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Codominant dimensions and Morita equivalences

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Introduction

Let ${}_RP$ be a projective left R-module with endomorphism ring S. Let A be a left R-module. We say that P-codominant dimension of A is $\geqq n$, denoted by P-codom. dim. $A\geqq n$, if there exists an exact sequence:

$$X_n\longrightarrow X_{n-1}\longrightarrow\cdots\longrightarrow X_2\longrightarrow X_1\longrightarrow A\longrightarrow 0$$

where X_i's are isomorphic to direct sums of P's.

It is clear that P-codom. dim. $A\geqq 1$ iff P generates A. It is also equivalent with the condition $TA=A$, where T is the trace ideal of ${}_RP$. In this paper it is shown that P-codom. dim. $A\geqq 2$ iff $P\otimes_S \mathrm{Hom}_R(P, A)$ and A are canonically isomorphic. Another some equivalent conditions for this are also obtained in § 2. On the other hand, let ${}_SB$ be a left S-module. Then it is shown that B and $\mathrm{Hom}_R(P, P\otimes_S B)$ are canonically isomorphic iff $\mathrm{Hom}_R(P, Q)$-dom. dim. $B\geqq 2$, where ${}_RQ$ is an injective cogenerator in ${}_R\mathfrak{M}$. Thus we see that the categories $\mathscr{C}_1=\{X\in{}_R\mathfrak{M}|P\text{-codom. dim. } X\geqq 2\}$ and $\mathscr{C}_2=\{Y\in{}_S\mathfrak{M}|\mathrm{Hom}_R(P, Q)\text{-dom. dim. } Y\geqq 2\}$ are (canonically) equivalent. In case where ${}_RP$ is a progenerator in ${}_R\mathfrak{M}$, we have $\mathscr{C}_1={}_R\mathfrak{M}$ and, since ${}_S\mathrm{Hom}_R(P, Q)$ is an injective cogenerator in ${}_S\mathfrak{M}$, $\mathscr{C}_2={}_S\mathfrak{M}$. Thus our result affords a generalization of Morita equivalence. Another variations of an equivalence of this type are also discussed in § 1 and § 4.

Since the trace ideal T of a projective module ${}_RP$ is an idempotent two-sided ideal of R, T induce a torsion theory $(\mathscr{T}, \mathscr{F})$ in the category of left R-modules: $\mathscr{T}=\{X\in{}_R\mathfrak{M}|TX=X$, or equivalently, P-codom. dim. $X\geqq 1\}$, $\mathscr{F}=\{X'\in{}_R\mathfrak{M}|TX'=0\}$. The condition under which $(\mathscr{T}, \mathscr{F})$ is hereditary, that is, $\mathscr{T}$ is closed under submodules were studied recently by some authors ([1], [6]). Here we add some other conditions for this in § 3. Some of them are the followings:

(1) *The class* $\{X\in{}_R\mathfrak{M}|P\text{-codom. dim. } X\geqq 1\}$ *coincides with the class* $\{X'\in{}_R\mathfrak{M}|P\text{-codom. dim. } X'\geqq 2\}$.

(2) $P\otimes_S \mathrm{Hom}_R(P, X)$ *and* TX *are canonically isomorphic for every left*

R-module X.

(3) *The functor* T: ${}_R\mathfrak{M}\rightarrow{}_R\mathfrak{M}$ *(the subfunctor of the identity functor on* ${}_R\mathfrak{M}$*) is exact.*

Finally, in § 5, we shall give some equivalent conditions under which the class $\{X\in{}_R\mathfrak{M}\mid TX=X\}$ is closed under submodules, direct products and injective envelopes.

In what follows we assume that all rings have an identity element and all modules are unital.

§ 1. Some generalizations of Morita equivalences

Let R, S be rings with an identity element. Let ${}_RA$ and ${}_RB$ be two left R-modules. We say that A-codominant dimension of B is $\geqq n$, denoted by A-codom. dim. $B\geqq n$, if there exists an exact sequence:

$$X_n\longrightarrow X_{n-1}\longrightarrow\cdots\longrightarrow X_1\longrightarrow B\longrightarrow 0,$$

where X_i's are isomorphic to direct sums of A's. Dually we say that A-dominant dimension of B is $\geqq n$[1], denoted by A-dom. dim. $B\geqq n$, if there exists an exact sequence:

$$0\longrightarrow B\longrightarrow Y_1\longrightarrow\cdots\longrightarrow Y_{n-1}\longrightarrow Y_n,$$

where Y_j's are isomorphic to direct products of A's. Dominant diemsion was introduced by K. Morita and H. Tachikawa and studied by them and some other authors.

In case ${}_RA_S$ is a two-sided R-S-module, there is a canonical homomorphism $\varepsilon_{A,B}$ of $A\underset{S}{\otimes}\mathrm{Hom}_R(A, B)$ into ${}_RB$ defined by

$$\varepsilon_{A,B}(a\otimes f)=f(a),\ a\in A,\ f\in\mathrm{Hom}_R(A, B).$$

Let ${}_SC$ be a left S-module. There is a canonical homomorphism $\eta_{A,C}$ of ${}_SC$ into $\mathrm{Hom}_R(A, A\underset{S}{\otimes}C)$ defined by

$$\left\{\eta_{A,C}(c)\right\}(a)=a\otimes c,\ c\in C,\ a\in A.$$

As is easily verified we have the following

LEMMA 1. *It holds the following relations*:

(1) $\mathrm{Hom}(1_A, \varepsilon_{A,B})\,\eta_{A,\,\mathrm{Hom}_R(A,\,B)}=1_{\mathrm{Hom}_R(A,B)}$

(2) $\varepsilon_{A,A\underset{S}{\otimes}C}(1_A\otimes\eta_{A,C})=1_{A\underset{S}{\otimes}C}$

LEMMA 2. $\varepsilon_{A,B}$ *is an isomorphism iff* A*-codom. dim.* $B\geqq 2$ *and* Hom

1) Cf. [8].

$(1_A, \varepsilon_{A,B})$ *is an isomorphism.*

PROOF. Assume that $\varepsilon_{A,B}$ is an isomorphism. Then clearly $\mathrm{Hom}(1_A, \varepsilon_{A,B})$ is an isomorphism. Let $\oplus S \to \oplus S \to \mathrm{Hom}_R(A, B) \to 0$ be an exact sequence of left S-modules, where $\oplus S$'s are free left S-modules. Then, by tensoring with A_S, we see that A-codom. dim. $A \otimes_S \mathrm{Hom}_R(A, B) = A$-codom. dim. $B \geqq 2$.

