha_t(B)\,dt$, then we have

$$\frac{1}{T}\int_0^T \alpha_t(B)\,dt$$

$$= \frac{1}{T}\Big(A+\alpha_1(A)+\cdots+\alpha_1^{N-1}(A)\Big) + \frac{1}{T}\int_N^{N+r} \alpha_t(B)\,dt$$

$$\leqq \frac{1}{N}\Big(A+\alpha_1(A)+\cdots+\alpha_1^{N-1}(A)\Big) + \frac{1}{T}(\alpha_1)^N\int_0^1 \alpha_t(B)\,dt$$

where N is the integral part of T and $r = T - N$. Since A and α_1 satisfy all assumptions of Lemma 5.4, and $\rho(A) = \rho(B)$, there exists an element C in M which is independent of T and satisfies the conclusions of Lemma 5.5.

Now we are in the position to prove Theorem B. For the proof, let us introduce a convenient notation. For the canonical decomposition $A = A_{(1)} - A_{(2)} + i(A_{(3)} - A_{(4)})$ of $A \in M$ into the positive elements put $[A] = A_{(1)} + A_{(2)} + A_{(3)} + A_{(4)}$. Then $0 \leq A_{(k)} \leq [A] \leq 4||A||1$ $(1 \leq k \leq 4)$.

PROOF OF THEOREM B.

Fix an arbitrary $A \in M$ and consider the element A_k in Lemma 5.3. Then we may assume that $||[A - A_k]|| \leq 1$ for all $k \geq 1$. By Lemma 5.5, for each integer $k \geq 1$, there exists C_k in M such that $0 \leq C_k \leq 2$, $\rho(C_k) \leq 4(\rho([A - A_k]))^{\frac{1}{2}}$ and $\frac{1}{T} \int_0^T \alpha_t([A - A_k]) \, dt \leq C_k + \frac{1}{T} 1$. Then, from Lemma 5.3, $A - A_k \to 0$ in the strong*-topology and hence $[A - A_k] \to 0$ strong ([12]). From the above inequality, $\rho(C_k) \to 0$, so $C_k \to 0$ strongly because ρ is faithful. Then by the Saito's non-commutative Egoroff's Theorem ([18]), for any $\delta > 0$ there exists a projection E_0 in M and a subsequence (we also denote this subsequence by $\{C_k\}$) of $\{C_k\}$ such that $\rho(1 - E_0) < \frac{\delta}{2}$ and $||C_k E_0|| \leq \frac{1}{k}$ for all $k \geq 1$. We also denote by $\{A_k\}$ the subsequence of $\{A_k\}$ corresponding to $\{C_k\}$. Then

$$\left\| \frac{1}{T} \int_0^T \alpha_t([A - A_k]) \, dt E_0 \right\| \leq \left(\frac{1}{k} + \frac{1}{T} \right)^{\frac{1}{2}}$$

so that

$$\left\| \frac{1}{T} \int_0^T \alpha_t(A - A_k) \, dt E_0 \right\|$$

$$\leq \sum_{i=1}^4 \left\| \frac{1}{T} \int_0^T \alpha_t((A - A_k)_{(i)}) \, dt E_0 \right\|$$

$$\leq \sum_{i=1}^4 \left\| \frac{1}{T} \int_0^T \alpha_t([A - A_k]) \, dt E_0 \right\|^{\frac{1}{2}}$$

$$\leq 4 \left(\frac{1}{k} + \frac{1}{T} \right)^{\frac{1}{4}} \quad \text{for all } T > 1 \text{ and } k \geq 1.$$

