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On quasi Dirichlet bounded harmonic functions

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In the present paper we denote by P, B, D ,H, SPH and SBH, positive, bounded, Dirichlet bounded, harmonic, superharmonic and subharmonic respectively. Let R be a Riemann surface $\notin O_g$ and let $\{R_n\}: n=0, 1, 2, \cdots$, be an exhaustion with compact analytic relative boundary ∂R_n. We call a domain G a subdomain, if ∂G consists of at most an enumerably infinite number of analytic curves clustering nowhere in R. In this note we use simply a domain G in the meaning that G is a sum of subdomains G_i such that $\Sigma\partial G_i$ clusters nowhere in R. Let R^∞ be the universal covering surface of R and map R^∞ onto $|\zeta|<1$. In the previous[1] paper we proved.

Let $U(z)$ be an HD function in R. Then $U(z)$ is a harmonic function $U(\zeta)$ in $|\zeta|<1$, $\varlimsup_{r\to 1}\int_0^{2\pi}|U(re^{i\theta})|^2\, d\theta<\infty$, $\zeta=re^{i\theta}$ and $U(\zeta)$ is representable by Poisson's integral. This is equivalent to $U(z)=U_1(z)-U_2(z)$, where $U_1(z)$ and $U_2(z)$ are positive *quasibounded harmonic* function (abbreviated by QHB).

The purpose of this paper is to extend the above theorem. Let G be a domain, if G is compact, we denote by H_g^G the solution of the Dirichlet problem with respect to the boundary value $g(z)$ on ∂G. If G is non compact and $g(z)\geqq 0$, we denote also by H_g^G the least positive $SPH \geqq g(z)$ on ∂G, *i. e.* $H_g^G=\lim_n H_{g_n}^{G\cap R_n}$, where $g_n(z)=g(z)$ on $\partial G\cap R_n$ and $g_n(z)=0$ on $\partial R_n\cap G$.

Let $G_2\subset G_1$ be domains. Let $w_{n,n+i}(z)$ be an HB in $G_1\cap R_{n+i}-G_2\cap(R_{n+i}-R_n)$ such that $w_{n,n+i}(z)=0$ on $\partial G_1\cap R_{n+i}+\partial R_{n+i}\cap(G_1-G_2)$, $=1$ on $(R_{n+i}-R_n)\cap G_2$. Then $w_{n,n+i}(z)\nearrow w_n(z)$ and $w_n(z)\downarrow : n\to\infty$. This limit is denote by $w(G_2\cap B, z, G_1)$ and is called $H. M.$[2] of $B\cap G_2$ relative to G_1. Let F be a closed set (or domain G). We denote $H_1^{CF}(H_1^{C\overset{o}{G}})$ by $w(F, z, R)$ $(w(G, z, R))$ simply. Let $U(z)$ be a positive SPH. If there exists no positive HB smaller than $U(z)$, we call $U(z)$ a *singular function.* If an HP is the limit of increasing sequence of HB functions, $U(z)$ is called *quasibounded harmonic function* (QHB). In this note we denote min $(M, U(z))$ by $U^M(z)$. Let $U(z)$ be a function (harmonic function). If

$$\lim_{\overline{M=\infty}} \frac{D(|U(z)|^M)}{M} = \alpha < \infty ,$$

i. e. there exists a sequence $\{M_i\}$ such that $\frac{D(|U|^{M_i})}{M_i} = \alpha_i, \alpha_i \to \alpha$. we call $U(z)$ a $QD(QHD)$ of *order* α denoted by $\alpha = \mathfrak{A}(U)$ (for $\{M_i\}$).

If a posistive SPH $U(z)$ is harmonic in R except at most a set of capacity zero and is a QD and *singular*, we call $U(z)$ a GG (*generalized Green function*).

LEMMA 1. *Let G be a domain and let R_0 be a compact disc. Then*

1) $w(G \cap B, z, R) = 0$ *if and only if* $w(G \cap B, z, R - R_0) = 0$.

2) *Let* $G^\delta = \{z \in G_1 : w(G_2 \cap B, z, G_1) < \delta < 1\}$. *Then* $w(B \cap G_2 \cap G^\delta, z, G_1) = 0$.

3) *A positive SPH $U(z)$ is singular, if and only if for any* $\varepsilon > 0$, $w(G_\varepsilon \cap B, z, R) = 0$: $G_\varepsilon = \{z \in R : U(z) > \varepsilon\}$.

4) *Let* $w'(z) = 1$ *on* $\bar{R}_0$ *and* $w'(z) = 1 - w(R \cap B, z, R - R_0)$. *Then* $w'(z)$ *is singular.*

5) *Let $G(z, p_0)$ be a Green function of R. Let $U(z)$ be singular and D be a domain and let* $D_\delta = \{z \in R : G(z, p_0) > \delta\}$. *Then*

$$H^{D \cap R_n}_{UM} \downarrow H^D_{UM} : n \to \infty \quad \text{and} \quad H^{D \cap D_\delta}_{UM} \downarrow H^D_{UM} : \delta \to 0 .$$

PROOF. 1) and 2) are proved in the previous paper[2)].

3) $H^{R_n}_{UM}$ is the greatest HP in R_n not larger than $U^M(z)$. Since $U^M(z)$ is an SPH, $H^{R_n}_{UM} \downarrow : n \to \infty$. $\lim_n H^{R_n}_{UM}$ is the greatest HP in R not larger than $U^M(z)$. Hence $U(z)$ is singular, if and only if $\lim_n H^{R_n}_{UM} = 0$ for any $M < \infty$. Suppose $w(G_\varepsilon \cap B, z, R) = 0$ for any $\varepsilon > 0$. Then $H^{R_n}_{UM} \leqq \varepsilon$ on $\partial R_n - G_\varepsilon$, $H^{R_n}_{UM} \leqq M = Mw((R - R_n) \cap G_\varepsilon, z, R)$ on $\partial R_n \cap G_\varepsilon$, whence

$$H^{R_n}_{UM} \leqq \varepsilon + Mw((R - R_n) \cap G_\varepsilon, z, R) .$$

Let $n \to \infty$ and then $\varepsilon \to 0$. Then $\lim_n H^{R_n}_{UM} = 0$ and $U(z)$ is singular. Suppose $U(z)$ is singular. Let $\Omega = R_{n+i} - (R_{n+i} - R_n) \cap G_\varepsilon$. Then $H^\Omega_{UM} > \varepsilon$ on $\partial((R_{n+i} - R_n) \cap G_\varepsilon)$. Since $\varepsilon w(G_\varepsilon \cap (R_{n+i} - R_n), z, R_{n+i})$ is the least HP in Ω larger than ε on $G_\varepsilon \cap (R_{n+i} - R_n)$,

$$H^{R_n}_{UM} \geqq H^\Omega_{UM} \geqq \varepsilon w(G_\varepsilon \cap (R_{n+i} - R_n), z, R_{n+i}) .$$

Let $i \to \infty$ and then $n \to \infty$. Then

$$0 = \lim_n H^{R_n}_{UM} \geqq \varepsilon w(G_\varepsilon \cap B, z, R) = 0 .$$

Hence we have 3).

4) Now $G_\varepsilon=\{z\in R: w'(z)>\varepsilon\}=\{z\in R: w(\mathrm{B}\cap \mathrm{R}, \mathrm{z}, \mathrm{R}-\mathrm{R}_0)<1-\varepsilon\}$. We have by 2) $w(G_\varepsilon\cap B, z, R)=0$ and $w'(z)$ is singular by 3).

5) Since $U^M(z)$ is an SPH, $H_{U^M}^{D\cap R_n}\downarrow: n\to\infty$. Let $V_n(z)$ be an HB in $D\cap R_n$ such that $V_n(z)=U^M(z)$ on $\partial D\cap R_n$ and $=0$ $\partial R_n\cap D$. Then

$$V_n(z)+H_{U^M}^{R_n}\geqq H_{U^M}^{D\cap R_n}\geqq V_n(z).$$

Let $n\to\infty$. Then $H_{U^M}^{R_n}\downarrow 0$. Thus

$$\lim_n H_{U^M}^{D\cap R_n}=\lim_n V_n(z)=H_{U^M}^D. \qquad (1)$$

Since $D\cap D_\delta\subset D$ and $U^M(z)$ is an SPH, by (1)

$$H_{U^M}^{D\cap D_\delta}=\lim_n H_{U^M}^{D\cap D_\delta\cap R_n}\geqq\lim_n H_{U^M}^{D\cap R_n}=H_{U^M}^D.$$

Now $H_{U^M}^{D\cap D_\delta}\downarrow: \delta\to 0$ and $\lim_{\delta\to 0} H_{U^M}^{D\cap D_\delta}\geqq H_{U^M}^D$.

Since $G(z, p_0)>0$ in R, there exists a number $\delta(n)$ such that $D_{\delta(n)}\supset R_n$ for any n. Then $D\cap R_n\cap D_\delta=D\cap R_n: \delta\leqq\delta(n)$. Hence

$$H_{U^M}^{D\cap R_n}=H_{U^M}^{D\cap D_\delta\cap R_n}\geqq H_{U^M}^{D\cap D_\delta}\geqq\lim_{\delta\to 0} H_{U^M}^{D\cap D_\delta}.$$

Let $n\to\infty$, then $H_{U^M}^D=\lim_{\delta\to 0} H_{U^M}^{D\cap D_\delta}$.

THEOREM 1. *Let $U(z)$ be an HP and let $G_M=\{z\in R: U(z)>M\}$. Then $Mw(G_M, z, R)\downarrow$ as $M\to\infty$. $S(z)=\lim_{M=\infty} Mw(G_M, z, R)$ $(\leqq U(z))$ is harmonic, singular and $\lim_{M=\infty} w(G_M, z, R)=0$.*

PROOF. Since $w(G_M, z, R)$ is the least positive SPH in R not smaller than 1 on G_M, $Mw(G_M, z, R)\leqq U(z)$ in CG_M. For $M_1\leqq M_2$, $M_2 w(G_{M_2}, z, R)\leqq U(z)=M_1=M_1 w(G_{M_1}, z, R)$ on ∂G_{M_1}. This implies $Mw(G_M, z, R)\downarrow$ as $M\to\infty$. Clearly $G_M\to$boundary of R as $M\nearrow\infty$, $S(z)=\lim_{M=\infty} Mw(G_M, z, R)$ is harmonic and

$$S(z)\leqq U(z) \quad\text{and}\quad \lim_{M=\infty} w(G_M, z, R)=0. \qquad (1)$$

Let $V_n^L(z)$ be an HB in R_n-G_L such that $V_n^L(z)=0$ on $\partial G_L\cap R_n$ and $V_n^L(z)=w(G_L, z, R)$ on ∂R_n-G_L. Then by the definition of $w(G_L, z, R)$

$$\lim_n V_n^L(z)=V^L(z)=0.$$

Let $\acute{H}_{S^M}^{R_n-G_L}(M<L)$ be the solution of the Dirichlet problem in R_n-G_L with boundary value $S^M(z)$ on ∂R_n-G_L and $=0$ on $\partial G_L\cap R_n$. Then

$$\acute{H}_{S^M}^{R_n-G_L}\leqq\acute{H}_{S}^{R_n-G_L}\leqq\acute{H}_{Lw(G_L,z,R)}^{R_n-G_L}=LV_n^L(z) \text{ and } \lim_n\acute{H}_{S^M}^{R_n-G_L}=0. \qquad (2)$$

By the maximum principle

$$H^{R_n}_{SM} \leqq H^{R_n - G_L}_{SM} + Mw(G_L, z, R) \text{ in } R_n .$$

Let $n \to \infty$ and then $L \nearrow \infty$. Then by (1) and (2), $\lim_n H^{R_n}_{SM} = 0$. Thus $S(z)$ is singular and $\leqq U(z)$.

