

## On tensor products of linear operators

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### Introduction

Let  $X$  and  $Y$  be Banach spaces and  $X \hat{\otimes}_\alpha Y$  the completion of the tensor product  $X \otimes Y$  with respect to a reasonable norm  $\alpha$ . Let  $A : D[A] \subset X \rightarrow X$  and  $B : D[B] \subset Y \rightarrow Y$  be densely defined closed linear operators in  $X$  and  $Y$  respectively with domains  $D[A]$  and  $D[B]$ . Define their tensor product  $A \otimes B$  by  $(A \otimes B)(x \otimes y) = Ax \otimes By$  for  $x \in D[A]$ ,  $y \in D[B]$ . It is a densely defined linear operator in  $X \hat{\otimes}_\alpha Y$  with domain  $D[A \otimes B] = D[A] \otimes D[B]$ .

The aim of this note is to study the closability of  $A \otimes B$  in  $X \hat{\otimes}_\alpha Y$ , and the range and null space of its closure  $A \hat{\otimes}_\alpha B$  if closable. The results will amplify [11, Theorem 4.2] and [12, Theorems 4.4, 4.5 and 4.6], which have been concerned with the problem of when  $A \hat{\otimes}_\alpha B - \lambda$  is Fredholm or semi-Fredholm except for  $\lambda = 0$ .

We shall use the notions of reasonable norms,  $\otimes$ -norms, (left- and right-) injective and projective  $\otimes$ -norms in the sense of Grothendieck [6]. The greatest reasonable norm  $\pi$  is a projective  $\otimes$ -norm and the smallest reasonable norm  $\epsilon$  an injective  $\otimes$ -norm. Schatten [14] denoted them by  $\gamma$  and  $\lambda$ , respectively. For each  $\otimes$ -norm  $\alpha$  there exist the greatest (resp. left-, resp. right-) injective  $\otimes$ -norm which is  $\leq \alpha$  and the smallest (resp. left-, resp. right-) projective  $\otimes$ -norm which is  $\geq \alpha$  (see [6], [1]). For other concrete  $\otimes$ -norms see Saphar [13] and Chevet [2].

In Section 1 we state our results, first when both  $X$  and  $Y$  are Banach spaces and next when they are Hilbert spaces. Section 2 contains two useful lemmas on quotient mappings, which are of independent interest. The results are proved in Section 3.

### 1. The Results

#### 1.1. *The Banach space case.*

Throughout,  $X$  and  $Y$  are Banach spaces. The identity operator in a Banach space  $Z$  is denoted by the same  $I$  or sometimes by  $I_Z$ .

*The closability.* It is known that  $A \otimes B$  is closable in  $X \hat{\otimes}_\alpha Y$  if  $\alpha$  is

a *faithful* reasonable norm on  $X \otimes Y$ , that is, the natural continuous linear mapping  $j^*: X \hat{\otimes}_\alpha Y \rightarrow X \hat{\otimes}_\alpha Y$  is one-to-one (see [9], [10]). We use the notion of the (*bounded*) *approximation property*, for short, (*b.*) *a. p.* (e. g. [4,] [5] and [6]).

THEOREM 1.1. *Let  $\alpha$  be a uniform reasonable norm on  $X \otimes Y$ .*

a) *If either  $X$  or  $Y$  has the b. a. p., then  $\alpha$  is faithful on  $X \otimes Y$ .*

b) *If  $\alpha$  is a left- or right-injective or an injective  $\otimes$ -norm, then  $\alpha$  is faithful on  $X \otimes Y$ .*

*Therefore, in these cases,  $A \otimes B$  is closable in  $X \hat{\otimes}_\alpha Y$ .*

REMARKS 1. Observe that Theorem 1.1 implies also the closability of polynomial operators associated with each polynomial  $P(\xi, \eta) = \sum_{jk} c_{jk} \xi^j \eta^k$ , provided  $A$  and  $B$  have nonempty resolvent sets (use [10, Theorem 1.1]).

2. The greatest reasonable norm  $\pi$  is faithful on  $X \otimes Y$  if either  $X$  or  $Y$  has only the a. p. [5, Chap. I, §5]; so, in this case,  $A \otimes B$  is closable in  $X \hat{\otimes}_\pi Y$ . However, the question is open whether  $A \otimes B$  is closable in  $X \hat{\otimes}_\alpha Y$  without the hypothesis of the b. a. p. or a. p. Note that Enflo [3] has constructed Banach spaces without the a. p.

*The range and null space.* For  $Z_1, Z_2$  Banach spaces, the range and null space of a linear operator (mapping)  $T: D[T] \subset Z_1 \rightarrow Z_2$  are denoted by  $R[T]$  and  $N[T]$ , respectively. A closed subspace  $M$  of a Banach space  $Z$  is said to be *complementary* if there is a continuous projection of  $Z$  onto  $M$ .

Since  $A$  and  $B$  are closed,  $N[A]$  and  $N[B]$  are closed subspaces of  $X$  and  $Y$ , respectively. It is evident that

$$(1.1) \quad R[A \otimes B] = R[A] \otimes R[B],$$

$$(1.2) \quad N[A \otimes B] = N[A] \otimes D[B] + D[A] \otimes N[B].$$

$R[A \hat{\otimes}_\alpha B]$  is the projection of the closure in  $(X \hat{\otimes}_\alpha Y) \times (X \hat{\otimes}_\alpha Y)$  of the graph  $G(A \otimes B)$  of  $A \otimes B$  into the second  $X \hat{\otimes}_\alpha Y$  and so is in general a dense subspace of the closure  $\bar{R}[A \otimes B]$  in  $X \hat{\otimes}_\alpha Y$ . The closure  $\bar{N}[A \otimes B]$  in  $X \hat{\otimes}_\alpha Y$  is obviously a subspace of  $N[A \hat{\otimes}_\alpha B]$ .

