

A theorem on group spaces

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(Received September 25, 1978)

§ 1. Introduction.

The papers [5] and [9] contain the classification of all finite doubly transitive permutation groups (Ω, G) which have the property that the stabiliser G_α on a point $\alpha \in \Omega$ contains a normal subgroup Q acting sharply transitively on $\Omega - \{\alpha\}$. This result has found many applications, especially in finite geometry. However in most of these applications the algebraic objects most immediately connected to the geometric situation are group spaces rather than permutation groups. Therefore it seems advisable to adapt the above result to the more general case of group spaces. To do this is the main purpose of this paper. Also, we investigate, which elements of a group G of the type considered here are the product of two elements lying in conjugates of Q (see Lemma 2.6). Furthermore, in § 3 we present an application of our result to collineation groups of projective planes. Our notation is the same as in [3] and [5]. The following lemma which shall be needed for the proof of our main theorem, may be of independent interest. Under the additional assumption that X is a 2-group or X is a p -group and A is sharply transitive on X/Y it has been proved in [7, Lemma 3.3] and [8, Lemma 2.2 and Lemma 2.3].

LEMMA 1.1. *Let X be a finite group, Y a normal subgroup of X of index at least 3 and A a group of automorphisms of X such that A centralises Y and acts transitively on $X/Y - \{1\}$. Clearly X/Y is an elementary abelian p -group for some prime p . Denote $|X/Y| = p^n$ and $C = [X, A]$. Then*

a) *C is a p -group, $X = YC$, $[Y, C] = 1$, $C' \leq \Phi(C) \leq C \cap Y \leq {}_3C$, ${}_3C = C \cap Y$ or ${}_3C = C$, and C' is elementary abelian of order at most p^{n-1} ;*

b) *if $N \leq C$ and N is invariant under A , then $N \leq C \cap Y$ or $N = C$;*

c) *if in addition $p \nmid |A|$, then $C' = \Phi(C) = C \cap Y$, A is faithfully represented on $C/C \cap Y$, and one of the following statements holds:*

I) $X = Y \times C$.

II) C is a quaternion group of order 8, $|A| = 3$.

III) $|X/Y| = p^2$, where $p \in \{11, 19, 29, 59\}$, C is an extraspecial group of order p^3 , $C \cap Y = {}_3C$ and $A^{(\infty)} \cong SL(2, 5)$.

- IV) $|X/Y|=p^2$, where $p \in \{3, 5, 7, 11, 23\}$, C is an extraspecial group of order p^3 , $C \cap Y = \mathfrak{z}C$ and A contains a normal subgroup isomorphic to a quaternion group of order 8.
- V) $|X/Y|=3^4$, $|C \cap Y|=3$, 3^2 or 3^3 , and A contains a normal subgroup isomorphic to an extraspecial group of order 32.

PROOF. Let G be the corresponding semidirect product of X and A . YC/Y is a normal subgroup of X/Y which is left invariant by A . Also, A is trivial on $X/Y/YC/Y$, so that $YC=X$. If $x \in X$ and $a \in A$, then $[x, a] = (a^{-1})^x a \in \mathfrak{C}_G Y$, as $Y \trianglelefteq G$ and hence $\mathfrak{C}_G Y \trianglelefteq G$. So $[Y, C] = 1$. Let N be a subgroup of C which is left invariant by A and assume that $N \not\leq C \cap Y$. Then YN/Y is a non-trivial subgroup of X/Y left invariant by A , so that $YN=X$. If $y \in Y$, $x \in N$ and $a \in A$, then $[yx, a] = (yx)^{-1}(yx)^a = x^{-1}y^{-1}y^a x^a = x^{-1}x^a \in N$. Hence $C \leq N$, and we have b). Let C_p be a Sylow p -subgroup of C . Then $C = (C \cap Y)C_p$, as $C/C \cap Y \cong CY/Y = X/Y$ is a p -group. Also, $C \cap Y \leq \mathfrak{z}C$ so that C_p is characteristic in C and therefore $C_p = C$ by b). Let $u, v \in C - Y$. As $C' \leq \Phi(C) \leq C \cap Y \leq \mathfrak{z}C$ and $C/C \cap Y$ is elementary abelian, we have $[u, v]^p = [u^p, v] = 1$ so that C' is elementary abelian too. Let $k_1, \dots, k_n \in C$ such that $C/C \cap Y = \langle k_1(C \cap Y), \dots, k_n(C \cap Y) \rangle$. Because of the transitivity there exists $a \in A$ such that $u^a Y = k_1 Y$. As A leaves invariant C , there exist $y_1, y_2 \in C \cap Y$ such that $u^a = k_1 y_1$ and $v^a = k_1^{e_1} \dots k_n^{e_n} y_2$ for suitable integers e_i . Now $[u, v] = [u, v]^a = [u^a, v^a] = [k_1 y_1, k_1^{e_1} \dots k_n^{e_n} y_2] = [k_1, k_2]^{e_2} \dots [k_1, k_n]^{e_n} \in \langle [k_1, k_2], \dots, [k_1, k_n] \rangle$. So $|C'| \leq p^{n-1}$, and we have a).

