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On cut loci of compact symmetric spaces

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Introduction. Let (G, K) be a compact riemannian symmetric pair. In the present note we shall be concerned with the cut locus of a point in $M=G/K$. If M is simply connected, then R. J. Crittenden has shown that the cut locus of a point coincides with the first conjugate locus (see also J. Cheeger [2], J. Cheeger and D. Ebin [3] for another proof). On the other hand Y-C. Wong has studied geodesics, cut loci, and conjugate loci of Grassmann manifolds by a geometric method not appealing to Lie group theory. Recently the author ([10]) has studied the cut locus of a point in a manifold of all Lagrangean subspaces of a real symplectic vector space via Crittenden's view point. In the present note, the method used in [10] will be generalized to compact riemannian symmetric pair. Our first main result is stated as follows (Theorem 2.5): the cut locus of o in $M=G/K$ is determined by the cut locus of o in a flat torus $\mathfrak{a}/\Gamma(G, K)$, where $\mathfrak{a}$ is a cartan subalgebra of (G, K) and $\Gamma(G, K)$ is the lattice of $\mathfrak{a}$ defined by $\Gamma(G, K):=\{A\in\mathfrak{a};\ \exp A\in K\}$. Next purpose of the present note is to study the cut loci and conjugate loci of a point of some standard non-simply connected riemannian symmetric spaces of compact type by using this method. We shall study the cut loci of unitary group $\boldsymbol{U}(n)$, special orthogonal group $\boldsymbol{SO}(n)$, real Grassmann manifold $\boldsymbol{SO}(n+m)/\{(\boldsymbol{O}(n)\times\boldsymbol{O}(m))\cap\boldsymbol{SO}(n+m)\}$, $\boldsymbol{U}(n)/\boldsymbol{O}(n)$, and $\boldsymbol{U}(2n)/\boldsymbol{Sp}(n)$.

§1. Preliminaries

1°. First we shall review the notion of cut locus and conjugate locus of a point in a compact riemannian manifold $(M, \langle\ ,\ \rangle)$. Let Exp_x denote the exponential mapping from the tangent space T_xM at x to M onto M. If X is a unit tangent vector at $x\in M$, then γ_X: $t\rightarrow\mathrm{Exp}_x tX$ is a geodesic parametrized by arc-length emanating from x with the initial direction X. Then t_0X (resp. $\mathrm{Exp}_x t_0X$) is called a tangent conjugate point (resp. conjugate point) of x along a geodesic γ_X, if there exists a non-zero Jacobi-field $J(t)$ along γ_X such that $J(0)=J(t_0)=0$. Next $\bar{t}_0X$ (resp. $\mathrm{Exp}_x\bar{t}_0X$) will be called a tangent cut point (resp. cut point) of x along γ_X, if the geodesic segment $\gamma_{X|[0,\bar{t}_0]}$ is a minimal geodesic segment, but $\gamma_{X|[0,s]}$ can not be a minimal geodesic for any $s>\bar{t}_0$. Then the following is well-known (see, e.g., [1]): Assume that Exp $\bar{t}_0 X$ is a cut point of x along γ_X which is not a conjugate point of x along γ_X, then there exists a unit vector $Y \in T_x M$, $Y \neq X$ such that Exp $\bar{t}_0 X=$ Exp $\bar{t}_0 Y$ holds. The set of (tangent) conjugate points (resp. (tangent) cut points) of x along all geodesics emanating from x is called a (tangent) conjugate locus (resp. (tangent) cut locus) of x. Finally the interior set Int (x) of x is defined as $M\backslash$cut locus of x. Let $\bar{t}_0(X)$ be a positive number such that $\bar{t}_0(X)X$ is the tangent cut point of x along γ_X. Then Exp_x maps $\bigcup_{X\in T_xM, \langle X,X\rangle=1} \{\mathrm{Exp}\ tX;\ 0\leqq t<\bar{t}_0(X)\}$ diffeomorphically onto Int (x). Thus Int (x) is an open cell for any $x\in M$. Cut locus is important because it contains all information about the topology of M. It is an interesting problem to study the relation between cut locus and conjugate locus. (e.g., see [1], [4], [6], [14]).

Now we shall be concerned with the cut locus of a point in a compact symmetric space which is not necessarily simply connected.

2°. Let (G, K) be a compact riemannian symmetric pair which consists of

(i) compact connected Lie group G and compact subgroup K of G.

(ii) involutive automorphism θ of G such that $G_\theta^0 \subset K \subset G_\theta$, where we put $G_\theta := \{x\in G;\ \theta(x)=x\}$ and G_θ^0 denotes the identity component of G_θ.

(iii) G-invariant riemannian structure $\langle\ ,\ \rangle$ on $M=G/K$.

If $\mathfrak{g}$ (resp $\mathfrak{k}$) denotes the Lie algebra of G (resp. K), then $(\mathfrak{g}, \mathfrak{k}, d\theta)$ is an orthogonal involutive Lie algebra corresponding to (G, K). Put $\mathfrak{m} := \{X\in \mathfrak{g};\ d\theta(X)=-X\}$. Then we have a vector space decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$. We may identify the tangent spac $T_o M$ with $\mathfrak{m}$ via the canonical projection $\pi: G\to G/K$ $(o=\pi(e))$. Note that every G-invariant riemannian structure $\langle\ ,\ \rangle$ on M may be induced from an inner product Q on $\mathfrak{m}$ which is invariant under AdK. In the following we shall choose an AdG-invariant and θ-invariant inner product Q on $\mathfrak{g}$ which is an extension of Q in $\mathfrak{m}$ and fix it. Now the geodesic $\gamma_X: t\to \mathrm{Exp}_0\ tX$ which emanates from o with the initial direction $X\in\mathfrak{m}$ is given by $\mathrm{Exp}_o\ tX=\pi\ \exp\ tX$, where exp denotes the exponential mapping of Lie group G. Next, the curvature tensor $R(X, Y)Z=\nabla_{[X,Y]}Z-[\nabla_X, \nabla_Y]Z$ at $T_o M$ is given by $R(X, Y)Z=[[X, Y], Z]$, $X, Y, Z\in\mathfrak{m}$ ([7]). Now let $\mathfrak{t}$ (resp. $\mathfrak{a}$) be a maximal abelian subalgebra of G (resp. Cartan subalgebra of (G, K)). We assume $\mathfrak{a}\subset\mathfrak{t}$. Let $\Sigma(G)$ (resp. $\Sigma(G, K)$) be the root space of G (resp. (G, K)), i.e.,

$$\Sigma(G) := \left\{\alpha\in\mathfrak{t} \mid \alpha\neq 0,\ \tilde{\mathfrak{g}}_\alpha\neq\{0\}\right\} \Big(\text{resp.}$$

$$\Sigma(G, K) := \left\{\gamma \in \mathfrak{a} \mid \gamma \neq 0,\ \mathfrak{g}_\gamma^c \neq \{0\}\right\}\Big), \text{ where}$$

$$\tilde{\mathfrak{g}}_\alpha := \{X \in \mathfrak{g}^c \text{ (complexification of } \mathfrak{g}) \mid [H, X] = 2\pi\sqrt{-1}\ \langle \alpha, H\rangle X \text{ for every } H \in \mathfrak{t}\}\ \Big(\text{resp.}$$

$$\mathfrak{g}_\gamma^c := \{X \in \mathfrak{g}^c \mid [H, X] = 2\pi\sqrt{-1}\langle \gamma, H\rangle X \text{ for every } H \in \mathfrak{a}\}\Big).$$