Assume, conversely, that A-codom. dim. $B \geqq 2$ and $\mathrm{Hom}(1_A, \varepsilon_{A,B})$ is an isomorphism. Let $\oplus A \to \oplus A \to B \to 0$ be an exact sequence of left R-modules. Applying to this the functors $\mathrm{Hom}_R(\ , B)$ and $\mathrm{Hom}_R(\ , A \otimes_S \mathrm{Hom}_R(A, B))$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_R(B, B) & \longrightarrow & \mathrm{Hom}_R(A, B) & \longrightarrow & \mathrm{Hom}_R(A, B) \\
& \uparrow {\scriptstyle \mathrm{Hom}(1_B, \varepsilon_{A,B})} & & \uparrow {\scriptstyle \prod \mathrm{Hom}(1_A, \varepsilon_{A,B})} & & \uparrow {\scriptstyle \prod \mathrm{Hom}(1_A, \varepsilon_{A,B})} \\
0 \longrightarrow & \mathrm{Hom}_R(B, A \otimes_S \mathrm{Hom}_R(A, B)) & \longrightarrow & \mathrm{Hom}_R(A, A \otimes_S \mathrm{Hom}_R(A, B)) & \longrightarrow & \mathrm{Hom}_R(A, A \otimes_S \mathrm{Hom}_R(A, B)).
\end{array}$$

Since $\prod \mathrm{Hom}(1_A, \varepsilon_{A,B})$'s are isomorphisms, so is $\mathrm{Hom}(1_B, \varepsilon_{A,B})$. Let $h \in \mathrm{Hom}_R(B, A \otimes_S \mathrm{Hom}_R(A, B))$ be such that $\varepsilon_{A,B} h = 1_B$. Then $\varepsilon_{A,B}$ is an epimorphism and $A \otimes_S \mathrm{Hom}_R(A, B) = h(B) \oplus \mathrm{Ker.}\ \varepsilon_{A,B}$. It follows that $\mathrm{Hom}_R(A, A \otimes_S \mathrm{Hom}(A, B)) = \mathrm{Hom}_R(A, h(B)) \oplus \mathrm{Hom}_R(A, \mathrm{Ker.}\ \varepsilon_{A,B})$. Since $\mathrm{Hom}(1_A, \varepsilon_{A,B})$ is an isomorphism and $\mathrm{Hom}(1_A, \varepsilon_{A,B}) \{\mathrm{Hom}_R(A, \mathrm{Ker.}\ \varepsilon_{A,B})\} = 0$, we have $\mathrm{Hom}_R(A, \mathrm{Ker.}\ \varepsilon_{A,B}) = 0$. If follows that $\mathrm{Ker.}\ \varepsilon_{A,B} = 0$, because $\mathrm{Ker.}\ \varepsilon_{A,B}$ is generated by ${}_RA$. Thus $\varepsilon_{A,B}$ is an isomorphism.

Let ${}_RQ$ be an injective cogenerator in ${}_R\mathfrak{M}$. We denote the left S-module $\mathrm{Hom}_R(A, Q)$ by A^*.

LEMMA 3. $\eta_{A,C}$ *is an isomorphism iff* A^**-dom. dim.* $C \geqq 2$ *and* $1_A \otimes \eta_{A,C}$ *is an isomorphism.*

PROOF. Suppose $\eta_{A,C}$ is an isomorphism. Then clearly $1_A \otimes \eta_{A,C}$ is an isomorphism. Let $0 \to A \otimes_S C \to \prod Q \to \prod Q$ be an exact sequence of left R-modules where $\prod Q$'s are direct products of Q's. Then, by applying functors $\mathrm{Hom}_R(A,\ )$, we have the following exact sequence of left S-modules:

$$0 \longrightarrow \mathrm{Hom}_R(A, A \otimes_S C) \longrightarrow \prod A^* \longrightarrow \prod A^*,$$

which means in turn that A^*-dom. dim. $C \geqq 2$.

Suppose, conversely, A^*-dom. dim. $C \geqq 2$ and $1_A \otimes \eta_{A,C}$ is an isomorphism. At first we note that $\mathrm{Hom}(\eta_{A,C}, 1_{A^*})$ is an isomorphism because it is the composition of the following isomorphisms: $\mathrm{Hom}_S(\mathrm{Hom}_R(A, A \otimes_S C),$

$A^*)\xrightarrow{\text{can.}}\mathrm{Hom}_R(A\underset{S}{\otimes}\mathrm{Hom}_R(A, A\underset{S}{\otimes}C), Q)\xrightarrow{\mathrm{Hom}(1_A\otimes\eta_{A,C}, 1_Q)}\mathrm{Hom}_R(A\underset{S}{\otimes}C, Q)\xrightarrow{\text{can.}}$ $\mathrm{Hom}_S(C, A^*)$. Let $0\to C\to\prod A^*\to\prod A^*$ be an exact sequence of left S-modules. By applying to this the functors $\mathrm{Hom}_S(C,\ )$ and $\mathrm{Hom}_S(\mathrm{Hom}_R(A, A\underset{S}{\otimes}C),\ )$ we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_S(C, C) & \longrightarrow & \mathrm{Hom}_S(C, A^*) & \longrightarrow & \mathrm{Hom}_S(C, A^*) \\ & \uparrow{\scriptstyle \mathrm{Hom}(\eta_{A,C}, 1_C)} & & \uparrow{\scriptstyle \prod\mathrm{Hom}(\eta_{A,C}, 1_{A^*})} & & \uparrow{\scriptstyle \prod\mathrm{Hom}(\eta_{A,C}, 1_{A^*})} \\ 0 \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(A, A\underset{S}{\otimes}C), C) & \rightarrow & \prod\mathrm{Hom}_S(\mathrm{Hom}_R(A, A\underset{S}{\otimes}C), A^*) & \rightarrow & \prod\mathrm{Hom}_S(\mathrm{Hom}_R(A, A\underset{S}{\otimes}C), A^*). \end{array}$$