We assume now that $p \nmid |A|$. Consider the action of A on C/C' : We have $C = [X, A] = [YC, A] = [C, A]$ and hence $C/C' = [C/C', A]$. Hence by a Lemma of Zassenhaus (see Huppert [6, III 13.4]) $C \cap Y/C' \leq \mathfrak{C}_{C/C'} A = 1$ so that $C' = \Phi(C) = C \cap Y$. Therefore we can assume that $C' \neq 1$, as otherwise I) holds. Then $C' = \Phi(C) = \mathfrak{z}C = C \cap Y$ by a), so that C is special, and A is symplectic on C/C' . Actually the automorphism group A acts trivially on Y and therefore is faithfully represented on C . Being a p' -group it is also faithfully represented on C/C' . Clearly, A is transitive on $C/C' - \{1\}$. To finish our proof we use the classification of transitive linear groups whose order is prime to their characteristic (see [4]; the result needed here can also be deduced from [2, Theorems 4.9, 3.9, 5.6, 5.10 and 5.13]; if A is sharply transitive on $X/Y - \{1\}$, one also can use the Zassenhaus [12] classification of finite nearfields). The exceptional cases are easily handled: If $n=2$, then $|C'|=p$ and C is extraspecial by a). For odd p we must have elements of order p in $C - C'$, as C is non-abelian. Hence each coset of C' contains an element of order p , and C' is of exponent p . Assume now that A induces a semi-linear group of dimension 1 on C/C' . Then

the linear part \bar{A} of A is a cyclic subgroup of order at least $(p^n - 1)/n$ which is semi-regular on $C/C' - \{1\}$. On the other hand A is symplectic so that $|\bar{A}| \leq p^{n/2} + 1$ (see [6, V, 17.13, a), (2)]). So $p^{n/2} - 1 \leq n$ and hence $n = 2$ or 4 , as n is even. If $n = 4$, then $p = 2$, and $p \nmid |A|$ implies $|\bar{A}| = 15$, a contradiction. If $n = 2$, then $p \leq 3$. Here $p = 2$ implies II). For $p = 3$ we have $|\bar{A}| \leq 4$ and hence $|A| = 8$. Clearly now A is a quaternion group, and C is extraspecial of exponent 3.

§ 2. On regular normal subgroups in pointstabilisers of doubly transitive group spaces.

Let (Ω, G) be a finite transitive group space with the property:
 (*) *There exists an element $\alpha \in \Omega$ such that G_α contains a normal subgroup Q which is sharply transitive on $\Omega - \{\alpha\}$.*

For this paragraph we introduce the following notation :

S is the normal closure of Q in G .

Φ is the representation of G on Ω determined by the group space (Ω, G) .
 K is the kernel of Φ and $Z = K \cap S$. If U is a subgroup or an element of G , then we denote $U^\Phi = \bar{U}$. Also, $\Omega(U)$ is the set of fixed points of U .

If $g \in G$ and $Q^g \subseteq G_\alpha$, then $\alpha \in \Omega(Q^g) = \Omega(Q)^g = \{\alpha^g\}$ so that $g \in G_\alpha$ and $Q^g = Q$. Hence for each $\xi \in \Omega$ the stabiliser G_ξ contains exactly one conjugate of Q , which we denote by Q_ξ .

