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On maximal hyperbolic sets

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In this paper we study structures of hyperbolic sets which are maximal invariant sets in their sufficiently small neighbourhoods. One of their important examples is an Anosov diffeomorphism (not necessarily assumed to be topologically transitive).

First we give some definitions.

DEFINITION. Suppose U is an open set of a manifold M, $f: U\rightarrow M$ is a diffeomorphism onto an open set of M. $\Lambda\subset U$ is called a hyperbolic set if Λ is a compact invariant set which satisfies the followings; $T_{\Lambda}M$ splits into a Whitney sum of Tf-invariant subbundles

$$T_{\Lambda}M=E^{s}\oplus E^{u}$$

such that there are $c>o$ and $o\leqq\lambda<1$ with

$$\|Tf^{n}v\|\leqq c\lambda^{n}\|v\| \qquad \text{if } v\in E^{s}$$

$$\|Tf^{-n}v\|\leqq c\lambda^{n}\|v\| \qquad \text{if } v\in E^{u}$$

for $n\geqq 0$.

Definition. Let $\mathscr{A}=\{A_{1}, A_{2}, \cdots A_{n}\}$ be a finite set, and $\mathscr{A}^{Z}$ be a space of functions from Z to $\mathscr{A}$ with the compact-open topology. (Here we assume $\mathscr{A}$ and Z have the discrete topologies.) Let

$$\rho:\mathscr{A}^{Z}\rightarrow \mathscr{A}^{Z}$$

be given by $\rho((a_{i})_{i\in Z})=(b_{i})_{i\in Z}$ where $b_{i}=a_{i+1}$. Let $T=(t_{ij})_{i,j=,\cdots,n}$ be a $n\times n$ 0--1 matrix. The ρ-invariant set

$$\Sigma=\left\{(a_{i})_{i\in Z}\in \mathscr{A}^{Z}|t_{n_{i}n_{i+1}}=1 \text{ were } a_{i}=A_{n_{i}}\right\}$$

is called a subshift of finite tyoe (on symbols $\mathscr{A}$ determined by T). ρ is called a shift transformation.

DEFINITION. A hyperbolic set Λ has a local product structure if there is a positive number δ such that for any $x\in\Lambda$

$$\phi: W_{\delta}^{s}(x:\Lambda)\times W_{\delta}^{u}(x:\Lambda)\longrightarrow\Lambda$$

is a homeomorphism onto a neighbourhood of x in Λ. Here ϕ is given by $\phi(y, z)=W_{2\delta}^{u}(y:\Lambda)\cap W_{2\delta}^{s}(z:\Lambda)$.

DEFINITION. A closed set $D\subset W_\delta^s(\Lambda)$ is called a proper fundamental domain for $W_\delta^s(\Lambda)$ if $\bigcup_{n\geqq 0} f^n(D)\supset W_\delta^s(\Lambda)-\Lambda$ and $D\cap\Lambda=\phi$. A proper fundamental domain for $W_\delta^u(\Lambda)$ is defined similarly. Where $W_\delta^s(\Lambda)=\bigcup_{x\in\Lambda} W_\delta^s(x)$.

We define for $\delta>o$ and $x\in\Lambda$,

$$W_\delta^s(x:\Lambda)=W_\delta^s(x)\cap\Lambda$$

and for $B\subset\Lambda$

$$W_\delta^s(B;\Lambda)=\bigcup_{x\in B} W_\delta^s(x:\Lambda)\,.$$

Similarly we define $W_\delta^u(x:\Lambda)$ and $W_\delta^u(B:\Lambda)$. $int_\Lambda B$ denotes the interior of B in Λ. $Per(f)$ is the set of periodic points of f.

In [3] we proved the following.