Since ε is normal, it is continuous in the strong*-topology on every bounded set ([16]). Hence $\varepsilon(A_k) \to \varepsilon(A)$ in the strong*-topology. Hence there exists again a projection $E \leq E_0$ in M and a subsequence $\{A - A_{k_n}\}_{n \geq 1}$ of $\{A - A_k\}_{k \geq 1}$ $(k \to \infty$ as $n \to \infty)$ such that $\rho(E_0 - E) < \frac{\delta}{2}$ and $||(\varepsilon(A) - \varepsilon(A_k))E|| \leq \frac{1}{n}$. Consequently we have

$$\left\|\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt - \varepsilon(A)\right)E\right\|$$

$$\leq \left\|\frac{1}{T}\int_0^T \alpha_t(A - A_{k_n})\,dtE\right\| + \left\|\left(\frac{1}{T}\int_0^T \alpha_t(A_{k_n})\,dt - \varepsilon(A_{k_n})\right)E\right\|$$

$$+ \left\|\left(\varepsilon(A) - \varepsilon(A_{k_n})\right)E\right\|$$

$$\leq 4\left(\frac{1}{k_n} + \frac{1}{T}\right)^{\frac{1}{4}} + \left\|\left(\frac{1}{T}\int_0^T \alpha_t(A_{k_n})\,dt - \varepsilon(A_{k_n})\right)E\right\| + \frac{1}{n}\ .$$

Hence for an arbitrary fixed $n \geq 1$, we have

$$\varlimsup_{T \to +\infty}\left\|\left(\frac{1}{T}\int_0^T \alpha_t(A)\,dt - \varepsilon(A)\right)E\right\| \leq 4\left(\frac{1}{k_n}\right)^{\frac{1}{4}} + \frac{1}{n}$$

which implies the existence of the limit. Thus all the proof of Theorem B is completed.

6. Proof of Theorem C

The method of the proof of Theorem C is almost the same as Theorem B. We shall need three lemmas, of which key ones are Lemma 6.2 and 6.3.

LEMMA 6.1. *Let* $\{T_t\}_{t \geq 0}$ *be a strongly continuous one-parameter semi-group of contractions on a Hilbert space H such that $T_0 = I$. Define f_k as follows:*

$$f_k(t) = \begin{cases} k & 0 \leq t \leq \dfrac{1}{k} \\[2mm] 0 & 0 > t \ \text{or}\ t > \dfrac{1}{k} \end{cases}$$

Then for every $\xi \in H$, $\int_0^\infty f_k(t)\,T_t\xi\,dt \to \xi$ *(as $k \to \infty$) in the norm of H.*

PROOF. The proof follows from the relation

$$\left\|\int_0^\infty f_k(t)\,T_t\xi\,dt - \xi\right\| = \left\|k\int_0^{\frac{1}{k}} T_t\xi\,dt - \xi\right\| \qquad (\xi \in H)\ .$$

LEMMA 6.2. *Let* $\alpha = \{\alpha_t\}_{t \geq 0}$ *be a dynamical semi-group on a von Neumann algebra M and f_k as in Lemma 6.1. For each $A \in M$, define $A_k = f_k*\alpha(A)$ $(k = 1, 2, 3, \cdots)$. Then $\|A_k\| \leq \|A\|$, $A_k \to A$ as $k \to \infty$ in the strong $*$-topology and* $\displaystyle\lim_{T \to +0}\left\|\frac{1}{T}\int_0^T \alpha_t(A_k)\,dt - A_k\right\| = 0$ *for each $k \geq 1$.*

PROOF. The inequality $\|A_k\| \leq \|A\|$ is obvious. From the definition of A_k, we have for each $k \geq 1$,

$$\left\| \frac{1}{T} \int_0^T \alpha_s(A_k) \, ds - A_k \right\|$$

$$= \left\| \frac{1}{T} \int_0^T \int_0^\infty f_k(t) \, \alpha_{t+s}(A) \, dt ds - \int_0^\infty f_k(t) \, \alpha_t(A) \, dt \right\|$$

$$= \left\| \int_0^\infty \left(\frac{1}{T} \int_0^T f_k(h-s) \, ds - f_k(h) \right) \alpha_h(A) \, dh \right\|$$

$$\leq \|A\| \int_0^\infty \left| \frac{1}{T} \int_0^T f_k(h-s) \, ds - f_k(h) \right| dh$$

Then it follows that the last term converges to 0 from direct computations. The remainder of the proof follows from Lemma 6.1 and the same arguments as in the proof of Lemma 5. 3. The proof is completed.