Let $G_2 \subset G_1$ be domains. Let $\omega_{n,n+i}(z)$ an HB in $G_1 \cap R_{n+i} - G_2 \cap (R_{n+i} - R_n)$ such that $\omega_{n,n+i}(z) = 0$ on $R_{n+i} \cap \partial G_1$, $\frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0$ on $\partial R_{n+i} \cap (G_1 - G_2)$ and $\omega_{n,n+i}(z) = 1$ on $G_2 \cap (R_{n+i} - R_n)$. If there exists a number n_0 and M such that $D(\omega_{n_0,n_0+i}(z)) \leqq M$ for $i = 1, 2, \cdots$. Then $\omega_{n,n+i}(z) \Rightarrow \omega_n(z) : i \to \infty$ ($\Rightarrow$means convergence and convergence in Dirichlet norm), $\omega_n(z) \Rightarrow$ a harmonic function denoted by $\omega(G_2 \cap B, z, G_1)$ called *C. P.*[2] of $G_2 \cap B$ *relative to* G_1. Let F be a closed set (or domain G) of G_1, we denote by $\omega(F, z, G_1)$ ($\omega(G, z, G_1)$) the HD function which is 1 on F (on G) and $=0$ on ∂G_1 and has *M. D. I.* (minimal Dirichlet integral) among all harmonic functions with the same value as $\omega(F, z, G_1)$ ($\omega(G, z, G_1)$) on $\partial F + \partial G_1 (\partial G + \partial G_1)$. In this case we also call $\omega(F, z, G_1)$ *C. P.* of F relative to G_1. Let $\omega(z)$ be a *C. P.* then

$$\int_{\partial G_M} \frac{\partial}{\partial n} \omega(z)\, ds = D(\omega(z)) \qquad \text{for almost } M:\ 0 < M < 1 . \qquad (2)$$

LEMMA 2. *Let $R \in O_g$ and $G_i : i = 1, 2, \cdots i_0$ be domains and $U_i(z)$ be an HD in G_i. Then there exists another exhaustion $\{R_m\}$ of R such that*

$$\int_{G_i \cap \partial R_m} \left| \frac{\partial}{\partial n} U_i(z) \right| ds \to 0 :\ m \to \infty \qquad \textit{for any } i .$$

Let G be a domain of one component and let $\tilde{G}$ be the double of G relative to ∂G. If $\tilde{G} \in O_g$, we denote by $G \in SO_g$. Let SO_{HB} be the class of domains such that there exists no HB vanishing on ∂G except for capacity zero, then it is well known the following facts[2]:

$$SO_g \subset SO_{HB}.\ G' \subset G \text{ and } G \in SO_g(SO_{HB}) \text{ implies } G' \in SO_g(SO_{HB}).$$

In this note if every component of $G \in SO_g(SO_{HB})$, we denote by $G \in SO_g(SO_{HB})$. Then it is clear the above facts are valid.

THEOREM 2. *Let $G(z)$ be a GG. Then*

1) *$\operatorname{Sup}_z G(z) = \infty$. Put $G_\delta = \{z \in R : G(z) > \delta\}$. Then $G_\delta \in SO_g$ and $G(z) = \delta w(G_\delta, z, R)$ in CG_δ.*

2) *$\int_{\partial G_M} \frac{\partial}{\partial n} G(z)\, ds = k$ for every M and $D(G^M(z)) = kM$. Such const. k is*

called mass of $G(z)$ and is denoted by $\mathfrak{M}(G(z))$.

3) *Let $U(z)$ be positively harmonic except at most a set of capacity zero and $\leqq G(z)$. Then $U(z)$ is a GG with*

$$\mathfrak{M}\big(U(z)\big) \leqq \mathfrak{M}\big(G(z)\big).$$

4) *Let Ω be a domain in G_M: $0<M<1$. Then*

$$D\big(w(\Omega, z, R)\big) \leqq \mathfrak{M}\big(G(z)\big)/M \text{ and } \int_{\partial\Omega} \frac{\partial}{\partial n} w(\Omega, z, R)\, ds \leqq \mathfrak{M}\big(G(z)\big)/M.$$

PROOF. 1), 2) and 3) are proved[1),4)]. We show 4). Let $0<\delta<M$. Since $G_\delta \in SO_g \subset SO_{HB}$, every HB in a domain D in G_δ is uniquely determined by the value on ∂D, then

$$G(z) = (M-\delta)\,\omega(G_M, z, G_\delta)+\delta = (M-\delta)\,w(G_M, z, G_\delta)+\delta \text{ in } G_\delta - G_M,$$

whence

$$D\big(w(G_M, z, G_\delta)\big) = \frac{1}{(M-\delta)^2} D_{G_\delta - G_M}\big(G(z)\big) = \mathfrak{M}\big(G(z)\big)/(M-\delta).$$

By $\Omega \subset G_M$, $\omega(\Omega, z, G_\delta) \leqq \omega(G_M, z, G_\delta)$ and by the Dirichlet principle

$$D\big(\omega(\Omega, z, G_\delta)\big) \leqq D\big(\omega(G_M, z, G_\delta)\big) \leqq \mathfrak{M}\big(G(z)\big)/(M-\delta).$$

$\omega(\Omega, z, G_\delta) = w(\Omega, z, G_\delta) \leqq w(\Omega, z, R)$ and by the definition of $w(\Omega, z, R)$

$$\lim_{\delta \to 0} w(\Omega, z, G_\delta) = w(\Omega, z, R).$$

Hence $D(w(\Omega, z, R)) \leqq \dfrac{\mathfrak{M}(G(z))}{M}$.

Now $G_\delta \in SO_g$, $\displaystyle\int_{C_\alpha} \frac{\partial}{\partial n}\omega(\Omega, z, G_\delta)\, ds = \text{const.}$ for any C_α by Lemma 2, where $C_\alpha = \{z \in R:\ \omega(\Omega, z, G_\delta) = \alpha\}$: $0 \leqq \alpha \leqq 1$, hence

$$-\int_{\partial\Omega} \frac{\partial}{\partial n}\omega(\Omega, z, G_\delta)\, ds = D\big(\omega(\Omega, z, G_\delta)\big) \leqq D\big(\omega(G_M, z, G_\delta)\big).$$

$-\dfrac{\partial}{\partial n} w(\Omega, z, G_\delta) \geqq -\dfrac{\partial}{\partial n} w(\Omega, z, R) \geqq 0$ on $\partial\Omega$, hence

$$-\int_{\partial\Omega} \frac{\partial}{\partial n} w(\Omega, z, R)\, ds \leqq \mathfrak{M}\big(G(z)\big)/M.$$

THEOREM 3. 1) *Let $G_1(z)$ and $G_2(z)$ be GG, s. Then $G(z) = G_1(z) + G_2(z)$ is a GG and $\mathfrak{M}(G(z)) = \mathfrak{M}(G_1(z)) + \mathfrak{M}(G_2(z))$.*

2) *Let* $G(z)$ *be a GG and let* $G'(z)$ *be an HP except at most a set of capacity zero and* $\leqq G(z)$, *then* $G(z)-G'(z)$ *is a GG and*

$$\mathfrak{M}\big(G(z)-G'(z)\big)=\mathfrak{M}\big(G(z)\big)-\mathfrak{M}\big(G'(z)\big).$$

PROOF. 1) $G(z)$ is evidently singular. Let $G_M=\{z\in R: G(z)>M\}$, $G_{i,M}=\{z\in R: G_i(z)>M\}$. Then

$$G_M\supset G_{1,M}+G_{2,M}.$$

$D(G^M(z))=\underset{CG_M}{D}(G(z))=\underset{CG_M}{D}(G_1(z)+G_2(z))\leqq 2(\underset{CG_M}{D}(G_1(z))+\underset{CG_M}{D}(G_2(z)))\leqq 2\underset{CG_{1,M}}{D}(G_1(z))+2\underset{CG_{2,M}}{D}(G_2(z))=2M\mathfrak{M}(G_1(z))+2M\mathfrak{M}(G_2(z))$. Hence $G(z)$ is a GG and by $D(G^M(z))=M\mathfrak{M}(G(z))$

$$\mathfrak{M}\big(G(z)\big)\leqq 2\Big(\mathfrak{M}\big(G_1(z)\big)+\mathfrak{M}\big(G_2(z)\big)\Big).$$

By $G_\delta\in SO_g\subset SO_{HB}$

$$G(z)=(M-\delta)\,\omega(G_M, z, G_\delta)+\delta \text{ in } G_\delta-G_M.$$

Similarly $G_i(z)=(M-\delta)\,\omega(G_{i,M}, z, G_{i,\delta})+\delta$ in $G_{i,\delta}\subset G_\delta$. Let $\tilde{G}_i(z)=(M-\delta)\,\omega(G_{i,M}, z, G_\delta)+\delta$. Then by $G_{i,\delta}\in SO_{HB}$, $\tilde{G}_i(z)\geqq G_i(z)$ in $G_{i,\delta}\subset G_\delta$. On the other hand, $\tilde{G}_i(z)=M=G_i(z)$ on $\partial G_{i,M}$, hence

$$0\leqq\int_{\partial G_\delta}\frac{\partial}{\partial n}\tilde{G}_i(z)\,ds=-\int_{\partial G_{i,M}}\frac{\partial}{\partial n}\tilde{G}_i(z)\,ds\leqq-\int_{\partial G_{i,M}}\frac{\partial}{\partial n}G_i(z)\,ds=\mathfrak{M}\big(G_i(z)\big). \quad (1)$$

