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On modules which are flat over their endomorphism rings

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Let ${}_RM$ be a left R-module over a ring R[1], and S be the endomorphism ring of ${}_RM$. Let ${}_RA$ be a left R-module. We say that M-codominant dimension of A is $\geqq n$, if there is an exact sequence:

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow A \longrightarrow 0\,,$$

where each X_i is isomorphic to a (finite or infinite) direct sum of copies of ${}_RM$. We denote by $\mathscr{C}_n$ the category of left R-modules whose M-codominant dimensions are $\geqq n$.

Recently T. Würfel has shown that, for a left R-module ${}_RM$, the following statements are equivalent:[2]

(a) The right S-module M_S is flat.

(b) ${}_RM$ generates the kernel of every homomorphism ${}_RM^m \rightarrow {}_RM^n$, where m, n are natural numbers. (Here one can also set $n=1$).

Further, R. W. Miller has proved that, in case where ${}_RM$ is finitely generated and projective, the above statements are equivalent to

(c) $\mathscr{C}_2 = \mathscr{C}_3$[3]

Here, regarding to the above results, we shall prove the following

THEOREM. *Let ${}_RM$ be left R-module with the endomorphism ring S, and Q an injective cogenerator in the category ${}_R\mathfrak{M}$ of all left R-modules. Then the following statements are equivalent:*

(1) *M_S is flat.*

(2) *The left S-module ${}_S\mathrm{Hom}_R(M, Q)$ is injective.*

(3) *${}_S\mathrm{Hom}_R(M, Q)$ is absolutely pure, that is, every homomorphism of a finitely generated submodule of ${}_SS^m$ to ${}_S\mathrm{Hom}_R(M, Q)$ is extended to that of ${}_SS^m$.*

(4) *${}_S\mathrm{Hom}_R(M, Q)$ is semi S-injective, that is, every homomorphism of a finitely generated left ideal of S to ${}_S\mathrm{Hom}_R(M, Q)$ is extended*

1) In what follows, we assume that every ring has an identity element and every module is unital.

2) Cf. [5], 1.14 Satz.

3) Cf. [2], Theorem 2.1*.

to that of S.

In case where $_RM$ satisfies the condition $TM=M$, where T is the trace ideal of $_RM$: $T=\sum_{f\in\mathrm{Hom}_R(M,R)} f(M)$, the above statements are equivalent to

(5) *Ker $\alpha\in\mathscr{C}_2$ for every homomorphism $\alpha: X\rightarrow Y$, where $X, Y\in\mathscr{C}_2$.*

(6) *Ker $\alpha\in\mathscr{C}_1$ for every homorphism $\alpha: X\rightarrow Y$, where $X, Y\in\mathscr{C}_2$.*

(7) *Ker $\alpha\in\mathscr{C}_1$ for every homorphism $\alpha: X\rightarrow Y$, where X, Y are direct sums of copies of $_RM$.*

Further, in case where $_RM$ is projective, the above statements are equivalent to

(8) *If $X\supseteqq Y$, and $X\in\mathscr{C}_2$, $Y\in\mathscr{C}_1$, then $Y\in\mathscr{C}_2$.*

(9) $\mathscr{C}_2=\mathscr{C}_3$

(10) *The class $\mathscr{C}_2$ forms an exact Grothendieck subcategory of $_R\mathfrak{M}$.*

(11) *The class $\mathscr{C}_2$ forms an exact abelian subcategory of $_R\mathfrak{M}$.*

PROOF. The implications (2)$\Rightarrow$(3)$\Rightarrow$(4) are clear. (4)$\Rightarrow$(1). It suffices to show (b) under the condition (4). Let $(s_1, s_2, \cdots, s_m)\in S^m$ be a homomorphism of $_RM^m$ to $_RM$, and K be its kernel. Let H be the trace of $_RM$ in K: $H=\sum_{f\in\mathrm{Hom}_R(M,K)} f(M)$. Suppose $H\subsetneqq K$. Let $(x_1, x_2, \cdots, x_m)$ be an element of K which is not contained in H. Then there is a homomorphism φ of M^m to Q such that $\varphi(H)=0$, $\varphi\{(x_1, x_2, \cdots, x_m)\}\neq 0$. Then, as is easily seen, the mapping

$$\delta:\ \sum_{i=1}^{m} Ss_i\ni\sum_i a_i s_i\longrightarrow\Big(M\ni x\longrightarrow\varphi\big\{(xa_1, xa_2, \cdots, xa_m)\big\}\in Q\Big),$$

is a well defined homorphism of $\sum_i Ss_i$ to $_S\mathrm{Hom}_R(M, Q)$. It follows, by assumption, that there is an element $g\in\mathrm{Hom}_R(M, Q)$ such that $g(\sum_i xa_i s_i)=\varphi\{(xa_1, xa_2, \cdots, xa_m)\}$ for $x\in M$. But this implies $0=g(\sum_i x_i s_i)=\varphi\{(x_1, x_2, \cdots, x_n)\}\neq 0$, a contradiction. Thus $H=K$, as asserted. The implications (5)$\Rightarrow$(6)$\Rightarrow$(7) are clear, because direct sums of copies of M have M-codominant dimensions $\geqq 2$.