PROPOSITION 2.1. *If $N \trianglelefteq S$ and $N \not\subseteq K$, then $S = QN$.*

PROOF. As S is doubly transitive, N must be transitive. Hence Q^N contains Q_ξ for all $\xi \in \Omega$, and therefore $Q^N = Q^G$.

PROPOSITION 2.2. $[K, S] = 1$.

PROOF. As K and Q are normal subgroups of G_α , we have $[K, Q] \leq K \cap Q = 1$.

PROPOSITION 2.3. *Let Q^* be any normal subgroup of G_α . Then $[KQ^*, S_\alpha] \leq Q^*$.*

PROOF. Let $k \in K$, $a \in Q^*$ and $s \in S_\alpha$. Then $k^{-1}s^{-1}k = s^{-1}$ by 2.2 so that $[ka, s] = a^{-1}k^{-1}s^{-1}kas = a^{-1}a^s \in Q^*$.

THEOREM 2.4. *Let (Ω, G) be a finite transitive group space, where $|\Omega| > 2$. Assume that for some $\alpha \in \Omega$ the stabiliser G_α contains a normal subgroup Q which is sharply transitive on $\Omega - \{\alpha\}$. If S is the normal closure of Q in G , then one of the following holds :*

(i) $S \cong PSL(2, q)$, $SL(2, q)$, $Sz(q)$, $PSU(3, q)$, $SU(3, q)$ or a group of Ree type of order $q^3(q^3 + 1)(q - 1)$, where q is a prime power and the degree

is $q+1$ in the linear case, q^2+1 in the Suzuki case and q^3+1 in the unitary and Ree case.

(ii) $S \cong PFL(2, 8)$ and $|\Omega|=28$.

(iii) (Ω, S) is sharply 2-transitive.

(iv) $|\Omega|=p^2$ for $p \in \{3, 5, 7, 11, 23, 29, 59\}$, $S=O_p(S) \cdot Q$, $O_p(S)$ is extraspecial of order p^3 and exponent p , ${}_3O_p(S)={}_3S$ is the kernel of (Ω, S) and S induces a sharply 2-transitive group on Ω .

PROOF. Assume at first that \bar{S} is sharply 2-transitive on Ω , and let X be the preimage in S of the sharply transitive normal subgroup of \bar{S} . Then Q acts transitively on $X/Z-\{1\}$, so that we can apply Lemma 1.1. Here $[X, Q] \trianglelefteq XQ=S$ and $X=[X, Q]$ by Proposition 2.1. Note that the exceptional case V) does not occur here because $|Q|=|X/Z|-1$. Also, $S \cong SL(2, 3)$ if X is a quaternion group. Suppose that $|Z|=19$. Then $Q^{(\infty)} \cong SL(2, 5)$ and $Q \cong C_3 \times SL(2, 5)$, so that the Sylow 3-subgroups of Q are elementary abelian. But this is impossible as \bar{S} is a Frobenius group.

Consider now the case that $\bar{S} \cong PFL(2, 8)$ and $|\Omega|=28$. Here $\bar{S}^{(\infty)} = \bar{S}^{(\infty)} \cong PSL(2, 8)$ so that $S^{(\infty)}$ is a Schur extension of $PSL(2, 8)$ and hence actually $S^{(\infty)} \cong PSL(2, 8)$. Also, $S/S^{(\infty)}$ contains $ZS^{(\infty)}/S^{(\infty)}$ as a central subgroup with cyclic factor group. Hence $S/S^{(\infty)}$ is abelian and $S^{(\infty)}=S'$. So $S' \cong PSL(2, 8)$. But $[Q, S_a] \leq Q \cap S'$ and $|[Q, S_a]| \geq |[\bar{Q}, \bar{S}_a]|=9$, so that $S=QS' \cong PFL(2, 8)$ by Proposition 2.1.

