Hokkaido Mathematical Journal Vol. 7 (1978) p. 271-288

On the normal p-structure of afinite group and related topics I

By Helmut BENDER

(Received July 9, 1977)

To Reinhold Baer, on his seventy-fifth birthday, 22 July 1977

1. Introduction.

In this note we describe (for any prime p) group theoretic properties p^{*} and $p^{*}p$ and the corresponding functors $O_{p^{*}}$ and $O_{p^{*}p}$.

They are in the class of all (finite!) groups what the functors $O_{p'}$ and $O_{p',p}$ are for solvable or, more precisely, for p-constrained groups.

Besides this analogy $\begin{matrix} p^{*}\leftrightarrow p' \\ p^{*}p\leftrightarrow p',p \end{matrix}$ there is another interesting analogy, namely between $O_{p^{*}}$, $O_{p^{*}p}$ and the well-known functors E and F^{*}, respectively. In order to exhibit this analogy most clearly we treat these four functors (and the corresponding properties) uniformly (in section 4).

This is done via the concept of the generalized centralizer $C_{G}^{*}(X)$ discussed in section 3, and by working with a prime set π which is the set of all primes or consists of our p only.

In sections 5 and 6 we specialize to these two cases, getting the (well known) elementary $E-F^{*}$-theory and our $p^{*}-p^{*}p$-theory.

In the theory of simple groups (general classification problems) one has reached a point where one is forced to handle nearly arbitrary (sub) groups H, and hence needs small subgroups of H conveniently structured which still control the structure of H somehow. It is exactly this what $E(H)$, $F^{*}(H)$, $O_{p^{*}}(H)$, $O_{p^{*}p}(H)$ and similar constructions are all about. In this field however, due to the structure of the known simple groups, $O_{p^{*}}(H)$ and $O_{p^{*}p}(H)$ appear in a certain special form, and then many of our results are contained in the work of Gorenstein and Walter, see [2], 17 2, [3], [4], [5]. These include our Theorem 6. 10 stating that $O_{p^{*}}(N_{H}(P))\subseteq O_{p^{*}}(H)$ for every p-subgroup P.

Part of this research initiated during the Group Theory Conference at Hokkaido University and Kyoto University in 1974. I wish to thank these institutions for their hospitality, and the Taniguchi Foundation for its support.

2. Notation.

We use the following notation —mostly standard or self-explanatory— in addition to that of [2].

A group theoretic property (=class of groups) X is a *radical property* if every group G has a maximal normal X-subgroup (the *X-radical* of G), usually denoted by $X(G)$ or $O_X(G)$.

This means that products of normal X-subgroups are X-groups. Moreover, since $N \lhd G$ implies $X(N) \lhd G$, *every subnormal X-subgroup of G lies in* $X(G)$.

Conversely, any functor X (assigning to every G a characteristic subgroup $X(G)$) yields a property which may also be denoted by X and is defined by "$G \in X \Leftrightarrow G = X(G)$".

We write $F_\pi(G)$ for $O_\pi(F(G))$, where π of course stands for a set of primes.

A property X is *residual* if every G has a unique smallest normal subgroup with X-factor group. This *X-residual* of G is often denoted by $O^X(G)$.

By $O^S(G)$, $O^F(G)$, $O^A(G)$, $S^\pi(G)$, $F^\pi(G)$, $A^\pi(G)$ we denote the solvable, nilpotent, abelian (π-)residual of G, respectively. Here we recall that properties inherited by subgroups and direct products are residual.

The symbol $O_{X_1,X_2,\dots}(G)$ we use quite freely. Any X_i may be a radical property or a functor, and X_1 is also allowed to be a normal subgroup of G. So for example we have $O_{2,F}(G)/O_2(G) = F(G/O_2(G)) = F_{2'}(G/O_2(G))$ and $O_{N,p}(G)/N = O_p(G/N)$ where $N \lhd G$.

Extensions of X-groups by Y-groups are called $X-Y$-groups. If X and Y are radical properties with Y inherited by factor groups, then $X-Y$ is a radical property and $O_{X-Y}(G)$ equals $O_{X,Y}(G)$.

If one —hence each— of the subgroups $S^\pi(G)$, $F^\pi(G)$, $A^\pi(G)$ equals G, G is called *π-perfect.*

A group is *π-solvable* (*π-nilpotent*) if all non-π'-chief-factors are abelian (central). Notice that π-nilpotent groups are nothing but π'-F-groups.

A product AB is *seminormal* if A or B is normal in AB.

A *centralizer-closed* subgroup K of G satisfies $C_G(K) \subseteq K$.

$[A]$ denotes the mapping $X \to [A, X]$.

A *semisimple* group is a direct product of non-abelian simple groups, a *quasisemisimple* group is a perfect group H with $H/Z(H)$ semisimple, and such an H is *quasisimple* if $H/Z(A)$ is non-abelian simple.

G is *constrained* if its Fitting subgroup is centralizer-closed, i. e. $C_G(F(G)) \subseteq F(G)$, and G is *π-constrained* if $G/O_{\pi'}(G)$ is constrained.

3. Nilpotent action and the generalized centralizer.

3. 1. Let A be an operator group on the group K. The following four conditions are equivalent. When they are satisfield, we say that A *acts nilpotently on* K.

(i) $[A]^n K=1$ *for some integer n.*

(ii) A *stabilizes some subgroup series* $1=K_0\subseteq K_1\subseteq\cdots\subseteq K_n=K$, *i.e.* $[A, K_{i+1}]\subseteq K_i$.

(iii) A *stabilizes some normal series* $1=K_0\lhd K_1\lhd\cdots\lhd K_n=K$.

(iv) A *is subnormal in the semidirect product* KA *of* K *and* A.

This situation has been studied by P. Hall in [6] (also for infinite groups). One of his results is the basic nilpotent action lemma stated below (in general, $A/C_A(K)$ is nilpotent and $[A, K]$ is locally nilpotent).

It allows (for finite groups) to add a fifth condition to the above:

(v) A *stabilizes a series* $1=K_0\subseteq K_1\subseteq\cdots\subseteq K_m=K$ *of normal subgroups of* K.

3. 2. Nilpotent action is inherited in many obvious ways:

(i) *If* A *is nilpotent on* K, *then on every* A*-invariant subgroup and factor group of* K.

(ii) *If* A *is nilpotent on all quotients* K_{i+1}/K_i *of some series* $1=K_0\lhd\cdots\lhd K_n=K$, *then* A *is nilpotent on* K.

(iii) By (i) and (ii), *if* A *is nilpotent on both factors of some seminor mal product* XY, *then* A *is nilpotent on* XY.

In particular, the A-group K has a unique largest normal subgroup on which A is nilpotent.

(iv) *Any seminormal product* A_0A_1 *of two subgroups of* A *nilpotennt on* K *is itself nilpotent on* K.

To prove (iv), let $A_0\lhd A_0A_1$ and observe that (by (i)) A_1 —hence A_0A_1— is nilpotent on every quotient $[A_0]^iK/[A_0]^{i+1}K$. Then apply (ii).