Let $G^*(z)=(\tilde{G}_1(z)+\tilde{G}_2(z))\,A+B$, where $A=\dfrac{M-\delta}{M-2\delta}$, $B=\dfrac{-\delta M}{M-2\delta}$. Then $G^*(z)=\delta=G(z)$ on ∂G_δ and $G^*(z)\geqq M$ on ∂G_M. By $G_\delta\in SO_{HB}$, $G^*(z)\geqq G(z)$ in $G_\delta-G_M$. Then by (1)

$$\mathfrak{M}\big(G(z)\big)=\int_{\partial G_\delta}\frac{\partial}{\partial n}G(z)\,ds\leqq\int_{\partial G_\delta}\frac{\partial}{\partial n}G^*(z)\,ds=\left(\frac{M-\delta}{M-2\delta}\right)$$
$$\left(\int_{\partial G_\delta}\frac{\partial}{\partial n}\big(\tilde{G}_1(z)+\tilde{G}_2(z)\big)\right)ds\leqq\left(\frac{M-\delta}{M-2\delta}\right)\Big(\mathfrak{M}\big(G_1(z)\big)+\mathfrak{M}\big(G_2(z)\big)\Big).$$

Let $\delta\to 0$. Then $\mathfrak{M}(G(z))\leqq\mathfrak{M}(G_1(z))+\mathfrak{M}(G_2(z))$. Now $\left(\dfrac{M}{M-\delta}\right)(G(z)-\delta)=M\omega(G_M, z, G_\delta)$. By $G_\delta\in SO_g$, $M\int_{\partial G_\delta}M\frac{\partial}{\partial n}\omega(G_M, z, G_\delta)\,ds=\underset{G_\delta-G_M}{D}(M\omega(G_M, z, G_\delta))=\left(\dfrac{M}{M-\delta}\right)^2\underset{G_\delta-G_M}{D}(G(z))=\left(\dfrac{M^2}{M-\delta}\right)\mathfrak{M}(G(z))$, *i.e.*

$$M\int_{\partial G_\delta}\frac{\partial}{\partial n}\omega(G_M, z, G_\delta)\,ds=\left(\frac{M}{M-\delta}\right)\mathfrak{M}\big(G(z)\big).$$

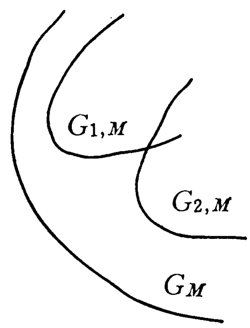

Fig. 1.

Hence for any $\varepsilon>0$, there exists a const. $\delta>0$ such that

$$0\leqq M\int_{\partial G_\delta}\frac{\partial}{\partial n}\omega(G_M, z, G_\delta)\,ds\leqq\mathfrak{M}\big(G(z)\big)+\varepsilon\,. \qquad (2)$$

$G_\delta\supset G_{i,\delta}$, whence by the Dirichlet principle $D(\omega(G_{i,M}, z, G_{i,\delta}))\geqq D(\omega(G_{i,M}, z, G_\delta))$. By the definition of $w(G_{i,M}, z, R)$, $G(z)=G_1(z)+G_2(z)\geqq Mw(G_{1,M}, z, R)+Mw(G_{2,M}, z, R)\geqq Mw(G_{1,M}, z, G_\delta)+Mw(G_{2,M}, z, G_\delta)=M\omega(G_{1,M}, z, G_\delta)+M\omega(G_{2,M}, z, G_\delta)$ in G_δ, whence $M\omega(G_M, z, G_\delta)=G(z)=M\geqq M\omega(G_{1,M}, z, G_\delta)+M\omega(G_{2,M}, z, G_\delta)$ on ∂G_M.

$$M\omega(G_M, z, G_\delta)=0=M\omega(G_{1,M}, z, G_\delta)+M\omega(G_{2,M}, z, G_\delta) \text{ on } \partial G_\delta\,. \qquad (3)$$

Hence by $G_\delta\in SO_{HB}$,

$$M\omega(G_M, z, G_\delta)\geqq M\omega(G_{1,M}, z, G_\delta)+M\omega(G_{2,M}, z, G_\delta) \text{ in } G_\delta-G_M\,. \qquad (4)$$

By (3) and (4)

$$0\leqq\int_{\partial G_\delta}\frac{\partial}{\partial n}\omega(G_M, z, G_\delta)\,ds\geqq\int_{\partial G_\delta}\frac{\partial}{\partial n}\omega(G_{1,M}, z, G_\delta)\,ds+\int_{\partial G_\delta}\frac{\partial}{\partial n}\omega(G_{2,M}, z, G_\delta)\,ds\,. \qquad (5)$$

Now

$$0<\int_{\partial G_\delta}\frac{\partial}{\partial n}\omega(G_{i,M}, z, G_\delta)\,ds=-\int_{\partial G_{i,M}}\frac{\partial}{\partial n}\omega(G_{i,M}, z, G_\delta)\,ds\,. \qquad (6)$$

Clearly $M\omega(G_{i,M}, z, G_\delta)=M=G_i(z)$ on $\partial G_{i,M}$ and $M\omega(G_{i,M}, z, G_\delta)\leqq G_i(z)$ in $G_\delta-G_{i,M}$ and

$$-M\int_{\partial G_{i,M}}\frac{\partial}{\partial n}\omega(G_{i,M}, z, G_\delta)\,ds\geqq-\int_{\partial G_{i,M}}\frac{\partial}{\partial n}G_i(z)\,ds=\mathfrak{M}\big(G_i(z)\big)\,. \qquad (7)$$

Hence by (7), (6), (5) and (2)

$$\mathfrak{M}\big(G_1(z)\big)+\mathfrak{M}\big(G_2(z)\big)\leqq\mathfrak{M}\big(G(z)\big)+\frac{\varepsilon}{M}\,.$$

Let $\varepsilon\to0$, then $\mathfrak{M}(G(z))=\mathfrak{M}(G_1(z))+\mathfrak{M}(G_2(z))$.

Let $G(z, p)$ be a Green function and let p_0 be a fixed point in R. Put $K(z, p)=\dfrac{G(z, p)}{G(p, p_0)}$. Then $K(z, p)=1$ at $z=p_0$. Let $\{p_i\}$ be a divergent sequence such that $\{K(z, p_i)\}$ converges to a positive harmonic function denoted by $K(z, p)$. Then we say that $\{p_i\}$ determines an ideal boundary point p. We denote by Δ_K the set of all the ideal boundary points. Then the distance $\delta(p_1, p_2)$ between p_1 and p_2 in $R+\Delta_K$ is defined as

$$\delta(p_1, p_2)=\sup_{z\in R_0}\left|\frac{K(z, p_1)}{1+K(z, p_1)}-\frac{K(z, p_2)}{1+K(z, p_2)}\right|.$$

The topology induced by this metric is called Martin's topology[5]. Let $N(z,p)$[2] be an N-Green function of $R-R_0$. We use $N(z,p)$ instead of $K(z,p)$ and we have N-Martin's topology on $R-R_0+\Delta_N$. Also we have G-Martin's topology on $R+\Delta_G$ by using $G(z,p)$. Then $\Delta_\alpha(\alpha=K, N, G)$ is compact. In G-Martin's topology the set of p such that $G(z,p)=0$ consists of only one point.

Representation of generalized Green functions. Let $G(z)$ be a GG with $\mathfrak{M}(G(z))=2\pi$. Then there exists a uniquely determined positive mass μ on R such that $G(z)-\int G(z,p)\,d\mu(p)=U(z)$ is an HP. Then by Theorem 2 $U(z)$ is also a GG with $\mathfrak{M}(U(z))\leqq 2\pi$. Let p_0 be a fixed point and let $D_\delta=\{z\in R: G(z,p_0)>\delta\}$. The following discussion is simpler than the previous one[4].

Let $U_{D_\delta\cap(R-R_n)}(z)$ be the least non negative SPH in R larger than $U(z)$ on $D_\delta\cap(R-R_n)$. Then $U_{D_\delta\cap(R-R_n)}(z)\downarrow U_{D_\delta\cap B}(z): n\to\infty$. Clearly $U_{D_\delta\cap B}(z)$ is also harmonic and a GG with $\mathfrak{M}(U_{D_\delta\cap B}(z))\leqq\mathfrak{M}(U(z))$. Then

$$(U_{D_\delta\cap B})_{D_\delta\cap B}=U_{D_\delta\cap B}\,, \tag{1}$$

where the operation "$U_{D_\delta\cap B}$ from U" is Martin's method[5]. Let $U'=U-U_{D_\delta\cap B}$. Then by Theorem 2 U' is also a GG with $\mathfrak{M}(U')\leqq\mathfrak{M}(U)$. Now

$$U_{D_\delta\cap B}=(U_{D_\delta\cap B}+U')_{D_\delta\cap B}=(U_{D_\delta\cap B})_{D_\delta\cap B}+U'_{D_\delta\cap B}$$

and

$$U'_{D_\delta\cap B}=0\,.$$

Let $\Omega_M=\{z\in R: U'(z)>M\}$. Then by Theorem 2, 1)

$$U'(z)=H_M^{C\Omega_M}=M w(\Omega_M, z, R) \quad \text{in} \quad C\Omega_M\,.$$

Let $T_M(z)=Mw(\Omega_M\cap D_\delta, z, R)$, then $T_M(z)\leqq U'(z)$. We show $T_M(z)\to 0$ as $M\to\infty$. Let $T_{M,n}(z)$ be an HB in $R_n-(\Omega_M\cap D_\delta)$ such that $T_{M,n}(z)=0$ on $\partial R_n-(\Omega_M\cap D_\delta)$, $=M$ on $\partial(\Omega_M\cap D_\delta)\cap R_n$. Let $M_0<M$ and $G_n(\zeta, z)$ and $G(\zeta, z)$ be Green functions of $R_n-(\Omega_{M_0}\cap D_\delta)$ and $R-(\Omega_{M_0}\cap D_\delta)$ respectively. Then $\frac{\partial}{\partial n}G_n(\zeta, z)\uparrow\frac{\partial}{\partial n}G(\zeta, z)$ on $\partial(\Omega_{M_0}\cap D_\delta)$. Also $T_{M,n}(z)\nearrow T_M(z): n\to\infty$, hence

$$T_M(z)=\frac{1}{2\pi}\int\limits_{\partial(\Omega_{M_0}\cap D_\delta)} T_M(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds=\left(T_M(z)\right)_{\Omega_{M_0}\cap D_\delta}.$$

Assume $\varlimsup\limits_{M\to\infty} T_M>0$. Then there exists a sequence $\{M_i\}$ such that $M_i\to\infty$ and $T_{M_i}\to$ an HP $T(z)>0$. Now $T_{M_i}(z)\leqq U'(z)$ and $T(z)\leqq U'(z)$. $\int\limits_{\partial(\Omega_{M_0}\cap D_\delta)} U'(\zeta)$

$\frac{\partial}{\partial n}G(\zeta, z)\,ds=U'_{\Omega_{M_0}\cap D_\delta}(z)\leqq U'(z)<\infty$. Hence by Lebesgue's theorem

$$T(z)=\lim_i T_{M_i}(z)=\frac{1}{2\pi}\int_{\partial(\Omega_{M_0}\cap D_\delta)}\lim_i T_{M_i}(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds=\frac{1}{2\pi}\int_{\partial(\Omega_{M_0}\cap D_\delta)}T(\zeta)\frac{\partial}{\partial n}$$

