Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation III

By Atsushi YOSHIKAWA

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§ 0. Introduction

Let

$$(0.0) \quad P = P(t, x, D_t, D_x) = D_t^2 - t^2 \sum_{j,k=1}^n a_{jk}(t, x) D_j D_k + \\ + b_0(t, x) D_t + \sum_{j=1}^n b_j(t, x) D_j + c(t, x).$$

Here $a_{jk}(t, x)$, $b_i(t, x)$, $c(t, x)$ are C^∞ functions of $(t, x) = (t, x_1, \dots, x_n) \in \mathbf{R} \times \mathbf{R}^n$, and

$$D_t = -i\partial/\partial t, \quad D_j = -i\partial/\partial x_j, \quad j = 1, \dots, n,$$

$i^2 = -1$, as usual. We assume that

$$(0.1) \quad a_{jk}(t, x) = a_{kj}(t, x), \quad j, k = 1, \dots, n, \quad \text{be real valued,}$$

$$(0.2) \quad a_{jk}(t, x) = \delta_{jk} \quad \text{for} \quad t \geq t_0, \quad |x| \geq r_0,$$

t_0, r_0 being some positive numbers, and

$$(0.3) \quad \sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k \geq c \sum_{j=1}^n \xi_j^2$$

for $(t, x) \in \mathbf{R}^{n+1}$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$, c being a positive constant independent of t, x, ξ .

Consider the following Cauchy problem :

$$(0.4) \quad Pv(t, x) = 0, \quad t > 0, \quad x \in \mathbf{R}^n,$$

$$(0.5) \quad v(\tau, x) = f(x), \quad D_t v(\tau, x) = g(x), \quad x \in \mathbf{R}^n.$$

Here $\tau \geq 0$ and f, g are given distributions in $\mathcal{E}'(\mathbf{R}^n)$.

In our previous paper [2], we have constructed good parametrices (cf. [2], Definition 0.1) for the problem (0.4) (0.5) when $a_{jk}(t, x) \equiv \delta_{jk}$. The purpose of the present paper is to extend our previous reasoning to cover the operator of the form (0.0) with (0.1)~(0.3). Note that the assumption (0.2) is just to avoid unnecessary complications and in fact is not restrictive at all.

Before stating our main result, we introduce several notations.

Let

$$(0.6) \quad a(t, x, \xi) = (\sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k)^{1/2}$$

for $(t, x, \xi) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \setminus 0$. Thus $a(t, x, \xi)$ is positive and positively homogeneous of degree 1 in the variables ξ . Assume as before [1], [2] that

$$(0.7) \quad \text{Im} \sum_{j=1}^n b_j(0, x) \xi_j \quad \text{be uniformly bounded}$$

for $x \in \mathbf{R}^n$ and ξ on the unit sphere $\mathbf{S}^{n-1}$. Then we set

$$(0.8) \quad m(\sigma) = -\frac{1}{4} + \frac{1}{4} \sup \left\{ \sigma \text{Im} \sum_{j=1}^n b_j(0, x) \xi_j / a(0, x, \xi) \right\}, \quad \sigma^2 = 1,$$

the superimum being taken over $(x, \xi) \in \mathbf{R}^n \times \mathbf{S}^{n-1}$. Finally, let $\phi^\sigma(t, \tau, x, \xi)$, $\sigma^2=1$, be respectively solutions of the following equations:

$$(0.9) \quad \phi_t^\sigma = \sigma t a(t, x, \phi_x^\sigma), \quad \sigma^2 = 1,$$

with

$$(0.10) \quad \phi^\sigma|_{t=\tau} = \langle x, \xi \rangle, \quad (\tau \geq 0).$$

To avoid unnecessary complications, we assume that $\phi^\sigma(t, \tau, x, \xi)$, $\sigma^2=1$, be well-defined everywhere in the large (cf. (0.2)).

Now our main result reads as follows.

THEOREM. *There exist symbols*

$$(0.11) \quad p_\sigma(t, \tau, x, \xi) \in S^{\epsilon, 2m(\sigma)+\epsilon, 2m(-\sigma)+1+\epsilon},$$

$$(0.12) \quad q_\sigma(t, \tau, x, \xi) \in S^{\epsilon-1, 2m(\sigma)+\epsilon, 2m(-\sigma)+\epsilon},$$

$$(0.13) \quad \tilde{p}_\sigma(t, \tau, x, \xi) \in S_+^{\epsilon, 2m(-\sigma)+1+\epsilon},$$

$$(0.14) \quad \tilde{q}_\sigma(t, \tau, x, \xi) \in S_+^{\epsilon-1, 2m(-\sigma)+\epsilon},$$

$$(0.15) \quad p_\sigma^\#(t, \tau, x, \xi) \in S_\infty^\epsilon,$$

$$(0.16) \quad q_\sigma^\#(t, \tau, x, \xi) \in S_\infty^{\epsilon-1},$$

$\sigma^2=1$, such that

$$(0.17) \quad \begin{aligned} [E_0(t, \tau)f](x) = & \sum_{\sigma=\pm 1} (2\pi)^{-n} \iint e^{i\{\phi^\sigma(t, \tau, x, \xi) - \langle y, \xi \rangle\}} p_\sigma(t, \tau, x, \xi) f(y) dy d\xi + \\ & + \sum_{\sigma=\pm 1} (2\pi)^{-n} \iint e^{i\{\phi^\sigma(t, \tau, x, \xi) - \langle y, \xi \rangle\}} \tilde{p}_\sigma(t, \tau, x, \xi) f(y) dy d\xi + \\ & + \sum_{\sigma=\pm 1} (2\pi)^{-n} \iint e^{i\{\phi^\sigma(t, \tau, x, \xi) - \langle y, \xi \rangle\}} p_\sigma^\#(t, \tau, x, \xi) f(y) dy d\xi \end{aligned}$$

and

$$\begin{aligned}
(0.18) \quad [E_1(t, \tau) g](x) = & \\
= \sum_{\sigma=\pm 1} (2\pi)^{-n} \iint e^{i\{\phi^\sigma(t, \tau, x, \xi) - \langle y, \xi \rangle\}} q_\sigma(t, \tau, x, \xi) g(y) dy d\xi + & \\
+ \sum_{\sigma=\pm 1} (2\pi)^{-n} \iint e^{i\{\phi^\sigma(t, \tau, x, \xi) - \langle y, \xi \rangle\}} \tilde{q}_\sigma(t, \tau, x, \xi) g(y) dy d\xi + & \\
+ \sum_{\sigma=\pm 1} (2\pi)^{-n} \iint e^{i\{\phi^\sigma(t, \tau, x, \xi) - \langle y, \xi \rangle\}} q_\sigma^\#(t, \tau, x, \xi) g(y) dy d\xi &
\end{aligned}$$

form a good parametrix for the problem (0.4) (0.5). Here the integrals in (0.17) and (0.18) are oscillatory ones over $\mathbf{R}^n \times \mathbf{R}^n$. The symbol classes in (0.11)~(0.16) are given in [2]. ε is an arbitrary positive number and may be omitted if $n=1$ and $b_1(0, x)/a_{11}(0, x)$ is independent of x .