THEOREM 1.2. *Let  $\alpha$  be a reasonable norm on  $X \otimes Y$ , and assume  $A \otimes B$  is closable in  $X \hat{\otimes}_\alpha Y$ . If  $R[A \hat{\otimes}_\alpha B]$  is closed, so are both  $R[A]$  and  $R[B]$ , and*

$$(1.3) \quad R[A \hat{\otimes}_\alpha B] = \bar{R}[A \otimes B].$$

*Further, if  $R[A \hat{\otimes}_\alpha B]$  is complementary, so are both  $R[A]$  and  $R[B]$ .*

THEOREM 1.3. *Let  $\alpha$  be a  $\otimes$ -norm, and assume  $A \otimes B$  is closable in*

$X \hat{\otimes}_\alpha Y$ . Suppose one of the following four conditions is satisfied:

- (i) either  $N[A]$  or  $N[B]$  is of finite codimension;
- (ii) both  $N[A]$  and  $N[B]$  are complementary;
- (iii)  $\alpha$  is a right- (resp. left-) projective  $\otimes$ -norm and  $N[A]$  (resp.  $N[B]$ ) is complementary;
- (iv)  $\alpha$  is a projective  $\otimes$ -norm.

Then  $R[A \hat{\otimes}_\alpha B]$  is complementary if and only if both  $R[A]$  and  $R[B]$  are complementary; in this case, one has (1.3) and

$$(1.4) \quad N[A \hat{\otimes}_\alpha B] = \bar{N}[A \otimes B],$$

and there exists a constant  $C > 0$  such that

$$(1.5) \quad \|(A \otimes B)u\|_\alpha \geq C \operatorname{dist}(u, N[A \otimes B]), \quad u \in D[A \otimes B].$$

In the next three theorems, note with Theorem 1.1 that  $A \otimes B$  and  $A \otimes I$  are closable.

**THEOREM 1.4.** Let  $\alpha$  be a  $\otimes$ -norm. Suppose one of the following two conditions is satisfied:

- (i)  $\alpha$  is a right- (resp. left-) injective  $\otimes$ -norm and  $R[A]$  (resp.  $R[B]$ ) is complementary;
- (ii)  $\alpha$  is an injective  $\otimes$ -norm.

Then  $A \hat{\otimes}_\alpha B$  is one-to-one and has closed range if and only if both  $A$  and  $B$  are one-to-one and have closed range.

**THEOREM 1.5.** Besides the hypothesis of Theorem 1.3, suppose that either  $X$  or  $Y$  and either  $X/N[A]$  or  $Y/N[B]$  have the b. a. p. Then one has (1.4).

**THEOREM 1.6.** Let  $\alpha$  be a uniform reasonable norm on  $X \otimes Y$ . Suppose  $Y$  has the b. a. p. Then  $N[A \hat{\otimes}_\alpha I] = \bar{N}[A \otimes I]$ .

**REMARKS 1.** The converse of Theorem 1.2 is not true in general, even if  $R[A]$  and  $R[B]$  are complementary. Theorem 1.3 (resp. Theorem 1.4) is not true in general if  $\alpha$  is an injective (resp. projective)  $\otimes$ -norm. In fact, if  $A$  (resp.  $B$ ) is a topological homomorphism of  $X$  (resp.  $Y$ ) onto itself,  $A \hat{\otimes}_\alpha B$  is not necessarily a topological homomorphism of  $X \hat{\otimes}_\alpha Y$  onto itself. If  $A$  (resp.  $B$ ) is a topological isomorphism of  $X$  (resp.  $Y$ ) into itself,  $A \hat{\otimes}_\alpha B$  is not necessarily a topological isomorphism of  $X \hat{\otimes}_\alpha Y$  into itself ([5, Chap. I, § 3, p. 92] and [12, Remark 2 to Prop. 1.2]).

2. Theorem 1.3 is known if  $A$  and  $B$  are continuous projections (e. g. [14, Theorem 3.10], [8, Theorems 4.1 and 4.2 where read  $\alpha$  as a  $\otimes$ -norm] and [11, Prop. 1.5]).

3. We don't know whether (1.4) always holds for every  $\otimes$ -norm  $\alpha$ . However, the statement is not true for a general polynomial operator (see [12, 3.5, 3°]).

1.2. *The Hilbert space case.*

Assume  $X$  and  $Y$  are Hilbert spaces. Then a rather complete result can be obtained since a Hilbert space has the b.a.p. and every closed subspace of it is complementary. The *prehilbertian norm*  $\sigma$  on  $X \otimes Y$  is by definition the norm induced on it by the inner product  $(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2)(y_1, y_2)$  for  $x_i \in X, y_i \in Y, i=1, 2$ .

THEOREM 1.7. *Let  $\alpha$  be a  $\otimes$ -norm or the prehilbertian norm. Then*

- a)  $A \otimes B$  is closable in  $X \hat{\otimes}_\alpha Y$ ;
- b) one always has (1.4);
- c)  $R[A \hat{\otimes}_\alpha B]$  is closed if and only if both  $R[A]$  and  $R[B]$  are closed; in this case, one has (1.3), (1.4) and (1.5).