$G(\zeta, z)\,ds=T_{\Omega_{M_0}\cap D_\delta}(z)$. Hence

$$T=T_{\Omega_M\cap D_\delta}\qquad\text{for any } M.$$

Since $U'(z)$ is harmonic in R, $\Omega_M\to$ boundary of R as $M\to\infty$. Hence for any n there exists a number $M(n)$ such that $\Omega_M\subset R-R_n$: $M>M(n)$ and

$$T(z)=\lim_{M=\infty}T_{\Omega_M\cap D_\delta}(z)\leqq T_{D_\delta\cap B}(z)\leqq U'_{D_\delta\cap B}(z)=0\,.$$

This is a contradiction. Hence

$$T_M(z)\to 0: M\to\infty\,. \tag{2}$$

$w(\Omega_M\cap C\overset{o}{D}_\delta, z, R)\leqq w(\Omega_M, z, R)\leqq w(\Omega_M\cap D_\delta, z, R)+w(\Omega_M\cap C\overset{o}{D}_\delta, z, R)$. By (2)

$$U'(z)=\lim_{M=\infty}Mw(\Omega_M\cap C\overset{o}{D}_\delta, z, R)=\lim_{M=\infty}Mw(\Omega_M, z, R)\,. \tag{3}$$

Let $\delta'<U'(p_0)$ $(\delta'<\delta)$, then $\Omega_{\delta'}\ni p_0$. Let $G'(z, p_0)$ be a Green function of $\Omega_{\delta'}$. Then $G'(z, p_0)<G(z, p_0)$ in CD_δ. Now $U'(z)$ is a GG. Let $\Omega_M^\delta=\Omega_M\cap C\overset{o}{D}_\delta$. Then by Theorem 2, 4)

$$U'^M(z)=Mw(\Omega_M, z, R)\geqq Mw(\Omega_M^\delta, z, R)\,,$$

$$D\big(Mw(\Omega_M^\delta, z, R)\big)\leqq D_{C\Omega_M}\big(U'^M(z)\big)=M\mathfrak{M}\big(U'(z)\big)\leqq 2\pi M\,.$$

$\Omega_{\delta'}\in SO_g$ and $D_{R-v(p_0)}(G'(z, p_0))<\infty$, where $v(p_0)$ is a neighbourhood of p_0. Hence by Lemma 2 there exists a D-exhaustion $\{R_m\}$ relative to $w(\Omega_M^\delta, z, R)$ in $\Omega_{\delta'}-\Omega_M^\delta$ and $G'(z, p_0)$ in $\Omega_{\delta'}$ such that

$$\int_{\partial R_m\cap(\Omega_{\delta'}-\Omega_M^\delta)}\left|\frac{\partial}{\partial n}w(\Omega_M^\delta, z, R)\right|ds\downarrow 0 \text{ and }\int_{\partial R_m\cap\Omega_{\delta'}}\left|\frac{\partial}{\partial n}G'(z, p_0)\right|ds\downarrow 0 \text{ as } m\to\infty\,. \tag{4}$$

Put $S(z)=Mw(\Omega_M^\delta, z, R)$. Then by Theorem 2, 4) $-\frac{\partial}{\partial n}S(z)\geqq 0$ on $\partial\Omega_M^\delta$ and $0\leqq-\int_{\partial\Omega_M^\delta}\frac{\partial}{\partial n}S(z)\,ds\leqq M\int_{\partial\Omega_M}\frac{\partial}{\partial n}w(\Omega_M, z, R)\,ds=\mathfrak{M}(U'(z))$, whence by $G'(z, p_0)$ $\leqq\delta$ on $\partial\Omega_M^\delta$,

$$-\int_{\partial\Omega_M^\delta\cap R_m}G'(z, p_0)\frac{\partial}{\partial n}S(z)\,ds\leqq\delta\mathfrak{M}\big(U'(z)\big)\qquad\text{for any } m\,.$$

By (4) we have $\int_{\partial R_m\cap(\Omega_{\delta'}-\Omega^\delta_M)} G'(z,p_0)\frac{\partial}{\partial n}S(z)\,ds\to 0: m\to\infty$, $\int_{\partial\Omega^\delta_M\cap R_m} S(z)\frac{\partial}{\partial n}G'(z,p_0)\,ds$ $=M\int_{\partial\Omega^\delta_M\cap R_m}\frac{\partial}{\partial n}G'(z,p_0)\,ds=-M\int_{\partial R_m\cap\Omega^\delta_M}\frac{\partial}{\partial n}G'(z,p_0)\,ds\to 0: m\to\infty$ for $S(z)=M$ on $\partial\Omega^\delta_M$.

$$\int_{\partial R_m\cap(\Omega_{\delta'}-\Omega^\delta_M)} S(z)\frac{\partial}{\partial n}G'(z,p_0)\,ds\to 0: m\to\infty .$$

On the other hand, by $S(z)\leqq U'(z)=\delta'$ on $\partial\Omega_{\delta'}$, $\int_{\partial\Omega_{\delta'}\cap R_m} S(z)\frac{\partial}{\partial n}G'(z,p_0)\,ds\leqq$ $2\pi\delta'$ and $\int_{\partial\Omega_{\delta'}\cap R_m} G'(z,p_0)\frac{\partial}{\partial n}S(z)\,ds=0$ by $G'(z,p_0)=0$ *on* $\partial\Omega_{\delta'}$. Let $M>U'(p_0)$ then $\Omega_{\delta'}-\Omega^\delta_M\ni p_0$. Then by Green's formula

$$\int_{\partial\Omega^\delta_M\cap R_m+\partial R_m\cap(\Omega_{\delta'}-\Omega^\delta_M)+R_m\cap\partial\Omega_{\delta'}+p_0} G'(z,p_0)\frac{\partial}{\partial n}S(z)\,ds$$
$$=\int_{\partial\Omega^\delta_M\cap R_m+\partial R_m\cap(\Omega_{\delta'}-\Omega^\delta_M)+R_m\cap\partial\Omega_{\delta'}+p_0} S(z)\frac{\partial}{\partial n}G'(z,p_0)\,ds .$$

We have $S(p_0)\leqq\frac{\mathfrak{M}(U'(z))\,\delta}{2\pi}+\delta'$. Let $\delta'\to 0$ and then $M\to\infty$. Then by (3) $S(p_0)\to U'(p_0)$ and

$$U'(p_0)\leqq\frac{\mathfrak{M}(U'(z))\,\delta}{2\pi}\leqq\delta .$$

Thus $U(z)=U_{D_\delta\cap B}(z)+U'(z)$ and $U'(z)\downarrow 0$ as $\delta\to 0$. Every positive harmonic function is represented by a mass on Martin's boundary Δ_K. Let $\Delta_{K,\delta}=\Delta_K\cap\bar{D}_\delta$, then $U_{D_\delta\cap B}(z)$ is represented by a mass on $\Delta_{K,\delta}$. Hence we have.

THEOREM 4. *The harmonic part of a generalized Green function with* $\mathfrak{M}(G(z))\leqq 2\pi$ *is represented by a mass on* $\bigcup_{\delta>0}\Delta_{K,\delta}$. *As a special case, if* $\Delta_{K,\delta}=0$ *for any* $\delta>0$, *the harmonic part is zero.*

Green potential whose total mass is bounded. We suppose G-Martin's topology is defined on $R+\Delta_G$. Let μ be a positive mass on $R+\Delta_G$. We consider

$$P^\mu(z)=\int G(z,p)\,d\mu(p),\qquad \int d\mu=1 .$$

Clearly if every point of Δ_G is regular and $\mu=0$ in R (*i. e.* $G(z,p)=0$ for $p\in\Delta_G$), $P^\mu=0$.

LEMMA 3. *Let Ω be a domain and let Ω_n be compact domain such that $\Omega_n \nearrow \Omega : n \to \infty$. Let U_n be a harmonic function in Ω_n such that $D_{\Omega_n}(U_n^L) \leqq M$ and for Ω_m, $U_n \to U$ in Ω_m for any m. Then $D_\Omega(U^L) \leqq M$.*

In fact, let $G_n = \{z \in \Omega_n : U_n(z) < L\}$ and $G^\varepsilon = \{z \in \Omega : U(z) < L - \varepsilon\}$. Then for any given $\varepsilon > 0$ and m, there exist a number $l(\varepsilon, m)$ such that

$$(G^\varepsilon \cap \Omega_m) \subset G_l : l > l(\varepsilon, m).$$

$U_n \to U$ implies $\frac{\partial}{\partial x} U_n \to \frac{\partial U}{\partial x}$ and $\frac{\partial}{\partial y} U_n \to \frac{\partial}{\partial y} U$. By Fatou's Lemma

$$D_{G^\varepsilon \cap \Omega_m}(U) \leqq \varliminf_{l} D_{G^\varepsilon \cap \Omega_m}(U_l) \leqq \varliminf_{l} \underset{G_l}{D}(U_l) = \varliminf_{l} D(U_l^L) \leqq M.$$

Let $\varepsilon \to 0$. Then since Ω_m is compact, $\lim_{\varepsilon \to 0} D_{G^\varepsilon \cap \Omega_m}(U) = \underset{\Omega_m}{D}(U^L) \leqq M$. Let $m \to \infty$. Then

$$\underset{\Omega}{D}(U^L) \leqq M.$$

We proved

LEMMA 4[6]. *Let G be a circular rectangle $\{r_1 \leqq |z| \leqq r_2,\ 0 \leqq \arg z \leqq \Theta\}$ with a finite number of radial slits. Let $\Gamma_1 = \{|z| = r_1,\ 0 \leqq \arg z \leqq \Theta\}$, $\Gamma_2 = \{|z| = r_2,\ 0 \leqq \arg z \leqq \Theta\}$. Let $U(z)$ be an HB in G and continuous on $G + \Gamma_1 + \Gamma_2$. Then*

$$D\big(U(z)\big) \geqq \frac{1}{\log \frac{r_2}{r_1}} \int_0^\Theta \left| U(r_1 e^{i\theta}) - U(r_2 e^{i\theta}) \right|^2 d\theta.$$

If $U(z)$ is not continuous at a finite number of points on $\Gamma_1 + \Gamma_2$, divide G into some circular rectangls, then we have the same conclusion.