The method to be employed is essentially the same as before [1], [2]. However, in our previous discussions we have largely relied on the simple form of the partial differential operator in consideration. In the present paper, a certain novelty appears in the construction of $p_\sigma(t, \tau, x, \xi)$ and $q_\sigma(t, \tau, x, \xi)$. Though the basic idea of using semi-homogeneous decomposition is common to [1], [2], we need several modifications of our previous arguments. The most important improvement in our present discussions is that we discuss directly the transport operators

$$(0.19) \quad T^\sigma = e^{-i\phi^\sigma} P e^{i\phi^\sigma}, \quad \sigma^2 = 1.$$

We give two decompositions of the transport operators T^σ , the semi-homogeneous one and the homogeneous one. Using the former decomposition, we construct $p_\sigma(t, \tau, x, \xi)$ and $q_\sigma(t, \tau, x, \xi)$. Although the underlying techniques employed in this stage is common to [1], [2], the present treatment makes clearer the meaning of $p_\sigma(t, \tau, x, \xi)$ and $q_\sigma(t, \tau, x, \xi)$. The remaining constructions of $\tilde{p}_\sigma(t, \tau, x, \xi)$, $\tilde{q}_\sigma(t, \tau, x, \xi)$, $p_\sigma^\#(t, \tau, x, \xi)$, $q_\sigma^\#(t, \tau, x, \xi)$ are done by the homogeneous decompositions of the transport operators and are thus just the same as before [2].

The content of the present paper is as follows:

- § 1. Transport operators and their decompositions;
- § 2. Constructions of $p_\sigma(t, \tau, x, \xi)$ and $q_\sigma(t, \tau, x, \xi)$, $\sigma^2 = 1$;
- § 3. End of the proof of Theorem.

§ 1. Transport operators and their decompositions.

Let $\tilde{x} = (t, x)$ and $\tilde{\xi} = (\xi_0, \xi)$, $x, \xi \in \mathbf{R}^n$. We set

$$(1.1) \quad P_2(\tilde{x}, \tilde{\xi}) = \xi_0^2 - t^2 \sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k,$$

$$(1.2) \quad P_1(\tilde{x}, \tilde{\xi}) = b_0(t, x) \xi_0 + \sum_{j=1}^n b_j(t, x) \xi_j,$$

$$(1.3) \quad P_2(\tilde{x}, \tilde{D}) = D_t^2 - t^2 \sum_{j,k=1}^n a_{jk}(t, x) D_j D_k,$$

$$(1.4) \quad P_1(\tilde{x}, \tilde{D}) = b_0(t, x) D_t + \sum_{j=1}^n b_j(t, x) D_j,$$

$$(1.5) \quad P_0(\tilde{x}, \tilde{D}) = c(t, x).$$

Then

$$(1.6) \quad P = P(\tilde{x}, \tilde{D}) = P_2(\tilde{x}, \tilde{D}) + P_1(\tilde{x}, \tilde{D}) + P_0(\tilde{x}, \tilde{D}).$$

As usual we write

$$(1.7) \quad P_2^{(k)}(\tilde{x}, \tilde{\xi}) = \partial P_2(\tilde{x}, \tilde{\xi}) / \partial \xi_k,$$

$$(1.8) \quad P_{2(l)}(\tilde{x}, \tilde{\xi}) = \partial P_2(\tilde{x}, \tilde{\xi}) / \partial x_l,$$

for $k, l = 0, 1, \dots, n$, ($x_0 = t$). The transport operators T^σ , $\sigma^2 = 1$, are then expressed in the following way:

$$(1.9) \quad \begin{aligned} T &= e^{-i\phi^\sigma} P e^{i\phi^\sigma} \\ &= \sum_{j=0}^n P_2^{(j)}(\tilde{x}, \phi_{\tilde{x}}^\sigma) D_j + i P_2(\tilde{x}, \tilde{D}) \phi^\sigma + P_1(\tilde{x}, \phi_{\tilde{x}}^\sigma) + P(\tilde{x}, \tilde{D}), \end{aligned}$$

where ϕ^σ , $\sigma^2 = 1$, are given by (0.9) (0.10) and $D_0 = D_t$, $\phi_{\tilde{x}}^\sigma = (\phi_t^\sigma, \phi_x^\sigma)$.

The transport operators T^σ , $\sigma^2 = 1$, admit semi-homogeneous decompositions. Namely, we have the following

PROPOSITION 1.1. *The transport operators T^σ , $\sigma^2 = 1$, are decomposed into the following formal sums:*

$$(1.10) \quad T^\sigma = \sum_{k=0}^{\infty} T_k^\sigma,$$

T_k^σ being semi-homogeneous of degree $1 - k/2$, $k = 0, 1, 2, \dots$. More precisely, for each k ,

$$(1.11) \quad T_k^\sigma = \sum_{l=0}^k \tau^l T_{k,l}^\sigma,$$

$T_{k,l}^\sigma$ being partial differential operators independent of τ and semi-homogeneous of degree $1 - k/2 + l/2$; namely,

$$(1.12) \quad \begin{aligned} T_0^\sigma &= T_{0,0}^\sigma = \\ &= D_t^2 + 2\sigma t a(0, x, \xi) D_t + \sum_{j=1}^n b_j(0, x) \xi_j - i\sigma a(0, x, \xi), \end{aligned}$$

$$(1.13) \quad T_{k,0}^\sigma = t^k |\xi| C(x, \xi) + t^{k-2} \{C(x, \xi) t D_t + C(x, \xi; D_x)\}$$

and

$$(1.14) \quad T_{k,l}^\sigma = t^{k-l} |\xi| \{C(x, \xi) + C(x, \xi) t D_t + C(x, \xi; D_x)\}$$

for $k = 1, 2, \dots$; $l = 1, 2, \dots, k$. Here $C(x, \xi)$ denote various suitable functions in $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n \setminus 0$ of homogeneous of degree zero, and $C(x, \xi; D_x)$ various

suitable partial differential operators in x with coefficients smooth functions of $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0$ of homogeneous of degree zero. In (1.13), the second term in the right hand side disappears when $k=1$.