Since $\prod\mathrm{Hom}(\eta_{A,C}, 1_{A^*})$'s are isomorphisms, so is $\mathrm{Hom}(\eta_{A,C}, 1_C)$. Let $h\in\mathrm{Hom}_S(\mathrm{Hom}_R(A, A\underset{S}{\otimes}C), C)$ be such that $h\,\eta_{A,C}=1_C$. Then $\eta_{A,C}$ is a monomorphism and we have $\mathrm{Hom}_R(A, A\underset{S}{\otimes}C)=\eta_{A,C}(C)\oplus\mathrm{Ker.}\ h$. It follows that $\mathrm{Hom}_S(\mathrm{Hom}_R(A, A\underset{S}{\otimes}C), A^*)=\mathrm{Hom}_S(\eta_{A,C}(C), A^*)\oplus\mathrm{Hom}_S(\mathrm{Ker.}\ h, A^*)$. *Since* $\mathrm{Hom}(\eta_{A,C}, 1_{A^*})$ is an isomorphism and $\mathrm{Hom}(\eta_{A,C}, 1_{A^*})\{\mathrm{Hom}_S(\mathrm{Ker.}\ h, A^*)\}=0$, $\mathrm{Hom}_S(\mathrm{Ker.}\ h, A^*)=0$. This implies that $\mathrm{Ker.}\ h=0$ because $\mathrm{Ker.}\ h$ is cogenerated by A^*. Thus $\eta_{A,C}$ is an epimorphism, whence an isomorphism.

From Lemma 2 and Lemma 3 we have the following

THEOREM 1. *Let ${}_RA_S$ be a two-sided R-S-module. Then there is a category isomorphism between the class* $\mathscr{C}=\{X\in{}_R\mathfrak{M}\mid A$*-codom. dim.* $X\geqq 2$ *and* $\mathrm{Hom}(1_A, \varepsilon_{A,X})$ *is an isomorphism*$\}$ *and the class* $\mathscr{D}=\{Y\in{}_S\mathfrak{M}\mid A^*$*-dom. dim.* $Y\geqq 2$ *and* $1_A\otimes\eta_{A,Y}$ *is an isomorphism*$\}$ *which is induced from the equivalent functors*:

$$F:\quad \mathscr{C}\ni X\longrightarrow F(X)=\mathrm{Hom}_R(A, X)\in\mathscr{D}$$
$$G:\quad \mathscr{D}\ni Y\longrightarrow G(Y)=A\underset{S}{\otimes}Y\in\mathscr{C}\,.$$

LEMMA 4. *Let ${}_RA$ be a left R-module with the endomorphism ring S. Let T be the trace ideal of ${}_RA$:* $T=\sum_{g\in\mathrm{Hom}_R(A,R)} g(A)$. *Then* T *Ker.* $\varepsilon_{A,X}=0$ *for every* $X\in{}_R\mathfrak{M}$.

PROOF. Let $\sum_i a_i\otimes f_i\in A\underset{S}{\otimes}\mathrm{Hom}_R(A, X)$ be in $\mathrm{Ker.}\ \varepsilon_{A,X}$. Then for every $g\in\mathrm{Hom}_R(A, R)$ and for every $a\in A$, we have $g(a)\sum_i a_i\otimes f_i=\sum_i a[g, a_i]\otimes f_i=a\otimes\sum_i [g, a_i]\cdot f_i=0$, where $[g, a_i]$'s denote the endomorphisms of ${}_RA$ defined by $x[g, a_i]=g(x)a_i$, $x\in A$. Thus we have $T\,\mathrm{Ker.}\ \varepsilon_{A,X}=0$, as asserted.

COROLLARY. *If* $TA=A$, *then Ker.* $\varepsilon_{A,X}$ *is small in* $A\underset{S}{\otimes}\mathrm{Hom}_R(A, X)$

for every $X \in {}_R\mathfrak{M}$.

Proof. Let $\mathrm{Ker.}\,\varepsilon_{A,X} + \mathfrak{u} = A \underset{S}{\otimes} \mathrm{Hom}_R(A, X)$, where $\mathfrak{u}$ is a submodule of $A \underset{S}{\otimes} \mathrm{Hom}_R(A, X)$. Then we have $\mathfrak{u} \supseteq T\mathfrak{u} = T\ \mathrm{Ker.}\,\varepsilon_{A,X} + T\mathfrak{u} = A \underset{S}{\otimes} \mathrm{Hom}_R(A, X)$. It follows that $\mathfrak{u} = A \underset{S}{\otimes} \mathrm{Hom}_R(A, X)$. *This proves the corollary.*

Theorem 2. *Let ${}_RA$ be a left R-module with the endomorphism ring S, such that $TA = A$. Then the class $\mathscr{C} = \{X \in {}_R\mathfrak{M} \mid A\text{-codom. dim. } X \geqq 2\}$ and the class $\mathscr{D} = \{Y \in {}_S\mathfrak{M} \mid A^*\text{-dom. dim. } Y \geqq 2\}$ are category equivalent in the way described in Theorem* 1.

Proof. By Theorem 1, it suffices to show that $\mathrm{Hom}(1_A, \varepsilon_{A,X})$ and $1_A \otimes \eta_{A,Y}$ are isomorphisms for every $X \in {}_R\mathfrak{M}$ and for every $Y \in {}_S\mathfrak{M}$. By Lemma 1, $\mathrm{Hom}(1_A, \varepsilon_{A,X})$ is an epimorphism. Let $\varphi \in \mathrm{Hom}_R(A, A \underset{S}{\otimes} \mathrm{Hom}_R(A, X))$ be in $\mathrm{Ker.\,Hom}(1_A, \varepsilon_{A,X})$. Then $\varphi(A) \subseteq \mathrm{Ker.}\,\varepsilon_{A,X}$, and, from which we have $\varphi(A) = T\varphi(A) \subseteq T\,\mathrm{Ker.}\,\varepsilon_{A,X} = 0$. It follows that $\varphi = 0$. Thus $\mathrm{Hom}(1_A, \varepsilon_{A,X})$ is a monomorphism, whence an isomorphism. Next, by Lemma 1, $\varepsilon_{A, A \underset{S}{\otimes} Y}$ is an epimorphism. Since $TA = A$, and $\mathrm{Ker.}\,\varepsilon_{A, A \underset{S}{\otimes} Y}$ is a direct summand of $A \underset{S}{\otimes} \mathrm{Hom}_R(A, A \underset{S}{\otimes} Y)$ we have $0 = T\,\mathrm{Ker.}\,\varepsilon_{A, A \underset{S}{\otimes} Y} = \mathrm{Ker.}\,\varepsilon_{A, A \underset{S}{\otimes} Y}$. Thus $\varepsilon_{A, A \underset{S}{\otimes} Y}$, or equivalently, $1_A \otimes \eta_{A,Y}$ is an isomorphism. This proves the theorem.