3. 3. By 3. 2. iv, *our operator group* A *on* K *has a unique largest normal subgroup* $C_A^*(K)$ *nilpotent on* K.

Since $B\lhd A$ implies $C_B^*(K)\lhd A$, it contains every subnormal subgroup of A nilpotent on K.

For any subgroup U of K we set $C_A^*(U):=C_{N_A(U)}^*(U)$.

This defines $C_A^*(U)$ *for arbitrary subgroups* A *and* U *of a group* G.

3. 4. Clearly, a group G is nilpotent if and only if $C_G^*(G)=G$. More generally, $C_G^*(G)$ equals $F(G)$.

A seminormal product KA (with K normal) of nilpotent groups K and A is nilpotent if and only if $C_A^*(K)=A$, see 3. 2. iv/iii.

More generally, if $K \lhd G$ with K and G/K nilpotent, then G is nilpotent if and only if $G=KC_G^*(K)$.

3.5. Nilpotent action lemma (P. Hall [6]): *Suppose A is nilpotent on K. Then $[A, K]$ and $A/C_A(K)$ are nilpotent and have the same prime divisors.*

PROOF. Set $\pi=\pi(A/C_A(K))$ and let $p\in\pi$. Then an S_p-subgroup A_p of A does not centralize K, i.e. $[A_p, K]\neq 1$.

Being nilpotent on K, A_p is subnormal in the semidirect product KA_p.

Hence $A_p\subseteq O_p(KA_p)$, so that $[A_p, K]\subseteq K\subset O_p(KA_p)=O_p(K)$.

This proves $\pi\subseteq\pi([A, K])$ and $[A, K]\subseteq F_\pi(K)$.

As for nilpotency of $A/C_A(K)$, consider distinct primes p and q in π. Then $[A_p, A_q]\subseteq C_A(K/O_p(K))\subset C_A(K/O_q(K))=C_A(K)$.

A more natural proof (not using semidirect products) is based on the following phenomenom.

3.6. *If $[X]^2Y=1$, then $[X, Y]$ is abelian, and for each $y\in Y$ the mapping $x\rightarrow[x, y]$ is an homomorphism from X into $[X, Y]$.*

This holds for groups X and Y whenever it makes sense.

3.7. In the situation of the nilpotent action lemma, $C_{Z(F(K))}(A)$ is a non-identity normal subgroup of K (unless $[A, K]=1$).

This yields condition (v) of 3.1.

3.8. As another immediate consequence of the nilpotent action lemma, *any p-subgroup P of a group G satisfies*

$$C_G^*(P)=O_{C_G(P),P}\bigl(N_G(P)\bigr)\,.$$

3.9. *Nilpotent action of A on K is guaranteed if A centralizes a centralizer-closed normal subgroup L of K*, because then $[A, K]\subseteq C_K(L)\subseteq L$.

If in addition $|A:A'|$ is prime to $|F(K)|$ —or just to $|[A, K]|$— then the nilpotent action lemma forces $[A, K]=1$.

By induction, *this is also true under the weaker assumption* (*on* L) *that there exists a series $L=K_1\lhd K_2\lhd\cdots\lhd K_n=K$ with $K_i=K_i^A$ centralizer-closed in K_{i+1}.*

3.10. *Assume now that our A-group K is nilpotent on* $[A, K]$. Then, in the semidirect product KA, K is nilpotent on $A[A, K]$.

So if $|K:K'|$ is prime to the order of $[A, K]$ (a subgroup of $F(K)$), *then* $[A, K]=1$, by the nilpotent action lemma.

Note that all the above assumptions are trivially satisfied when $[A, K]\subseteq Z(K)$ and K is perfect (in which case one can also apply 3.6).

3.11. If K is a centralizer-closed subgroup of a group G, then the nilpotent action lemma forces every subgroup of $N_G(K)$ nilpotent on K to be nilpotent. In particular, $C_G^*(K)=F(N_G(K))$.

3.12. Thus a group G is constrained if and only if $G_G^*(F(G))=F(G)$. It follows that *G is π-constrained if and only if an S_π-subgroup P of $O_{\pi,F}(G)$ satisfies $C_G^*(P)\subseteq O_{\pi,F}(G)$.*

4. $\pi^*\pi$ and π^*.

4.1. In this section π is either the set of all primes or consists of a single prime p. It suffices to keep in mind that *the Sylow π-theorem holds in every group.*

By X_π we usually mean an S_π-subgroup of a group X. By the Frattini argument we have $H=XN_H(X_\pi)$ and hence

(i) *$XC_H(X_\pi)\lhd H$ whenever $X\lhd H$.*

We often use the fact that, by the nilpotent action lemma,

(ii) $S^\pi(C_H^*(P))\subseteq F^\pi(C_H^*(P))\subseteq C_H(P)$

for any π-subgroup P of a group H. We also mention that

(iii) *$X\lhd H=XU$ implies $H=XS^\pi(U)$ if H is π-perfect.*

Notation introduced here will be used only in the case $\pi=p$. In the other case we shall write E for π^* and O_{π^*}, and F^* for $\pi^*\pi$ and $O_{\pi^*\pi}$.

4.2. Definition of $\pi^*\pi$ and π^*: *H is a $\pi^*\pi$-group if $H=XC_H^*(X_\pi)$ for every $X\lhd H$; and π^*-groups are π-perfect $\pi^*\pi$-groups.*

By 4.1.iii/ii, *a π^*-group H satisfies $H=XS^\pi(C_H(X_\pi))$ for every normal subgroup X.*

4.3. *Both $\pi^*\pi$ and π^* are inherited by factor groups.*

Since π-perfectness is inherited by factor groups, it suffices to prove this for $\pi^*\pi$. So it suffices to verify that

$$X/M\lhd H/M=:\bar{H} \text{ implies } \overline{C_H^*(X_\pi)}\subseteq C_{\bar{H}}^*(\bar{X}_\pi)\,,$$

and this will be clear when $\overline{C_H^*(X_\pi)}\lhd N_{\bar{H}}(\bar{X}_\pi)$, which amounts to $C_H^*(X_\pi)M\lhd N_H(X_\pi M)$.

Now observe that, by the Frattini argument, $N_H(X_\pi M)=N_H(X_\pi)\,M$, and remember $C_H^*(X_\pi)\lhd N_H(X_\pi)$.

4.4. Since in a $\pi^*\pi$-group H an abelian (or nilpotent) normal π-subgroup H_1 satisfies $H=C_H^*(H_1)$, it follows from 4.3 that *a $\pi^*\pi$-group H is nilpotent on every abelian, hence on every solvable π-factor H_1/H_0* (with $H_i\lhd H$). *Thus H_1/H_0 is nilpotent.*

In particular, $S^\pi(H)=F^\pi(H)$.

If H is a π^*-group, then the nilpotent action lemma forces $H=F^\pi(H)$

to centralize all such solvable π-factors.