We have the root space decomposition $\mathfrak{g}^c = \tilde{\mathfrak{g}}_0 + \sum_{\alpha \in \Sigma(G)} \tilde{\mathfrak{g}}_\alpha\ (\tilde{\mathfrak{g}}_0 = \mathfrak{t}^c)$ and $\mathfrak{g}^c = \mathfrak{g}_0^c + \sum_{\gamma \in \Sigma(G,K)} \mathfrak{g}_\gamma^c$. If we put $\Sigma_0(G) = \Sigma(G) \cap \mathfrak{b}$, where $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, then we get $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{a}$ and $\mathfrak{g}_\gamma^c = \sum_{\bar{\alpha} = \gamma} \tilde{\mathfrak{g}}_\alpha$, where $\bar{\alpha}$ denotes the orthogonal projection of $\alpha \in \mathfrak{t}$ to $\mathfrak{a}$. Especially we get $\Sigma(G, K) = \{\bar{\alpha} : \alpha \in \Sigma(G) - \Sigma_0(G)\}$. Next let $\Sigma^+(G)$ (resp. $\Sigma^+(G, K)$) be the set of positive roots with respect to a σ-order, then we have $\Sigma^+(G, K) = \{\bar{\alpha} : \alpha \in \Sigma^+(G) - \Sigma_0(G)\}$. Put $\mathfrak{k}_\gamma := \mathfrak{k} \cap (\mathfrak{g}_\gamma^c + \mathfrak{g}_{-\gamma}^c)$, $\mathfrak{m}_\gamma := \mathfrak{m} \cap (\mathfrak{g}_\gamma^c + \mathfrak{g}_{-\gamma}^c)$ for $\gamma \in \Sigma^+(G, K)$ and let $\mathfrak{k}_0 = \{X \in \mathfrak{k} : [X, \mathfrak{a}] = 0\}$ be the centralizer of $\mathfrak{a}$. Then we have $\mathfrak{k} = \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+(G,K)} \mathfrak{k}_\gamma$, $\mathfrak{m} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(G,K)} \mathfrak{m}_\gamma$. Now the next lemma is useful in the following.

LEMMA 1.1. [11]. *For every $\alpha \in \Sigma^+(G) - \Sigma_0(G)$, there exist $S_\alpha \in \mathfrak{k}$, $T_\alpha \in \mathfrak{m}$ such that*

(i) *for every $\gamma \in \Sigma^+(G, K)$ $\{S_\alpha;\ \bar{\alpha} = \gamma\}$ froms a basis of $\mathfrak{k}_\gamma$ and $\{T_\alpha;\ \bar{\alpha} = \gamma\}$ forms a basis of $\mathfrak{m}_\gamma$.*

(ii) $[H, S_\alpha] = 2\pi\langle \alpha, H\rangle T_\alpha$, $[H, T_\alpha] = -2\pi\langle \alpha, H\rangle S_\alpha$ *for* $H \in \mathfrak{a}$.

(iii) $\mathrm{Ad}(\exp H)\ S_\alpha = \cos 2\pi\langle \alpha, H\rangle S_\alpha + \sin 2\pi\langle \alpha, H\rangle T_\alpha$ *for* $H \in \mathfrak{a}$.
$\mathrm{Ad}(\exp H)\ T_\alpha = -\sin 2\pi\langle \alpha, H\rangle S_\alpha + \cos 2\pi\langle \alpha, H\rangle T_\alpha$ *for* $H \in \mathfrak{a}$.

(iv) $[S_\alpha, T_\alpha] = \pi\ H_\gamma(:= \pi\gamma/2(\gamma, \gamma))$, *where* $\gamma = \bar{\alpha} \in \mathfrak{a}$, and $\langle T_\alpha, T_\alpha\rangle = (4(\gamma, \gamma))^{-1}$.

3°. Now the following lemma gives a characterization of tangent conjugate points of o along a geodesic of M in terms of root system and well-known. For other proofs see Helgason [7], Crittenden [4].

LEMMA 1.2. *Let (G, K) be a compact riemannian symmetric pair. Then for $H \in \mathfrak{a}$, $\langle H, H\rangle = 1$, $t_0 H$ is a conjugate point of o along a geodesic $\gamma_H : t \to \mathrm{Exp}_o\ tH$ if and only if there exists an $\alpha \in \Sigma^+(G) - \Sigma_0(G)$ such that $t_0\langle \alpha, H\rangle = \frac{m}{2}$, $m \in \boldsymbol{Z} - \{0\}$ does hold.*

Proof. ($\Longrightarrow$). By definition there exists a non-zero Jacobi-field $J(t)$ with $J(0) = 0$, $J(t_0) = 0$. Since $M = G/H$ is of compact type the eigenvalues $\lambda_1 = 0, \lambda_2, \cdots, \lambda_n$ of the symmetric linear transformation $X \in \mathfrak{m} \to R(H, X)\ H = -(\mathrm{ad}\ H)^2 X$ is non-negative. Let $E_1 = H, E_2, \cdots, E_n$ be the corresponding eigenvectors and $E_i(t)$ $(i=1, \cdots, n)$ be the parallel translation of E_i along the geodesic γ_H. If we put $J(t)=\Sigma f_i(t)\, E_i(t)$, then the Jacobi equation $\nabla\nabla J(t)+R(\dot\gamma_H(t), J(t))\,\dot\gamma_H(t)=0$ reduces to $f_i''(t)+\lambda_i f_i(t)=0$ $(i=1, \cdots, n)$ and consequently we have

$$f_i(t)=\begin{cases}\dfrac{a_i}{\sqrt{\lambda_i}}\sin\sqrt{\lambda_i}\,t & \text{if } \lambda_i>0,\ a_i=f_i'(0)\\ a_i t & \text{if } \lambda_i=0,\ a_i=f_i'(0).\end{cases}$$