Remark. *The condition $TA = A$ holds, for example, when ${}_RA$ is a projective module.*

Theorem 3. *Let ${}_RA$ be a finitely generated quasi-projective module with the endomorphism ring S. Then the class $\{X \in {}_R\mathfrak{M} \mid A\text{-codom. dim. } X \geqq 2\}$ and ${}_S\mathfrak{M}$ are equivalent in the way described in Theorem* 1.

Proof. Let ${}_RX$ be a left R-module such that A-codom. dim. $X \geqq 2$. We want to show that $\varepsilon_{A,X}$ is an isomorphism. For this purpose. let $\oplus A \to \oplus A \to X \to 0$ be an exact sequence of left R-modules. Applying to this $\mathrm{Hom}_R(A, \ )$ we have the following exact sequence of left S-modules: $\oplus S \to \oplus S \to \mathrm{Hom}_R(A, X) \to 0$, because ${}_RA$ is finitely generated and quasiprojetive. Then, by applying to this $A \underset{S}{\otimes} -$, we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
\oplus A & \longrightarrow & \oplus A & \longrightarrow & X & \longrightarrow & 0 \\
\alpha \big\uparrow & & \beta \big\uparrow & & \big\uparrow \varepsilon_{A,X} & & \\
\oplus A & \longrightarrow & \oplus A & \longrightarrow & A \underset{S}{\otimes} \mathrm{Hom}_R(A, X) & \longrightarrow & 0,
\end{array}
$$

where α, β are the canonical isomorphisms. It follows that $\varepsilon_{A,X}$ is an isomorphism.

Next, let ${}_SY$ be a left S-module and $\bigoplus S \to \bigoplus S \to Y \to 0$ be an exact sequence of left S-modules. Then, as above, applying $A \underset{S}{\otimes} -$ and then $\mathrm{Hom}_R(A, \ )$ we see that $\eta_{A,Y}$ is an isomorphism. Our theorem is thus proved.

COROLLARY 1. *Let ${}_RA$ be as in Theorem 3 and ${}_RQ$ be an injective cogenerator in ${}_R\mathfrak{M}$. Then $A^* = \mathrm{Hom}_R(A, Q)$ is a cogenerator in ${}_S\mathfrak{M}$.*

PROOF. By Theorem 1 and Theorem 3, A^*-dom. dim. $Y \geqq 2$ for every $Y \in {}_S\mathfrak{M}$. But this implies that A^* is a cogenerator in ${}_S\mathfrak{M}$.

COROLLARY 2. *For a left R-modules X, A-codom. dim. $X \geqq 2$ iff $\varepsilon_{A,X}$ is an isomorphism.*

PROOF. This follows also directly from Theorem 1 and Theorem 3.

COROLLARY 3 (K. Morita). *Let ${}_RA$ be a progenerator (=finitely generated projective and generator) with the endomorphism ring S. Then ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$ are category equivalent in the way described in Theorem* 1.

§ 2. Modules whose codominant dimensions are $\geqq 2$

Let ${}_RP$ be a projective module with the endomorphism ring S. Let T be the trace ideal of ${}_RP$. For a left R-module X, it is clear that P-codom. dim. $X \geqq 1$, that is, X generated by P iff $TX = X$.

THEOREM 4. *For a left R-module X, the following statements are equivalent:*

(1) *P-codom. dim.* $X \geqq 2$

(2) $TX = X$, *and, for every left R-module Y such that* $TY = Y$ *and for every epimorphism f of Y onto X,* $T\,\mathrm{Ker}.f = \mathrm{Ker}.f$

(3) *For every exact sequence* $0 \to A \to B \xrightarrow{g} C \to 0$ *of left R-modules such that* $TA = 0$ *and for every homomorphism h of X into C, there exists a unique homomorphism j of X into B such that* $gj = h$, *or equivalently,* $\mathrm{Hom}(1_X, g)\colon \mathrm{Hom}_R(X, B) \to \mathrm{Hom}_R(X, C)$ *is an isomorphism*

(4) $\varepsilon_{P,X}\colon P \underset{S}{\otimes} \mathrm{Hom}_R(P, X) \to X$ *is an isomorphism*

PROOF. (1)$\Rightarrow$(2). Let P-codom. dim. $X \geqq 2$. Then clearly $TX = X$. Let $\bigoplus P \to \bigoplus P \to X \to 0$ be an exact sequence. Combining this with the exact sequence $0 \to \dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f} \xrightarrow{i} \dfrac{Y}{T\,\mathrm{Ker}.f} \xrightarrow{\nu} \dfrac{\mathrm{Ker}.f}{Y} \to 0$, where i and ν denote the natural injection and epimorphism, respectively, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
0\to\prod \mathrm{Hom}_R\left(P,\dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}\right) & \to & \prod \mathrm{Hom}_R\left(P,\dfrac{Y}{T\,\mathrm{Ker}.f}\right) & \to & \prod \mathrm{Hom}_R\left(P,\dfrac{Y}{\mathrm{Ker}.f}\right)\to 0 \\
\uparrow & & \uparrow & & \uparrow \\
0\to\prod \mathrm{Hom}_R\left(P,\dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}\right) & \to & \prod \mathrm{Hom}_R\left(P,\dfrac{Y}{T\,\mathrm{Ker}.f}\right) & \to & \prod \mathrm{Hom}_R\left(P,\dfrac{Y}{\mathrm{Ker}.f}\right)\to 0 \\
\uparrow & & \uparrow & \mathrm{Hom}\,(1_X,\nu) & \uparrow \\
0\longrightarrow \mathrm{Hom}_R\left(X,\dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}\right) & \longrightarrow & \mathrm{Hom}_R\left(X,\dfrac{Y}{T\,\mathrm{Ker}.f}\right) & \longrightarrow & \mathrm{Hom}_R\left(X,\dfrac{Y}{\mathrm{Ker}.f}\right) \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}$$