PROOF. The proposition follows from (1.9) and the Taylor expansions around $(t, \tau) = (0, 0)$ of $P_2^{(j)}(\tilde{x}, \phi_{\tilde{x}}^\sigma)$, $j = 0, \dots, n$, $P_2(\tilde{x}, \tilde{D})\phi^\sigma$, $P_1(\tilde{x}, \phi_{\tilde{x}}^\sigma)$ and the coefficients of $P(\tilde{x}, \tilde{D})$.

REMARK 1.2. In our previous paper [2], due to the simple form of the operator P for that case, the terms corresponding to $T_{k,l}^\sigma$, $l \geq 1$, ($k=1, 2, \dots$), did not appear. The presence of these terms causes some new complexity. This complexity can be handled in the frame of our previous discussions with a little modification as will be shown in the next section.

We end this section by the following obvious proposition.

PROPOSITION 1.3. *The transport operators T^σ , $\sigma^2=1$, admit the following decompositions in homogeneous parts:*

$$(1.15) \quad T^\sigma = \tilde{T}_1^\sigma + \tilde{T}_0^\sigma,$$

$$(1.16) \quad \tilde{T}_1^\sigma = \sum_{j=0}^n P_2^{(j)}(\tilde{x}, \phi_{\tilde{x}}^\sigma) D_j + iP_2(\tilde{x}, \tilde{D})\phi^\sigma + P_1(\tilde{x}, \phi_{\tilde{x}}^\sigma),$$

$$(1.17) \quad \tilde{T}_0^\sigma = P(\tilde{x}, \tilde{D}).$$

$\tilde{T}_j^\sigma$, $j=0, 1$, are homogeneous of degree j , respectively.

§ 2. Constructions of $p_\sigma(t, \tau, x, \xi)$ and $q_\sigma(t, \tau, x, \xi)$, $\sigma^2=1$.

We begin by the following observation.

PROPOSITION 2.1. *Let $p(\tau, x, \xi)$ and $q(\tau, x, \xi)$ be given. Then the solutions $u_\sigma(t, \tau, x, \xi)$, $\sigma^2=1$, of the equations.*

$$(2.1) \quad T_0^\sigma u_\sigma = 0, \quad \sigma^2 = 1,$$

with the initial conditions

$$(2.2) \quad \sum_{\sigma=\pm 1} u_\sigma|_{t=\tau} = p(\tau, x, \xi),$$

$$(2.3) \quad \sum_{\sigma=\pm 1} \left\{ \sigma \tau a(0, x, \xi) + D_t \right\} u_\sigma|_{t=\tau} = q(\tau, x, \xi)$$

are given by

$$(2.4) \quad u_\sigma(t, \tau, x, \xi) = -\frac{\sigma i}{2} \phi_\sigma \left(a(0, x, \xi)^{1/2} t, B \right) \times \\ \times \left[p(\tau, x, \xi) \left\{ i \sigma a(0, x, \xi)^{1/2} \tau \phi_{-\sigma} \left(a(0, x, \xi)^{1/2} \tau, B \right) - \right. \right.$$

$$-\phi'_{-\sigma}\left(a(0, x, \xi)^{1/2}\tau, B\right)\} + \\ + ia(0, x, \xi)^{-1/2}q(\tau, x, \xi)\phi_{-\sigma}\left(a(0, x, \xi)^{1/2}\tau, B\right)\},$$

where

$$(2.5) \quad B = B(x, \xi) = \sum_{j=1}^n b_j(0, x) \xi_j / a(0, x, \xi)$$

and ϕ_σ , $\sigma^2=1$, are the entire functions defined by Proposition 2.1 in [1].

PROOF. This is an immediate consequence of Corollary 2.4 in [1].

Now let us set formally

$$(2.6) \quad p_\sigma(t, \tau, x, \xi) \sim \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} p_{j\sigma}^m(t, \tau, x, \xi)$$

and

$$(2.7) \quad q_\sigma(t, \tau, x, \xi) \sim \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} q_{j\sigma}^m(t, \tau, x, \xi).$$

Here $p_{j\sigma}^m(t, \tau, x, \xi)$ and $q_{j\sigma}^m(t, \tau, x, \xi)$ are respectively semihomogeneous of degree $-(m+j)/2$ and $-(m+j+1)/2$. Namely,

$$(2.8) \quad p_{j\sigma}^m(\lambda^{-1/2}t, \lambda^{-1/2}\tau, x, \lambda\xi) = \lambda^{-(m+j)/2} p_{j\sigma}^m(t, \tau, x, \xi)$$

and

$$(2.9) \quad q_{j\sigma}^m(\lambda^{-1/2}t, \lambda^{-1/2}\tau, x, \lambda) = \lambda^{-(m+j+1)/2} q_{j\sigma}^m(t, \tau, x, \xi)$$

for $\lambda > 0$.

We determine $p_{j\sigma}^m$ and $q_{j\sigma}^m$ by the following equations (Compare with the discussions in [2] before Proposition 2.2).

For $m=0$, we solve $p_{0\sigma}^0$ from the equations

$$(2.10) \quad T_0^\sigma p_{0\sigma}^0 = 0, \quad \sigma^2 = 1,$$

with the initial conditions

$$(2.11) \quad \sum_{\sigma=\pm 1} p_{0\sigma}^0|_{t=\tau} = 1,$$

$$(2.12) \quad \sum_{\sigma=\pm 1} \{\sigma t a(0, x, \xi) + D_t\} p_{0\sigma}^0|_{t=\tau} = 0.$$

Then we determine $p_{j\sigma}^0$, $j \geq 1$, from the equations

$$(2.13) \quad T_0^\sigma p_{j\sigma}^0 = - \sum_{k=1}^j T_{k,0}^\sigma p_{j-k\sigma}^0, \quad \sigma^2 = 1.$$

This is done as before [2] via the asymptotic expansion of the right hand side in the sector $S = \{t \in \mathbb{C} \setminus 0; |\arg t| < \pi/4\}$. For each j and σ , we then have a formal series solution of (2.13) and we can thus specify $p_{j\sigma}^0$ as the solution, entire in t , which has this formal solution as the asymptotic expansion in the sector S .