Thus if $t_0 H$ is a tangent conjugate point, then there exists an eigenvalue $\lambda>0$ of $X\to-(\operatorname{ad}H)^2X$ such that $\sqrt{\lambda}\,t_0=m\pi$ for $m\in\boldsymbol{Z}-\{0\}$. Let $X\in\mathfrak{m}$ be an eigenvector coresponding to λ. If we put $X=A+\Sigma a_\alpha T_\alpha$, $A\in\mathfrak{a}$, then

$$\begin{aligned}(\operatorname{ad}H)^2X &= -\textstyle\sum \mathrm{a}_\alpha(2\pi\langle\alpha, H\rangle)^2T_\alpha\\ &= -\lambda(A+\textstyle\sum a_\alpha T_\alpha).\end{aligned}$$

Thus we get $A=0$ and for some $\alpha\in\Sigma^+(G)-\Sigma_0(G)$ we have $(2\pi\langle\alpha, H\rangle)^2=\lambda=\left(\dfrac{m}{t_0}\pi\right)^2$ and consequently $\langle\alpha, H\rangle t_0=\dfrac{m}{2}$, $m\in\boldsymbol{Z}-\{0\}$.

($\Longleftarrow$). By the assumption and lemma 1 (ii), we get

$$\begin{aligned}(\operatorname{ad}H)^2T_\alpha &= -(2\pi\langle\alpha, H\rangle)^2T_\alpha.\\ &= -\left(\frac{m}{t_0}\right)^2T_\alpha.\end{aligned}$$

Let $E(t)$ be the parallel vector field along γ_H such that $E(0)=T_\alpha$. If we put $J(t)=\sin\dfrac{m\pi}{t_0}t\,E(t)$, then $\nabla\nabla\ J(t)+R(\dot\gamma_H(t), J(t))\,\dot\gamma_H(t)=-\left(\dfrac{m\pi}{t_0}\right)^2\sin\dfrac{m\pi}{t_0}t\,E(t)+\left(\dfrac{m\pi}{t_0}\right)^2\sin\dfrac{m\pi}{t_0}t\,E(t)=0$ and $J(t)$ is a non-zero Jacobi field with $J(0)=J(t_0)=0$. Thus t_0H is the tangent conjugate point of o.

REMARK 1.3. Since $\operatorname{Ad}K\,\mathfrak{a}=\mathfrak{m}$ and $\operatorname{Ad}K$ acts on $\mathfrak{m}$ as an isometry, lemma 2 characterizes the conjugate points of o in a compact reimannian symmetric pair. For $H\in\mathfrak{m}$, γ_H has no conjugate point if and only if H belongs to the center of $\mathfrak{g}$.

COROLLARY 1.4. *Let $X=\operatorname{Ad}kH\in\mathfrak{m}\,(H\in\mathfrak{a},\ k\in K)$, $\langle H, H\rangle=1$. Then the first conjugate point $t_0(X)\,X$ of o along a geodesic γ_X is determined by*

$$t_0(X)=\operatorname*{Min}_{\alpha\in\Sigma^+(G)-\Sigma^0(G)}\frac{1}{2|\langle\alpha, H\rangle|}.$$

§2. Tangent cut locus.

1°. Now we shall turn to tangent cut points.

PROPOSITION 2.1. *Let (G, K) be a compact riemannian symmetric pair and $\mathfrak{a}$ be a Cartan subalgebra of (G, K). For a unit vector $X \in \mathfrak{a}$, assume that $t_0 X$ is a cut point of o along γ_X. Then either $t_0 X$ is the first tangent conjugate point of o along γ_X or there exists a unit vector $Y \in \mathfrak{a}$, $Y \neq X$ such that* $\mathrm{Exp}_o\, t_0 X = \mathrm{Exp}_o\, t_0 Y$ *does hold.*

Proof. Suppose that $t_0 X$ is not a conjugate point of o along γ_X, then there exists a unit vector $Z \in \mathfrak{m}$, $Z \neq X$ such that $\mathrm{Exp}\; t_0 X = \mathrm{Exp}\; t_0 Z$. We shall show $[X, Z] = 0$. In fact if $[X, Z] \neq 0$, then clearly $Z \notin \mathfrak{a}$. Since $\mathrm{Exp}\; t_0 X = \pi \exp t_0 X = \pi \exp t_0 Z$ holds, we have

$$\exp t_0 X = \exp t_0 Z\, k \text{ for some } k \in K.$$

Then $\exp(-t_0 X) \exp sZ \exp(t_0 X) = k^{-1} \exp sZ\, k$ holds and consequently we get

$$\mathrm{Ad}\left(\exp(-t_0 X)\right) Z = \mathrm{Ad}\, k^{-1} Z.$$

On the other hand by lemma 1.1 Z may be written in the form

$$Z = Z_0 + \sum_{\gamma \in \Sigma^+(G,K)} \sum_{\bar{\alpha}=\gamma} a_\alpha T_\alpha \text{ with } Z_0 \in \mathfrak{a}.$$

Note that at least one a_α can not be zero. Thus we get

$$\begin{aligned} &\mathrm{Ad}(\exp(-t_0 X))\,(Z_0 + \sum_{\gamma \in \Sigma^+(G,K)} \sum_{\bar{\alpha}=\gamma} a_\alpha T_\alpha) \\ &= Z_0 + \sum_{\gamma \in \Sigma^+(G,K)} \sum_{\gamma=\bar{\alpha}} a_\alpha \{\cos 2\pi\, t_0 \langle \alpha, X\rangle T_\alpha + \sin 2\pi\, t_0 \langle \alpha, X\rangle S_\alpha\}. \end{aligned}$$

On the other hand, since

$$\mathrm{Ad}(k^{-1})\,(Z_0 + \sum_{\gamma \in \Sigma^+(G,K)} \sum_{\bar{\alpha}=\gamma} a_\alpha T_\alpha) \in \mathfrak{m}, \text{ we have } \sin 2\pi\, t_0 \langle \alpha, X\rangle = 0$$

for some $\alpha \in \Sigma^+(G) - \Sigma_0(G)$. On the other hand, since $[X, Z] = -\Sigma\, 2\pi\, a_\alpha \langle \alpha, X\rangle S_\alpha \neq 0$ holds, we know that $t_0 X$ is a conjugate point of o along γ_X by lemma 1.2 which is a contradiction. Thus we get $[X, Z] = 0$. Now let $\mathfrak{a}'$ be a Cartan subalgebra which contains X and Z. Then there exist $k \in K$, $Y \in \mathfrak{a}$ (unit vector, $Y \neq X$) such that $X = \mathrm{Ad}\, kX$, $Z = \mathrm{Ad}\, kY$. So we have

$$\begin{aligned} \tau_k\, \mathrm{Exp}\; t_0 X &= \mathrm{Exp}\; t_0\, \mathrm{Ad}\, kX = \mathrm{Exp}\; t_0 X = \mathrm{Exp}\; t_0 Z \\ &= \mathrm{Exp}\; t_0\, \mathrm{Ad}\, kY = \tau_k\, \mathrm{Exp}\; t_0 Y, \end{aligned}$$

where τ_k denotes the left translation by $k \in K$. Thus we have $\mathrm{Exp}\; t_0 X = \mathrm{Exp}\; t_0 Y$ for some unit $Y \in \mathfrak{a}$, $Y \neq X$. Q. E. D.