THEOREM 0. *Suppose U is a neighbourhood of a hyperbolic set Λ. Then there is a hyperbolic set Λ' with $\Lambda\subset\Lambda'\subset U$ satisfying the followings. There are a subshift of finite type Σ and a surjection $\pi:\Sigma\to\Lambda'$ with $f\pi=\pi\rho$. Here ρ denotes a shift transformation. (π is called a semi-conjugacy from Σ to Λ'.)*

Theorem 0 implies

COROLLARY. *Suppose Λ is a hyperbolic set which is a maximal invariant set in its sufficiently small neighbourhood. Then there are a subshift of finite type Σ and a semi-conjugacy $\pi:\Sigma\to\Lambda$.*

LEMMA 1. *Suppose Λ is a hyperbolic set. Then the followings are equivalent.*

(1) *Λ is a maximal invariant set in its sufficiantly small neighbourhood U.*

(2) *Λ has a local product structure.*

(3) *Λ has a proper fundamental domain.*

PROOF. Suppose Λ satisfies (1). Let $\delta>0$ be sufficiently small such that $2\delta<d(\Lambda, M-U)$ and $W_\delta^s(x)\subset W_\delta^u(y)=$one point for any x, $y\in\Lambda$. It is sufficient to prove $W_\delta^s(x)\cap W_\delta^u(y)=\{z\}\subset\Lambda$. Because $d(f^n(x), f^n(z))<\delta$ and $d(f^{-n}(y), f^{-n}(z))<\delta$ for $n\geqq 0$, we have $cl\bigcup_{n\in Z} f^n(z)\subset U$. Because Λ is a maximal invariant set in U, we have $cl\bigcup_{n\in Z} f^n(z)\subset\Lambda$. Thus $z\in\Lambda$. We proved that (1) implies (2). (2) implies (3) ([5]), and (3) implies (1) (cf. [4]).

LEMMA 2. *Let Λ be a hyperbolic set. Suppose there are a subshift of finite type Σ, and a semi-conjugacy $\pi:\Sigma\to\Lambda$. Then there are periodic points $x_1,\cdots,x_l\in\Lambda$ such that*

$$\Lambda=\bigcup_{i=1}^{l} clW^s(x_i:\Lambda)\,.$$

REMARK. In the above we may assume $clW^s(x_i:\Lambda)$ $(i=1,\cdots,l)$ contains interior points in Λ.

PROOF. Foe any $(a_i)\in\Sigma$ and a positive integer m, there are integers n and $n+k$ such that

$$a_n=a_{n+k}$$
$$m\leqq n<n+k\leqq m+N+1\,,$$

here N is a number of symbols of Σ. Then there is an element $(b_i)\in\Sigma$ defined by

$$b_i=a_i \qquad \text{for } i\leqq n+k$$
$$b_{n+ik+j}=a_{n+j} \quad \text{for } i\geqq 1\,, \quad 0\leqq j\leqq k-1.$$

So $\pi((b_i))\in W^s(\pi(c))$, where $c\in\Sigma$ is the periodic point with period k determind by the segment $(a_n,\cdots,a_{n+k})$. Because the number of periodic points with period less than $N+1$ is finite, the lemma is proved.

THEOREM 1. *Suppose Λ is a hyperbolic set which is a maximal invariant set in its sufficiently small neighbourhood. Then*
(1) *there are closed invariant subsets $\Omega_1,\cdots,\Omega_n\subset\Lambda$ such that $\Omega_i=clPer(f|\Omega_i)$, $\Lambda=\bigcup_{i=1}^{n} clW^s(\Omega_i:\Lambda)$, $W^s(\Omega_i:\Lambda)$ is an open set in Λ, and $W^u(\Omega_i:\Lambda)=\Omega_i$ for $i=1,\cdots,n$. Moreover $W^s(\Omega_i:\Lambda)\cap W^s(\Omega_j:\Lambda)=\phi$ if $i\neq j$.*
(2) *There are closed invariant subsets $\Omega'_1,\cdots,\Omega'_m\subset\Lambda$ such that $\Omega'_i=clPer(f|\Omega'_i)$, $\Lambda=\bigcup_{i=1}^{m} clW^u(\Omega'_i:\Lambda)$, $W^u(\Omega'_i:\Lambda)$ is an open set in Λ and $W^s(\Omega'_i:\Lambda)=\Omega'_i$ for $i=1,\cdots,m$. Moreover $W^u(\Omega'_i:\Lambda)\cap W^u(\Omega'_j:\Lambda)=\phi$ if $i\neq j$.*