For $m \geq 1$, we solve $p_{0\sigma}^m$ from the equations

$$(2.14) \quad T_0^\sigma p_{0\sigma}^m = - \sum_{k=1}^m \tau^k T_{k,k}^\sigma p_{0\sigma}^{m-k}$$

with the initial conditions

$$(2.15) \quad \sum_{\sigma=\pm 1} p_{0\sigma}^m|_{t=\tau} = - \sum_{j=1}^m p_{j\sigma}^{m-j}|_{t=\tau},$$

$$(2.16) \quad \begin{aligned} \sum_{\sigma=\pm 1} \{ \sigma t a(0, x, \xi) + D_t \} p_{0\sigma}^m|_{t=\tau} = \\ = - \sum_{\sigma=\pm 1} \{ \sigma t a(0, x, \xi) + D_t \} \sum_{j=1}^m p_{j\sigma}^{m-j}|_{t=\tau} - \\ - \sum_{\sigma=\pm 1} \sigma \sum_{k=1}^m \tau^{k+1} |\xi| c_k^\sigma(x, \xi) \sum_{j=0}^{m-k} p_{m-j-k\sigma}^j|_{t=\tau}. \end{aligned}$$

Here $c_k^\sigma(x, \xi)$, $k=1, 2, \dots$, are functions of $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n \setminus 0$, homogeneous of degree zero, and determined by the Taylor expansion of $\phi_i^\sigma(\tau, \tau, x, \xi)$ around $\tau=0$:

$$(2.17) \quad \begin{aligned} \phi_i^\sigma(\tau, \tau, x, \xi) = \sum_{k=0}^\infty \sigma \tau^{k+1} |\xi| c_k^\sigma(x, \xi), \quad \sigma^2 = 1, \\ |\xi| c_0^\sigma(x, \xi) = a(0, x, \xi). \end{aligned}$$

We then determine $p_{j\sigma}^m$, $j \geq 1$, from the equations

$$(2.18) \quad T_0^\sigma p_{j\sigma}^m = - \sum_{k=1}^j T_{k,0}^\sigma p_{j-k\sigma}^m - \sum_{l=1}^m \sum_{k=0}^j \tau^l T_{k+l,l}^\sigma p_{j-k\sigma}^{m-l}$$

as the solutions, entire in t , with the prescribed asymptotic behaviors in the sector S .

In a similar manner, we solve $q_{0\sigma}^0$ from the equations

$$(2.19) \quad T_0^\sigma q_{0\sigma}^0 = 0, \quad \sigma^2 = 1,$$

with the initial conditions

$$(2.20) \quad \sum_{\sigma=\pm 1} q_{0\sigma}^0|_{t=\tau} = 0,$$

$$(2.21) \quad \sum_{\sigma=\pm 1} \{ \sigma t a(0, x, \xi) + D_t \} q_{0\sigma}^0|_{t=\tau} = 1.$$

We then determine $q_{j\sigma}^0$, $j \geq 1$, from the equations

$$(2.22) \quad T_0^\sigma q_{j\sigma}^0 = - \sum_{k=1}^j T_{k,0}^\sigma q_{j-k\sigma}^0, \quad \sigma^2 = 1,$$

as the solutions, entire in t , with the prescribed asymptotic behaviors in the sector S .

For $m \geq 1$, we solve $q_{0\sigma}^m$ from the equations

$$(2.23) \quad T_0^\sigma q_{0\sigma}^m = - \sum_{k=1}^m \tau^k T_{k,k}^\sigma q_{0\sigma}^{m-k}$$

with the initial conditions

$$(2.24) \quad \sum_{\sigma=\pm 1} q_{0\sigma}^m|_{t=\tau} = - \sum_{j=1}^m q_{j\sigma}^{m-j}|_{t=\tau},$$

$$\begin{aligned}
(2.25) \quad \sum_{\sigma=\pm 1} \{ \sigma t a(0, x, \xi) + D_t \} q_{0\sigma}^m|_{t=\tau} &= \\
&= - \sum_{\sigma=\pm 1} \{ \sigma t a(0, x, \xi) + D_t \} \sum_{j=1}^m q_{j\sigma}^{m-j}|_{t=\tau} - \\
&\quad - \sum_{\sigma=\pm 1} \sigma \sum_{k=1}^m \tau^{k+1} |\xi| c_k^\sigma(x, \xi) \sum_{j=0}^{m-k} q_{m-k-j\sigma}^j|_{t=\tau}.
\end{aligned}$$

We finally determine $q_{j\sigma}^m$, $j \geq 1$, from the equations

$$\begin{aligned}
(2.26) \quad T_0^\sigma q_{j\sigma}^m &= - \sum_{k=1}^j T_{k,0}^\sigma q_{j-k\sigma}^m - \\
&\quad - \sum_{l=1}^m \sum_{k=0}^j \tau^l T_{k+l,l}^\sigma q_{j-k\sigma}^{m-l}
\end{aligned}$$

as the solutions, entire in t , with the prescribed asymptotic behaviors in the sector S .

We can thus recurrently determine all the $p_{j\sigma}^m$ and $q_{j\sigma}^m$. The growth estimates of $p_{j\sigma}^m$ and $q_{j\sigma}^m$ are given by the following

PROPOSITION 2.2. For $|\xi|=1$,

$$\begin{aligned}
(2.27) \quad p_{j\sigma}^m(t, \tau, x, \xi) &= \\
&= C t^{-2\tilde{\mu}_\sigma+j} (\log t)^N (1 + o_t(1)) \tau^{-2\tilde{\mu}_{-\sigma}+m+1} (\log \tau)^M (1 + o_\tau(1)),
\end{aligned}$$

$$\begin{aligned}
(2.28) \quad q_{j\sigma}^m(t, \tau, x, \xi) &= \\
&= C t^{-2\tilde{\mu}_\sigma+j} (\log t)^N (1 + o_t(1)) \tau^{-2\tilde{\mu}_{-\sigma}+m} (\log \tau)^M (1 + o_\tau(1)),
\end{aligned}$$

$$(2.29) \quad p_{j\sigma}^m(\tau, \tau, x, \xi) = C \tau^{j+m} (\log \tau)^M (1 + o_\tau(1)),$$

$$(2.30) \quad q_{j\sigma}^m(\tau, \tau, x, \xi) = C \tau^{j+m-1} (\log \tau)^M (1 + o_\tau(1))$$

as $t, \tau \rightarrow \infty$ in the sector S . Here

$$(2.31) \quad \tilde{\mu}_\sigma = \frac{1}{4} + \sigma \frac{i}{4} B(x, \xi), \quad \sigma^2 = 1,$$

$B(x, \xi)$ being that of (2.5), and C stands for various functions of x, ξ appropriate to each case, M, N various non-negative integers. $o_t(1)$ and $o_\tau(1)$ denote various functions of t and τ , respectively, in $O^{-1+\varepsilon}(S, \mathbf{R}^n \times \mathbf{S}^{n-1})$, $\varepsilon > 0$.