2°. By this proposition, to determine the tangent cut point $\tilde{t}_0(X) X$ of o along γ_X, $X \in \mathfrak{a}$, we must search for the minimum positive value $\tilde{t}_0(X)$

such that $\operatorname{Exp} \tilde{t}_0(X) X = \operatorname{Exp} \tilde{t}_0(X) Y$ holds for some unit $Y \in \mathfrak{a}$, $Y \neq X$. For that purpose we put $\Gamma(G, K) := \{A \in \mathfrak{a}; \exp A \in K\}$. Then we get

LEMMA 2.2. $\tilde{t}_0(X) = \operatorname*{Min}_{A \in \Gamma(G,K)-\{0\}} \dfrac{\langle A, A\rangle}{2|\langle X, A\rangle|}$

Proof. $\operatorname{Exp} t_0 X = \operatorname{Exp} t_0 Y$ holds for unit vectors $X, Y (X \neq Y)$ in $\mathfrak{a}$ if and only if $\exp t_0(X - Y) \in K$ i.e. $t_0(X - Y) \in \Gamma(G, K)$. If we put $A = t_0(X - Y)$, then we get $Y = X - \dfrac{1}{t_0} A$ and from $\langle X, X\rangle = \langle Y, Y\rangle = 1$ it is easy to see $t_0 = \dfrac{\langle A, A\rangle}{2\langle X, A\rangle}$.

REMARK 2.3. $\Gamma(G, K)$ is a lattice in $\mathfrak{a}$. Note that $\Gamma(G, K) \subset \{A \in \mathfrak{a}; \exp 2A = e\}$ and $\Gamma(G, K) = \{A \in \mathfrak{a}; \exp 2A = e\}$ if $K = G_\theta$. If we consider the torus $\mathfrak{a}/\Gamma(G, K)$ with a flat riemannian structure, then $\{\tilde{t}_0(X) X, X \in \mathfrak{a}\}$ is nothing but the tangent cut locus of o in $\mathfrak{a}/\Gamma(G, K)$. In fact for $A \in \Gamma(G, K)$, $\dfrac{(A, A)}{2(X, A)} X$ is the vector in $\mathfrak{a}$ whose terminal point is determined as the intersection of the line generated by X and the hyperplane passing through the point $A/2$ perpendicularly to the direction A.

REMARK 2.4. If the lattice $\Gamma(G, K)$ is generated by the vectors $\{A_1, \cdots, A_r\}$ with $A_i \perp A_j (i \neq j)$, then $\tilde{t}_0(X) = 1/(2 \operatorname{Max}|x_i|)$, where $X = \sum x_i A_i$ is a unit vector in $\mathfrak{a}$. In fact, if we put $\alpha_i^2 = \langle A_i, A_i\rangle$, then $\sum (x_i)^2 (\alpha_i)^2 = 1$. Now for $A = \sum m_i A_i \in \Gamma(G, K)$ we get

$$\frac{\langle A, A\rangle}{2|\langle X, A\rangle|} = \frac{m_1^2\alpha_1^2 + \cdots + m_r^2\alpha_r^2}{2|m_1 x_1 \alpha_1^2 + \cdots + m_r x_r \alpha_r^2|} \geqq$$
$$\frac{m_1^2\alpha_1^2 + \cdots + m_r^2\alpha_r^2}{2\operatorname{Max}|x_i|(|m_1|\alpha_1^2 + \cdots + |m_r|\alpha_r^2)} \geqq 1/(2\operatorname{Max}|x_i|).$$

Summing up we have the following theorem.

THEOREM 2.5. *Let (G, K) be a compact riemannian symmetric pair. Then the tangent cut locus of o is given by* $\operatorname{Ad} K(\cup\{\tilde{t}_0(X) X; X \in \mathfrak{a}, |X| = 1\})$, *i.e., the tagent cut locus of o is determined by the tangent cut locus of o in the flat torus* $\mathfrak{a}/\Gamma(G, K)$.

Proof. Let $t_0(X) X$ be the first tangent conjugate point of $X \in \mathfrak{a}$. Then by lemma 1.2 there exists an $\alpha \in \Sigma^+(G) - \Sigma_0(G)$ such that T_α is an eigenvector of the symmetric transformation $Y \to R(X, Y) X$ of $\mathfrak{m}$ with eigenvalue $\lambda = (\pi/t_0(X))^2$. Now $1/\sqrt{\lambda}\, R(X, T_\alpha)$ is an element of the holonomy algebra of $M = G/K$ and acts on $\mathfrak{m}$ as a skewsymmetric linear transformation. Thus there exists a one-parameter subgroup h_s in K whose adjoint representation $s \to \mathrm{Ad}\ h_s$ on $\mathfrak{m}$ is generated by $1/\sqrt{\lambda}\, R(X, T_\alpha)$ see [16]. Then $(t, s) \to h_s(c(t))$ is a variation of the geodesic $c(t) := \mathrm{Exp}\ tX$. Its variation vector field $Y(t): t \to \frac{\partial}{\partial s} h_s(c(t))|_{s=0}$ is a Jacobi field along c with $Y(0) =0$, $\nabla Y(0)=\partial/\partial s\ \mathrm{Ad}\ h_s\ \dot{c}(0)|_{s=0}=1/\sqrt{\lambda}\ R(X, T_\alpha)\, X=\sqrt{\lambda}\ T_\alpha$. By the proof of lemma 1. 2, we get $Y(t)=\sin \sqrt{\lambda}\ t\ E(t)$, where $E(t)$ is the parallel translation of T_α along c. Since $Y(t_0(X))=0$ and h_s is a one parameter subgroup in K, $h_s c(t_0(X))=\mathrm{Exp}\ t_0\,\mathrm{Ad}\ h_s X$ reduces to a point $c(t_0(X))$ for every s.