REMARK. Theorem 1.7 implies that for  $X$  and  $Y$  Hilbert spaces, every  $\otimes$ -norm and the prehilbertian norm have the *h-* and *i-properties*, which are the notions introduced in [12].

In the proofs where there might be some confusion, for a pair of Banach spaces  $Z_1$  and  $Z_2$ , a  $\otimes$ -norm  $\alpha$  and  $u \in Z_1 \hat{\otimes}_\alpha Z_2$ , we shall use the symbol  $\alpha(u; Z_1, Z_2)$  instead of  $\|u\|_\alpha$ .

2. Lemmas

The proof of Theorem 1.3 uses the following lemmas.

LEMMA 2.1. *Let  $\alpha$  be a projective  $\otimes$ -norm. Then, for every pair of Banach spaces  $X$  and  $Y$  and their respective closed subspaces  $M$  and  $N$  with quotient mappings  $f: X \rightarrow \hat{X} = X/M$  and  $g: Y \rightarrow \hat{Y} = Y/N$ , the canonical linear mapping*

$$(2.1) \quad f \hat{\otimes}_\alpha g : X \hat{\otimes}_\alpha Y \rightarrow \hat{X} \hat{\otimes}_\alpha \hat{Y}$$

*is a surjective metric homomorphism and so induces an isometry of  $X \hat{\otimes}_\alpha Y / N[f \hat{\otimes}_\alpha g]$  onto  $\hat{X} \hat{\otimes}_\alpha \hat{Y}$ .  $N[f \hat{\otimes}_\alpha g]$  is the closure of*

$$(2.2) \quad N[f \otimes g] = M \otimes Y + X \otimes N.$$

Lemma 2.1 may be implicitly contained in the definition of projective  $\otimes$ -norms [6, p. 25]; it yields immediately the formula

$$(2.3) \quad \alpha((f \hat{\otimes}_\alpha g)u; \hat{X}, \hat{Y}) = \inf \{ \alpha(u-v; X, Y); v \in N[f \hat{\otimes}_\alpha g] \}, u \in X \hat{\otimes}_\alpha Y,$$

for every pair of Banach spaces  $X$  and  $Y$  and their respective closed

subspaces  $M$  and  $N$ . However, as the assertion concerning the null space of  $f \hat{\otimes}_\alpha g$  does not seem to be evident except when either  $X$  or  $Y$  is of finite dimension, we give a proof of it.

PROOF of LEMMA 2.1. By continuity of  $f \hat{\otimes}_\alpha g$  it suffices to show that there is a constant  $C > 0$  such that

$$(2.4) \quad \alpha((f \otimes g)u; \hat{X}, \hat{Y}) \geq C^2 \inf \{ \alpha(u-v; X, Y); v \in N[f \otimes g] \}, \quad u \in X \otimes Y.$$

To show (2.4) we have only to show both

$$(2.5) \quad \alpha((f \otimes I)u; \hat{X}, Y) \geq C \inf \{ \alpha(u-w; X, Y); w \in M \otimes Y \}, \quad u \in X \otimes Y,$$

and

$$(2.6) \quad \alpha((I \otimes g)v; \hat{X}, \hat{Y}) \geq C \inf \{ \alpha(v-w; \hat{X}, Y); w \in \hat{X} \otimes N \}, \quad v \in \hat{X} \otimes Y.$$

Note (2.2) and that  $f \otimes I$  maps  $X \otimes Y$  onto  $\hat{X} \otimes Y$  with null space  $M \otimes Y$ .

We show (2.5) and (2.6), in fact, as equalities (and hence (2.4), too) with  $C=1$ . By the property of  $\otimes$ -norms ([6, p. 11] or [1, p. 162]),

$$\begin{aligned} & \alpha((f \otimes I)u; \hat{X}, Y) \\ &= \inf \{ \alpha((f \otimes I)u; \hat{E}, F); \hat{E} \in \mathcal{F}(\hat{X}), F \in \mathcal{F}(Y), \hat{E} \otimes F \ni (f \otimes I)u \}. \end{aligned}$$

Here  $\mathcal{F}(Z)$  denotes the family of all finite-dimensional subspaces of a Banach space  $Z$ . If  $\hat{E} \in \mathcal{F}(\hat{X})$  set  $E = f^{(-1)}(\hat{E})$ . Then  $E$  is a closed subspace of  $X$ , and  $M$  is a closed subspace of  $E$  with finite codimension. We have  $\hat{E} = E/M$ . The restriction  $f_E$  of  $f$  to  $E$  is the canonical quotient mapping of  $E$  onto  $\hat{E}$ . By the left-projectivity of  $\alpha$ ,  $f_E \hat{\otimes}_\alpha I_F$  is a metric homomorphism of  $E \hat{\otimes}_\alpha F$  onto  $\hat{E} \hat{\otimes}_\alpha F$  and so induces an isometry of  $E \hat{\otimes}_\alpha F / N[f_E \hat{\otimes}_\alpha I_F]$  onto  $\hat{E} \hat{\otimes}_\alpha F$ . Note that  $E \hat{\otimes}_\alpha F = E \otimes F$  and  $\hat{E} \hat{\otimes}_\alpha F = \hat{E} \otimes F$ , and so

$$N[f_E \hat{\otimes}_\alpha I_F] = N[f_E \otimes I_F] = M \otimes F,$$

because  $F$  is of finite dimension. Consequently, we have

$$\begin{aligned} & \alpha((f \otimes I)u; \hat{E}, F) \\ & \geq \inf \{ \alpha(u-w; E, F); w \in M \otimes F \} \\ & = \inf_{w \in M \otimes F} \inf_{G \in \mathcal{F}(E), G \otimes F \ni u-w} \alpha(u-w; G, F) \\ & \geq \inf_{w \in M \otimes Y} \inf_{G \in \mathcal{F}(X), H \in \mathcal{F}(Y), G \otimes H \ni u-w} \alpha(u-w; G, H) \\ & = \inf \{ \alpha(u-w; X, Y); w \in M \otimes Y \}. \end{aligned}$$

Thus we obtain altogether the inequality (2.5) with  $C=1$ . Once this is established, it is easy to see with (2.3) that it is actually equality.