Assume that $_RM$ satisfies the condition $TM=M$. Let X be a left R-module. It is shown in [3] that $X\in\mathscr{C}_2$ iff $M\otimes_S\mathrm{Hom}_R(M, X)$ and X are naturally isomorphic under the mapping $\varepsilon_{M,X}$: $M\otimes_S\mathrm{Hom}_R(M, X)\ni\sum_i m_i\otimes f_i\rightarrow\sum_i f_i(m_i)\in X$.[4)]

(1)$\Rightarrow$(5). Let M_S be flat and α be a homomorphism $X\rightarrow Y$, where X, $Y\in\mathscr{C}_2$. Applying the functor $M\otimes_S\mathrm{Hom}_R(M, X)$ to the exact sequence: $0\rightarrow \mathrm{Ker}\,\alpha\xrightarrow{\iota} X\xrightarrow{\alpha} Y$, we have the following commutative diagram with exact rows:

4) Cf. [3], Theorem 2.

$$\begin{array}{ccccc} 0 \longrightarrow \operatorname{Ker}\alpha & \longrightarrow & X & \longrightarrow & Y \\ \uparrow{\scriptstyle \varepsilon_{M,\operatorname{Ker}\alpha}} & & \uparrow{\scriptstyle \varepsilon_{M,X}} & & \uparrow{\scriptstyle \varepsilon_{M,Y}} \\ 0\longrightarrow M\otimes_S \operatorname{Hom}_R(M,\operatorname{Ker}\alpha) & \longrightarrow & M\otimes_S\operatorname{Hom}_R(M,X) & \longrightarrow & M\otimes_S \operatorname{Hom}_R(M,Y)\end{array}$$

Since $\varepsilon_{M,X}$, $\varepsilon_{M,Y}$ are isomorphisms, so is also $\varepsilon_{M,\operatorname{Ker}\alpha}$. (7) implies (1), because (7) implies (b).

Assume that ${}_RM$ is projective. Then ${}_RM$ satisfies the condition $TM=M$. (5)⇒(8). Let $X\supseteq Y$ be such that $X\in\mathscr{C}_2$, $Y\in\mathscr{C}_1$. Then there is an exact sequence: $\oplus M\to X\xrightarrow{\nu} X/Y\to 0$. Applying to this the functor $M\otimes_S \operatorname{Hom}_R(M,\ )$, we see that $X/Y\in\mathscr{C}_2$. It follows by (5) that $Y\in\mathscr{C}_2$, because Y is the kernel of ν. (8)⇒(9). Let $X\in\mathscr{C}_2$. Then there is an exact sequence: $\oplus M\xrightarrow{\alpha}\oplus M\to X\to 0$. Since $\oplus M\in\mathscr{C}_2$ and $\operatorname{Im}\alpha\in\mathscr{C}_1$, we have $\operatorname{Im}\alpha\in\mathscr{C}_2$ by (8). It follows that $\operatorname{Ker}\alpha\in\mathscr{C}_1$ by [3], Therorem 4. This implies that $X\in\mathscr{C}_3$. Thus we have $\mathscr{C}_2=\mathscr{C}_3$. (9)⇒(7). Consider a homomorphism $\alpha:\oplus M\to\oplus M$. Then we have the following exact sequence: $0\to\operatorname{Ker}\alpha\xrightarrow{\iota}\oplus M\xrightarrow{\alpha}\oplus M\xrightarrow{\nu}\oplus M/\operatorname{Im}\alpha\to 0$. On the other hand, since $\oplus M/\operatorname{Im}\alpha\in\mathscr{C}_2$, whence $\in\mathscr{C}_3$, there is a following exact sequence: $0\to L\to\oplus M\to\oplus M\to\oplus M/\operatorname{Im}\alpha\to 0$, where $L\in\mathscr{C}_1$. Applying Schanuel's lemma[5] to the above sequences, we see that $\operatorname{Ker}\alpha\in\mathscr{C}_1$. (1)⇒(10). Let M_S be flat and α be a homomorphism $X\to Y$, X, $Y\in\mathscr{C}_2$. Then by (5) $\operatorname{Ker}\alpha\in\mathscr{C}_2$. Applying $M\otimes_S\operatorname{Hom}_R(M,\ )$ to the exact sequence: $X\xrightarrow{\alpha}Y\xrightarrow{\nu}Y/\operatorname{Im}\alpha\to 0$, we see that $Y/\operatorname{Im}\alpha\in\mathscr{C}_2$. Further, from the exact sequence: $\operatorname{Ker}\alpha\to X\to\operatorname{Im}\alpha\to 0$, where X, $\operatorname{Ker}\alpha\in\mathscr{C}_2$, we see as above that $\operatorname{Im}\alpha\in\mathscr{C}_2$. Since $\mathscr{C}_2$ is closed under direct sums, and has M as a generator, it follows that $\mathscr{C}_2$ is an exact Grothendieck subcategory of ${}_R\mathfrak{M}$. The assertions (10)⇒(11)⇒(5) are clear.

5) Cf. [4], Theorem 3.41.

References

[1] V. P. CAMILLO and K. R. FULLER: Rings whose faithful modules are flat over their endomorphism rings, Arch. Math. 27 (1976), 522–525.

[2] R. W. MILLER: Finitely generated projective modules and TTF-classes, Pacific J. Math. Vol. 64, No. 2 (1976), 505–515,

[3] T. ONODERA: Codominant dimensions and Morita equivalences, Hokkaido Math. J. 6 (1977), 169–182.

[4] J. J. ROTMAN: Notes on homological algebra, Van Nostrand Reinhold Mathematical Studies 26 (1970).

[5] T. WÜRFEL: Über absolut reine Ringe, Algebra Bericht Nr. 4, Math. Inst. d. Univ. München (1973).

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