Since $TP=P$ and $TX=X$ we have $\mathrm{Hom}_R\left(P,\dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}\right)=\mathrm{Hom}_R\left(X,\dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}\right)=0$. It follows that $\mathrm{Hom}\,(1_X,\nu)$: $\mathrm{Hom}_R\left(X,\dfrac{Y}{T\,\mathrm{Ker}.f}\right)\to\mathrm{Hom}_R\left(X,\dfrac{Y}{\mathrm{Ker}.f}\right)$ is an isomorphism. Let $\bar{f}$ be the induced isomorphism of $\dfrac{Y}{\mathrm{Ker}.f}$ to X and let $g\in\mathrm{Hom}_R\left(X,\dfrac{Y}{T\,\mathrm{Ker}.f}\right)$ be such that $\nu\cdot g=\bar{f}^{-1}$. Then we have $\dfrac{Y}{T\,\mathrm{Ker}.f}=g(X)\oplus\dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}$. But since $TY=Y$ this implies that $\mathrm{Ker}.\,f=T\ \mathrm{Ker}.\,f$.

(2)⇒(1). Assume (2). Then since $TX=X$ there exists an epimorphism f of $\oplus P$ onto X, and, again by assumption, the kernel of f is generated by P. Thus there exists an exact sequence $\oplus P\to\oplus P\to X\to 0$, that is P-codom. dim. $X\geqq 2$.

(1)⇒(3). Assume (1) and let $0\to A\to B\overset{g}{\to}C\to 0$ be an exact sequence such that $TA=0$, or equivalently, $\mathrm{Hom}_R(P,A)=0$. Then, just as in the proof for (1)⇒(2), we see that $\mathrm{Hom}\,(1_X,g):\mathrm{Hom}_R(X,B)\to\mathrm{Hom}_R(X,C)$ is an isomorphism. Thus (3) holds.

(3)⇒(1). Assume (3). Then from the trivial exact sequence $0\to\dfrac{X}{TX}\to\dfrac{X}{TX}\to 0\to 0$, we have that $\mathrm{Hom}\,(1_X,0):\mathrm{Hom}_R\left(X,\dfrac{X}{TX}\right)\to\mathrm{Hom}_R(X,0)\,(=0)$ is an isomorphism. It follows that $X=TX$. Let $f:\oplus P\to X$ be an epimorphism and let $0\to\dfrac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}\overset{\iota}{\to}\dfrac{\oplus P}{T\,\mathrm{Ker}.f}\overset{\nu}{\to}\dfrac{\oplus P}{\mathrm{Ker}.f}\to 0$ be the canonical exact sequence. Let h be an isomorphism of X onto $\dfrac{\oplus P}{\mathrm{Ker}.f}$. Then, by assumption, there is a homomorphism g of X into $\dfrac{\oplus P}{T\,\mathrm{Ker}.f}$ such that $\nu g=h$. It follows that $\frac{\oplus P}{T\,\mathrm{Ker}.f}=g(X)\oplus\frac{\mathrm{Ker}.f}{T\,\mathrm{Ker}.f}$, and, from which we have Ker. $f=T\mathrm{Ker}.f$ because $TP=P$. Thus we see that P-codom. dim. $X\geqq 2$.

(1)$\Leftrightarrow$(4). This follows directly from Theorem 1 and Theorem 2.

Corollary. *If X has a projective cover $P_0\overset{\varepsilon}{\to}X\to 0$, then P-codom.* dim. *$X\geqq 2$ iff $TX=X$ and T Ker. ε=Ker. ε.*

Proof. Assume that $TX=X$ and $T\,\mathrm{Ker}.\,\varepsilon=\mathrm{Ker}.\,\varepsilon$. Let $\oplus P\overset{f}{\to}X\to 0$ *be an exact sequence. Then there is a homomorphism $g:\oplus P\to P_0$ such* that $\varepsilon g=f$. Since Ker. ε is small in P_0, it follows that g is an epimorphism and, since P_0 is projective, there is a monomorphism $h:P_0\to\oplus P$ such that $gh=1_{P_0}$. Thus we have $\oplus P=h(P_0)\oplus\mathrm{Ker}.\,g$. Since, as is easily verified, Ker. $g\subseteq\mathrm{Ker}.f$ and $h(\mathrm{Ker}.\,\varepsilon)=h(P_0)\subset\mathrm{Ker}.f$, $h(P_0)\overset{f_{h(P_0)}}{\longrightarrow}X\to 0$ is also a projective cover for X. Now we have $T\,\mathrm{Ker}.f=T((h(P_0)\cap\mathrm{Ker}.f)\oplus\mathrm{Ker}.\,g)=T(h(\mathrm{Ker}.\,\varepsilon)\oplus\mathrm{Ker}.\,g)=h(\mathrm{Ker}.\,\varepsilon)\oplus\mathrm{Ker}.\,g=\mathrm{Ker}.f$. It follows that P-codom. dim. $X\geqq 2$. The converse part of the proof follows direct from Theorem 4.

§ 3. On projective self-generators

A module is called self-generator if it generates all its submodules[2].

Let ${}_RP$ be a projective left R-module with the trace ideal T. Since T is an idempotent two-sided ideal of R, it induces the torsion theory $(\mathscr{T},\mathscr{F})$, where $\mathscr{T}=\{X\in{}_R\mathfrak{M}\mid TX=X\}$ and $\mathscr{F}=\{Y\in{}_R\mathfrak{M}\mid TY=0\}$. Further, let S be the endomorphism ring of ${}_RP$. Following characterizations for ${}_RP$ to be a self-generator are due to [1], [2], [6].

Theorem 5. *For a projective module ${}_RP$ the following statements are equivalent*:

(1) *${}_RP$ is a self-generator*

(2) *The class $\{X\in{}_R\mathfrak{M}\mid P\text{-codom. dim. } X\geqq 1\}$ is closed under submodules, that is, the torsion theorey $(\mathscr{T},\mathscr{F})$ is hereditary*

(3) *The right R-module $\left(\frac{R}{T}\right)_R$ is flat*

(4) *$Tp\ni p$ for every element $p\in P$*

(4)′ *$Ann_R(p)+T=R$ for every element $p\in P$, where $Ann_R(p)=\{r\in R\mid rp=0\}$, the annihilator left ideal of p in R.*

(4)″ *$\bigcap_{i=1}^{n}Ann_R(p_i)+T=R$ for every finite set of elements $p_1, p_2,\cdots,p_n\in P$.*

2) Cf. [10].