We now use the classification of Shult [9] and Hering, Kantor, Seitz [5] of finite groups with a split BN -pair of rank 1. It follows that in all remaining cases \bar{S} is simple. Also $[\bar{Q}, \bar{S}_a]=\bar{Q}$, so that $Q \leq S'$ and hence $S=S'$ by Proposition 2.1. So S is a Schur extension of \bar{S} , and we can apply [1]. Suppose that a prime p divides $(|Z|, |\Omega|)$ and let P be a Sylow p -subgroup of Z . Then $P \times Q$ is a Sylow p -subgroup of S . By a theorem of Gaschütz (see [6, I. 17.4]) there exists $C \leq S$ such that $S=P \times C$. Now $S' \leq C$ implies $P=1$, a contradiction.

Finally, a remark on the uniqueness of Q .

LEMMA 2.5. *If Q is not unique, then \bar{S} is sharply 2-transitive or $S \cong PFL(2, 8)$.*

PROOF. Assume that G_a contains a second normal subgroup Q^* sharply transitive on $\Omega-\{a\}$, and that \bar{S} is not sharply 2-transitive. Then \bar{S}_a contains only one sharply transitive normal subgroup, so that $KQ^*=KQ$ and $[KQ, S_a] \leq Q^* \cap Q$ by Proposition 2.3. As $|[KQ, S_a]| \geq |[\bar{Q}, \bar{S}_a]|$, the lemma follows.

LEMMA 2.6. *Let a be an element in $S-Z$ of prime order r . Then*

a is the product of two elements lying in conjugates of Q unless possibly if $S \cong SU(3, q)$ or $PSU(3, q)$ for $q \neq 2$, and r is an odd prime dividing $q^3 + 1$.

PROOF. We consider at first the cases (iii) and (iv) in Theorem 2.4. Then $|\Omega|$ is a power of a prime p . Let $\xi \in \Omega - \{\alpha\}$ and $x \in Q_\xi - \{1\}$. As $S = QO_p(S)$, there exists elements $y \in Q$ and $a \in O_p(S)$ such that $x = ya$ and $y^{-1}x = a$. Here $a \in O_p(S) - Z$, as otherwise x fixes α . Also, all elements in the coset Za are conjugate to a under $O_p(S)$, and all cosets of Z in $O_p(S)$ different from Z are conjugate to Za under Q . Hence $O_p(S) - Z = a^s$, and each element in $O_p(S) - Z$ is the product of two elements conjugate to elements in Q . Assume now that $a \in S - O_p(S)$ and a has prime order r . Then $r \neq p$, so that a fixes a point ξ . But $S_\xi = Q_\xi \times Z$, and $|Z| \nmid p$. So actually $a \in Q_\xi$.

Case (ii) is no problem because $PFL(2, 8)$ contains Frobenius groups of order 12 and 21. If $S \cong SL(2, q)$, $PSL(2, q)$ or $Sz(q)$, then we can apply [3, Lemma 2.7] or [10, Theorem 9] respectively. Note that the groups $SU(3, q)$ and $PSU(3, q)$ contain subgroups isomorphic to $SL(2, q)$, and groups of Ree type of order $q^3(q^3 + 1)(q - 1)$ contain $PSL(2, q)$. So we can again apply [3, Lemma 2.7]. Also, we have the description of groups of Ree type in [11].

§ 3. An application to perspectivities of projective planes.

THEOREM 3.1. Let \mathfrak{p} be a projective plane, l a line of \mathfrak{p} , P a point of \mathfrak{p} not incident with l and G a finite group of collineations of \mathfrak{p} fixing P and l . Let $\Omega = \{X \in l \mid G(X, XP) \neq 1\}$, $q = |\Omega| - 1$, and assume that $q \geq 2$ and there exists a point $Z \in \Omega$ such that $G(Z, ZP)$ is transitive on $\Omega - \{Z\}$. Then the subgroup S generated by all elations in G is isomorphic to $SL(2, q)$, and \mathfrak{p} contains desarguesian subplanes of order q .

PROOF. Obviously G leaves invariant Ω , and we can apply Theorem 2.4 to the group space (Ω, G) . Here $|S| \geq (q - 1)q(q + 1)$ by [3, (2.3)]. Hence $S \cong SL(2, q)$, and \mathfrak{p} contains desarguesian subplanes of order q by [3, Theorem 2.8].

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