In particular, $S^{\pi}(K)=F^{\pi}(K)=A^{\pi}(K)$ *for every* $K \lhd H$, *and all solvable normal π-subgroups of H lie in $Z(H)$.*

4.5. *A solvable π-subgroup A and a π^*-subgroup K normalizing each other necessarily centralize each other.*

For $[A, K]$ is a solvable normal π-subgroup of the π^*-group K, hence lies in $Z(K)$; and since K is π-perfect, this implies $[A, K]=1$, see 3.10.

4.6. $K \lhd \lhd H \in \pi^*$ *implies* $O^{\pi'}(K) \lhd H$, *hence* $H=KC_H(K_\pi)$.

Proof: Proceeding by induction, we may assume $H=XC_H(X_\pi)$ for some $X \lhd \lhd H$ such that $K \lhd X$.

Then $K_\pi :=K \cap X_\pi$ satisfies $\langle K_\pi^H \rangle = \langle K_\pi^X \rangle = \langle K_\pi^K \rangle = O^{\pi'}(K)$.

4.7. We show that X, $K \lhd H$ *with* $K \in \pi^*\pi$ *implies*

$$K=(X \cap K)\, C_K^*(X_\pi).$$

Let $D :=X \cap K$ and $D_\pi = X_\pi \cap K$. Then $K=DK_0$ with $K_0 :=C_K^*(D_\pi)$. The Frattini argument applied to $(K_0 \cap X)\, X_\pi = K_0 X_\pi \cap X \lhd K_0 X_\pi$ yields $K_0 X =(K_0 \cap X)\, X_\pi N_{K_0 X_\pi}(X_\pi)$, hence, by Dedekind's modular law, $K_0=(K_0 \cap X)\, N_{K_0}(X_\pi)$. It follows that $K=DK_0=(X \cap K)\, N_{K_0}(X_\pi)$.

Now observe that $N_{K_0}(X_\pi)$ is nilpotent on X_π (because $[N_{K_0}(X_\pi), X_\pi]$ lies in $X_\pi \cap K=D_\pi$) and is normal in $N_H(X_\pi)$, thus lies in $C_K^*(X_\pi)$.

4.8. *Both $\pi^*\pi$ and π^* are radical properties, i.e. any normal product $H=K_1K_2$ of $\pi^*\pi$-groups (π^*-groups) K_1* and K_2 is a $\pi^*\pi$-group (π^*-group).

To prove this, consider some $X \lhd H$. By 4.7, $K_i=(X \cap K_i)\, C_{K_i}^*(X_\pi)$ and hence $H=K_1K_2=XC_H^*(X_\pi)$. Thus H is a $\pi^*\pi$-group. As for π^*, note that π-perfectness is inherited by normal products.

The above defines $O_{\pi^*\pi}(G)$ and $O_{\pi^*}(G)$ *for every group G.*

Being simple, hence $\pi^*\pi$-groups, *all minimal subnormal subgroups of G lie in* $O_{\pi^*\pi}(G)$. In particular, $O_{\pi^*\pi}(G)\neq 1$ *unless* $G=1$.

4.9. By 4.7 and 4.1. iii/ii, X, $K \lhd H$ *with* $K \in \pi^*$ *implies*

$$K=(X \cap K)\, C_K(X_\pi)=(X \cap K)\, S^{\pi}\big(C_K(X_\pi)\big).$$

This allows to improve 4.8 in the π^*-case: *Let $H=K_1K_2$ with π^*-subgroups K_1 and K_2 normalized by suitable S_π-subgroups of H. Then H is a π^*-group.*

To verify this, let $X \lhd H$ and choose X_π so that it normalizes both K_i. Apply the above to $X_\pi K_i$ in place of H. This yields $K_i=(X \cap K_i)\, C_{K_i}(X_\pi)$, hence $H=XC_H(X_\pi)$.

4.10. We proceed to prove the following basic result:

If H/K and K are $\pi^*\pi$-groups, with $H=KC_H^*(K_\pi)$, then H is a $\pi^*\pi$-group.

We have to show that $H=XC_H^*(X_\pi)$ for every $X\lhd H$.

By 4.7, $K=(X\cap K)\,C_K^*(X_\pi)$. Let $K_\pi=(X\cap K)_\pi S=(X_\pi\cap K)\,S$ with $S=C_K^*(X_\pi)_\pi$. The Frattini argument yields

$$N_H(X_\pi)=C_K^*(X_\pi)\,N_{N_H(X_\pi)}(S)=C_K^*(X_\pi)\,N_H(X_\pi S)=C_K^*(X_\pi)\,N_H(X_\pi K_\pi)\,.$$

It follows that $C_K^*(X_\pi)\,C_H^*(X_\pi K_\pi)$ is normal in $N_H(X_\pi)$ and hence lies in $C_H^*(X_\pi)$. Thus it suffices to verify $H=XKC_H^*(X_\pi K_\pi)$.

Hence, replacing X by XK, we may assume that $K\subseteq X$.

Let $H^*:=C_H^*(K_\pi)$, $K^*:=K\cap H^*$, and $X^*:=X\cap H^*$.

Then $H=KH^*$, $X=KX^*$, whence we may assume $X_\pi=K_\pi X_\pi^*$, and $H^*/K^*\simeq H/K\in\pi^*\pi$.

The latter yields $H^*=X^*U$ with $U:=C_{H^*}^*(X_\pi^*K^*/K^*)$.

The Frattini argument applied to $(U\cap X)\,X_\pi=UX_\pi\cap X\lhd UX_\pi$ yields $UX_\pi=(U\cap X)\,X_\pi N_{UX_\pi}(X_\pi)$, hence $U=(U\cap X)\,N_U(X_\pi)$.

Being normal in $N_H(X_\pi)$ and nilpotent on X_π, $N_U(X_\pi)$ lies in $C_H^*(X_\pi)$.

Now $H=KH^*=KX^*U=KX^*(U\cap X)\,N_U(X_\pi)\subseteq XC_H^*(X_\pi)$, as required.

4.11. We prove that *$\pi^*\pi$ is inherited by normal subgroups*:

Let $H\lhd G\in\pi^*\pi$. Proceeding by induction, we may assume that $H/K\in\pi^*\pi$, where $K=O_{\pi^*\pi}(H)$; here we have to remember that $K\neq 1$ unless $H=1$, see 4.8, and that $G/K\in\pi^*\pi$, by 4.3.

Clearly, $G=KC_G^*(K_\pi)$ implies $H=K(H\cap C_G^*(K_\pi))=KC_H^*(K_\pi)$.

Then 4.10 yields $H\in\pi^*\pi$.

4.12. *If* $K\lhd H\in\pi^*$, then, by 4.4 and 4.11, *$A_\pi(K)$ is a π-perfect $\pi^*\pi$-group, i.e. a π^*-group.* This is the π^*-analogue of 4.11.