Similarly, the right-projectivity of  $\alpha$  proves (2.6) as equality with  $C=1$ . Q.E.D.

LEMMA 2.2. *Let  $\alpha$  be a  $\otimes$ -norm. Let  $M$  and  $N$  be closed subspaces of  $X$  and  $Y$  with quotient mappings  $f: X \rightarrow \hat{X} = X/M$  and  $g: Y \rightarrow \hat{Y} = Y/N$ , and suppose one of the following three conditions is satisfied:*

- (i) *either  $M$  or  $N$  is of finite codimension;*
- (ii) *both  $M$  and  $N$  are complementary;*
- (iii)  *$\alpha$  is a right- (resp. left-) projective  $\otimes$ -norm and  $M$  (resp.  $N$ ) is complementary.*

*Then the canonical linear mapping (2.1) is a surjective topological homomorphism and so induces a topological isomorphism of  $X \hat{\otimes}_\alpha Y / N[f \hat{\otimes}_\alpha g]$  onto  $\hat{X} \hat{\otimes}_\alpha \hat{Y}$ .  $N[f \hat{\otimes}_\alpha g]$  is the closure of (2.2).*

PROOF. We establish (2.4) in each of the three cases.

(i) Assume that  $\text{codim } M < \infty$ , so that  $r = \dim \hat{X} = \infty$ . Choose a sequence  $\{e_i\}_{i=1}^r$  in  $X$  of linearly independent elements such that  $\{f(e_i)\}_{i=1}^r$  forms a normalized basis of  $\hat{X}$ . Let  $u \in X \otimes Y$ . There is a sequence  $\{y_i\}_{i=1}^r$  in  $Y$  such that

$$(f \otimes g)u = \sum_{i=1}^r f(e_i) \otimes g(y_i).$$

Since  $\dim \hat{X} < \infty$ , all  $\otimes$ -norms on  $\hat{X} \otimes \hat{Y}$  are equivalent, and  $\hat{X} \hat{\otimes}_\alpha \hat{Y} = \hat{X} \otimes \hat{Y}$ . Therefore  $\alpha((f \otimes g)u; \hat{X}, \hat{Y})$  is equivalent to  $\varepsilon((f \otimes g)u; \hat{X}, \hat{Y})$  and hence to

$$\sum_{i=1}^r \|g(y_i)\| = \inf \left\{ \sum_{i=1}^r \|y_i + z_i\|; z_i \in N[g], i=1, 2, \dots, r \right\},$$

which is equivalent to

$$\begin{aligned} & \inf \left\{ \alpha \left( \sum_{i=1}^r e_i \otimes (y_i + z_i); X, Y \right); z_i \in N[g], i=1, 2, \dots, r \right\} \\ & \geq \inf \left\{ \alpha(u - v; X, Y); v \in N[f \otimes g] \right\}. \end{aligned}$$

This establishes (2.4).

(ii) In this case, we can decompose  $X$  and  $Y$  into the topological direct sums  $X = M \oplus X_0$  and  $Y = N \oplus Y_0$ . Consequently,  $X \otimes Y$  is also decomposed into the topological direct sum

$$X \otimes Y = (M \otimes N) \oplus (M \otimes Y_0) \oplus (X_0 \otimes N) \oplus (X_0 \otimes Y_0)$$

under the norm  $\alpha$ . Denote the projection of  $X$  (resp.  $Y$ ) onto  $M$  (resp.  $N$ ) along  $X_0$  (resp.  $Y_0$ ) by  $P$  (resp.  $Q$ ). Then the restriction  $f_0$  (resp.  $g_0$ ) of  $f$

(resp.  $g$ ) to  $X_0$  (resp.  $Y_0$ ) is a topological isomorphism of  $X_0$  (resp.  $Y_0$ ) onto  $\hat{X}$  (resp.  $\hat{Y}$ ). Since  $\alpha$  is a  $\otimes$ -norm, we have for  $u \in X \otimes Y$

$$\begin{aligned} & \alpha\left(\left[(I-P) \otimes (I-Q)\right] u ; X_0, Y_0\right) \\ &= \alpha\left(\left(f_0^{-1} \otimes g_0^{-1}\right)\left(f_0 \otimes g_0\right)\left[(I-P) \otimes (I-Q)\right] u ; X_0, Y_0\right) \\ &\leq \|f_0^{-1}\| \|g_0^{-1}\| \alpha\left(\left(f_0 \otimes g_0\right)\left[(I-P) \otimes (I-Q)\right] u ; \hat{X}, \hat{Y}\right) \\ &= \|f_0^{-1}\| \|g_0^{-1}\| \alpha\left(\left(f \otimes g\right) u ; \hat{X}, \hat{Y}\right). \end{aligned}$$

On the other hand,  $\alpha\left(\left[(I-P) \otimes (I-Q)\right] u ; X_0, Y_0\right)$  is equivalent to  $\alpha\left(\left[(I-P) \otimes (I-Q)\right] u ; X, Y\right)$  (e. g. [11, Prop. 1.5]), which is in turn greater than or equal to the right-hand side of (2.4) with the constant  $C^2$  removed.