In this section we shall add some other characterizations of projective self-generators.

THEOREM 5 (continued). *The following statements are equivalent to the statements* (1)~(4)″ *in the theorem above*:

(5) *The class* $\{X\in{}_R\mathfrak{M}\mid P\text{-codom. dim. } X\geqq 1\}$ *coincides with the class* $\{Y\in{}_R\mathfrak{M}\mid P\text{-codom. dim. } Y\geqq 2\}$.

(6) $\varepsilon_{P,X}: P\underset{S}{\otimes}\mathrm{Hom}_R(P, X)\to TX$ *is an isomorphism for every* $X\in{}_R\mathfrak{M}$.

(7) $\mathrm{Hom}_R\left(P, \dfrac{\mathfrak{v}}{\mathfrak{u}}\right)\neq 0$ *for every submodules* $\mathfrak{v}$, $\mathfrak{u}$ *of* P *such that* $0\subseteq\mathfrak{u}\subsetneqq\mathfrak{v}\subseteq P$.

(8) $TE(\mathfrak{m})=0$ *for every simple left* R*-module* $\mathfrak{m}$ *such that* $T\mathfrak{m}=0$. *Here* $E(\mathfrak{m})$ *denotes, as usual, the injective envelope of* $\mathfrak{m}$.

(9) *Every homomorphic image of* P *is* Q*-torsionless, where* $Q=E(\oplus\mathfrak{m}_\alpha)$, $\mathfrak{m}_\alpha$ *ranging over all (non-isomorphic) simple left* R*-modules such that* $T\mathfrak{m}_\alpha=\mathfrak{m}_\alpha$.

(10) *Every left* R*-module* X *such that* $TX=X$ *is* Q*-torsionless.*

(11) *The functor* $T: {}_R\mathfrak{M}\ni X\to TX\in{}_R\mathfrak{M}$, $Tf=f_{TX}$ *(the restriction of* f *to* TX*), where* X, $Y\in{}_R\mathfrak{M}$, $f\in\mathrm{Hom}_R(X, Y)$, *is exact.*[3)]

PROOF. (2)⇒(5). Assume (2) and let X be a left R-module such that P-codom. dim. $X\geqq 1$. Then, since every submodule of a direct sum of P's is generated by P, we see that P-codom. dim. $X\geqq 2$. Thus (5) holds.

(5)⇒(1). Assume (5) and let $\mathfrak{u}$ be a submodule of P. Then, by assumption, P-codom. dim. $\dfrac{P}{\mathfrak{u}}\geqq 2$. Consider the following exact sequence:

$$0\longrightarrow\frac{\mathfrak{u}}{T\mathfrak{u}}\xrightarrow{\iota}\frac{P}{T\mathfrak{u}}\xrightarrow{\nu}\frac{P}{\mathfrak{u}}\longrightarrow 0\,.$$

where ι and ν are the canonical injection and epimorphism, respectively. Then, by Theorem 4, there is a homomorphism $f\in\mathrm{Hom}_R\left(\dfrac{P}{\mathfrak{u}}, \dfrac{P}{T\mathfrak{u}}\right)$ such that $\nu f=1_{\frac{P}{\mathfrak{u}}}$. It follows that $\dfrac{P}{T\mathfrak{u}}=f\left(\dfrac{P}{\mathfrak{u}}\right)\oplus\dfrac{\mathfrak{u}}{T\mathfrak{u}}$, and from which we can easily deduce $\dfrac{\mathfrak{u}}{T\mathfrak{u}}=0$, that is $\mathfrak{u}=T\mathfrak{u}$. Thus P is a self-generator.

(5)⇒(6). Assume (5) and let X be an arbitrary left R-module. Then, since TX is generated by P, P-codom. dim. $TX\geqq 2$. It follows that $0=\mathrm{Ker.}\ \varepsilon_{P,TX}=\mathrm{Ker.}\ \varepsilon_{P,X}$ by Theorem 4. Thus (6) holds.

3) In Theorem 5, the equivalences (1)⇔(2)⇔(5) hold also for quasiprojective modules (Cf. [2], Lemma 2.2).

(6)$\Rightarrow$(5). This follows direct from the fact that P-codom. dim. $P\otimes_S Y\geqq 2$ for every $Y\in{}_S\mathfrak{M}$.

(1)$\Rightarrow$(7). Assume (1). Then, since P generates $\mathfrak{v}$, there exists a homomorphism $f\in\mathrm{Hom}_R(P,\mathfrak{v})$ such that $f(P)\not\subseteq\mathfrak{u}$. Then νf is a non-zero homomorphism of P into $\frac{\mathfrak{v}}{\mathfrak{u}}$, where ν is the canonical epimorphism of onto $\frac{\mathfrak{v}}{\mathfrak{u}}$. Thus $\mathrm{Hom}_R\left(P,\frac{\mathfrak{v}}{\mathfrak{u}}\right)\neq 0$.

(7)$\Rightarrow$(1). Assume (7). Suppose there is a submodule $\mathfrak{v}$ of P such that $T\mathfrak{v}\subsetneqq\mathfrak{v}$. Then there exists a non-zero homomorphism $f\in\mathrm{Hom}_R\left(P,\frac{\mathfrak{v}}{T\mathfrak{v}}\right)$. But then we have $f(P)=f(TP)=Tf(P)=0$, a contradiction. It follows that P is a self-generator.

(2)$\Rightarrow$(8). Assume (2) and let $\mathfrak{m}$ be a simple left R-module such that $T\mathfrak{m}=0$. Suppose $TE(\mathfrak{m})=0$. Then since $TE(\mathfrak{m})$ is generated by P and contains $\mathfrak{m}$, $\mathfrak{m}$ is generated by P. It follows that $T\mathfrak{m}=\mathfrak{m}$, a contradiction. Thus (8) holds.