On the other hand we have $[S_\alpha, T_\alpha]=\pi H_\gamma \in \mathfrak{a}$ by lemma 1. 1. (iv). Now an easy calculation gives the following:

$$R(X, T_\alpha)\, X = 4\pi^2 \langle \alpha, X\rangle^2 T_\alpha$$
$$R(X, T_\alpha)\, T_\alpha = -2\pi^2 \langle \alpha, X\rangle H_\gamma$$
$$R(X, T_\alpha)\, H_\gamma = 2\pi^2 \langle \alpha, X\rangle T_\alpha\,.$$

So $R(H, X)$ leaves the subspace $\mathfrak{h}$ generated by $\{X, T_\alpha, H_\gamma\}$ invariant and $s \to \mathrm{Ad}\ h_s$ induces a one-parameter subgroup of $SO(3)$. Thus $s \to \mathrm{Ad}\ h_s X$ is a (small) circle in the unit sphere in $\mathfrak{h}$ which passes through X with the tangent $\sqrt{\lambda}\ T_\alpha$. Since T_α is orthogonal to $\mathfrak{a}$ this circle intersects a plane in $\mathfrak{a}$ which is generated by X and H_γ. Note that this circle is a great circle defined by X and T_α if and only if X and H_γ are linearly dependent. Thus we know that there exists a unit vector $Y \neq X$ in $\mathfrak{a}$ such that $\mathrm{Exp}\ t_0(X)\, X=\mathrm{Exp}\ t_0(X)\ Y=c(t_0(X))$, i. e., $\tilde{t}_0(X) \leqq t_0(X)$ does hold. Finally Ad k, $k \in K$, scts on $\mathfrak{m}$ as an isometry and $\mathrm{Ad}\ K\ \mathfrak{a}=\mathfrak{m}$. This completes the proof of theorem. Q. E. D.

REMARK 2. 6. In [9] Rauch has asserted that the sphere of radius $t_0(X)$ in the subspace spanned by X and the eigenspace $\mathfrak{n}$ of $Y \to R(X, Y)\ X$ of $\mathfrak{m}$ corresponding to the eigenvalue λ is mapped into the one point $c(t_0(X))$. But this seems to be incorrect. In fact if dim $\mathfrak{a}=1$ then this holds if and only if X and H_γ are linearly dependent.

3°. Now let C be the center of $\mathfrak{g}$ and $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ be the semi-simple part of $\mathfrak{g}$. Then we have the Lie algebra direct sum $\mathfrak{g}=C \oplus \mathfrak{g}'$ and $\mathfrak{a}= C_m+\mathfrak{a}'$, where we put $\mathfrak{a}'=\mathfrak{a} \cap \mathfrak{g}'$, $C_m := C \cap \mathfrak{m}$. Note that C_m and $\mathfrak{a}'$ are orthogonal with respect to an Ad K-invariant inner product Q. Let $\{A_1, \cdots, A_r\}$ be generator of $\Gamma(G, K)$ and put $g_{jk}=\langle A_j, A_k\rangle$. Next let $\{c_\alpha\}_{\alpha=1,\cdots,s}$ be generator of $\Gamma(C_m, X)=\{A \in C_m\,;\ \exp A \in K\}$ which is a lattice of C_m. Then we have $c_\alpha=\sum_{j=1}^{r} m_\alpha^j A_j$ with $m_\alpha^j \in \boldsymbol{Z}\ (\alpha=1, \cdots, s)$. If we denote by $(c^{\alpha\beta})$ the inverse matrix of $(c_{\alpha\beta})$ where we set $c_{\alpha\beta}=\langle c_\alpha, c_\beta\rangle=\sum m_\alpha^j m_\beta^k g_{jk}$, then for $X=\sum x_i A_i \in \mathfrak{a}$, its C_m-component X_c is given by $X_c= \sum_{\alpha,\beta,i,j} c^{\alpha\beta} m_\beta^j x^i g_{ij} c_\alpha$.

Next since $\mathfrak{g}'$ is compact semi-simple, the restriction of Killing form B to $\mathfrak{g}'\cap\mathfrak{m}$ is negative-definite and in the following we shall consider the case $Q|_{\mathfrak{a}'}=-\lambda_2 B|_{\mathfrak{a}'}$ holds for some positive constant λ (e. g., if M is irreducble). In this case, for a unit vector $X=\Sigma x_i A_i\in\mathfrak{a}$ we get

$$\bar{t}_0(X) = \underset{(m_1,\cdots,m_r)\in Z^r-\{0\}}{\mathrm{Min}} \Big[\sum_{\alpha,\beta,i,j,k,l} c^{\alpha\beta} m_\alpha^i m_\beta^j m^k m^l g_{ik} g_{jl}$$
$$+\pi^2\lambda^2 \sum_{\alpha\in\Sigma(G)-\Sigma_0(G)}\Big(\sum_{i=1}^r m_i n_i(\alpha)\Big)^2\Big]\Big/\Big[2\Big(\sum_{\alpha,\beta,i,j,k,l} c^{\alpha\beta} m_\alpha^i m_\beta^j m^k x^l g_{ik} g_{jl}\Big)$$
$$+\pi^2\lambda^2 \sum_{\alpha\in\Sigma(G)-\Sigma_0(G)}\Big(\sum_{i=1}^r m_i n_i(\alpha)\Big)\Big(\sum_{i=1}^r x_i n_i(\alpha)\Big)\Big],$$

where $n_i(\alpha):=\langle\alpha, 2A_i\rangle$ $(i=1,\cdots,r)$ are integers because of lemma 1. 1. (iii). In fact, let X' (resp. A') be the $\mathfrak{a}'$-component of X (resp. A). Then we have

$$Q(X', A') = (2\pi\lambda)^2 \sum_{\alpha\in\Sigma(G)-\Sigma_0(G)} \langle\alpha, X\rangle\langle\alpha, A\rangle$$
$$= \pi^2\lambda^2\sum_\alpha\Big(\sum_{i=1}^r x_i n_i(\alpha)\Big)\Big(\sum_{i=1}^r m_i n_i(\alpha)\Big),$$
$$Q(A', A') = (2\pi\lambda)^2 \sum_{\alpha\in\Sigma(G)-\Sigma_0(G)} \langle\alpha, A\rangle^2$$
$$= \pi^2\lambda^2\sum_\alpha\Big(\sum_{i=1}^r m_i n_i(\alpha)\Big)^2,$$

where $A=\sum_{i=1}^r m_i A_i$.