THEOREM 5. *Let R be a Riemann surface $\notin O_g$, $G(z, p_0)$ be a Green function and let $D_\delta = \{z \in R : G(z, p_0) > \delta\}$. Let $U(z)$ be a positive SPH such that $U(z)$ is harmonic in $CD_\kappa = R - D_\kappa$, singular and there exists a const. α such that $\underset{CD_\kappa}{D}(U^M(z)) \leqq M\alpha : 0 < M < \infty$. Then for any given M and ε, we can find a compact domain $\Omega \ni p_0$ depending only on M, κ and α but not on $U(z)$ such that*

$$H^\Omega_{U^M}(p_0) < 3\varepsilon.$$

PROOF. At first we investigate $D_{CD_\delta}(U^M(z))$. Let $0 < \lambda' < \lambda < \delta < \kappa$ and let $D^\delta_\lambda = \{z \in R : \lambda < G(z, p_0) < \delta\}$. Since $D_{\lambda'} \in SO_g \subset SO_{HB}$, $H^{D^\delta_{\lambda'}}_{U^M}$ is uniquely determined and by the Dirichlet principle $D_{D^\delta_{\lambda'}}(U^M(z)) \geqq D_{D^\delta_{\lambda'}}(H^{D^\delta_{\lambda'}}_{U^M})$ and $D_{CD_\delta}(U^M(z)) \geqq D_{D^\delta_\lambda}(H^{D^\delta_{\lambda'}}_{U^M})$. Let $\lambda' \to 0$. Then by Lemma 1, 5) $H^{D^\delta_{\lambda'}}_{U^M} \to H^{CD_\delta}_{U^M}$. Now

$H_{U^M}^{D_{\lambda'}^\delta}$ is harmonic in $D_{\lambda'}^\delta$, whece by Fatou's Lemma $D_{CD_\delta}(U^M)\geqq \underline{\lim}_{\lambda'\to 0} D_{D_{\lambda'}^\delta}(H_{U^M}^{D_{\lambda'}^\delta}) \geqq D_{D_\lambda^\delta}(H_{U^M}^{CD_\delta})$. Let $\lambda\to 0$. Then

$$M\alpha \geqq D_{CD_\delta}(U^M) \geqq D_{CD_\delta}(H_{U^M}^{CD_\delta}). \tag{1}$$

Let $\delta<\frac{3\pi\varepsilon^3}{16M^2\alpha}$ and find D_δ, D_δ is non compact generally. Since $D_\delta\in SO_g$, $w(D_\delta\cap(R-R_n), z, R)\to 0$ as $n\to\infty$. Let n_1 be a number such that

$$w\Big(D_\delta\cap(R-R_{n_1}), p_0, R\Big)<\frac{\varepsilon}{M}. \tag{2}$$

Also $D_\lambda\left(\lambda<\delta \text{ and } \lambda<\frac{\delta\varepsilon}{2M}\right)\in SO_g$, whence $\omega_n(z)\Rightarrow G(z, p_0)$ as $n\to\infty$, where $\omega_n(z)$ is an HB in $D_\lambda^\delta\cap R_n$ such that $\omega_n(z)=\lambda$ on $\partial D_\lambda\cap R_n$, $=\delta$ on $\partial D_\delta\cap R_n$ and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\partial R_n\cap D_\lambda^\delta$. *i. e.* $\lim_n \omega_n(z)=\lambda+(\delta-\lambda)\,\omega(D_\delta, z, D_\lambda)$. By $D_\delta\in SO_g$, $\int_{\partial D_\delta}\frac{\partial}{\partial n}G(z, p_0)\,ds=2\pi$, $\frac{\partial}{\partial n}\omega_n(z)\to\frac{\partial}{\partial n}G(z, p_0)$ on $\partial D_\delta+\partial D_\lambda$. Let $n_2(>n_1)$ be a number such that

$$\int_{\partial D_\delta\cap R_{n_2}}\frac{\partial}{\partial n}G(z, p_0)\,ds\geqq 2\pi-\varepsilon.$$

Let n_3 be a number such that

$$\int_{\partial D_\delta\cap R_{n_2}}\left|\frac{\partial}{\partial n}\omega_n(z)-\frac{\partial}{\partial n}G(z, p_0)\right|ds<\frac{\pi\varepsilon}{2M}:\ n\geqq n_3. \tag{3}$$

Assume there exists a set F on $\partial D_\delta\cap R_{n_2}$ such that $U^M(z)>\varepsilon$ on F and $\int_F\frac{\partial}{\partial n}G(z, p_0)\,ds\geqq\frac{2\pi\varepsilon}{M}$. Then by (3)

$$\int_F\frac{\partial}{\partial n}\omega_n(z)\,ds\geqq\frac{3\pi\varepsilon}{2M}:\ n\geqq n_3. \tag{4}$$

$H_{U^M}^{CD_\delta}=U^M(z)\leqq M$ and $G(z, p_0)=\delta$ on ∂D_δ. Hence by the definition of $H_{U^M}^{CD_\delta}$, $H_{U^M}^{CD_\delta}\leqq\frac{M}{\delta}G(z, p_0)$ in CD_δ and

$$H_{U^M}^{CD_\delta}\leqq\frac{M\lambda}{\delta}\quad\text{on}\quad\partial D_\lambda, \tag{5}$$

Map $D_\lambda^\delta\cap R_n:\ n\geqq n_3$ onto a circular rectangle G with a finite number of radial slits by $\zeta=\exp(\omega_n(z)+i\,\tilde{\omega}_n(z))$, where $\tilde{\omega}_n(z)$ is a conjugate function of $\omega_n(z)$,

$$G=\left\{e^{\lambda}\leqq|\zeta|\leqq e^{\delta},\ 0\leqq\arg\zeta\leqq\Theta\right\}.\quad \Theta=\int\limits_{\partial D_\delta\cap R_n}\frac{\partial}{\partial n}\omega_n(z)\,ds\leqq 2\pi\,.$$

Then F is mapped onto a set F_ζ on $|\zeta|=e^{\delta}$ of angular measure $\geqq\dfrac{3\pi\varepsilon}{2M}$ by (4). $U^M(e^{\delta+i\theta})\geqq\varepsilon$ on F_ζ and $H^{CD_\delta}_{U^M}(e^{\lambda+i\theta})\leqq\dfrac{\lambda M}{\delta}<\dfrac{\varepsilon}{2}$ by (5). Hence by (1) and Lemma 4

$$M\alpha\leqq D_{CD_\delta}(U^M)\geqq D(H^{CD_\delta}_{U^M})\geqq\int\limits_{F'}\frac{\left|U^M(e^{\delta+i\theta})-\dfrac{\lambda M}{\delta}\right|^2}{\delta-\lambda}\,d\theta\geqq\frac{3\left(\dfrac{\varepsilon}{2}\right)^2\pi\varepsilon}{2\delta M}\geqq 2M\alpha:\ F'=\{\theta:$$

$e^{\delta+i\theta}\in F_\zeta\}$. This is a contradiction. Hence $U^M(z)<\varepsilon$ on $\partial D_\delta\cap R_{n_2}$ except at most a set F on $\partial D_\delta\cap R_{n_2}$ such that $\displaystyle\int\limits_F\frac{\partial}{\partial n}G(z,p_0)\,ds<\frac{2\pi\varepsilon}{M}$ and $U^M(z)\leqq M$ on F. Let $V(z)$ be an HB in D_δ such that $V(z)=U^M(z)$ on $\partial D_\delta\cap R_{n_2}$, and $=0$ on $\partial D_\delta\cap(R-R_{n_2})$. Then by $D_\delta\in SO_{HB}$ such $V(z)$ is uniquely determined and

$$V(p_0)<\varepsilon+\frac{M}{2\pi}\int\limits_F\frac{\partial}{\partial n}G(z,p_0)\,ds\leqq 2\varepsilon\,.$$

Evidently $V(z)+Mw(D_\delta\cap(R-R_{n_2}),z,R)\geqq H^{C_\delta\cap R_{n_2}}_{U^M}$. Hence by (2) $H^{G_\delta\cap R_{n_2}}_{U^M}<3\varepsilon$ at p_0. Let $\Omega=D_\delta\cap R_{n_2}$. Then Ω is a required domain.

THEOREM 6. 1) *Let P^μ be the potential of a positive mass μ on $R+\Delta_G$ such that $\int d\mu=1$. Then P^μ is singular.*

2) *If $\mu=0$ on R, P^μ is a GG with $\mathfrak{M}(P^\mu)\leqq 2\pi$.*

3) *As a special case, $G(z,p):p\in\Delta_G$ is a GG. i.e. Let $\{p_0\}$ be a sequence such that $p_i\to$ boundary of R and $G(z,p_i)$ converges to an HP $G(z,p)$. Then $G(z,p)$ is a GG.*

PROOF. 1) Let $\varepsilon>0$, $\delta<\varepsilon$, $D_\delta=\{z\in R: G(z,p_0)>\delta\}$ and let $\Delta_{G,\delta}=\Delta_G\cap\bar{D}_\delta$. Let μ_1, μ_2 and μ_3 be the restriction of μ on R, on $\Delta_{G,2\delta}$ and on $\Delta_G-\Delta_{G,2\delta}$ respectively. Then it is well known $\int G(z,p)\,d\mu_1(p)$ is singular. Let $U(z)=\int G(z,p)\,d\mu_2(p)$ and $U'(z)=\int G(z,p)\,d\mu_3(p)$. Since $G(z,p):p\in R+\Delta_G$ is continuous with respect to p in any compact set in R and $U(z)$ is uniformly approximated by a sequence $U_n=\sum^n c_iG(z,p_i): p_i\in D_\delta\cap R$, $c_i>0$, $\sum c_i=\int d\mu_2=\alpha$. Clearly $D(U^M_n)=2\pi M\alpha$ and $U_n(z)$ is harmonic in CD_δ and singular. Hence by Theorem 5 for any M and ε there exists a compact domain Ω such that $H^\Omega_{U^M_n}(p_0)<\varepsilon$. Since $\partial\Omega$ is compact $U_n(z)\to U(z)$ on $\partial\Omega$ and $H^\Omega_{U^M}(p_0)\leqq\varepsilon$. Also $U(z)$ is an SPH, $H^\Omega_{U^M}\geqq H^{R_n}_{U^M}$ for $R_n\supset\Omega$. Then by $G(p,p_0)<2\delta: p\notin\Delta_{G,2\delta}$ and $U'(p_0)<2\varepsilon$,

$$H^{R_n}_{(U+U')^M}\leqq H^{R_n}_{U^M}+U'(z)<3\varepsilon\quad\text{at}\quad z=p_0\,.$$

Then $H^{R_n}_{(U+U')^M}\downarrow$: $n\to\infty$ and $\lim_n H^{R_n}_{(U+U')^M}<3\varepsilon$ at p_0. Let $\varepsilon\to 0$, then $\lim_n H^{R_n}_{(U+U')^M}=0$. Thus $U(z)+U'(z)$ and $\int G(z,p)\,d\mu(p)$ is singular 2).