PROOF. This is proved essentially in the same way as Proposition 2.3 of [2]. The apparition of M, N in the present case is due to the forms of $T_{k,l}^\sigma$ (cf. (1.14), in particular).

In view of the semi-homogeneity requirements on $p_{j\sigma}^m$ and $q_{j\sigma}^m$, we can interpret the above result in terms of the symbol classes in [2]. Thus we obtain

COROLLARY 2.3. For any m and j , we have

$$(2.32) \quad p_{j\sigma}^m(t, \tau, x, \xi) \in S^{\epsilon, 2m(\sigma)+j+\epsilon, 2m(-\sigma)+m+1+\epsilon},$$

$$(2.33) \quad q_{j\sigma}^m(t, \tau, x, \xi) \in S^{\epsilon-1, 2m(\sigma)+j+\epsilon, 2m(-\sigma)+m+\epsilon},$$

$$(2.34) \quad p_{j\sigma}^m(\tau, \tau, x, \xi) \in S^{\epsilon, 2m(\sigma)+2m(-\sigma)+m+j+1+2\epsilon},$$

$$(2.35) \quad q_{j\sigma}^m(\tau, \tau, x, \xi) \in S^{\epsilon-1, 2m(\sigma)+2m(-\sigma)+m+j+2\epsilon},$$

$\sigma^2=1$. Here ϵ is an arbitrary positive number and

$$m(\sigma) = -\frac{1}{4} + \frac{1}{4} \sup \{ \sigma \operatorname{Im} \sum_{j=1}^n b_j(0, x) \xi_j / a(0, x, \xi) \}$$

as defined by (0.8).

Now let us apply Proposition 1.8 and Corollary 1.9 of [2] to $p_{j\sigma}^m$ and $q_{j\sigma}^m$. First we have

$$(2.36) \quad p_{j\sigma}^m(t, \tau, x, \xi) \in S^{\epsilon, 2m(\sigma)+\epsilon, 2m(-\sigma)+m+1+\epsilon}$$

and

$$(2.37) \quad q_{j\sigma}^m(t, \tau, x, \xi) \in S^{\epsilon-1, 2m(\sigma)+\epsilon, 2m(-\sigma)+m+\epsilon},$$

$\sigma^2=1$, such that, for all $N>0$,

$$(2.38) \quad p_{j\sigma}^m - \sum_{j<N} p_{j\sigma}^m \in S^{\epsilon, 2m(\sigma)+N+\epsilon, 2m(-\sigma)+m+1+\epsilon}$$

and

$$(2.39) \quad q_{j\sigma}^m - \sum_{j<N} q_{j\sigma}^m \in S^{\epsilon-1, 2m(\sigma)+N+\epsilon, 2m(-\sigma)+m+\epsilon}.$$

We then have

$$(2.40) \quad p_{\sigma}(t, \tau, x, \xi) \in S^{\epsilon, 2m(\sigma)+\epsilon, 2m(-\sigma)+1+\epsilon}$$

and

$$(2.41) \quad q_{\sigma}(t, \tau, x, \xi) \in S^{\epsilon-1, 2m(\sigma)+\epsilon, 2m(-\sigma)+\epsilon},$$

$\sigma^2=1$, such that, for all $N>0$,

$$(2.42) \quad p_{\sigma} - \sum_{m<N} p_{\sigma}^m \in S^{\epsilon, 2m(\sigma)+\epsilon, 2m(-\sigma)+N+1+\epsilon}$$

and

$$(2.43) \quad q_{\sigma} - \sum_{m<N} q_{\sigma}^m \in S^{\epsilon-1, 2m(\sigma)+\epsilon, 2m(-\sigma)+N+\epsilon}.$$

Let us set

$$(2.44) \quad p_{00}(\tau, x, \xi) = \sum_{\sigma=\pm 1} p_{\sigma}(\tau, \tau, x, \xi) - 1,$$

$$(2.45) \quad q_{00}(\tau, x, \xi) = \sum_{\sigma=\pm 1} q_{\sigma}(\tau, \tau, x, \xi),$$

$$(2.46) \quad p_{01}(\tau, x, \xi) = \sum_{\sigma=\pm 1} \{ \phi_{\tau}^{\sigma}(t, \tau, x, \xi) + D_{\tau} \} p_{\sigma}(t, \tau, x, \xi) |_{t=\tau},$$

$$(2.47) \quad q_{01}(\tau, x, \xi) = \sum_{\sigma=\pm 1} \left\{ \phi_t^\sigma(t, \tau, x, \xi) + D_t \right\} q_\sigma(t, \tau, x, \xi)|_{t=\tau} - 1.$$

Then we have the following

PROPOSITION 2.4

$$(2.48) \quad p_{00}(\tau, x, \xi) \in S^{\epsilon, \infty},$$

$$(2.49) \quad q_{00}(\tau, x, \xi) \in S^{\epsilon-1, \infty},$$

$$(2.50) \quad p_{01}(\tau, x, \xi) \in S^{\epsilon+1, \infty},$$

$$(2.51) \quad q_{01}(\tau, x, \xi) \in S^{\epsilon, \infty}.$$

PROOF. We prove (2.48) and (2.50). (2.49) and (2.51) can be shown analogously. For any $M > 0$,

$$\begin{aligned} p_{00}(\tau, x, \xi) &= \left\{ \sum_{\sigma=\pm 1} \sum_{m=1}^M \sum_{j=0}^M p_{j\sigma}^m(\tau, \tau, x, \xi) - 1 \right\} + \\ &\quad + \sum_{\sigma=\pm 1} \sum_{m=0}^M \left\{ p_\sigma^m(\tau, \tau, x, \xi) - \sum_{j=0}^M p_{j\sigma}^m(\tau, \tau, x, \xi) \right\} + \\ &\quad + \sum_{\sigma=\pm 1} \left\{ p_\sigma(\tau, \tau, x, \xi) - \sum_{m=0}^M p_\sigma^m(\tau, \tau, x, \xi) \right\}. \end{aligned}$$