(8)$\Rightarrow$(1). Assume (8). Suppose there is a submodule $\mathfrak{u}$ of P such that $T\mathfrak{u}\subsetneqq\mathfrak{u}$. Let $\mathfrak{u}'$, $\mathfrak{u}''$ be submodules of P such that $T\mathfrak{u}\subseteq\mathfrak{u}'\subsetneqq\mathfrak{u}''\subseteq\mathfrak{u}$ and $\frac{\mathfrak{u}''}{\mathfrak{u}'}$ is simple. Then since $T\mathfrak{u}''\subseteq T\mathfrak{u}\subseteq\mathfrak{u}'$ we have $T\frac{\mathfrak{u}''}{\mathfrak{u}'}=0$. It follows that $TE\left(\frac{\mathfrak{u}''}{\mathfrak{u}'}\right)=0$. On the other hand, since $E\left(\frac{\mathfrak{u}''}{\mathfrak{u}'}\right)$ is injective, the natural epimorphism $\nu\,(\neq 0):\mathfrak{u}''\to\frac{\mathfrak{u}''}{\mathfrak{u}'}$ is extended to a homomorphism $\tilde{\nu}:P\to E\left(\frac{\mathfrak{u}''}{\mathfrak{u}'}\right)$. This is a contradiction. Thus P is a self-generator.

(1)$\Rightarrow$(9). Assume that P is a self-generator. Let $\mathfrak{u}$ be a submodule of P and p be an element of P such that $p\notin\mathfrak{u}$. Let $\mathfrak{m}$ be a simple epimorphic image of $\frac{\mathfrak{u}+Rp}{\mathfrak{u}}$. Then since $T(\mathfrak{u}+Rp)=\mathfrak{u}+Rp$ we see that $T\mathfrak{m}=\mathfrak{m}$. It follows that there exists a homomorphism f of $\frac{P}{\mathfrak{u}}$ into Q such that $f(p+\mathfrak{u})\neq 0$. This implies that $\frac{P}{\mathfrak{u}}$ is Q-torsionless.

(9)$\Rightarrow$(1). Assume (9). Suppose there exists a submodule $\mathfrak{u}$ of P such that $T\mathfrak{u}\neq\mathfrak{u}$. Let $x\in\mathfrak{u}$ be such that $x\notin T\mathfrak{u}$. Then there is a homomorphism $f\in\mathrm{Hom}_R\left(\frac{P}{T\mathfrak{u}},Q\right)$ such that $f(x+T\mathfrak{u})=0$. Since $Rf(x+T\mathfrak{u})$ contains a simple submodule of Q, $Tf(x+T\mathfrak{u})\neq 0$. On the other hand, we have $Tf(x+T\mathfrak{u})=0$ because $x\in\mathfrak{u}$. This is a contradiction. Thus P is a self-generator.

(9)$\Rightarrow$(10). Assume (9) and let X be a left R-module such that $TX=X$. Let x be a non-zero element of X and $\mathfrak{m}$ be a simple epimorphic image of Rx. Then, by (2), we have $T\mathfrak{m}=\mathfrak{m}$. It follows that there exists a homomorphism f of X to Q such that $f(x)\neq 0$. Thus X is Q-torsionless.

(10)$\Rightarrow$(9). This is trivial.

(6)$\Rightarrow$(11). Assume (6) and let $0\rightarrow X\xrightarrow{f} Y\xrightarrow{g} Z\rightarrow 0$ be an exact sequence of left R-modules. Then since ${}_RP$ is projective and P_S is flat[4] we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & P\otimes\mathrm{Hom}_R(P, X) & \xrightarrow{P\otimes\mathrm{Hom}(P,f)} & P\otimes\mathrm{Hom}_R(P, Y) & \xrightarrow{P\otimes\mathrm{Hom}(P,g)} & P\otimes\mathrm{Hom}_R(P, Z) & \longrightarrow 0 \\
& \uparrow{\scriptstyle \varepsilon_{P,X}} & Tf & \uparrow{\scriptstyle \varepsilon_{P,Y}} & Tg & \uparrow{\scriptstyle \varepsilon_{P,Z}} & \\
& TX & \longrightarrow & TY & \longrightarrow & TZ, &
\end{array}$$

where $\varepsilon_{P,X}$, $\varepsilon_{P,Y}$ and $\varepsilon_{P,Z}$ are all isomorphism. It follows that the sequence:

$$0\longrightarrow TX\xrightarrow{Tf} TY\xrightarrow{Tg} TZ\longrightarrow 0$$

is exact. Thus T is exact.

(11)$\Rightarrow$(1). Assume T is exact. Let $\mathfrak{u}$ be a submodule of P and consider the following canonical exact sequence:

$$0\longrightarrow \frac{\mathfrak{u}}{T\mathfrak{u}}\xrightarrow{\iota}\frac{P}{T\mathfrak{u}}\xrightarrow{\nu}\frac{P}{\mathfrak{u}}\longrightarrow 0\,.$$

Then we have the exact sequence:

$$0\longrightarrow 0\longrightarrow \frac{P}{T\mathfrak{u}}\xrightarrow{T\nu}\frac{P}{\mathfrak{u}}\longrightarrow 0\,.$$

But this implies that $T\mathfrak{u}=\mathfrak{u}$. It follows that P is a self-generator.

Thus we have completed all of our proofs.

If R is a commutative ring or regular ring, then every projective R-module is necessarily a self-generator[5].

A ring R is called left V-ring if every simple left R-module is injective, or equivalently, if every left R-module has a zero (Jacobson-) radical. By Corollary to Lemma 4 and Theorem 5 we have the following

PROPOSITION 1. *Let R be a left V-ring. Then every projective left R-module is a self-generator.*

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4) Cf. [2], Lemma 2.1.

5) Cf. [10], THEOREM 3.1.

§ 4. Further variations of Morita equivalences

From Theorem 3 and Theorem 5 we can deduce direct the following

THEOREM 6 (K. Fuller).[6] *Let $_RP$ be a finitely generated quasi-projective self-generator with the endomorphism ring S, and let $\mathscr{C}$ be the class $\{X \in {}_R\mathfrak{M} \mid P\text{-codom. dim. } X \geqq 1\}$. Then we have the following category isomorphism between $\mathscr{C}$ and $_S\mathfrak{M}$:*

$$\mathscr{C} \underset{P\otimes_S}{\overset{\mathrm{Hom}_R(P,\ )}{\rightleftarrows}} {}_S\mathfrak{M}.$$

An example (G. Azumaya): Let S be a ring and P_S be a projective generator in $\mathfrak{M}_S$. Set $R=\mathrm{End}(P_S)$. Then the left R-module $_RP$ is finitely generated projective and $\mathrm{End}(_RP)=S$. Further $_RP$ is a self-generator[7]. Thus we have the category isomorphism between $\{X \in {}_R\mathfrak{M} \mid P\text{-codom. dim. } X \geqq 1\}$ and $_S\mathfrak{M}$ in the way described in Theorem 6.