2) Let $\mu=0$ in R and $\int d\mu=1$. Then $P^\mu=\int G(z,p)\,d\mu(p)$ is approximated by a sequence $U_n=\sum^n c_i G(z,p_i)$, $\sum c_i=1$, $p_i\in R$. Evidently $D(U_n^M)=2\pi M$. Hence by Lemma 3 $D((P^\mu)^M)\leqq 2\pi M$ and by 1) P^μ is singular. Thus P^μ is a GG.

A REMARK ON 3). If R is a compact domain, 3) is trivial, because $G(z,p)>0$ if and only if $p_i\to p\in\partial R$ and p is irregular for the Dirichlet problem. There is no continuum of ∂R containing p and $G(z,p)=0$ on $\partial R-p$.

Quasi Dirichlet bounded harmonic functions.

A REMARK ON DIRICHLET INTEGRALS. Let $U(z)$ be an HD in a domain Ω and $V(z)$ be its conjugate. Ω will be a simply connected domain Ω' by cutting along some curves. Then $f(z)=U+iV$ is a one valued analytic function in Ω'. Then the area of $f(\Omega')=\int_\Omega |f'(z)|^2 dxdy=D(U)$. We take as a local parameter $\zeta=n+is$ at z, where n is an inner normal and s is a tangent of C_t at z: $C_t=\{z\in\Omega: U(z)=t\}$. Then $D(U)=\int\int|f'(z)|ds|f'(z)|dn$. Now $|f'(z)|=\frac{\partial V}{\partial s}=\frac{\partial U}{\partial n}$ at $z\in C_t$ and $|f'(z)|=\left|\frac{\partial U}{\partial n}\right|$ along C_t, $|f'(z)|dn=\frac{\partial U}{\partial n}dn=dU$ along the normal. Hence we have

LEMMA 5. $D(U)=\int_a^b L_t dt$: $L_t=\int_{C_t}\frac{\partial}{\partial n}U ds$, $a=\inf_z U(z)$, $b=\sup_z U(z)$.

LEMMA 6. A. *Let $\Omega\subset\Omega'$ be compact domains with analytic relative boundary. Let $U(z)$ be an HP on $\bar\Omega$ such that $U(z)=0$ on $\partial\Omega\cap\Omega'$ and let $U'(z)$ be an HP on $\bar\Omega'$ such that $U'(z)=U(z)$ on $\partial\Omega\cap\partial\Omega'$ and $U'(z)=0$ on $\partial\Omega'-\partial\Omega$. Let $G_L=\{z\in\Omega: U(z)>L\}$, $G'_L=\{z\in\Omega': U'(z)>L\}$, $\Gamma_L=\partial G_L\cap\bar\Omega$ $\Gamma'_L=\partial G'_L\cap\bar\Omega'$. Then*

1) $U(z)\leqq U'(z)$.

2) $0<\int_{\Gamma'_L}\frac{\partial}{\partial n}U' ds\leqq\int_{\Gamma_L}\frac{\partial}{\partial n}U ds$ *and* $D(U'^L)\leqq D(U^L)$.

PROOF. 1) is evident. 2) Since $U(z)=U'(z)$ on $\partial\Omega\cap\partial\Omega'$, $\frac{\partial}{\partial n}U'\geqq\frac{\partial}{\partial n}U$ on $\bar G_L\cap\partial\Omega-\Gamma_L$. By $U'(z)=L$ on $(\bar G'_L-\bar G_L)\cap\partial\Omega'$, $\frac{\partial}{\partial n}U'\geqq 0$ on $(\bar G'_L-\bar G_L)\cap\partial\Omega'$, where inner normals are with respect to G_L and G'_L. Now $\int_{\Gamma_L}\frac{\partial}{\partial n}U ds$

$$=-\int_{\tilde{G}_L\cap\partial\Omega-\Gamma_L}\frac{\partial}{\partial n}U\,ds,\ \int_{\Gamma'_L}\frac{\partial}{\partial n}U'\,ds=-\int_{\tilde{G}'_L\cap\partial\Omega-\Gamma'_L}\frac{\partial}{\partial n}U'\,ds=-\int_{\tilde{G}_L\cap\partial\Omega-\Gamma_L}\frac{\partial}{\partial n}U'\,ds-\int_{(\tilde{G}'_L-\tilde{G}_L)\cap\partial\Omega'}\frac{\partial}{\partial n}U'\,ds.$$

Hence

$$0\leqq\int_{\Gamma'_L}\frac{\partial}{\partial n}U'\,ds\leqq\int_{\Gamma_L}\frac{\partial}{\partial n}U\,ds\,.$$

Let $L_t=\int_{\Gamma_t}\frac{\partial}{\partial n}U\,ds$ and $L'_t=\int_{\Gamma'_t}\frac{\partial}{\partial n}U'\,ds.$ Then $L'_t\leqq L_t$ and by Lemma 5 we have

$$D(U'^L)=\int_0^L L'_t\,dt\leqq\int_0^L L_t\,dt=D(U^L)\,.$$

Similarly we have the following

LEMMA 6. B. *Let Ω and Ω' be domains in Lemma 6. A. Let $U(z)$ be an HP on $\bar{\Omega}$ such that $U(z)=\max U(z)=M$ on $\partial\Omega\cap\Omega'$ and let $U'(z)$ be an HP on $\bar{\Omega}'$ such that $U'(z)=U(z)$ on $\partial\Omega\cap\partial\Omega'$ and $U'(z)=M$ on $\partial\Omega'-\bar{\Omega}$. Then*

1) $U'(z)\leqq U(z)$.

2) $0\leqq\int_{\Gamma'_L}\frac{\partial}{\partial n}U'\,ds\leqq\int_{\Gamma_L}\frac{\partial}{\partial n}U\,ds.$

3) $D(U'^L)\leqq D(U^L)$.

THEOREM 7. *Let $U(z)$ be a QHD such that $\mathfrak{A}(U)=\alpha$ for $\{M_i\}$ i.e. $\frac{D(|U|^{M_i})}{M_i}=\alpha_i,\ \alpha_i\to\alpha.$ Then there exist positive QHD's U_1 and U_2 with $\mathfrak{A}(U_1)\leqq\alpha$, $\mathfrak{A}(U_2)\leqq\alpha$ for $\{M_i\}$ and $U=U_1-U_2$, $U_1\geqq U^+=\max(0, U)$ and $U_2\geqq -U^-$: $U^-=\min(0, U)$.*

PROOF. If $U(z)>0$, our assertion is trivial. We suppose $\inf U(z)<0$. For any $\delta'>0$, consider $D(U)$ over $\{z\in R: 0<U(z)<\delta'\}<\infty$. By Lemma 5, there exists a const. δ such that $0<\delta<\delta'$ and $\int_{C_\delta}\frac{\partial}{\partial n}U(z)\,ds=K<\infty$: $C_\delta=\{z\in R: U(z)=\delta\}$. Let D_0 be a disc in $\{z\in R: U(z)<0\}\cap R_{n_0}$, where n_0 is a suitable number. Let $\tilde{U}(z)=(U(z)-\delta)^+$ and $\tilde{G}_L=\{z\in R: \tilde{U}(z)>L\}$. Then $0<\int_{\partial\tilde{G}_0}\frac{\partial}{\partial n}\tilde{U}\,ds\leqq K.$ In Lemma 6, A, let $\Omega=R_n\cap\tilde{G}_0$, $\Omega'=R_n-D_0$: $n>n_0$. Then $\Omega\subset\Omega'$. Let $\tilde{U}'_n(z)=H^{\Omega'}_{\tilde{U}}=H^{R_n-D_0}_{\tilde{U}}$. Then $\tilde{U}'_n(z)=0$ on ∂D_0. Let $\tilde{G}'_L=\{z\in R: \tilde{U}'_n(z)>L\}$. Then by Lemma 6, A, 2)

$$\int_{\partial D_0} \frac{\partial}{\partial n} \tilde{U}'_n ds \leqq \int_{\partial R_n - \tilde{G}_0 + \partial D_0} \frac{\partial}{\partial n} \tilde{U}'_n ds \leqq \int_{\partial \tilde{G}_0} \frac{\partial}{\partial n} \tilde{U} ds \leqq K. \qquad (1)$$

Since $\tilde{U}$ is an SBH, $H^{R_n - D_0}_{\tilde{U}} \nearrow$ as $n \to \infty$. By (1) $\lim_n H^{R_n - D_0}_{\tilde{U}} = \tilde{\tilde{U}} < \infty$. Let $w'(z) = 1 - w(B \cap R, z, R - D_0)$ and $w'(z) = 1$ on $\bar{D}_0$. Since $\tilde{\tilde{U}}$ and $w'(z)$ are harmonic on ∂D_0, $\max_{z \in \partial D_0} \frac{\partial}{\partial n} \tilde{\tilde{U}}(z) / \min_{z \in \partial D_0} \left(-\frac{\partial}{\partial n} w'(z) \right) \leqq K' < \infty$. Then $K' w'(z) + \tilde{\tilde{U}} + \delta$ is an $SPH \geqq U^+$. Let $\Omega' = R_n$ and $\Omega = R_n \cap G_0$: $G_0 = \{z \in R : U(z) > 0\}$ and let $U^*_n = H^{R_n}_{U^+}$. Since U^+ is an SBH,

$$(U^+ \leqq) \; U^*_n \nearrow U^* \leqq K' w'(z) + \tilde{\tilde{U}} + \delta < \infty .$$

On the other hand, by Lemma 6. A. 2)

$$D_{R_n}(U^{*M_i}_n) \leqq D_{R_n}\big((U^+)^M\big) \leqq M_i \alpha_i \qquad \text{for any } n .$$

Also by Lemma 3 $D_R(U^{*M_i}) \leqq \varliminf_n D_{R_n}(U^{*M_i}_n) \leqq M_i \alpha_i$. Let $U_1 = U^*$. Then U_1 is a required function. Similarly let $V^*_n = H^{R_n}_{-U^-}$. Then $V^*_n \nearrow V^* < \infty$ and $U^*_n - V^*_n = H^{R_n}_{U^+} - H^{R_n}_{-U^-} = H^{R_n}_{U} = U$, whence $U_1 - U_2 = U$, where $U_2 = V^*$. Thus we have the theorem.