(iii) Assume that  $\alpha$  is a left-projective  $\otimes$ -norm and  $N$  is complementary. By the same proof as in the proof of Lemma 2.1, the left-projectivity of  $\alpha$  implies (2.5). (2.6) can be shown in the same way as in (ii) above. This proves (2.4). Q.E.D.

### 3. Proofs of Theorems

PROOF OF THEOREM 1.1. a) Suppose  $Y$  has the b.a.p. Let  $u \in X \hat{\otimes}_\alpha Y$  and  $j_i^\alpha(u) = 0$ . We show  $u = 0$ . Choose a sequence  $\{u_n\}_{n=1}^\infty$  in  $X \otimes Y$  which converges to  $u$  in  $X \hat{\otimes}_\alpha Y$ . Each  $u_n$  has a representation

$$(3.1) \quad u_n = \sum_{i=1}^{r_n} x_i^{(n)} \otimes y_i^{(n)},$$

where we may assume that the sequence  $\{y_i^{(n)}\}_{i=1}^{r_n} \subset Y$  is linearly independent and the sequence  $\{x_i^{(n)}\}_{i=1}^{r_n} \subset X$  consists of linearly independent unit vectors such that there is a sequence  $\{f_i^{(n)}\}_{i=1}^{r_n}$  of unit vectors in  $X'$ , the topological dual space of  $X$ , with  $\langle x_i^{(n)}, f_j^{(n)} \rangle = \delta_{ij}$ ,  $i, j = 1, \dots, r_n$ . Set

$$K = \left\{ \langle u_n, x' \rangle_X ; x' \in X', \|x'\| \leq 1, n = 1, 2, \dots \right\},$$

with  $\langle u_n, x' \rangle_X = \sum_{i=1}^{r_n} \langle x_i^{(n)}, x' \rangle y_i^{(n)}$ ,  $x' \in X'$ . Note that  $K$  is relatively compact in  $Y$  (e.g. [7]) and each  $y_i^{(n)}$  belongs to  $K$ .

By the b.a.p. of  $Y$ , there exists a uniformly bounded sequence  $\{T_l\}_{l=1}^\infty$  of bounded linear operators on  $Y$  of finite rank such that  $(T_l - I)y \rightarrow 0$  uniformly on  $K$  as  $l \rightarrow \infty$ . By uniformness of  $\alpha$ ,  $\|I \hat{\otimes}_\alpha T_l\| = \|T_l\| \leq \lambda$ ,  $l = 1, 2, \dots$ . Set  $v_l = (I \hat{\otimes}_\alpha T_l)u$ . We show that  $v_l = 0$  for all  $l$  and  $v_l \rightarrow u$  in  $X \hat{\otimes}_\alpha Y$  as  $l \rightarrow \infty$ . In fact,  $v_l \in X \otimes Y$  and by continuity,

$$j_i^\alpha(v_l) = j_i^\alpha\left((I \hat{\otimes}_\alpha T_l)u\right) = (I \hat{\otimes}_\alpha T_l)\left(j_i^\alpha(u)\right) = 0.$$

Since  $j_i^\alpha$  is the identity mapping on  $X \otimes Y$ , we have  $v_l = 0$ . For each  $n$ ,

$$\begin{aligned} \|v_l - u\|_\alpha &\leq \left\| (I \hat{\otimes}_\alpha T_l)(u - u_n) \right\|_\alpha + \left\| (I \hat{\otimes}_\alpha T_l) u_n - u_n \right\|_\alpha + \|u_n - u\|_\alpha \\ &\leq (\lambda + 1) \|u - u_n\|_\alpha + \sum_{i=1}^{r_n} \|x_i^{(n)}\| \left\| (T_l - I) y_i^{(n)} \right\|. \end{aligned}$$

Since  $y_i^{(n)} \in K$ , we obtain  $\limsup_{l \rightarrow \infty} \|v_l - u\|_\alpha \leq (\lambda + 1) \|u - u_n\|_\alpha$ . On tending  $n \rightarrow \infty$ , we have  $v_l \rightarrow u$  in  $X \hat{\otimes}_\alpha Y$  as  $l \rightarrow \infty$ .

b) We have only to consider the case where  $\alpha$  is a left-injective  $\otimes$ -norm. Let  $U^0$  be the closed unit ball of  $X'$  equipped with the weak\* topology.  $U^0$  is then compact. Let  $C(U^0)$  be the Banach space of the continuous functions on  $U^0$  and let  $j: X \rightarrow C(U^0)$  be the canonical injection. Then the following diagram is commutative:

$$\begin{array}{ccc} X \hat{\otimes}_\alpha Y & \xrightarrow{j_i^\alpha} & X \hat{\otimes}_\epsilon Y \\ j \hat{\otimes}_\alpha I \downarrow & & \downarrow j \hat{\otimes}_\epsilon I \\ C(U^0) \hat{\otimes}_\alpha Y & \xrightarrow{J_i^\alpha} & C(U^0) \hat{\otimes}_\epsilon Y \end{array}$$

Both  $j \hat{\otimes}_\alpha I$  and  $j \hat{\otimes}_\epsilon I$  are isometries since both  $\alpha$  and  $\epsilon$  are left-injective  $\otimes$ -norms. The natural continuous linear mapping  $J_i^\alpha$  of  $C(U^0) \hat{\otimes}_\alpha Y$  into  $C(U^0) \hat{\otimes}_\epsilon Y$  is by a) one-to-one, because  $C(U^0)$  has the b.a.p. It follows that the natural continuous linear mapping  $j_i^\alpha$  of  $X \hat{\otimes}_\alpha Y$  into  $X \hat{\otimes}_\epsilon Y$  is also one-to-one, that is,  $\alpha$  is faithful on  $X \otimes Y$ . Q.E.D.