The last two sums belong to $S^{\epsilon, 2m(\sigma)+2m(-\sigma)+M+2+2\epsilon}$ by virtue of (2.38) and (2.42). Now by (2.11) and (2.15), the first term reduces to

$$\sum_{\sigma=\pm 1} \sum_{m+j > M} p_{j\sigma}^m(\tau, \tau, x, \xi),$$

which also belongs to $S^{\epsilon, 2m(\sigma)+2m(-\sigma)+M+2+2\epsilon}$ because of (2.32). This proves (2.48). On the other hand, for any $M > 0$,

$$\begin{aligned} p_{01}(\tau, x, \xi) &= \\ &= \sum_{\sigma=\pm 1} \left\{ \phi_t^\sigma(t, \tau, x, \xi) + D_t \right\} (p_\sigma - \sum_{m=0}^M p_\sigma^m)|_{t=\tau} + \\ &\quad + \sum_{\sigma=\pm 1} \left\{ \phi_t^\sigma(t, \tau, x, \xi) + D_t \right\} \sum_{m=0}^M (p_\sigma^m - \sum_{j=0}^M p_{j\sigma}^m)|_{t=\tau} + \\ &\quad + \sum_{\sigma=\pm 1} \left\{ \phi_t^\sigma(t, \tau, x, \xi) + D_t \right\} \sum_{j, m=0}^M p_{j\sigma}^m|_{t=\tau}. \end{aligned}$$

The first two sums belong to $S^{\epsilon+1, 2m(\sigma)+2m(-\sigma)+M+1+2\epsilon}$ in view of (2.38) and (2.42). Now by the Taylor expansion,

$$\begin{aligned} \phi_t^\sigma(\tau, \tau, x, \xi) &= \sigma \tau a(0, x, \xi) + \sigma \sum_{k=1}^M \tau^{k+1} |\xi| c_k^\sigma(x, \xi) + \\ &\quad + \sigma \tau^{M+2} |\xi| c_{M+1}^\sigma(\tau, x, \xi), \end{aligned}$$

$c_{M+1}^\sigma(\tau, x, \xi)$ being homogeneous of degree zero. Then by (2.12) and (2.16),

$$\begin{aligned} \sum_{\sigma=\pm 1} \left\{ \phi_t^\sigma(t, \tau, x, \xi) + D_t \right\} \sum_{j, m=0}^M p_{j\sigma}^m|_{t=\tau} &= \\ &= \sum_{\sigma=\pm 1} \sigma \tau^{M+2} |\xi| c_{M+1}^\sigma(\tau, x, \xi) \sum_{j, m=0}^M p_{j\sigma}^m|_{t=\tau} + \end{aligned}$$

$$+ \sum_{\sigma=\pm 1} \left\{ \sigma t a(0, x, \xi) + D_t \right\} \sum_{j+m > M} p_{j\sigma}^m |_{t=\tau} + \\ + \sum_{\sigma=\pm 1} \sigma \sum_{k=1}^M \tau^{k+1} |\xi| c_k^\sigma(x, \xi) \sum_{j+m+k > M} p_{j\sigma}^m |_{t=\tau}$$

belongs to $S^{\epsilon+1, 2m(\sigma)+m(-\sigma)+M+1+2\epsilon}$. This proves (2.50).

Let us set, for $\sigma^2=1$,

$$(2.52) \quad r_\sigma(t, \tau, x, \xi) = T^\sigma p_\sigma(t, \tau, x, \xi)$$

and

$$(2.53) \quad s_\sigma(t, \tau, x, \xi) = T^\sigma q_\sigma(t, \tau, x, \xi).$$

Then we have the following proposition, which has no analogue in our previous discussions [2].

PROPOSITION 2.5. For $\sigma^2=1$,

$$(2.54) \quad r_\sigma(t, \tau, x, \xi) \in S^{\epsilon+1, 2m(\sigma)+\epsilon, 2m(-\sigma)+1+\epsilon}$$

and

$$(2.55) \quad s_\sigma(t, \tau, x, \xi) \in S^{\epsilon, 2m(\sigma)+\epsilon, 2m(-\sigma)+\epsilon}.$$

Furthermore, for any $N, J > 0$, there exist $r_{\sigma, N, J}(t, \tau, x, \xi)$ and $s_{\sigma, N, J}(t, \tau, x, \xi)$ such that

$$(2.56) \quad r_{\sigma, N, J}(t, \tau, x, \xi) \in S^{\epsilon+1, J+1, 2m(-\sigma)+1+\epsilon},$$

$$(2.57) \quad r_\sigma - r_{\sigma, N, J} \in S^{\epsilon+1, 2m(\sigma)+\epsilon, 2m(-\sigma)+N+2+\epsilon}$$

and

$$(2.58) \quad s_{\sigma, N, J}(t, \tau, x, \xi) \in S^{\epsilon, J+1, 2m(-\sigma)+\epsilon},$$

$$(2.59) \quad s_\sigma - s_{\sigma, N, J} \in S^{\epsilon, 2m(\sigma)+\epsilon, 2m(-\sigma)+N+1+\epsilon}.$$

PROOF. (2.54) and (2.55) follow from (2.40) and (2.41) in view of Propositions 1.5, 1.6 of [2] and Proposition 1.3. We prove (2.56) and (2.57). (2.58) and (2.59) are shown analogously.

Let

$$p_{\sigma, N} = \sum_{m=0}^N p_\sigma^m.$$

Then by (2.42) and Propositions 1.5, 1.6 of [2],

$$(2.60) \quad T^\sigma(p_\sigma - p_{\sigma, N}) \in S^{\epsilon+1, 2m(\sigma)+\epsilon, 2m(-\sigma)+2+N+\epsilon}.$$

Let us show

$$(2.61) \quad T^\sigma p_{\sigma, N} \in S^{\epsilon+1, \infty, 2m(-\sigma)+1+\epsilon}$$

modulo $S^{\epsilon+1, 2m(\sigma)+\epsilon, 2m(-\sigma)+2+N+\epsilon}$.