PROOF. Suppose that $x_1,\cdots,x_n$ are periodic points given in lemma 2 and $int_\Lambda clW^s(x_i:\Lambda)\neq\phi$ for $i=1,\cdots,n$. Let $x\in W^s(x_i:\Lambda)$ be an interior point of $clW^s(x_i:\Lambda)$ in Λ. Suppose $\delta>0$ is sufficiently small. Then for a positive integer n with $d(f^n(x),\,x_i)\geqq\delta$, the map

$$\phi: W^u_\delta(f^n(x):\Lambda)\longrightarrow W^u_{3\delta}(x_i:\Lambda)$$

given by $\phi(y)=W^s_{3\delta}(y:\Lambda)\cap W^u_{3\delta}(x_i:\Lambda)$ is a homeomorphism onto a neighbourhood of x_i in $W^u_{3\delta}(x_i:\Lambda)$. Therefore

$$W^u(x_i:\Lambda)\subset int_\Lambda clW^s(x_i:\Lambda)\,.$$

If $int_\Lambda clW^s(x_i:\Lambda)\cap int_\Lambda clW^s(x_j:\Lambda)\neq\phi$, it follows that $clW^s(x_i:\Lambda)=clW^s(x_j:\Lambda)$. So we may assume $clW^s(x_i:\Lambda)\cap clW^s(x_j:\Lambda)=\phi$ if $i\neq j$.

Define $\Omega_i = clW^u(x_i : \Lambda)$. Because $W^u(x_i : \Lambda)$ contains dense homoclinic points, we have $clPer(f|\Omega_i) = \Omega_i$.

$W^s(\Omega_i : \Lambda)$ is open in Λ, becasue for any $x \in W^s(\Omega_i : \Lambda)$ there are a positive integer n, a positive number δ and a point $p \in \Omega_i$ such that $W^s_\delta(p : \Lambda) \times W^u_\delta(p : \Lambda)$ is homeomorphic to an open set in Λ which contains $f^n(x)$.

Because we have $clW^s(x_i : \Lambda) \supset \Omega_i$, $clW^s(x_i : \Lambda) \supset W^s(\Omega_i : \Lambda) \supset W^s(x_i : \Lambda)$. Then

$$W^s(\Omega_i : \Lambda) = int_\Lambda clW^s(x_i : \Lambda)$$

and

$$\begin{aligned} \Lambda &= \bigcup_i clW^s(x_i : \Lambda) \\ &= \bigcup_i clW^s(\Omega_i : \Lambda) . \end{aligned}$$

This completes the proof of (1). the proof of (2) is similar.

THEOREM 2. *Suppose Λ is a hyperbolic set which is maximal invariant set in its sufficiently small neighbourhood. Let $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_l$ is the minimal decomposition into disjoint closed invariant sets.*

Then for any i $(i=1, \cdots, l)$ Λ_i satisfies one of the followings.

(1) *$Per(f|\Lambda_i)$ is dense in Λ_i,*

(2) *$Per(f|\Lambda_i)$ is nowhere dense in Λ_i.*

PROOF. Let $\Omega_i, \cdots, \Omega_n$, and $\Omega'_l, \cdots, \Omega'_m$ be given in theorem 1. Let $\Lambda_1, \cdots, \Lambda_l$ be the minimal decomposition into disjoint closed sets such that for any i $\Lambda_i \cap clW^s(\Omega_j : \Lambda) \neq \phi$ implies $\Lambda_i \supset clW^s(\Omega_j : \Lambda)$ and $\Lambda_i \cap clW^u(\Omega'_j : \Lambda) \neq \phi$ implies $\Lambda_i \supset clW^u(\Omega'_j : \Lambda)$.

If $\Omega_i \cap W^u(\Omega'_j : \Lambda) = \phi$, we have that $clW^s(\Omega_i : \Lambda) = clW^u(\Omega'_j : \Lambda)$ and $clW^s(\Omega_i : \Lambda)$ has dense periodic points. If $clW^s(\Omega_i : \Lambda)$ has dense periodic points and $clW^s(\Omega_i : \Lambda) \cap clW^s(\Omega_j : \Lambda) \neq \phi$, it follows that $clW^s(\Omega_j : \Lambda)$ has dense periodic points.

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