4.13. By 4.4 and 4.11 again, *$H\in\pi^*\pi$ implies $F^\pi(H)\in\pi^*$.* Thus

$$O_{\pi^*}(G)=F^\pi\big(O_{\pi^*\pi}(G)\big)\qquad \textit{for every group } G\,.$$

4.14. By the nilpotent action lemma, *any π-subgroup P satisfies*

$$O_{\pi^*}\big(C_G(P)\big)=O_{\pi^*}\big(C_G^*(P)\big)\,.$$

By 4.5, *any solvable π-subgroup P satisfies*

$$O_{\pi^*}\big(C_G(P)\big)=O_{\pi^*}\big(N_G(P)\big)\,.$$

4.15. *Assume* $K\lhd H=KU\in\pi^*$. By 4.4, $K/F^\pi(K)$ is centralized by H. Hence $F^\pi(K)\,U$ is a normal subgroup with nilpotent π-factor group, and therefore equals H. By 4.12, $F^\pi(K)$ equals $O_{\pi^*}(K)$.

By 4. 1. iii, U can be replaced by $F^{\pi}(U)$ which in case $U \in \pi^*\pi$ equals $O_{\pi^*}(U)$. *So we get* $H=O_{\pi^*}(K)\,O_{\pi^*}(U)$ *if* $U\in\pi^*\pi$.

4. 16. *Assume* $H/K\in\pi^*\pi$. If $H=C_H^*(K_\pi)$, then 4. 10 forces H to be a $\pi^*\pi$-group. *It follows that,* more generally, $C_H^*(K_\pi)$ *is a* $\pi^*\pi$-group (because $C_H^*(K_\pi)/C_K^*(K_\pi)\simeq C_H^*(K_\pi)\,K/K \lhd H/K\in\pi^*\pi$ and $\pi^*\pi$ is inherited by normal subgroups).

4. 17. It also follows that $C_K^*(K_\pi)\in\pi^*\pi$ *in* 4. 7. *So if* $K\in\pi^*$ *in* 4. 7, we conclude from 4. 14 (forcing $O_{\pi^*}(C_K^*(X_\pi))=O_{\pi^*}(C_K(X))$ and 4. 15 that

$$K=O_{\pi^*}(X\cap K)\,O_{\pi^*}\Big(C_K(X_\pi)\Big)$$

4. 18. Applying 4.7/17 with $K:=O_{\pi^*\pi}(G)$ and $K:=O_{\pi^*}(G)$ we get

$$O_{\pi^*\pi}(G)=XO_{\pi^*\pi}\Big(C_G^*(X_\pi)\Big)\qquad \textit{for any normal } \pi^*\pi\textit{-subgroup } X$$

and

$$O_{\pi^*}(G)=XO_{\pi^*}\Big(C_G(X_\pi)\Big)\qquad \textit{for any normal } \pi^*\textit{-subgroup } X.$$

Observe that these two products are normal (because $G=XN_G(X_\pi)$) $\pi^*\pi$-groups (by 4. 10).

4. 19. We begin to characterize $O_{\pi^*\pi}(G)$ and $O_{\pi^*}(G)$ from above:

$O_{\pi^*\pi}(G)$ *is the unique smallest normal subgroup* K *of* G *satisfying* $C_G^*(K_\pi)\subseteq K$.

Proof: Let $K:=O_{\pi^*\pi}(G)$ and $X\lhd G$ with $C_G^*(X_\pi)\subseteq X$. Then 4. 7 yields $K=(X\cap K)C_K^*(X_\pi)\subseteq XC_G^*(X_\pi)\subseteq X$.

So consider $L:=KC_G^*(K_\pi)$. With H defined by $H/K:=O_{\pi^*\pi}(L/K)$, 4. 10 yields $H\in\pi^*\pi$, hence $H\subseteq O_{\pi^*\pi}(G)=K$, i. e. $O_{\pi^*\pi}(L/K)=1$, hence $L/K=1$, i. e. $C_G(K_\pi)\subseteq K$, as required.

4. 20. As mentioned in section 3, *a group* G *is* π*-constrained if and only if* $K:=O_{\pi',F}(G)$ *satisfies* $C_G^*(K_\pi)\subseteq K$.

By 4. 19, *this is equivalent to* $O_{\pi^*\pi}(G)=O_{\pi',F}(G)$, *hence also equivalent to* $O_{\pi^*}(G)=O_{\pi'}(G)$.

4. 21. *Each group* G *has a unique smallest normal subgroup* L *(namely* $L=O^{\pi'}(O_{\pi_*}(G)))$ *with* $C_G(L_\pi)$ *—or equivalently* $C_G^*(L_\pi)$*—* π*-constrained.*

Furthermore, L *is a* π^**-group and satisfies*

$$O_{\pi^*}(G)=LO_{\pi'}\Big(C_G(L_\pi)\Big)$$

and

$$O_{\pi^*\pi}(G)=LO_{\pi',F}\Big(C_G^*(L_\pi)\Big).$$

PROOF: Let $H:=O_{\pi^*}(G)$ and $L:=O^{\pi'}(H)$. Since $L/F^{\pi}(L)$ lies in $Z(H/F^{\pi}(L))$, see 4.4, Burnside's transfer theorem forces $H/F^{\pi}(L)$ to be π-nilpotent, hence to be a π'-group.

Thus L is π-perfect, hence is a π^*-group (because $L \lhd H \in \pi^*$).

Now 4.18 yields

$$H=O_{\pi^*}(G)=LO_{\pi^*}\left(C_G(L_\pi)\right)$$

and

$$O_{\pi^*\pi}(G)=LO_{\pi^*\pi}\left(C_G^*(L_\pi)\right).$$

Since $L_\pi=H_\pi$, it follows that $O_{\pi^*}(C_G(L_\pi))=O_{\pi^*}(C_G^*(L_\pi))$ has a central S_π-subgroup, hence is π-nilpotent (again by Burnside's transfer theorem), hence is a π'-group.

By 4.20, this means that $C_G(L_\pi)$ and $C_G^*(L_\pi)$ are π-constrained and also that $O_{\pi^*\pi}(C_G^*(L_\pi))=O_{\pi',F}(C_G^*(L_\pi))$.

So it only remains to verify that any $X \lhd G$ with $C_G(X_\pi)$ π-constrained (i.e. $O_{\pi^*}(C_G(X_\pi))=O_{\pi'}(C_G(X_\pi))$) contains L.

By 4.17, $L=(X\cap L)\,O_\pi^*(C_L(X_\pi))=(X\cap L)\,O_{\pi'}(C_L(X_\pi))$, hence $X\cap L\supseteq O^{\pi'}(L)=L$, as required.

4.22. *Let P be an S_π-subgroup of $O_{\pi^*\pi}(G)$, and K a P-invariant π^*-subgroup of G.*

Applying 4.9 (or 4.17) to $H:=N_G(K)$ with $X:=H\cap O_{\pi^*\pi}(G)$, we get $K=(X\cap K)\,C_K(P)$; and by 4.19, $C_G(P)\subseteq O_{\pi^*\pi}(G)$.