4°. Now we shall consider the simply connected case. In this case R. Crittenden has given a following result. We shall give our proof for completenes.

THEOREM 2. 7. ([2], [3] [4]). *Let G be a compact simply connected Lie group and (G, K) be a compact riemannian symmetric pair with connected K. Then the (tangent) cut locus of o in $M=G/K$ coincides with the first (tangent) conjugate locus of o.*

Proof. From the assumption of theorem we know that M is simply connected and $\Gamma(G,K)\cap\overline{F(G,K)}=\{0\}$, where $F(G,K)$ is any component of $\mathfrak{a}$-{diagram of (G,K)} whose closure $\overline{F(G,K)}$ contains the origin ([11]). Now for a unit vector $X\in\mathfrak{a}$, we assume that the tangent cut point $\bar{t}_0(X)X$ is not a tangent conjugate point along γ_X. Then from Th. 2. 5. we know that there exists a non-zero $A\in\Gamma(G,K)$, such that $\bar{t}_0(X)=\dfrac{\langle A, A\rangle}{2|\langle X, A\rangle|}<$

$\dfrac{1}{2\underset{\alpha\in\Sigma(G)-\Sigma_0(G)}{\mathrm{Max}}|\langle\alpha, X\rangle|}$, i. e., $|\langle\alpha, \bar{t}_0(X)X\rangle|<\dfrac{1}{2}$ for all $\alpha\in\Sigma(G)-\Sigma_0(G)$. Next put $Y=X-\frac{1}{\bar{t}_0(X)}A$, then Y is a unit vector in $\mathfrak{a}$ and we get $\mathrm{Exp}\ \bar{t}_0(X)X=\mathrm{Exp}\ \bar{t}_0(X)\,Y$. Then $\gamma_Y: t\rightarrow \mathrm{Exp}\ tY$; $0\leqq t\leqq \bar{t}_0(X)$ is a minimal geodesic segment and we have also $|\langle\alpha, \bar{t}_0(X)\,Y\rangle|\leqq\frac{1}{2}$ for all $\alpha\in\Sigma(G)-\Sigma_0(G)$. So we have $|\langle\alpha, 2A\rangle|=|\langle\alpha, 2\bar{t}_0(X)\,X\rangle-\langle\alpha, 2\bar{t}_0(X)\,Y\rangle|<2$ for every $\alpha\in\Sigma(G)-\Sigma_0(G)$. Since we assume K is connected, we have $K=G_\theta$ and consequently $\Gamma(G, K)=\{H/2;\ H\in\mathfrak{a},\ \exp H=e\}$ and $\langle\alpha, 2A\rangle$ is an integer for every $\alpha\in\Sigma(G)$ (Lemma 1. 1). So we get $|\langle\alpha, 2A\rangle|=0$ or 1 for every $\alpha\in\Sigma(G)$ from which we know that $A\in\Gamma(G, K)-\{0\}$ is contained in some $\overline{F(G, K)}$, a contradiction.

Q. E. D.

§ 3. Compact Lie groups

1°. Let H be a compact Lie group with a bi-invariant riemannian structure $\langle\ ,\ \rangle$. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{a}^*$ be a maximal abelian subalgebra of $\mathfrak{h}$. If we put $G=H\times H$, $K=\{(x, x)|\ x\in H\}$, $\theta(x, y)=(y, x)$, then (G, K) is a compact riemannian symmetric pair. Note that $\mathfrak{k}=\{(X, X);\ x\in\mathfrak{h}\}$ is the Lie algebra of K, $\mathfrak{m}:=\{\widetilde{X}\in\mathfrak{g};\ d\theta(\widetilde{X})=-\widetilde{X}\}$ is equal to $\{(X, -X);\ X\in\mathfrak{h}\}$ and $\mathfrak{a}:=\{(Y, -Y);\ Y\in\mathfrak{a}^*\}$ is a Cartan subalgebra of (G, K). We may identify G/K (resp. $\mathfrak{m}$) with H (resp. $\mathfrak{h}$) via the correspondence $(x, y)\rightarrow xy^{-1}$ (resp. $(X, -X)\rightarrow 2X$). $\mathfrak{a}$ may be identified with $\mathfrak{a}^*$ by this correspondence. In this case geodesic emanating from e with the initial direction $X\in\mathfrak{h}$ is given by the one-parameter subgroup $t\rightarrow\exp tX$ and the curvature tensor in $\mathfrak{h}:=T_eH$ is given by $R(X, Y)\,Z=1/4[[X, Y], Z]$ ([8]). Now the argument of § 1 and §2 gives the following:

LEMMA 3. 1. *For any unit vector $X\in\mathfrak{a}^*$, t_0X is a conjugate point of o along a geodesic $\gamma_X: t\rightarrow\exp tX$ if and only if there exists a root $\alpha\in\Sigma(H)$ such that $t_0\langle\alpha, X\rangle\in\boldsymbol{Z}-\{0\}$.*

COROLLARY 3. 2. *Let $t_0(X)\,X$ be the first conjugate point of e along a geodesic γ_X, where $X\in\mathfrak{a}^*$ is a unit vector. Then we have*

$$t_0(X)=\min_{\alpha\in\Sigma^+(G)}\frac{1}{|\langle\alpha, X\rangle|}.$$

On the other hand put $\Gamma(G):=\{A\in\mathfrak{a}^*|\exp A=e\}$ which is a lattice in $\mathfrak{a}^*$. Then if $A\in\Gamma(G)$ we have $\langle\alpha, A\rangle\in\boldsymbol{Z}$ for any $\alpha\in\Sigma(H)$ (lemma 1. 1 (iii)).

Now we define for a unit vector $X\in\mathfrak{a}^*$, $\tilde{t}_0(X):=\mathrm{Min}\,\{t_0>0;\ \exp t_0X=\exp t_0Y$ holds for some unit $Y\in\mathfrak{a}^*$ $Y\neq X\}$. Then we get

LEMMA 3. 3. $\tilde{t}^0(X)=\underset{A\in\Gamma(G)-\{0\}}{\operatorname{Min}} \dfrac{\langle A, A\rangle}{2|\langle X, A\rangle|}$, $X\in\mathfrak{a}^*$, $\langle X, X\rangle=1$.

Now as in theorem 2. 5 $\tilde{t}_0(X)$ determine the cut point $\tilde{t}_0(X)\,X$ of o along a geodesic γ_X. In the following we shall show that this method is valid to determine the cut loci of $\boldsymbol{U}(n)$ and $\boldsymbol{SO}(n)$.