The following lemma is considered as a non-commutative maximal ergodic theorem.

LEMMA 6. 3.   Let $\alpha = \{\alpha_t\}_{t \geq 0}$ be a dynamical semi-group on a von Neumann algebra M and ρ a α-invariant faithful normal state of M. Suppose that B is a positive element in the unit ball of M. Then there exists a positive element $C \in M$ such that $0 \leq C \leq 2$, $\rho(C) \leq 4(\rho(B))^{\frac{1}{2}}$ and $\frac{1}{T} \int_0^T \alpha_t(B) \, dt \leq C$ for all T in $(0, 1]$.

PROOF.   Put $D_n = \left\{ \frac{k}{2^n} \; ; \; k = 1, 2, 3, \cdots, 2^n \right\}$ for each integer $n \geq 1$, then $D_1 \subset D_2 \subset D_3 \subset \cdots \subset D_n \subset \cdots$. For an arbitrary fixed integer $n \geq 1$, define B_n and β_n as follows: $B_n = \int_0^{\frac{1}{2^n}} \alpha_t(B) \, dt$, $\beta_n = \alpha_{\frac{1}{2^n}}$. Then $2^n B_n$ and β_n satisfy all assumptions of Lemma 5. 4.   Since $\frac{1}{s} \int_0^s \alpha_t(B) \, dt = \frac{1}{N} \left(2^n B_n + \cdots + \beta_n^{N-1}(2^n B_n) \right)$ for $s = \frac{N}{2^n} \in D_n$ ($1 \leq N \leq 2^n$), by Lemma 5. 4, there exists a positive element $C_n \in M$ (which depends on n but not on N ($1 \leq N \leq 2^n$)) such that $0 \leq C_n \leq 2$, $\rho(C_n) \leq 4\rho(2^n B_n)^{\frac{1}{2}} = 4(\rho(B))^{\frac{1}{2}}$ and $\frac{1}{s} \int_0^s \alpha_t(B) \, dt \leq C_n$ for all $s \in D_n$. If $C \in M$ is a weak cluster point of $\{C_n\}$, then $0 \leq C \leq 2$, $\rho(C) \leq 4(\rho(B))^{\frac{1}{2}}$ and $\frac{1}{s} \int_0^s \alpha_t(B) \, dt \leq C$ for all $s \in \bigcup_{n=1}^\infty D_n$ because $\{D_n\}$ is the monotone increasing family. Moreover, $\bigcup_{n=1}^\infty D_n$ is dense in $(0, 1]$, thus we have $\frac{1}{T} \int_0^T \alpha_t(B) \, dt \leq C$ for every T in $(0, 1]$.

Now we shall show the local ergodic theorem.

PROOF OF THEOREM C.   Fix an arbitrary $A \in M$ and consider the ele-

ment A_k in Lemma 6. 2. By the same arguments as in the proof of Theorem B, we have a projection $E \in M$ such that $\rho(1-E) < \delta$ and

$$\varlimsup_{T \to +0} \left\| \left(\frac{1}{T} \int_0^T \alpha_t(A) \, dt - A \right) E \right\| \leq 4 \left(\frac{1}{k_n} \right)^{\frac{1}{4}} + \frac{1}{n}$$

for every $n \geq 1$, where $\{k_n\}$ is a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. Thus we have

$$\lim_{T \to +0} \left\| \left(\frac{1}{T} \int_0^T \alpha_t(A) \, dt - A \right) E \right\| = 0$$

which is the desired conclusion. The proof is completed.

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