The operations E and I[7]. Let U be an HP in R. Let $I[U]$ be the greatest SBH not larger than U in $R - R_0$ vanishing on ∂R_0. Let U_n be a harmonic function in $R_n - R_0$ such that $U_n = 0$ on ∂R_0, $U_n = U$ on ∂R_n and $U_n = U$ in $R - R_n$. Then U_n is an SPH, whence $U_n \geqq I[U]$, $U_n \downarrow$ as $n \to \infty$ and $\lim_n U_n \geqq I[U]$. $\lim_n U_n$ is harmonic and $\leqq U$ and $\lim_n U_n \leqq I[U]$. Hence $\lim_n U_n = I[U]$. From the construction of U_n, we see easily

$$I[U] = U - H^{R - R_0}_{U} .$$

Let U_0 be an HP in $R - R_0$ vanishing on ∂R_0. Let $E[U_0]$ be the least positive SPH in R not smaller than U_0. Let $U^* = U_0 + K(1 - w(B \cap R, z, R - R_0))$ in $R - R_0$ and K in R_0, then U^* is an SPH, where $K = \max_{z \in \partial R_0} \frac{\partial}{\partial n} U_0(z) / \min_{z \in \partial R_0} \frac{\partial}{\partial n} w(R \cap B, z, R - R_0)$. Let U_n be a function in R such that $U_n = H^{R_n}_{U_0}$ in R_n and $U = U_0$ in $R - R_n$, then U_n is an $SBH \geqq U_0$ and $U_n \nearrow \leqq U^* < \infty$. Similarly as $I[U]$ we have $\lim_n U_n = E[U_0]$ and

$$E[U_0] = U_0 + H^{R - R_0}_{E[U_0]} \qquad \text{in } R - R_0 .$$

Define E for $I[U]$. Let $U_n = H^{R_n}_{I[U]}$ then $U_n = H^{R_n}_{U} - H^{R_n}_{S}$, where $S = H^{R - R_0}_{U} \leqq K(1 - w(R \cap B, z, R - R_0))$: $K = \max_{z \in \partial R_0} U(z)$. By 4) of Lemma 1, $\lim_n H^{R_n}_{S} = 0$.

Then by $H_U^{R_n}=U$ we have $EI[U]=U$. Similarly we have $IE[U_0]=U_0$. Thus

$$EI[U]=U \quad \text{and} \quad IE[E_0]=U_0 .$$

If an HP U is a limit of increasing sequence of HB's, U is called a QHB (*quasibounded harmonic function*),

LEMMA 7. 1). *Let U be a positive QHD in R with $\mathfrak{A}(U)=\alpha$ for $\{M_i\}$. Then $I[U]$ is also a QHD and $\mathfrak{A}(I[U])=\alpha$ for M_i+k, where k is a const. Let U_0 be a positive QHD in $R-R_0$ vanishing on ∂R_0 and $\mathfrak{A}(U_0)=\alpha$ for $\{M_i\}$, then $E[U]$ is also a QHD with $\mathfrak{A}(E[U])=\alpha$ for M_i+k', where k' is a const.*

2) *If an HP U is singular in R, $I[U]$ is singular in $R-R_0$. If U_0 is singular in $R-R_0$ and $U=0$ on ∂R_0, $E[U_0]$ is singular in R.*

3) *If U is a QHB, $I[U]$ is also a QHB in $R-R_0$. If U_0 is a QHB vanishing on ∂R_0, $E[U]$ is a QHB in R.*

4) *If $G(z)$ is a GG, $I[G]$ is a GG in $R-R_0$. If $G(z)$ is a GG in $R-R_0$ vanishing on ∂R_0, $E[G]$ is a GG in R.*

PROOF. 1). Let $I[U]=V$ and $H_U^{R-R_0}=T$. Then $0\leqq T\leqq k\leqq K_1=\max_{z\in\partial R_0} U(z)$. Since T is harmonic on ∂R_0,

$$D(T)\leqq K_1<\infty . \tag{1}$$

$$\{z: U(z)<L\}=G_U^L\subset\{z:\ V(z)<L\}\subset G_U^{L+k}=\{z:\ U(z)<L+k\}.$$

$$D_{G_U^L}(U)-2D_{G_U^L}(U, T)+D_{G_U^L}(T)\leqq D(V^L)\leqq D_{G_U^{L+k}}(U)-2D_{G_U^{L+k}}(U, T)+D_{G_U^{L+k}}(T) . \tag{2}$$

By Schwarz's inequality

$$D(U,T)\leqq\sqrt{D(U)\,D(T)} . \tag{3}$$

If $\lim_{L=\infty} D(U^L)<\infty$, we have by (1), (2) and (3), $D(V)<\infty$ and $\mathfrak{A}(U)=0=\mathfrak{A}(I[U])$. If $\lim_{L=\infty} D(U^L)=\infty$, $2D(U^L, T)/L\to 0$ as $L\to\infty$. Hence by (2) we have

$$\lim_{L=\infty} D(U^L)/L\leqq\lim_{L=\infty} D(V^L)/L\leqq\lim_{L=\infty} D(U^{L+k})/L ,$$

$\mathfrak{A}(U)=\mathfrak{A}(I[U])$ for M_i and M_i+k respectively.
Now $E[U_0]=U_0+H_{E[U_0]}^{R-R_0}$ and $H_{E[U_0]}^{R-R_0}\leqq K_2$, $D(H_{E[U_0]}^{R-R_0})\leqq K_3$. The latter part is proved similarly.

2) If U is singular, $I[U]$ is clearly singular in $R-R_0$. Let U_0 be singular in $R-R_0$. $E[U_0]=U_0+H_{E[U_0]}^{R-R_0}$. Let $E[U_0]=T$ and $V=H_{E[U_0]}^{R-R_0}$. Then

$$H_{T^M}^{R_n}\leqq H_{U_0^M}^{R_n}+H_V^{R_n} .$$

Let $w'(z)=1$ on $\bar{R}_0$ and $w'(z)=1-w(R\cap B, z, R-R_0)$ in $R-R_0$. Then $w'(z)$ is an SPH and singular by Lemma 1, 4). Now $V\leqq K_4 w'(z)$ in $R-R_0$, $K_3=\max_{z\in\partial R_0} E[U_0]$. Hence $H_V^{R_n}\leqq K_4 H_{W'}^{R_n}\to 0$ as $n\to\infty$. Let $\acute{H}_{U_0^M}^{R_n-R_0}$ be the solution of Dirichlet problem with value $=0$ on ∂R_0 and $=U_0^M$ on ∂R_n. Then

$$H_{U_0^M}^{R_n}\leqq \acute{H}_{U_0^M}^{R_n-R_0}+K_4 w'(z) \text{ in } R_n-R_0\,.$$

Since U_0 is singular in $R-R_0$, $\lim_n \acute{H}_{U_0^M}^{R_n-R_0}=0$, whence $\lim_n H_{U_0^M}^{R_n}\leqq K_4 w'(z)$. $\lim_n H_{U_0^M}^{R_n}$ is an HB and $w'(z)$ is singular, whence $\lim_n H_{U_0^M}^{R_n}=0$. Thus $\lim H_{T^M}^{R_n}=0$ and $E[U_0]$ is singular. 3) is evident by the expression of I and E. 4) is a direct consequence of 1) and 2).

LEMMA 8. 1). *Let Ω be a compact domain with analytic relative boundary. Let $P(z)$ be a positive continuous SPH on $\bar{\Omega}$ such that $\frac{\partial}{\partial x}P$, $\frac{\partial}{\partial y}P$ are continuous except analytic curves in Ω and $D(P^M)\leqq M\alpha$. Let $U=H_P^\Omega$. Then*

$$D(U^M)\leqq M\alpha\,.$$

2) *Let $P(z)$ be an SPH satisfying the same condition as* 1) *in R. Let $U_n=H_P^{R_n}$. Then $U_n\downarrow U$ and $D(U^M)\leqq M\alpha$.*

PROOF. Let $D^M=\{z\in\Omega: U(z)<M\}$ and $\acute{D}^M=\{z\in\Omega: P(z)<M\}$. Then $D^M\supset\acute{D}^M$ and $P^M(z)=M$ in $\bar{D}^M-\acute{D}^M$ and $D_{D^M-D'^M}(P^M)=0$. Since $P^M(z)$ and $U(z)$ has the same boundary value on ∂D^M, by the Dirichlet principle

$$\left(D(U^M)=\right) D_{D^M}(U)\leqq D_{D^M}(P^M)=D_{D'^M}(P^M)\left(=D(P^M)\right).$$

Thus we have 1). Let $R_n=\Omega$. Then by 1) $D(U_n^M)\leqq D(P^M)\leqq M\alpha$. Since $P(z)$ is an SPH, $U_n\downarrow U$. By Lemma 3 we have at once 2).

***QHBD* functions.** Let $U(z)$ be an HP. If there exist increasing sequence of HB's $U_n(z)$: $n=1, 2, 3, \cdots$ and a sequence $\{M_i\}$ such that $U(z)=\lim U_n(z)$ and

$$\frac{D(U_n^{M_i})}{M_i}\leqq\alpha_i \qquad \text{for any } n \text{ and } \alpha_i\to\alpha\,,$$

we call $U(z)$ a $QHBD$ with $\mathfrak{A}(U)\leqq\alpha$ for $\{M_i\}$.

We see at once by Lemma 3 $U(z)$ is a QHD with $\mathfrak{A}(U)\leqq\alpha$ for $\{M_i\}$ and evidently $U(z)$ is a QHB.

Fullsuperharmonic functions ($FSPH$)[2),5)]. Let $U(z)$ be an HP in $R-R_0$

such that $U(z)=0$ on ∂R_0 and $D(U^M)<\infty$: $0<M<\infty$. Let D be a domain such that ∂D is compact. Let $U_D(z)$ be a function such that $U_D(z)=U(z)$ on D and $U_D(z)$ has M. D. I. over $R-D-R_0$ among all harmonic functions with the same value as $U(z)$ on $\partial R_0+\partial D$. If for any D, $U_D(z)\leqq U(z)$, $U(z)$ is called an *FSPH*. Then it is well known, $U_D(z)$ is given as

$$U_D(z)=\int_{\bar{D}} N(z,p)\,d\mu(p),\ \int d\mu=\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}U_D(z)\,ds\leqq\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds=\frac{\alpha}{2\pi},$$

where $\bar{D}$ is the closure of D with respect to *N-Martin's* topology.