2° $\boldsymbol{U}(n)\,(n\geqq 2)$. Let $\mathfrak{u}(n):=\{X\in\mathfrak{M}_n(\boldsymbol{C})\,|\,{}^t\bar{X}+X=0\}$ be the Lie algebra of $G=\boldsymbol{U}(n)$. We put

$$B_{ij}:=E_{ij}-E_{ji}\,,\quad C_{ij}=\sqrt{-1}\,(E_{ij}+E_{ji}),\quad \text{and } A_i=1/\sqrt{2}\,C_{ii}\,,$$

where E_{ij} denotes the $n\times n$-matrix whose r-th row and s-th column is given by $\delta_{ir}\,\delta_{js}$. Then

$$\left\{B_{ij}(1\leqq i<j\leqq n),\quad C_{ij}(1\leqq i<j\leqq n),\quad A_i(1\leqq i\leqq n)\right\}$$

forms an orthonormal basis for $\mathfrak{u}(n)$ with respect to Q which is a bi-invariant riemannian structure on $\boldsymbol{U}(n)$ defined by $Q(X, Y)=-\dfrac{1}{2}$ tr XY, $X, Y\in\mathfrak{u}(n)$. Now $\mathfrak{a}^*=\{A_1, \cdots, A_n\}$ forms a maximal abelian subalgebra in $\mathfrak{u}(n)$. Then we get

$$R\Big(\sum_{i=1}^n x^i A_i, B_{jk}\Big)\sum_{l=1}^n x^l A_l=\frac{(x^j-x^k)^2}{2}\,B_{jk}\,,$$

$$R\Big(\sum_{i=1}^n x^i A_i, C_{jk}\Big)\sum_{l=1}^n x^l A_l=\frac{(x^j-x^k)^2}{2}\,C_{jk}\,,$$

$$R\Big(\sum_{i=1}^n x^i A_i, A_j\Big)\sum_{l=1}^n x^l A_l=0\,.$$

Thus for $X=\Sigma x^i A_i\in\mathfrak{a}^*\,(\Sigma x_i^2=1)$, $\{B_{jk}, C_{jk}\}$ defines the eigenspace of the symmetric transformation $Z\to R(X, Z)\,X$ corresponding to the eigenvalue $\dfrac{(x^j-x^k)^2}{2}$ and so for $X=\Sigma x^i A_i\,(\Sigma x_i^2=1)\in\mathfrak{a}^*$ the first conjugate point $t_0(X)\,X$ of e along a geodesic γ_X is given by $t_0(X)=\dfrac{\sqrt{2}\,\pi}{\underset{j<k}{\operatorname{Max}}|x_j-x_k|}$. Next we shall consider the tangent cut locus of e in $\boldsymbol{U}(n)$. In this case $\Gamma(G)=\{\sum_i\sqrt{2}\,m_i\pi A_i\,;\ m_i\in\boldsymbol{Z}\}$ and for $X=\Sigma x^i A_i\in\mathfrak{a}^*\,(\Sigma x_i^2=1)$, we get

$$\tilde{t}_0(X)=\underset{A\in\Gamma(G)-\{0\}}{\operatorname{Min}}\frac{\langle A, A\rangle}{2|\langle X, A\rangle|}=\underset{(m_i)\in Z^n-\{0\}}{\operatorname{Min}}\frac{2\Sigma m_i^2\pi^2}{2\sqrt{2}\,|\Sigma m_i x_i\pi|}$$

$$=\frac{\pi}{\sqrt{2}}\,\frac{1}{\operatorname{Max}|x_i|}\,.$$

Since obviously $2\underset{1\leqq i\leqq n}{\operatorname{Max}}|x_i|\geqq\underset{\leqq j<k\geqq n}{\operatorname{Max}}|x_j-x_k|$ holds, we know that the tan-

gent cut locus of e

$$:=\mathrm{Ad}G\left\{\sum_{i=1}^{n}\frac{\pi}{\sqrt{2}}\frac{x_i}{\mathrm{Max}|x_i|}A_i\,;\ \Sigma x_i^2=1\right\}.$$

REMARK 3.4. The cut point $\bar{t}_0 X$ of e alog γ_X, $X=\Sigma x^i A_i$, $\Sigma x_i^2=1$, is the first conjugate point if and only if there exist some $j<k$ such that $|x_j|=|x_k|=\mathrm{Max}\,|x_i|$ and $x_j+x_k=0$ hold. Note that $\pi_1(\boldsymbol{U}(n))\cong\boldsymbol{Z}$.

Now let F be a complex vector space of complex dimension n, and $\|\ \|$ be the hermitian norm in F. Put $E=F\oplus F$ and ψ be the non-degenerate hermitian form on E defined by

$$\psi\left[(x,y)\right]=\|x\|^2-\|y\|^2.$$

Put $U(E;\psi):=\{P\subset E;$ subspace of complex dimension n such that $\psi_{|P}\equiv 0\}$. Then $U(E;\psi)$ has a natural topology as a subspace of complex Grassmann manifold. If we define a map $\Phi:\boldsymbol{U}(n)\to U(E,\psi)$ by $\Phi(U)=\{(x,Ux)|x\in F\}$, then Φ is a homeomorphism and consequently $U(E,\psi)$ has a structure of a compact riemannian manifold whose structure is derived from that of $\boldsymbol{U}(n)$. Put $\varDelta$ (resp. $\varDelta^{\perp})=\Phi(I)$ (resp. $\Phi(-I)$) where $I:F\to F$ denotes the identity mppaing. Then we have

THEOREM 3.5. *For $P\in U(E,\psi)$ define Γ_P by $\Gamma_P:=\left\{Q\in U(E,\psi);\ P\cap Q\neq\{0\}\right\}$. Then $\Gamma_{\varDelta^{\perp}}$ is the cut locus of $\varDelta$.*