PROOF of THEOREM 1.2. We may assume that neither  $A \subset O$  nor  $B \subset O$ . Therefore  $R[B] \neq \{0\}$  and  $R[B'] \neq \{0\}$ , where  $B'$  is the adjoint of  $B$ . By closedness of  $B$ ,  $\langle Y, D[B'] \rangle$  forms a dual system. Apply the Hahn-Banach theorem to choose  $y_0 \in D[B]$  and  $y'_0 \in D[B']$  with  $\|By_0\| = \langle By_0, y'_0 \rangle = \langle y_0, B'y'_0 \rangle = 1$ .

To show the first half of the theorem, assume  $R[A \hat{\otimes}_\alpha B]$  is closed. Then there exists a constant  $C > 0$  such that

$$\left\| (A \hat{\otimes}_\alpha B) u \right\|_\alpha \geq C \operatorname{dist}(u, N[A \hat{\otimes}_\alpha B]), \quad u \in D[A \hat{\otimes}_\alpha B].$$

Hence, for  $x \in D[A]$ ,

$$\begin{aligned} \|Ax\| &= \|Ax\| \|By_0\| = \|Ax \otimes By_0\|_\alpha = \left\| (A \hat{\otimes}_\alpha B)(x \otimes y_0) \right\|_\alpha \\ &\geq C \inf \left\{ \|x \otimes y_0 - v\|_\alpha; v \in N[A \hat{\otimes}_\alpha B] \right\} \\ &\geq C \inf \left\{ \|x \otimes y_0 - j_i^\alpha(v)\|_\epsilon; v \in N[A \hat{\otimes}_\alpha B] \right\}. \end{aligned}$$

Here we have used the fact that  $\epsilon \leq \alpha$ . If  $j_i^\alpha(v) = \sum_{i=1}^\infty x_i \otimes y_i$ , we set



$$\langle j_i^\alpha(v), y' \rangle_Y = \sum_{i=1}^{\infty} \langle y_i, y' \rangle x_i.$$

Then

$$\begin{aligned} \|Ax\| &\geq C \inf_{v \in N[A \hat{\otimes}_\alpha B]} \sup_{y' \in Y', \|y'\|=1} \left\| \langle y_0, y' \rangle x - \langle j_i^\alpha(v), y' \rangle_Y \right\| \\ &\geq C_1 \inf \left\{ \left\| \langle y_0, B' y'_0 \rangle x - \langle j_i^\alpha(v), B' y'_0 \rangle_Y \right\|; v \in N[A \hat{\otimes}_\alpha B] \right\} \\ &\geq C_1 \inf \left\{ \|x - z\|; z \in N[A] \right\} \\ &= C_1 \operatorname{dist}(x, N[A]), \quad C_1 = C \|B' y'_0\|^{-1}. \end{aligned}$$

Here the last inequality above is due to the fact that  $\langle j_i^\alpha(v), B' y'_0 \rangle_Y$  belongs to  $N[A]$ ; to see this we have only to note that  $D[A']$  is total by closedness of  $A$  and for  $x' \in D[A']$

$$\begin{aligned} 0 &= \langle (A \hat{\otimes}_\alpha B) v, x' \otimes y'_0 \rangle = \langle v, A' x' \otimes B' y'_0 \rangle \\ &= \langle j_i^\alpha(v), A' x' \otimes B' y'_0 \rangle = \langle \langle j_i^\alpha(v), B' y'_0 \rangle_Y, A' x' \rangle \\ &= \langle A \langle j_i^\alpha(v), B' y'_0 \rangle_Y, x' \rangle. \end{aligned}$$

This proves closedness of  $R[A]$ . Similarly,  $R[B]$  is closed.

Next, we show the second half. Assume  $R[A \hat{\otimes}_\alpha B]$  is complementary. Then there exists a continuous projection  $R$  of  $X \hat{\otimes}_\alpha Y$  onto  $R[A \hat{\otimes}_\alpha B]$ . Set  $z_0 = B y_0$ . Then  $X \otimes [z_0]$  is a closed subspace of  $X \hat{\otimes}_\alpha Y$  which is isometric to  $X$ .  $R[A] \otimes [z_0]$  is a closed subspace of  $X \otimes [z_0]$ . Consider the linear mapping  $Q$  of  $X \hat{\otimes}_\alpha Y$  onto  $X \otimes [z_0]$  defined by

$$Qu = \langle j_i^\alpha(u), y'_0 \rangle_Y \otimes z_0, \quad u \in X \hat{\otimes}_\alpha Y.$$

Then  $Q$  is a continuous projection with  $\|Q\| \leq \|y'_0\|$ , since  $Q(x \otimes z_0) = x \otimes z_0$  for every  $x \in X$  and

$$\|Qu\|_\alpha = \left\| \langle j_i^\alpha(u), y'_0 \rangle_Y \right\| \|z_0\| \leq \|j_i^\alpha(u)\| \|y'_0\| \leq \|y'_0\| \|u\|_\alpha.$$