Let $J > 0$ and

$$T^\sigma = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \tau^l \hat{T}_{k,l}^\sigma.$$

Here $\hat{T}_{k,l}^\sigma = T_{k,l}^\sigma$ for $l=0, \dots, k$; $k=0, \dots, j+2N+1$ and $\hat{T}_{k,l}^\sigma = 0$ for $k \geq J+2N+3$. $\hat{T}_{j+2N+2,l}^\sigma$, $l=0, \dots, J+2N+2$, correspond to the remainder terms in the Taylor expansion of T^σ so that

$$(2.62) \quad \hat{T}_{J+2N+2,0}^\sigma = t^{J+2N+2} |\xi| C(t, \tau, x, \xi, D_x) + t^{J+2N} C(t, \tau, x, \xi, tD_t, D_x)$$

and

$$(2.63) \quad \hat{T}_{J+2N+2,l}^\sigma = t^{J+2N+2-l} |\xi| C(t, \tau, x, \xi, tD_t, D_x),$$

$l=1, \dots, J+2N+2$, $C(t, \tau, x, \xi, tD_t, D_x)$ being suitable partial differential operators in tD_t, D_x with coefficients homogeneous of degree zero. Let

$$\hat{p}_{j\sigma}^m = p_{j\sigma}^m \text{ for } j=0, \dots, J+m; m=0, \dots, N,$$

$$\hat{p}_{j\sigma}^m = 0 \text{ for } j > J+m+1 \text{ if } m \leq N \text{ or for any } j \text{ if } m \geq N+1,$$

and

$$\hat{p}_{j+m+1\sigma}^m = p_\sigma^m - \sum_{j < J+m+1} p_{j\sigma}^m, \quad m=0, \dots, N.$$

Then $p_{\sigma,N} = \sum_{m=0}^N \sum_{j=0}^{\infty} \hat{p}_{j\sigma}^m$ and

$$\begin{aligned} T^\sigma p_{\sigma,N} &= \sum_{m=0}^{\infty} \sum_{j=m}^{\infty} \sum_{l=0}^m \sum_{k=0}^{j-m} \tau^l \hat{T}_{j-m-k+l,l}^\sigma \hat{p}_{k\sigma}^{m-l} = \\ &= \sum_{m=0}^N \sum_{j=m}^{J+m} \sum_{l=0}^m \sum_{k=0}^{j-m} \tau^l \hat{T}_{j-m-k+l,l}^\sigma \hat{p}_{k\sigma}^{m-l} + \\ &\quad + \sum_{m=0}^N \sum_{j=J+m+1}^{\infty} \sum_{l=0}^m \sum_{k=0}^{j-m} \tau^l \hat{T}_{j-m-k+l,l}^\sigma \hat{p}_{k\sigma}^{m-l} + \\ &\quad + \sum_{m=N+1}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^m \sum_{k=0}^{j-m} \tau^l \hat{T}_{j-m-k+l,l}^\sigma \hat{p}_{k\sigma}^{m-l}. \end{aligned}$$

The first sum vanishes by virtue of (2.10), (2.13), (2.14) and (2.18). The second sum belongs to $S^{1+\varepsilon, 2m(\sigma)+J+1+\varepsilon, 2m(-\sigma)+1+\varepsilon}$ since

$$\tau^l \hat{T}_{j-m-k+l,l}^\sigma \hat{p}_{k\sigma}^{m-l} \in S^{1+\varepsilon, 2m(\sigma)+j-m+\varepsilon, 2m(-\sigma)+m+1+\varepsilon}$$

by means of (1.12), (1.13), (1.14), (2.32), (2.38), (2.62) and (2.63). Similarly, the last sum, which is finite, belongs to $S^{1+\varepsilon, 2m(\sigma)+\varepsilon, 2m(-\sigma)+N+2+\varepsilon}$. Therefore, for any J ,

$$T^\sigma p_{\sigma,N} \in S^{1+\varepsilon, 2m(\sigma)+J+1+\varepsilon, 2m(-\sigma)+1+\varepsilon}$$

modulo a term in $S^{1+\varepsilon, 2m(\sigma)+\varepsilon, 2m(-\sigma)+N+2+\varepsilon}$. This shows (2.61). (2.56) and (2.57) now result from (2.60) and (2.61) by setting

$$r_{\sigma,N,J} = T^\sigma p_{\sigma,N} - \sum_{m=N+1}^{\infty} \sum_{j=J+m+1}^{\infty} \sum_{l=0}^m \sum_{k=0}^{j-m} \tau^l \hat{T}_{j-m-k+l,l}^\sigma \hat{p}_{k\sigma}^{m-l}.$$

From (2.56), (2.57) and (2.58) (2.59), we have the following

COROLLARY 2.6. *There exist symbols $r_\sigma^0(t, \tau, x, \xi)$, $r_\sigma^1(t, \tau, x, \xi)$, $r_\sigma^2(t, \tau, x, \xi)$*

and $s_\sigma^0(t, \tau, x, \xi)$, $s_\sigma^1(t, \tau, x, \xi)$, $s_\sigma^2(t, \tau, x, \xi)$, $\sigma^2=1$, such that

$$(2.64) \quad r_\sigma(t, \tau, x, \xi) = \sum_{j=0}^2 r_\sigma^j(t, \tau, x, \xi),$$

$$(2.65) \quad s_\sigma(t, \tau, x, \xi) = \sum_{j=0}^2 s_\sigma^j(t, \tau, x, \xi)$$

with

$$(2.66) \quad r_\sigma^0(t, \tau, x, \xi) \in S^{\epsilon+1, \infty, 2m(-\sigma)+1+\epsilon},$$

$$(2.67) \quad s_\sigma^0(t, \tau, x, \xi) \in S^{\epsilon, \infty, 2m(-\sigma)+1+\epsilon},$$

$$(2.68) \quad r_\sigma^1(t, \tau, x, \xi) \in S_\infty^{1+\epsilon},$$

$$(2.69) \quad s_\sigma^1(t, \tau, x, \xi) \in S_\infty^\epsilon,$$

$$(2.70) \quad r_\sigma^2(t, \tau, x, \xi), s_\sigma^2(t, \tau, x, \xi) \in S_+^{-\infty}.$$

PROOF. This follows immediately from the definitions of the symbol classes. In fact, $S^{\mu, \epsilon, \infty} \subset S_\infty^\mu$.

§ 3. End of the proof of Theorem.