This proves $K\subseteq O_{\pi^*\pi}(G)$, hence $K\subseteq O_{\pi^*}(G)$.

As an immediate consequence, *every $X \lhd G$ satisfies*

$$O_{\pi^*}\left(N_G(X_\pi)\right)\subseteq O_{\pi^*}(G),$$

which together with 4.17 yields

$$O_{\pi^*}(G)=O_{\pi^*}(X)\,O_{\pi^*}\left(C_G(X_\pi)\right).$$

4.23. Next we generalize some important action properties of π-groups on π'-groups.

First let P be an arbitrary group acting on a π^*-group K. We show that $K_0:=[P,K]$ *is a π^*-group*: Being normal in K, it is a $\pi^*\pi$-group, whence $K_0/O_{\pi^*}(K_0)$ is a central π-subgroup of $K/O_{\pi^*}(K_0)$. Since K is π-perfect, 3.9 forces P to centralize $K/O_{\pi^*}(K_0)$, as required.

Now let P be a π-group. We mainly show that

$$K=[P,K]\,O_{\pi^*}\left(C_K(P)\right) \quad \textit{and} \quad [P]^2K=[P,K].$$

In the semidirect product KP we consider the normal subgroup $X := [P, K]P = K_0P$. Since $X \cap K$ equals K_0, a π^*-group, 4.17 yields

$$K = K_0 O_{\pi^*}\left(C_K(X_\pi)\right).$$

Choose $X_\pi \supseteq P$. Then $K = K_0 C_K(P)$ and likewise $K_0 = [P, K_0]\, C_{K_0}(P)$, hence $K = [P, K_0]\, C_K(P)$, so that $[P, K] \subseteq [P, K_0]$, i.e. $K_0 = [P, K_0]$.

For the proof of $K = K_0 O_{\pi^*}(C_K(P))$, choose X_π so that it contains (in addition to P) an S_π-subgroup P_0 of $C_{K_0}(P)$.

Then we apply 4.16 with $K^* := C_K(P)$ and $K_0^* := C_{K_0}(P)$ in place of H and K, and get $C_{K^*}(P_0) \in \pi^*\pi$, hence

$$O_{\pi^*}\left(C_K(X_\pi)\right) \subseteq F^\pi\left(C_{K^*}(P_0)\right) = O_{\pi^*}\left(C_{K^*}(P_0)\right).$$

As P_0 is an S_π-subgroup of some normal subgroup of K^*, 4.22 yields $O_{\pi^*}(C_{K^*}(P_0)) \in O_{\pi^*}(K^*)$. It follows that $O_{\pi^*}(C_K(X_\pi))$ lies in $O_{\pi^*}(K^*) = O_{\pi^*}(C_K(P))$.

Now we have $K = K_0 O_{\pi^*}(C_K(X_\pi)) = K_0 O_{\pi^*}(C_K(P))$, as required.

4.24. *Now let P be a π-group acting on a $\pi^* - \pi$-group K.* We partially generalize 3.9:

Let $L = K_1 \lhd K_2 \lhd \cdots \lhd K_n = K$ with K_i centralizer-closed in K_{i+1} and P-invariant. If $[P, O_{\pi^*}(L)] = 1$, then $[P, O_{\pi^*}(K)] = 1$.

Since $\pi^* - \pi$ is inherited by normal subgroups, we may assume $K = K_2$. Choose L_π to be P-invariant. By 4.23, $[L_\pi, O_{\pi^*}(K)]$ is a π^*-group, hence lies in $O_{\pi^*}(L)$, hence is centralized by P. Thus, also by 4.23, it suffices to show that $Q := [P, O_{\pi^*}(C_{O_{\pi^*}(K)}(L_\pi))] = 1$. Centralizing both $O_{\pi^*}(L)$ (because P does) and L_π, Q lies in $C_K(O_{\pi^*}(L)\, L_\pi) = C_K(L) = Z(L)$. By 4.23, Q is a π^*-group and equals $[P, Q]$. So we get $[P, Q] = Q \subseteq O_{\pi'}(L) \subseteq C_L(P)$, hence $Q = 1$.

4.25. *Let A be a centralizer-closed normal subgroup of G_π, and K an A-invariant π^*-subgroup of G.*

We prove that K *lies in* $O_{\pi^*}(G)$, i.e. in $O_{\pi^*\pi}(G)$:

Since $Z(A)$ is an S_π-subgroup of $C_G(A)$, Burnside's transfer theorem forces $C_G(A)$ to be π-nilpotent. Hence $O_{\pi^*}(C_K(A)) \subseteq O_{\pi'}(C_G(A)) \subseteq O_{\pi^*}(G)$, the latter inclusion being due to $A \lhd G_\pi$, see 4.22.

So 4.23 allows to assume $K = [A, K]$.

Let $P = G_\pi \cap O_{\pi^*\pi}(G)$ and $P_0 = A \cap P$.

Since $[P_0, K] \subseteq [P_0, G] \subseteq O_{\pi^*\pi}(G)$, we may assume —again by 4.23— that $K \subseteq C_G(P_0)$.

We have $O_{\pi^*}(N_G(P_0)) \subseteq O_{\pi^*}(G)$, because $P_0 \lhd G_\pi$. Hence we may assume $G = N_G(P_0)$.

Since P is an S_π-subgroup of $O_{\pi^*\pi}(G)$, the Frattini argument yields $O_{\pi^*\pi}$

$(G)\,C_G(P/P_0) \lhd G$, and the fundamental 4.19 gives $C_G^*(P) \subseteq O_{\pi^*\pi}(G)$. By definition of π^*, $O_{\pi^*\pi}(G) \subseteq P_0 C_G^*(P_0)$.

So, using Dedekind's modular law, we get (note that $A \subseteq C_G(P/P_0)$)

$$\begin{aligned} K = [A, K] &\subseteq O_{\pi^*\pi}(G)\, C_G(P/P_0) \cap P_0 C_G^*(P_0) \\ &= O_{\pi^*\pi}(G)\big(C_G(P/P_0) \cap P_0 C_G^*(P_0)\big) \\ &= O_{\pi^*\pi}(G)\, P_0\big(C_G(P/P_0) \cap C_G^*(P_0)\big) \subseteq O_{\pi^*\pi}(G)\, C_G^*(P) = O_{\pi^*\pi}(G)\,, \end{aligned}$$

as required.

4.26. *Let $H:=O_{\pi^*\pi}(G)$, and P a π-subgroup of G subnormal in PH_π (for some $H_\pi = H_\pi^P$). Then*

$$O_{\pi^*}\big(N_G(P)\big) \subseteq O_{\pi^*}(G)\,.$$

PROOF: Suppose false. Choose K minimal among normal π^*-subgroups of $N_G(P)$ not lying in $O_{\pi^*}(G)$, i.e. not in H.

By 4.22, $H_\pi \not\subseteq P$. Thus $P \subsetneqq Q \lhd\lhd PH_\pi = QH_\pi$ for some Q, and we may assume that the assertion is true for Q, i.e. $O_{\pi^*}(N_G(Q))$ lies in $O_{\pi^*}(G)$.