THEOREM 8. *Let $U(z)$ be an HP such that $U(z)=0$ on ∂R_0 and $U(z)$ is an FSPH in $R-R_0$. Then $U(z)=V(z)+G(z)$, where $V(z)$ is a QHBD with $\mathfrak{A}(V)\leqq\alpha$ for any $\{M\}$ and $G(z)$ is a GG with $\mathfrak{M}(G)\leqq\alpha$: $\alpha=\int_{\partial R_0}\frac{\partial}{\partial n}U ds$.*

PROOF. Since $U(z)$ is an *FSPH*, $U_{R_m}(z)\nearrow U(z)$ as $m\to\infty$. $U_{R_m}(z)$ is given as $U_{R_m}(z)=\int N(z,p)\,d\mu_m(p)$, $\int d\mu_m\leqq\frac{1}{2\pi}\int_{\partial R_0}\frac{\partial}{\partial n}U(z)\,ds=\frac{\alpha}{2\pi}$. Since $U_{R_m}(z)$ is harmonic except ∂R_m, $\mu_m>0$ only on ∂R_m. In any compact set in $R-R_0-\partial R_m$, $U_{R_m}(z)$ is uniformly approximated by a sequence of the form $\Sigma c_i N(z,p_i)$: $\Sigma c_i\leqq\frac{\alpha}{2\pi}$. Clearly $D((\Sigma c_i N(z,p_i))^M)\leqq\alpha M$. Hence by Lemma 3

$$D(U^M_{R_m})\leqq\alpha M. \tag{1}$$

By $N(z,p)=G(z,p)+U(z,p)$: $p\in R-R_0$, $U_{R_m}(z)=\int G(z,p)\,d\mu_m(p)+\int U(z,p)\,d\mu_m(p)$. Let $P^m=\int G(z,p)\,d\mu_m(p)$ and $J_m=\int U(z,p)\,d\mu_m(p)$. Then P^m and J_m are bounded and P^m is singular, whence $\lim_n H^{R_n}_{P^m}=0$. Since $U_{R_m}(z)$ has *M. D. I.* over $R-R_m$, $J_m(z)\leqq U_{R_m}(z)\leqq\max_{z\in\partial R_m}U(z)$ in $R-R_m$ and J_m is an *HB*. Let $T_{m,n}=H^{R_n-R_0}_{J_m}$. Since J_m is harmonic, $T_{m,n}=J_m$ and

$$H^{R_n-R_0}_{U_{R_m}}=H^{R_n-R_0}_{P^m}+H^{R_n-R_0}_{J_m}.$$

Let $n\to\infty$. Then $\lim_n H^{R_n-R_0}_{U_{R_m}}=J_m$. Hence by Lemma 8 and by (1)

$$D(J^M_m)\leqq D(U^M_{R_m})\leqq\alpha M\quad\text{for any } m. \tag{2}$$

By $U_{R_m}\nearrow U$, $J_m\nearrow J$. Thus J is a *QHBD* with $\mathfrak{A}(J)\leqq\alpha$ for any $\{M\}$. Since $\int d\mu_m\leqq\frac{\alpha}{2\pi}$, we can find a squence $\{\mu_{m'}\}$ such that $\mu_{m'}\to\mu$ and $P^{m'}\to P$. Then

$$U(z)=P+J.$$

Now μ lies on only Δ_G, whence by Theorem 6, 2) P is a GG with $\mathfrak{M}(P)\leqq\alpha$ and we have the theorem.

The following is well known and we state without proof.

LEMMA 9. *Let U be an HP. Then U is divided into uniquely determined two parts, the quasibounded part U^Q and the singular part U^S and $U=U^Q+U^S$. This implies if HP, s U_1 and U_2 satisfy $U_1\geqq U_2$, then $U_1^Q\geqq U_2^Q$, $U_1^S\geqq U_2^S$.*

THEOREM 9. 1). *Let $U(z)$ be a positive QHD with $\mathfrak{A}(U)\leqq\alpha$ for $\{M_i\}$. Then*

$$U(z)=V(z)+G(z)\,,$$

where $V(z)$ is a positive $QHBD$ with $\mathfrak{A}(V)\leqq\alpha$ for $\{M_i'\}$: $M_i'=M_i+$ const., and $G(z)$ is a GG with $\mathfrak{M}(G(z))\leqq\alpha$.

2) *Let $U(z)$ be a QHD with $\mathfrak{A}(|U|)\leqq\alpha$ for $\{M_i\}$. Then*

$$U(z)=\Big(V_1(z)+G_1(z)\Big)-\Big(V_2(z)+G_2(z)\Big)\,,$$

where $V_i(z)$ is a positive $QHBD$ with $\mathfrak{A}(V_i)\leqq\alpha$ for $M_i'=M_i+$ const., and $G_i(z)$ is a GG with $\mathfrak{M}(G_i)\leqq\alpha$.

3) *In the representation of $U(z)$ in 2), if $\mathfrak{A}(|U|)=0$, $G_i(z)=0$ by Theorem 2. If every boundary points of R is regular ($\Delta_{G,\delta}=0$ for $\delta>0$), $G_i(z)=0$.*

4) *As a special case, if $U(z)$ is a QHD in $|z|<1$, then $U(z)$ is Poisson's integrable.*

PROOF. Let $\tilde{U}=I[U]$, where I is the operation from R into $R-R_0$. Then by Lemma 7 $\tilde{U}$ is also a QHD in $R-R_0$ with $\mathfrak{A}(\tilde{U})\leqq\alpha$ for $\{M_i'\}$: $M_i'=M_i+\max_{z\in\partial R_0}U(z)$. Let $G_L=\{z\in R-R_0:\tilde{U}(z)>L\}$ and let $\Omega'=R_{n+i}-R_0-G_L\cap(R_{n+i}-R_n)$, $\Omega=R_{n+i}-R_0-G_L$.
Then $\Omega'\supset\Omega$. Let $\tilde{T}_{L,n,n+i}=H^{\Omega'}_{\tilde{U}_L}$ and $\tilde{T}^*=H^{\Omega}_{\tilde{U}}=\tilde{U}$. Then $\tilde{T}_{L,n,n+i}=L=\max\tilde{T}_{L,n,n+i}$ on $G_L\cap\partial R_n$ and $T^*=L$ on $\partial G_L\cap R_{n+i}$. Hence by Lemma 6, B

$$D(\tilde{T}^{M_i'}_{L,n,n+i})\leqq D(\tilde{U}^{M_i'})\leqq M_i'\alpha_i,\ (M_i'\leqq L)\,.$$

Since $\tilde{U}$ is an SPH in $R-R_0-G_L$, $\tilde{T}_{L,n,n+i}\downarrow\tilde{T}_{L,n}$ as $i\to\infty(\leqq\tilde{U})$. Let $\tilde{T}_{L,n}=L$ in G_L-R_n. Then $\tilde{T}_{L,n}$ is an SPH in $R-R_0$ and $\tilde{T}_{L,n}\downarrow\tilde{T}_L$ as $n\to\infty$. Now $\tilde{T}_L$ is an HB in $R-R_0$ and by Lemma 3 $D(\tilde{T}^{M_i'}_L)\leqq M_i'\alpha_i$: $M_i'\leqq L$. Clearly $\tilde{T}_{L,n,n+i}\leqq\tilde{T}_{L',n,n+i}$ for $L<L'$ and $\tilde{T}_L\nearrow\tilde{T}\leqq\tilde{U}$ as $L\to\infty$. Hence $\tilde{T}$ is a $QHBD$ with

$$\mathfrak{A}(\tilde{T})\leqq\alpha\qquad\text{for }\{M_i'\}\,. \tag{1}$$

Let $S_L=Lw(G_L,z,R-R_0)$. Then by Theorem 1 $\lim_{L=\infty}S_L=S$ is singular and

$$S \leqq \tilde{U}. \tag{2}$$

By the maximum principle $\tilde{T}_{L,n,n+i} \leqq \tilde{U} \leqq \tilde{T}_{L,n,n+i} + S_L$. By letting $i \to \infty$, $n \to \infty$ and then $L \to \infty$,

$$\tilde{T} \leqq \tilde{U} \leqq \tilde{T} + S. \tag{3}$$

By (2), (3) and by Lemma 9 we have

$$\tilde{U} = \tilde{T} + S.$$

We show S is not only singular but also a CG. Let $A_L(z) = L\omega(G_L, z, R-R_0)$. Then since $A_L(z)$ has $M.D.I.$ over CG_L with same value as $\tilde{U}(z)$ on $\partial R_0 + \partial G_L$,

$$D\left(A_{M'_i}(z)\right) \leqq D(\tilde{U}^{M'_i}) \leqq M'_i \alpha_i \quad \text{and} \quad \int_{\partial R_0} \frac{\partial}{\partial n} A_{M'_i}(z)\, ds \leqq \alpha_i\,.$$

Consier $A_{M'_i}(z)$ with respect to N-Martin's topology. Clearly $A_{M'_i}(z)$ is an $FSPH$ and there exists a positive mass μ_i such that $A_{M'_i}(z) = \int_{\bar{G}_{M'_i}} N(z, p)\, d\mu_i(p)$, $\int d\mu_i \leqq \frac{\alpha_i}{2\pi}$, $\left\{\int d\mu_i\right\}$ are uniformly bounded, hence there exists a sequence $\{\mu_{i'}\}$ such that $\mu_{i'} \to \mu$ and $A_{M'_i}(z) \to A(z)$ (clearly $\mu = 0$ in R) and $A(z) = \int N(z, p)\, d\mu(p)$. Hence by Theorem 8

$$A(z) = A'(z) + A''(z)\,,$$

where $A'(z)$ is a $QHBD$ with $\mathfrak{A}(A'(z)) \leqq \alpha$ for any $\{M\}$ and $A''(z)$ is a GG with $\mathfrak{M}(A''(z)) \leqq \alpha$. By $\omega(G_L, z, R-R_0) \geqq w(G_L, z, R-R_0)$, $S \leqq A'(z) + A''(z)$. Also by Lemma 9 $S \leqq A''(z)$. Hence by Theorem 2. 3) S is an GG in $R-R_0$ with $\mathfrak{M}(S) \leqq \alpha$.

Now

$$U = E[\tilde{U}] = E[\tilde{T}] + E[S]\,.$$

By Lemma 7 $E[S]$ is a GG in R with $\mathfrak{M}(E[S]) \leqq \alpha$. Let $T_L = E[\tilde{T}_L]$. Then $T_L = \lim_n H^{R_n}_{\tilde{T}_L}$ and $\leqq L$. By putting $\Omega = R_n - R_0$, $\Omega' = R_n$, we have by Lemma 6. A $\underset{R_n}{D}((H^{R_n}_{\tilde{T}_L})^{M'_i} \leqq \underset{R_n - R_0}{D}(\tilde{T}^{M'_i}_L) \leqq M'_i \alpha_i : M'_i \leqq L$. Also by Lemma 3

$$D(T^{M'_i}_L) \leqq M'_i \alpha_i : M'_i \leqq L \quad \text{and} \quad T_L \leqq L\,. \tag{4}$$

$\lim_{L=\infty} E[\tilde{T}_L] \geqq \lim_L \tilde{T}_L = \tilde{T}$. Since $E[\tilde{T}]$ is the least positive SPH larger than $\tilde{T}$, $\lim_{L=\infty} E[\tilde{T}_L] \geqq E[\tilde{T}]$. On the other hand, clearly $\lim_L E[\tilde{T}_L] \leqq E[\tilde{T}]$. Hence

$T_L \nearrow E[\tilde{T}] = \lim_{L=\infty} E[\tilde{T}_L]$. Thus by (4) $E[\tilde{T}]$ is a $QHBD$ with order $\leqq \alpha$ for $\{M'_i\}$ and we have 1). 2) is obtained by 1) and by Theorem 7. 3) If $G(z) > 0$, $\mathfrak{M}(G(z)) > 0$ and the former part is evident. By Theorem 4, $\Delta_{G,\delta} = 0$ for $\delta > 0$ implies $G(z) = 0$ and 3) is proved. 4) is clear by 2) and 3).

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