Set  $P = QR$ . Then  $P$  is a continuous projection of  $X \hat{\otimes}_\alpha Y$  onto  $R[A] \otimes [z_0]$ , which obviously implies that  $R[A]$  is complementary. In fact, it suffices to check  $P$  is a projection. If  $u \in X \hat{\otimes}_\alpha Y$  then  $Ru = (A \hat{\otimes}_\alpha B)w$  for some  $w \in D[A \hat{\otimes}_\alpha B]$ . Then

$$\begin{aligned} Pu &= Q(Ru) = Q((A \hat{\otimes}_\alpha B)w) \\ &= \langle j_i^\alpha((A \hat{\otimes}_\alpha B)w), y'_0 \rangle_Y \otimes z_0 \end{aligned}$$

$$\begin{aligned}
 &= A \langle j'_\alpha(\omega), B' y'_0 \rangle_Y \otimes B y_0 \\
 &= (A \hat{\otimes}_\alpha B) \left( \langle j'_\alpha(\omega), B' y'_0 \rangle_Y \otimes y_0 \right),
 \end{aligned}$$

which belongs to  $R[A \hat{\otimes}_\alpha B]$ , and hence  $P^2 u = Pu$ . Thus  $P$  is a projection. Similarly,  $R[B]$  is complementary. Q. E. D.

PROOF of THEOREM 1.3. The one direction follows from Theorem 1.2. To show the other direction we have only to establish (1.5), assuming both  $R[A]$  and  $R[B]$  complementary. In fact, in view of the theorem of bipolars, (1.5) yields (1.3) and (1.4); by [11, Prop. 1.5], (1.3) yields that  $R[A \hat{\otimes}_\alpha B]$  is complementary.

Now, to establish (1.5) let  $u \in D[A \otimes B]$ . Set  $\hat{X} = X/N[A]$  and  $\hat{Y} = Y/N[B]$ . Let  $f: X \rightarrow \hat{X}$  and  $g: Y \rightarrow \hat{Y}$  be the quotient mappings. Define a linear operator  $\hat{A}: D[\hat{A}] \subset \hat{X} \rightarrow X$ . The domain  $D[\hat{A}]$  is the space of all  $\hat{x} \in \hat{X}$  such that every  $x \in \hat{x}$  belongs to  $D[A]$  and  $\hat{A}\hat{x} = Ax$ .  $\hat{A}$  is densely defined and closed. It has bounded inverse since  $R[\hat{A}] = R[A]$  is closed. Similarly, define a closed linear operator  $\hat{B}: D[\hat{B}] \supset \hat{Y} \rightarrow Y$  with  $B$ .  $\hat{A}^{-1}$  is a bounded linear operator of  $R[A]$  into  $\hat{X}$  and  $\hat{B}^{-1}$  of  $R[B]$  into  $\hat{Y}$ . It is readily seen that  $fD[A] = D[\hat{A}]$  and  $gD[B] = D[\hat{B}]$ . As  $\alpha$  is a  $\otimes$ -norm, we have

$$\begin{aligned}
 \alpha((f \otimes g)u; \hat{X}, \hat{Y}) &= \alpha((\hat{A}^{-1} \otimes \hat{B}^{-1})(\hat{A} \otimes \hat{B})(f \otimes g)u, \hat{X}, \hat{Y}) \\
 &= \alpha((\hat{A}^{-1} \otimes \hat{B}^{-1})(A \otimes B)u; \hat{X}, \hat{Y}) \\
 &\leq \|\hat{A}^{-1}\| \|\hat{B}^{-1}\| \alpha((A \otimes B)u; R[A], R[B]).
 \end{aligned}$$

Since both  $R[A]$  and  $R[B]$  are complementary,  $\alpha((A \otimes B)u; R[A], R[B])$  is equivalent to  $\alpha((A \otimes B)u; X, Y)$  (e.g. [11, Prop. 1.5]). On the other hand, both  $D[A]$  and  $D[B]$  are dense and so  $N[A \otimes B]$  is dense in  $N[f \otimes g]$ . Consequently,  $N[f \hat{\otimes}_\alpha g]$  is, by Lemmas 2.1 and 2.2, the closure of  $N[A \otimes B]$  in  $X \hat{\otimes}_\alpha Y$ . By the same lemmas again,  $f \hat{\otimes}_\alpha g$  is a topological homomorphism of  $X \hat{\otimes}_\alpha Y$  onto  $\hat{X} \hat{\otimes}_\alpha \hat{Y}$ . Hence  $\alpha((f \otimes g)u; \hat{X}, \hat{Y})$  is equivalent to  $\text{dist}(u, N[A \otimes B])$  in  $X \hat{\otimes}_\alpha Y$ . This establishes (1.5). Q. E. D.

PROOF of THEOREM 1.4. The one direction follows from Theorem 1.2. We show the other direction. Since  $\alpha$  is a  $\otimes$ -norm and  $A^{-1}$  is a bounded linear operator of  $R[A]$  into  $X$  and  $B^{-1}$  of  $R[B]$  into  $Y$ , we have for  $u \in D[A \otimes B]$

$$\begin{aligned}
 \alpha(u; X, Y) &= \alpha((A^{-1} \otimes B^{-1})(A \otimes B)u; X, Y) \\
 &\leq \|A^{-1}\| \|B^{-1}\| \alpha((A \otimes B)u; R[A], R[B]),
 \end{aligned}$$

which is equivalent to  $\|A^{-1}\| \|B^{-1}\| \alpha((A \otimes B)u; X, Y)$  by the hypothesis (i) or (ii). Hence follows that  $A \hat{\otimes}_\alpha B$  is one-to-one and has closed range. Q.E.D.