By 4.23 and minimal choice of K, K equals $[P, K]$ or $C_K(P)$.

In the first case, $K = [P, K] \subseteq P$ and hence $K \subseteq O_{\pi^*}(P) \subseteq O_{\pi^*}(Q) \subseteq O_{\pi^*}(N_G(Q)) \subseteq O_{\pi^*}(G)$.

Thus P centralizes K. Hence $[Q, K] = [P(Q \cap H_\pi), K] = [Q \cap H_\pi, K] \subseteq H$, so that —again by 4.23— it only remains to show that $O_{\pi^*}(C_K(Q)) \subseteq H$. Since $C_G(Q) \subseteq N_G(P) \subseteq N_G(K)$, we have $C_K(Q) \lhd C_G(Q)$and hence $O_{\pi_*}(C_K(Q)) \subseteq O_{\pi^*}(C_G(Q)) \subseteq O_{\pi^*}(G)$.

4.27. Finally we come to components. *Let K be a π^*-group with* $O_{\pi'}(K) = 1$ (replace K by $K/O_{\pi'}(K)$).

First we remark that *any π-solvable subnormal subgroup of K* (being a $\pi^*\pi$-group) *is π-nilpotent* (see 4.20), *hence lies in* $F_\pi(K) = Z(K)$. Similarly, *any π-solvable factor group of K is a π'-group.*

By 4.21, $L := O^{\pi'}(K)$ is a π^*-group satisfying $K = LO_{\pi'}(C_K(L_\pi))$.

Let $L_1, \cdots, L_n$ be the minimal non-π-solvable subnormal subgroups of K, the components of K. We have $n=0$ only when K is a π'-group.

By the above, a maximal normal subgroup of L_i is $L_i \cap Z(K) = Z(L_i)$. *Thus $L_i/Z(L_i)$ is simple, so that L_i is quasisimple.*

We have $L_i = O^{\pi'}(L_i) \lhd K$, see 4.6.

So for $i \neq j$ we have $[L_i, L_j] \subseteq L_i \cap L_j \subseteq Z(K)$, and this implies $[L_i, L_j] = 1$ because L_i is perfect.

In particular, $L_i \cap \prod_{j \neq i} L_j \subseteq Z(L_i)$, so that $L^* = L_1 L_2 \cdots L_n$ cannot be the product of a proper subset of the L_i.

Let P be a group acting on K. By 4.23, $[P, K]$ is a π^*-group (normal in K). Since $[P, K]$ centralizes every L_i centralized by P, no such L_i lies in $[P, K]$. Conversely, every L_i not lying in $[P, K]$ is centralized by P: If L_0^* is the product of these $L_i \not\subseteq [P, K]$, then $[P, L_0^*]$, a normal π^*-subgroup of L_0^* has no component, hence is a π^*-group, thus lies in $O_{\pi'}(K)=1$.

In short: *The components of $[P, K]$ are the components of K not centralized by P (and $[P, L^*]$ is the product of these components).*

By induction (applied to $K/[P, L^*]$), $[P, L]$ *equals* $[P, K] \cap L$ *and satisfies* $[P]^2 L=[P, L]$.

Since $C_K(L^*)$ contains no component, it is π-solvable. *So if $C_L(L_\pi^*)$ induces a π-solvable automorphism group on L^**, then $C_L(L_\pi^*)$ is π-solvable, hence L/L^* is (because $L=L^*C_L(L_\pi^*)$), hence is a π'-group. *Thus $L=L^*$ (because $L=O^{\pi'}(K)$), i.e. $L=L_1 L_2 \cdots L_n$.*

5. F^* and E.

This section corresponds to the case π=set of all primes in section 4. Thus "π-group" has to be read as "group", "π'-group" as "1", "π-perfect" as "perfect", etc. As mentioned before, we write F^* for $\pi^*\pi$ and $O_{\pi^*\pi}$ and E for π^* and O_{π^*}.

LEMMA 5.1. *Each of the following statements means that H is an F^*-group.*

(i) *$H=NC_H^*(N)$ for every $N \lhd H$.*

(ii) *Every factor group of H is an F^*-group.*

(iii) *Every normal subgroup of H is an F^*-group.*

(iv) *H is a product of normal F^*-subgroups.*

(v) *H has a normal F^*-subgroup K with H/K an F^*-group and $H=KC_H^*(K)$.*

PROOF. See 4.2, 3, 11, 8, 10.

LEMMA 5.2. *Each of the following statements means that H is an E-group.*

(i) *H is a perfect F^*-group.*

(ii) *$H=H'$ and $H=NC_H(N)$ for every $N \lhd H$.*

(iii) *Every factor group of H is an E-group.*

(iv) *$H=H'$ and K' is an E-group for every $K \lhd H$.*

(v) *H is quasisemisimple, i.e. $H=H'$ and $H/Z(H)$ is semisimple.*

(vi) *H is the central product of its quasisimple subnormal subgroups, the components of H.*

PROOF. See 4.2, 3, 12, 27. That a quasisemisimple group is an E-group follows from 5.1.v.

REMARK 5.3. *In an E-group all subnormal subgroups are normal, and a proper subset of components generates a proper subgroup.*

If $K \lhd H$, H an E-group, then K is the product of $K \cap Z(H)$ and the components lying in K.

LEMMA 5.4. *$F^*(G)$ and $E(G)$ can be characterized in many ways:*

$F^*(G) =$ *the largest normal F^*-subgroup of G*

$=$ *the smallest normal subgroup satisfying $C_G^*(F^*(G)) \subseteq F^*(G)$*

$= E(G)\,F(G)$, *a central product.*

$E(G) =$ *the largest normal E-subgroup of G*

$=$ *the smallest normal subgroup with $C_G(E(G))$ constrained*

$= O^F(F^*(G))$.

PROOF. By 4.21, $F^*(G) = E(G)\,F(G)$; and this product is central (4.5). For the rest see 4.8, 13, 19, 21.

REMARK. $F^*(G)$ is also the smallest centralizer-closed normal subgroup containing $F(G)$, the set of elements of G inducing inner automorphisms on every chief factor, and the full inverse image of the product of all minimal normal subgroups of $C_G(F(G))\,F(G)/F(G)$.

LEMMA 5.5. *A solvable subgroup and an E-subgroup normalizing each other must centralize each other.*

More generally, if a group A acts on an E-group K, and centralizes $K/Z(K) = K/S(K)$, then $[A, K] = 1$.

LEMMA 5.6. *If A is a group acting on an E-group K, then $[A, K]$ equals $[A, [A, K]]$ and is an E-group, namely the product of components of K not centralized by A.*

PROOF. See 4.27 or 4.23.

LEMMA 5.7. *$F^*(G) \subseteq H \subseteq G$ implies $E(H) = E(G)$.*

PROOF. See 4.22.