In view of (2.66) and (2.67) we may assume that

$$(3.1) \quad r_\sigma^0(t, \tau, x, \xi) \in S_+^{\epsilon+1, 2m(-\sigma)+1+\epsilon}, \quad \sigma^2=1,$$

$$(3.2) \quad s_\sigma^0(t, \tau, x, \xi) \in S_+^{\epsilon, 2m(-\sigma)+\epsilon}, \quad \sigma^2=1,$$

and

$$(3.3) \quad r_\sigma^0(t, \tau, x, \xi) \text{ and } s_\sigma^0(t, \tau, x, \xi) \text{ be flat at } t=0.$$

Then we have, just as before [1] [2], the following

PROPOSITION 3.1. *There exist symbols*

$$(3.4) \quad \tilde{p}_\sigma(t, \tau, x, \xi) \in S_+^{\epsilon, 2m(-\sigma)+1+\epsilon},$$

$$(3.5) \quad \tilde{q}_\sigma(t, \tau, x, \xi) \in S_+^{\epsilon-1, 2m(-\sigma)+\epsilon},$$

$\sigma^2=1$, such that

$$(3.6) \quad T^\sigma \tilde{p}_\sigma(t, \tau, x, \xi) + r_\sigma^0(t, \tau, x, \xi) \in S_+^{-\infty},$$

$$(3.7) \quad T^\sigma \tilde{q}_\sigma(t, \tau, x, \xi) + s_\sigma^0(t, \tau, x, \xi) \in S_+^{-\infty},$$

$$(3.8) \quad \tilde{p}_\sigma(t, \tau, x, \xi) \text{ and } \tilde{q}_\sigma(t, \tau, x, \xi) \text{ are flat at } t=0.$$

PROOF. This can be shown in the same way as § 3 of [2] in view of Proposition 1.3.

Let us set

$$(3.9) \quad \tilde{p}_{00}(\tau, x, \xi) = \sum_{\sigma=\pm 1} \tilde{p}_{\sigma}(\tau, \tau, x, \xi),$$

$$(3.10) \quad \tilde{q}_{00}(\tau, x, \xi) = \sum_{\sigma=\pm 1} \tilde{q}_{\sigma}(\tau, \tau, x, \xi),$$

$$(3.11) \quad \tilde{p}_{01}(\tau, x, \xi) = \sum_{\sigma=\pm 1} \left\{ \phi_{\sigma}^{\sigma}(t, \tau, x, \xi) + D_t \right\} \tilde{p}_{\sigma}(t, \tau, x, \xi)|_{t=\tau},$$

$$(3.12) \quad \tilde{q}_{01}(\tau, x, \xi) = \sum_{\sigma=\pm 1} \left\{ \phi_{\sigma}^{\sigma}(t, \tau, x, \xi) + D_t \right\} \tilde{q}_{\sigma}(t, \tau, x, \xi)|_{t=\tau}.$$

Then Proposition 3.1 immediately implies the following

COROLLARY 3.2.

$$(3.13) \quad \tilde{p}_{00}(\tau, x, \xi) \in S^{s, \infty},$$

$$(3.14) \quad \tilde{q}_{00}(\tau, x, \xi) \in S^{s-1, \infty},$$

$$(3.15) \quad \tilde{p}_{01}(\tau, x, \xi) \in S^{s+1, \infty},$$

$$(3.16) \quad \tilde{q}_{01}(\tau, x, \xi) \in S^{s, \infty}.$$

Now set

$$(3.17) \quad p_0(\tau, x, \xi) = -p_{00}(\tau, x, \xi) - \tilde{p}_{00}(\tau, x, \xi),$$

$$(3.18) \quad q_0(\tau, x, \xi) = -q_{00}(\tau, x, \xi) - \tilde{q}_{00}(\tau, x, \xi),$$

$$(3.19) \quad p_1(\tau, x, \xi) = -p_{01}(\tau, x, \xi) - \tilde{p}_{01}(\tau, x, \xi),$$

$$(3.20) \quad q_1(\tau, x, \xi) = -q_{01}(\tau, x, \xi) - \tilde{q}_{01}(\tau, x, \xi).$$

The following result is then immediate from Proposition 2.4 and Corollary 3.2.

COROLLARY 3.3.

$$(3.21) \quad p_0(\tau, x, \xi) \in S^{s, \infty},$$

$$(3.22) \quad q_0(\tau, x, \xi) \in S^{s-1, \infty},$$

$$(3.23) \quad p_1(\tau, x, \xi) \in S^{s+1, \infty},$$

$$(3.24) \quad q_1(\tau, x, \xi) \in S^{s, \infty}.$$

Then just in the same way as the proofs of Propositions 4.3 and 4.4 of [2], we have the following

PROPOSITION 3.4. *There exist symbols*

$$(3.25) \quad p_{\sigma}^{\#}(t, \tau, x, \xi) \in S_{\infty}^s$$

and

$$(3.26) \quad q_{\sigma}^{\#}(t, \tau, x, \xi) \in S_{\infty}^{s-1},$$

$\sigma^2=1$, such that

$$(3.27) \quad T^\sigma p_\sigma^\#(t, \tau, x, \xi) + r_\sigma^1(t, \tau, x, \xi) \in S_\infty^{-\infty},$$

$$(3.28) \quad T^\sigma q_\sigma^\#(t, \tau, x, \xi) + s_\sigma^1(t, \tau, x, \xi) \in S_\infty^{-\infty}$$

and

$$(3.29) \quad \sum_{\sigma=\pm 1} p_\sigma^\#(\tau, \tau, x, \xi) - p_0(\tau, x, \xi) \in S^{-\infty},$$

$$(3.30) \quad \sum_{\sigma=\pm 1} \left\{ \phi_i^\sigma(t, \tau, x, \xi) + D_i \right\} p_\sigma^\#(t, \tau, x, \xi) \Big|_{t=\tau} - p_1(\tau, x, \xi) \in S^{-\infty},$$

$$(3.31) \quad \sum_{\sigma=\pm 1} q_\sigma^\#(\tau, \tau, x, \xi) - q_0(\tau, x, \xi) \in S^{-\infty},$$

$$(3.32) \quad \sum_{\sigma=\pm 1} \left\{ \phi_i^\sigma(t, \tau, x, \xi) + D_i \right\} q_\sigma^\#(t, \tau, x, \xi) \Big|_{t=\tau} - q_1(\tau, x, \xi) \in S^{-\infty}.$$

The proof of Theorem is now complete.

References

- [1] YOSHIKAWA, A.: Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation I, Hokkaido Math. J., 6 (1977), 313-344.
- [2] YOSHIKAWA, A.: Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation II, Hokkaido Math. J., 7 (1978), 1-26.

Department of Mathematics
Hokkaido University