PROOF of THEOREM 1.5. We use the same notations as in the proof of Theorem 1.3. Analogous arguments used in the proof of Theorem 1.1 utilizing the b.a.p. of  $X$  or  $Y$  show that  $\hat{A} \otimes \hat{B}$  is a densely defined, closable linear operator of  $D[\hat{A}] \otimes D[\hat{B}] \subset \hat{X} \hat{\otimes}_\alpha \hat{Y}$  into  $X \hat{\otimes}_\alpha Y$ . Denote its closure by  $\hat{A} \hat{\otimes}_\alpha \hat{B}$ . Then it is easy to verify that

$$(A \hat{\otimes}_\alpha B)u = (\hat{A} \hat{\otimes}_\alpha \hat{B})(f \hat{\otimes}_\alpha g)u, \quad u \in D[A \hat{\otimes}_\alpha B].$$

$\hat{A}$  and  $\hat{B}$  are one-to-one, and so their adjoints  $(\hat{A})'$  and  $(\hat{B})'$  have dense range in  $(\hat{X})'$  and  $(\hat{Y})'$ , respectively. By assumption and Theorem 1.1,  $\alpha$  is faithful on  $\hat{X} \otimes \hat{Y}$ . Therefore,  $R[(\hat{A})'] \otimes R[(\hat{B})']$  and hence  $R[(\hat{A} \hat{\otimes}_\alpha \hat{B})']$  is dense in  $(\hat{X} \hat{\otimes}_\alpha \hat{Y})'$  in the weak topology defined by the dual system  $\langle \hat{X} \hat{\otimes}_\alpha \hat{Y}, (\hat{X} \hat{\otimes}_\alpha \hat{Y})' \rangle$ . It follows that  $\hat{A} \hat{\otimes}_\alpha \hat{B}$  is one-to-one, and

$$N[A \hat{\otimes}_\alpha B] = N[f \hat{\otimes}_\alpha g] \cap D[A \hat{\otimes}_\alpha B] = \bar{N}[A \otimes B].$$

Q.E.D.

PROOF of THEOREM 1.6. The proof is analogous to the proof of Theorem 1.1. a). Let  $u \in N[A \hat{\otimes}_\alpha I]$ . We must show that  $u \in \bar{N}[A \otimes I]$ . Choose a sequence  $\{u_n\}_{n=1}^\infty$  in  $D[A] \otimes Y$  which converges to  $u$  in  $X \hat{\otimes}_\alpha Y$ . Each  $u_n$  has a representation (3.1); in this case,  $\{x_i^{(n)}\}_{i=1}^r \subset D[A]$ . With the same  $K$ , choose the same sequence  $\{T_l\}_{l=1}^\infty$  as before. Set  $v_l = (I \hat{\otimes}_\alpha T_l)u$ . The same argument shows that  $v_l \in N[A] \otimes Y$  for all  $l$  and  $v_l \rightarrow u$  in  $X \hat{\otimes}_\alpha Y$  as  $l \rightarrow \infty$ . Q.E.D.

PROOF of THEOREM 1.7. a) From Theorem 1.1.

b) From Theorem 1.5. c) From Theorem 1.3. Q.E.D.

### References

- [1] AMEMIYA, I. and SHIGA, K.: On tensor products of Banach spaces. Kodai Math. Sem. Rep. 9, 161-178 (1959).
- [2] CHEVET, S.: Sur certains produits tensoriels topologiques d'espaces de Banach. Z. Wahrscheinlichkeitstheorie Verw. Geb. 11, 120-138 (1969).
- [3] ENFLO, P.: A counterexample to the approximation problem in Banach spaces. Acta Math. 130, 309-317 (1973).
- [4] FIGIEL, T. and JOHNSON, W. B.: The approximation property does not imply the bounded approximation property. Proc. Amer. Math. Soc. 41, 197-200 (1973).
- [5] GROTHENDIECK, A.: Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc. No. 16, 1955.

- [6] GROTHENDIECK, A.: Résumé de la théorie métrique des produits tensoriels topologiques. Boletim da sociedade de matemática de São Paulo 8 (1956).
- [7] HOLUB, J. R.: Compactness in topological tensor products and operator spaces. Proc. Amer. Math. Soc. 36, 398–406 (1972).
- [8] HOLUB, J. R.: Tensor product mappings. Math. Ann. 188, 1–12 (1970).
- [9] ICHINOSE, T.: On the spectra of tensor products of linear operators in Banach spaces. J. Reine Angew. Math. 244, 119–153 (1970).
- [10] ICHINOSE, T.: Operators on tensor products of Banach spaces. Trans. Amer. Math. Soc. 170, 197–219 (1972).
- [11] ICHINOSE, T.: Spectral properties of tensor products of linear operators. Trans. Amer. Math. Soc. (to appear)
- [12] ICHINOSE, T.: Spectral properties of tensor products of linear operators. II: The approximate point spectrum and Kato essential spectrum. Trans. Amer. Math. Soc. (to appear)
- [13] SAPHAR, P.: Produits tensoriels d'espaces de Banach et classes d'applications linéaires. Studia Math. 38, 71–100 (1970).
- [14] SCHATTEN, R.: A Theory of Cross-Spaces. Ann. of Math. Studies. No. 26. Princeton: Princeton Univ. Press 1950.

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