REMARK 5.8. *By 5.6, if $A \lhd\lhd H$, then $E(A)$ is the product of components of $E(H)$ not centralized by A.*

In particular, $E(H)$ normalizes A.

LEMMA 5.9. *Let $A \subseteq G$ and $A \lhd\lhd AF^*(G)$. Then $E(N_G(A)) = E(G)$.*

PROOF. See 4.26 and Remark 5.8.

REMARK 5.10. Since F^* is inherited by normal subgroups, *a subnormal* *subgroup A of $F^*(G)$ is just the product of a subgroup of $F(G)$ and some components of $E(G)$.* Thus A is centralizer-closed in $F^*(G)$ if and only if $E(G)$ lies in A (i.e. $E(G)=E(A)$) and $A\cap F(G)=F(A)$ is centralizer-closed in $F(G)$.

LEMMA 5.11. *Let B be an operator group on a group G with $|B:B'|$ prime to $|F(G)|$.*

If B centralizes some centralizer-closed subnormal subgroup of $F^*(G)$, then $[B, G]=1$.

PROOF. Since $C_G(F^*(G))\subseteq F^*(G)$, and by 5.10 normalizers of subnormal subgroups of F^*-groups are also subnormal, 3.9 applies.

LEMMA 5.12. *Let A be a centralizer-closed subnormal subgroup of $F^*(G)$, and K an A-invariant E-subgroup of G. Then $K\subseteq E(G)$.*

PROOF. By 5.6, $K=[A, K]\,E(C_K(A))$; and by 5.11, $B:=E(C_K(A))$ must be 1. Thus $K=[A, K]\subseteq F^*(G)$, i.e. $K\subseteq O^F(F^*(G))=E(G)$.

REMARK 5.13. For any component E_1 of $E(G)$ and any $g\in G$ we have $E_1^g=E_1$ or $E_1^g\cap E_1\subseteq Z(E_1)$. For $p\in\pi(E_1)$ and an S_p-subgroup P of $E(G)$, $P\cap E_1$ is a non-central S_p-subgroup of E_1. It follows that $C_G(P)$ normalizes E_1.

6. p^*p and p^*.

This main section corresponds to the case when our π in section 4 consists of a single prime p.

LEMMA 6.1. *Each of the following statements means that H is a p^*p-group.*

(i) *$H=NC_H^*(N_p)$ for every $N\lhd H$.*

(ii) *Every factor group of H is a p^*p-group.*

(iii) *Every normal subgroup of H is a p^*p-group.*

(iv) *H is a product of normal p^*p-groups.*

(v) *H has a normal p^*p-subgroup K with H/K a p^*p-group and $H=KC_H^*(K_p)$.*

PROOF. See 4.2, 3, 11, 8, 10.

We recall that for a p-subgroup P, $C_H^*(P)$ equals $O_{C_H(P),p}(N_H(P))$.

LEMMA 6.2. *Each of the following statements means that H is a p^*-group.*

(i) *H is a p-perfect p^*p-group.*

(ii) *$H=O^p(H)$ and $H=NC_H(N_p)$ for every $N\lhd H$.*

(iii) *Every factor group of H is a p^*-group.*

(iv) *$H=O^p(H)$ and $O^p(K)$ is a p^*-group for every $K\lhd H$.*

(v) *H is a product of normal p^*-subgroups.*

PROOF. See 4. 2, 3, 12, 8.

REMARK 6. 3. By 4. 6, *$K \lhd\lhd H \in p^*$ implies $O^{p'}(K) \lhd H$. Hence* 6. 2. ii *holds also for subnormal subgroups N.*

THEOREM 6. 4. *$L:=O^{p'}(O_{p^*}(G))$ is the smallest normal subgroup of G with $N_G(L_p)$ p-constrained. Furthermore, L is a p^*-group, and*

$$\begin{aligned} O_{p^*p}(G) &= \text{the largest normal } p^*p\text{-subgroup of } G \\ &= \text{the smallest normal subgroup } H \text{ of } G \text{ satisfying } C_G^*(H_p) \subseteq H \\ &= LO_{p',p}\left(N_G(L_p)\right), \end{aligned}$$

$$\begin{aligned} O_{p^*}(G) &= \text{the largest normal } p^*\text{-subgroup of } G \\ &= O^p\left(O_{p^*p}(G)\right) \\ &= LO_{p'}\left(N_G(L_p)\right). \end{aligned}$$

PROOF. See 4. 21, 19, and notice that for any subgroup K, $C_G(K_p)$ is p-constrained if and only if $N_G(K_p)$ is, and that $C_G(K_p) \subseteq K$ implies $C_G^*(K_p) = O_{C_G(K_p),p}(N_G(K_p)) = O_{p',p}(N_G(K_p))$.

LEMMA 6. 5. *A p-subgroup and a p^*-subgroup normalizing each other must centralize each other.*

LEMMA 6. 6. (p^*-action lemma). *Let A be a group acting on a p^*-group K. Then $[A, K]$ is a p^*-group.*

If A is a p-group, then $[A]^2K=[A, K]$ and

$$K=[A, K]\, O_{p^*}\left(C_K(A)\right).$$

PROOF. See 4. 23.

REMARK 6. 7. Generalizing the situation of 6. 5, consider an arbitrary subgroup A and a p^*-subgroup K normalizing each other. Then, by 6. 6, $[A, K] \subseteq O_{p^*}(A)$. Thus $O_{p^*}(A)=1$ would imply $[A, K]=1$.

Secondly, if A in 6. 6 just satisfies $A=O^{p'}(A)$, then $[A, K]$ equals the product of all $[P, K]$, with $P \in \mathrm{Syl}_p(A)$, and hence by 6. 6 again satisfies $[A]^2 K=[A, K]$.

It follows that $O^{p'}(A)=A \lhd\lhd G$ implies $[A, O_{p^*}(G)] \subseteq O_{p^*}(A)$.

THEOREM 6. 8. *Let $C_{G_p}(A) \subseteq A \lhd G_p$. Then every A-invariant p^*-subgroup of G lies in $O_{p^*}(G)$.*

PROOF. See 4. 25.

REMARK 6. 9. Since E-groups are p^*-groups by definition, p'-E-groups are i. e. $O_{p',E}(G) \subseteq O_{p^*}(G)$. Hence an S_p-subgroup P of $O_{p',E}(G)$ satisfies $O_{p^*}(G)=O_{p',E}(G)\, C_{O_{p^*}(G)}(P)$.

It follows that $O_{p^*}(G)$ fixes each component of $E(G/O_{p'}(G))$, see 5.13. Thus theorem 6.8 sharpens and generalizes theorem 1 of [7].

MAIN THEOREM 6.10. *For every p-subgroup P of a group G,* $O_{p^*}(N_G(P))=O_{p^*}(C_G(P))$ *lies in* $O_{p^*}(G)$.

More generally, it lies in $O_{p^*}(C_G(P_0))$ *for every subgroup* $P_0 \subseteq P$.