Let $_RP$ be a finitely generated projective left R-module with the endomorphism ring S. Let T be the trace ideal of $_RP$. Let further Q be the injective envelope of $\bigoplus \mathfrak{m}_\alpha$, where $\mathfrak{m}_\alpha$ ranges over the class of all (non-isomorphic) simple left R-modules such that $T\mathfrak{m}_\alpha=\mathfrak{m}_\alpha$. Then $_S\mathrm{Hom}_R(P, Q)$ is an injective cogenerator in $_S\mathfrak{M}$, and we have the following category isomorphism between the class $\{X \in {}_R\mathfrak{M} \mid Q\text{-dom. dim. } X \geqq 2\}$ and $_S\mathfrak{M}$:[8]

$$\{X \in {}_R\mathfrak{M} \mid Q\text{-dom. dim. } X \geqq 2\} \underset{\mathrm{Hom}_S(P^*,\ )}{\overset{\mathrm{Hom}_R(P,\ )}{\rightleftarrows}} {}_S\mathfrak{M},$$

where P^* is the R-dual of $_RP$: $P^*=\mathrm{Hom}_R(P, R)$.

Combining this with our Theorem 3 we have the following

THEOREM 7. *In the setting above we have the following category isomorphism:*

$$\{X \in {}_R\mathfrak{M} \mid P\text{-codom. dim. } X \geqq 2\} \underset{P\otimes_S \mathrm{Hom}_R(P,\ )}{\overset{\mathrm{Hom}_S(P^*,\ \mathrm{Hom}_R(P,\ ))}{\rightleftarrows}} \{Y \in {}_R\mathfrak{M} \mid Q\text{-dom. dim. } Y \geqq 2\}$$

§ 5. Supplementaries

Let $_RP$ be a projective left R-module with the trace ideal T. Let I

6) Cf. [2], Theorem 2.6.

7) Cf. [9], Satz 4.

8) Cf. [4], Theorem 2, Theorem 4.

be the annihilator ideal of ${}_RP$.

LEMMA 5. *The class* $\mathscr{C}=\{X\in{}_R\mathfrak{M}\,|\,TX=X\}$ *is closed under submodules and direct products iff* $T+I=R$.

PROOF. Suppose $\mathscr{C}$ is closed under submodules and direct products. Consider the direct product $\prod_{m\in M} A_m$, where $A_m=M$ for each $m\in M$. Let $x=\prod m$, $m\in A_m$. Then by assumption Rx, where $\frac{R}{I}$, is generated by P. It follows that $T+I=R$.

Conversely, suppose $T+I=R$. Then, since $IT=0$, for a left R-module X we see that $TX=X$ iff $IX=0$. Thus it is easy to see that $\mathscr{C}$ is closed under submodules and direct products.

PROPOSITION 2. *If R is a semi-perfect ring, then ${}_RP$ is a self-generator iff $T+I=R$. Further, in this case, I is the smallest left ideal of R with respect to this property.*

PROOF. By Theorem 5 it suffices to show that if ${}_RP$ is a self-generator then $T+I=R$. Suppose ${}_RP$ is a self-generator. Let $\mathfrak{l}_0$ be a left ideal of R such that $\mathfrak{l}_0+T=R$ and $\mathfrak{l}_0$ is minimal with respect to this property[9]. Then we have $I\mathfrak{l}_0=I\subseteq\mathfrak{l}_0$. Let p be an element of P. Then by Theorem 5 we see that $\frac{R}{\mathrm{Ann}_R(p)\cap\mathfrak{l}_0}$ is generated by P. It follows that $T+\mathrm{Ann}_R(p)\cap\mathfrak{l}_0=R$. Then by the minimality of $\mathfrak{l}_0$ we have $\mathfrak{l}_0\subseteq\mathrm{Ann}_R(p)$. Since this is true for every element p of P, we see that $\mathfrak{l}_0\subseteq I$. Thus we have $\mathfrak{l}_0=I$, whence $T+I=R$. The last assertion follows from the fact that if $\mathfrak{l}+T=R$, $\mathfrak{l}$ a left ideal of R, then $I\mathfrak{l}=I\subseteq\mathfrak{l}$.

Let Q be an injective envelope of $\oplus\mathfrak{m}_\alpha$ where $\mathfrak{m}_\alpha$ ranges over the class of all (non-isomorphic) simple left R-modules such that $T\mathfrak{m}_\alpha=\mathfrak{m}_\alpha$.

PROPOSITION 3. *The following statements are equivalent*:

(1) *The class $\mathscr{C}$ is closed under submodules, direct products and injective envelopes.*

(2) *$I\oplus T=R$ (direct sum).*

(3) *The class $\mathscr{C}$ coincides with the class $\{Y\in{}_R\mathfrak{M}\,|\,Q\text{-dom. dim. } Y\geqq 1\}$.*

PROOF. (1)$\Rightarrow$(2). Assume (1). Then by the lemma above we have $T+I=R$. It follows that I is an idempotent two-sided ideal of R and $\mathscr{C}$ coincides with the class $\{Y\in{}_R\mathfrak{M}\,|\,IY=0\}$, the torsionfree class corresponding to I. Because $\mathscr{C}$ is closed under injective envelope, $\left(\frac{R}{I}\right)_R$ is flat as a right R-module.[10] It follows that $I\subset T=IT=0$. Thus we have $I\oplus T=R$.

9) Cf. [3], Satz.

10) Cf. [1], Theorem 6.

(2)$\Rightarrow$(3). Suppose $I\oplus T=R$. Then by Theorem 5 we see that $\mathscr{C}\subseteq\{Y\in{}_R\mathfrak{M}\mid Q\text{-dom. dim. } Y\geqq 1\}$. On the other hand we have $IQ=0$. For, if $IQ\neq 0$ then IQ contains a simple submodule $\mathfrak{m}$ such that $T\mathfrak{m}=0$. But this is a contradiction. It follows that we have $IY=0$, that is $TY=Y$, for every Y such that Q-dom. dim. $Y\geqq 1$. Thus $\mathscr{C}=\{Y\in{}_R\mathfrak{M}\mid Q\text{-dom. dim. } Y\geqq 1\}$.

(3)$\Rightarrow$(1). This is almost clear.

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