PROOF. Let $P_0 \lhd P_1 \lhd \cdots \lhd P_n = P$. By 4.26, $O_{p^*}(C_G(P)) \subseteq O_{p^*}(G)$ and likewise $O_{p^*}(C_G(P_{i+1})) \subseteq O_{p^*}(N_G(P_i)) = O_{p^*}(C_G(P_i))$.

LEMMA 6.11. *Let P be a p-group acting on a p^*-group K. Suppose P centralizes $O_{p^*}(L)$ for some centralizer-closed subnormal subgroup L of $F^*(K)$. Then P centralizes K.*

PROOF. This is a special case of 4.24. *Observe in this connection that* $O_{p^*}(L)=O^p(L)=E(L)\,F_{p'}(L)$ *for any F^*-group L.*

Thus our assumption $[P, O_{p^*}(L)]=1$ just means that P centralizes $E(K)$ and some centralizer-closed subgroup of $F_{p'}(K)$.

LEMMA 6.12. *The following conditions are equivalent.*

(i) *G is p-constrained.*

(ii) $O_{p',E}(G)=O_{p'}(G)$.

(iii) $O_{p^*}(G)=O_{p'}(G)$.

PROOF. See 5.4 and 6.4.

LEMMA 6.13. $X \lhd G$ *implies* $O_{p^*}(G)=O_{p^*}(X)\,O_{p^*}(C_G(X_p))$.

PROOF. See 4.22.

PROPOSITION 6.14. *The following properties of a group G, with* $K:=O_{p^*}(G)$, $L:=O_{p',E}(G)$, $\bar{G}:=G/O_{p'}(G)$, *are equivalent.*

In case they are valid, we call p^* regular (on G).

(i) *K/L is a p'-group, i.e.* $O^{p'}(K)=O^{p'}(L)$.

(ii) *$C_G(L_p)$ is p-constrained, i.e.* $O_{p^*}(C_G(L_p))=O_{p'}(C_G(L_p))$.

(iii) *$C_G(P)$ is p-constrained for every p-subgroup P of G such that $C_L(P\cap L)$ is p-constrained.*

(iv) $K=LO_{p'}(C_G(L_p))$.

(v) $K=LO_{p'}(C_G(M_p))$, *where* $M:=O_{p',F^*}(G)$.

(vi) *$C_K(L_p)$ induces a p-solvable automorphism group on $\bar{L}$.*

(vii) *$C_K(L_p)$ induces a p-solvable automorphism group on each component X (i.e. on $X/Z(X)$, for X is perfect) of $\bar{L}$.*

PROOF. For (i)→(ii) see 6.4. Assuming (i), we derive (iii): By 6.10,

$$O^{p'}\Big(O_{p^*}\big(C_G(P)\big)\Big) \subseteq O^{p'}\Big(O_{p^*}\big(C_K(P\cap L)\big)\Big) \subseteq O_{p^*}\big(C_L(P\cap L)\big)$$
$$= O_{p'}\big(C_L(P\cap L)\big),$$

whence $O_{p^*}(C_G(P))$ is a p'-group, i. e. $C_G(P)$ is p-constrained.

Since certainly $C_L(L_p)$ is p-constrained, (iii) implies $K=LO_{p'}(C_G(X_p))$ whenever $X \lhd G$ and $O_{p^*}(X)=L$, see 6. 13. Thus (iii) implies (iv) and (v), which obviously yield (i). Trivially, (i) implies (vi), and that (vi) implies (i) has already been noticed in the last paragraph of 4. 27.

Finally, (vi) and (vii) are equivalent because p-solvability is residual and, with $A:=C_K(L_p)$, $C_A(\bar{L})=\cap C_A(X)$, X component of $\bar{L}$.

REMARK 6. 15. *Each of the following hypotheses implies that p^* is regular.*

(1) *If is a simple group of order divisible by p, then*

(i) $C_{\mathrm{Aut}(X)}(X_p)$ *is p-solvable, or*

(ii) *every p-automorphism of X centralizing X_p is inner, or*

(iii) *X satisfies Schreier's conjecture, i. e. its outer automorphism group is solvable.*

(2) *If A acts on a group L, then $C_A(L_p)$ induces a p-solvable automorphism group on $L/O_{p'}(L)$.*

Indeed, each of these four hypotheses implies the first one (1i), hence implies condition (vii) of 6. 14.

All known simple groups satisfy (1iii), and for odd p they also seem to satisfy (1ii).

Hypothesis (2) *is satisfied by $p=2$ (whence 2^* is regular)*, due to the following.

Theorem of Glauberman [1] : *Let $O_{2'}(G)=1$. Then $C_{\mathrm{Aut}(G)}(G_2)$ is 2-nilpotent and has abelian S_2-subgroups.*

By Remark 1 of [1], the p-analogue of this theorem holds provided the p-analogue of his famous Z^*-theorem holds. As an exercise, the reader may verify that (1ii) implies Glauberman's theorem (for p) with the addition that $C_{\mathrm{Aut}(G)}(G_p)$ has central S_p-subgroups.

Concluding remarks: We have already indicated in the introduction that when we restrict ourselves to a class of groups on which p^* is regular, many of our results are contained in the work of Gorenstein and Walter, namely in [4] (as point out in section 5 of [5], arguments in the relevant parts of [4] work also for odd primes). Notationally we have

$$L=E \qquad L_{p',p}=O_{p',F^*}$$

$$L_{p'}=O^{p,}O_{p',E} \qquad L^*_{p',p}=O_{p^*}\ (\text{if } p^* \text{ is regular}).$$

Since $L_{p'}(H)$ equals $O^{p'}(O_{p^*}(H))$ for regular p^*, it seems natural to denote the latter important subgroup (see 6. 4) by $L_{p'}(H)$ in general.

Then from 6.10 we immediately get $L_{p'}(N_H(P)) \subseteq L_{p'}(H)$ for every p-subgroup P, and $L_{p'}$ = "class of p^*-groups X satisfying $O^{p'}(X) = X$" is obviously a radical property.

References

[1] G. GLAUBERMAN: On the automorphism group of a finite group having no non-identity normal subgroups of odd order. Math. Z. 93 (1966), 154–160.

[2] D. GORENSTEIN: Finite Groups. New York, Harper and Row 1968.

[3] D. GORENSTEIN and J. H. WALTER: The π-layer of a finite group. Ill. J. Math. 15 (1971), 555–565.

[4] D. GORENSTEIN and J. H. WALTER: Balance and generation in finite groups. J. Algebra 33 (1975), 224–287.

[5] D. GORENSTEIN and R. LYONS: Nonsolvable signalizer functors on finite groups. Proc. London Math. Soc. (3) 35 (1977), 1–33.

[6] P. HALL: Some sufficient conditions for a group to be nilpotent. Ill. J. Math. 2 (1958). 787–801.

[7] J. G. THOMPSON: Envelopes and p-signalizers on finite groups. Ill. J. Math. 13 (1969), 87–90.

Mathematisches Seminar
Universität Kiel
2300 Kiel
West Germany