Proof. We put $\Gamma_P^k:=\{Q\in U;\ \dim P\cap Q=k\}$ $(k=0,1,2,\cdots,n)$. Firstly we shall show that $\Phi(\exp\mathrm{Ad}G\bigcup_{X\in\mathfrak{a}^*,\langle X,X\rangle=1}\{tX;\ 0\leqq t<\bar{t}_0(X)\})\subset\Gamma_{\varDelta^\perp}^0$, and Φ(exp $\mathrm{Ad}G\cup\{\bar{t}_0(X)\,X|X=\Sigma x^i A_i\in\mathfrak{a}^*,\ \Sigma x_i^2=1,\ \mathrm{Max}\,|x_i|=|x_{i_1}|=\cdots=|x_{i_k}|\})\subset\Gamma_{\varDelta^\perp}^k$ $(k=1,2,\cdots,n)$. In fact, if a unit $X=\Sigma x_i A_i$ satisfies $\mathrm{Max}\,|x_i|=|x_{i_1}|=\cdots=|x_{i_k}|$, we may assume $\mathrm{Max}\,|x_i|=|x_1|=\cdots=|x_k|>|x_{k+1}|\geqq\cdots\geqq|x_n|$. Then we get

$$\Phi(\exp\bar{t}_0(X)\,X)=\left\{\left(\begin{pmatrix}u_1\\ \vdots\\ u_k\\ \\ u_{k+1}\\ \vdots\\ u_n\end{pmatrix},\begin{pmatrix}-u_1\\ \vdots\\ -u_k\\ \\ \exp\frac{x_{k+1}}{|x_1|}\pi\sqrt{-1}\cdot u_{k+1}\\ \vdots\\ \exp\frac{x_n}{|x_1|}\pi\sqrt{-1}\cdot u_n\end{pmatrix}\right)\right\}\ \text{and}$$

obviously this subspace has a k-dimensional intersection with $\varDelta^\perp$. Since $\mathrm{Ad}G$ leaves invariant $\varDelta$, $\varDelta^\perp$, we have $\Phi(\exp(\mathrm{Ad}G\,\bar{t}_0(X)\,X))\subset\Gamma_{\varDelta^\perp}^k$. Similarly we get $\Phi(\exp\mathrm{Ad}G\bigcup_{X\in\mathfrak{a}^*,\langle X,X\rangle=1}\{tX;\ 0\leqq t<\bar{t}_0(X)\}\subset\Gamma_{\varDelta^\perp}^0$. Finally we get $U(E,$

$\psi)=\bigcup_{k=0}^{n}\Gamma_{\Delta^\perp}^{k}\supset\mathrm{Int}(e)\cup$ Cut locus of $e=U(E,\psi)$ and our assertion follows.

Q. E. D.

REMARK 3. 6. Γ_P plays an important role in a work of Edwards ([5]). Note that $\Gamma_P^k=\Phi(\exp\mathrm{Ad}G\cup\{\tilde{t}_0(X)\,X\,;\ X=\Sigma x_i A_i,\ \Sigma x_i^2=1,\ \mathrm{Max}\,|x_i|=|x_{i_1}|=\cdots=|x_{i_k}|\})$ defines a submanifold of codimension k^2 in $U(E,\psi)$.

3°. $\boldsymbol{SO}(n)$, $(n\geqq 3)$.

Let $\mathfrak{so}(n)=\{X\in\mathfrak{M}_n(R)|{}^tX+X=0\}$ be the Lie algebra of $G=\boldsymbol{SO}(n)$. If we put $e_{ij}=E_{ij}-E_{ji}$, where E_{ij} denotes the $n\times n$-matrix whose r-th row and s-th column is given by $\delta_{ir}\,\delta_{js}$, then $\{e_{ij}\}_{i<j}$ forms an orthonormal basis for $\mathfrak{so}(n)$ with respect to a bi-invariant riemannian structure defined by $\langle X, Y\rangle:=-1/2\ \mathrm{tr}XY$. By the Lie multiplication table $[e_{ij}, e_{kl}]=-\delta_{ik}e_{jl}+\delta_{il}\,e_{jk}+\delta_{jk}\,e_{il}-\delta_{jl}\,e_{ik}$ we know that $\mathfrak{a}^*:=\{e_{12}, e_{34},\cdots, e_{2m-1,2m}\}$ defines a maximal abelian subalgebra of $\mathfrak{so}(n)$, where either $n=2m$ or $n=2m+1$ $\left(m=\left[\frac{n}{2}\right]\right)$. Moreover we get for $X=\sum_{i=1}^{m}x^i e_{2i-1,2i}$,

$$R\left(X,\frac{e_{2a-1\,2b-1}+e_{2a\,2b}}{\sqrt{2}}\right)X=\frac{(x^a-x^b)^2}{4}\,\frac{e_{2a-1\,2b-1}+e_{2a\,2b}}{\sqrt{2}}\ (1\leqq a<b\leqq m),$$

$$R\left(X,\frac{e_{2a\,2b}-e_{2a-1\,2b-1}}{\sqrt{2}}\right)X=\frac{(x^a+x^b)^2}{4}\,\frac{e_{2a\,2b}-e_{2a-1\,2b-1}}{\sqrt{2}}\ (1\leqq a<b\leqq m),$$

$$R\left(X,\frac{e_{2a-1\,2b}-e_{2a\,2b-1}}{\sqrt{2}}\right)X=\frac{(x^a-x^b)^2}{4}\,\frac{e_{2a-1\,2b}-e_{2a\,2b-1}}{\sqrt{2}}\ (1\leqq a<b\leqq m),$$

$$R\left(X,\frac{e_{2a-1\,2b}+e_{2a\,2b-1}}{\sqrt{2}}\right)X=\frac{(x^a+x^b)^2}{4}\,\frac{e_{2a-1\,2b}+e_{2a\,2b-1}}{\sqrt{2}}\ (1\leqq a<b\leqq m),$$

$$R(X, e_{2a-1,2a})\,X=0\ (1\leqq a\leqq m),$$

and if $n=2m+1$,

$$R(X, e_{2a,2m+1})\,X=\frac{(x^a)^2}{4}\,e_{2a,2m+1}\ (1\leqq a\leqq m),$$

$$R(X, e_{2a-1,2m+1})\,X=\frac{(x^a)^2}{4}\,e_{2a-1,2m+1}\ (1\leqq a\leqq m).$$

Thus the tangent conjugate points of e along $\gamma_X: t\to\exp tX$, $X=\Sigma x^a e_{2a-1,2a}$, $\Sigma(x^a)^2=1$ are $\frac{2r\pi}{|x^a\pm x^b|}X$ (with mulipltaicity 2) and $\frac{2r\pi}{|x^a|}X$ (with multiplicity 2 and if $n=2m+1$), and the first tangent conjugate point of e along γ_X is given by $\frac{2\pi}{\mathrm{Max}_{a<b}|x^a\pm x^b|}X$. On the other hand, since $\Gamma(\boldsymbol{SO}(n)):=\{X\in\mathfrak{a}^*:\exp X=e\}=\{\sum_k 2\pi m_k\,e_{2k-1,2k}\,;\ (m_k)\in\boldsymbol{Z}^k\}$, we get for $X=\Sigma x^a e_{2a-1,2a}$, $\Sigma(x^a)^2=1$,

$$\tilde{t}_0(X) = \frac{\pi}{\mathrm{Max}|x^a|} \left(\leqq \frac{2\pi}{\underset{a<b}{\mathrm{Max}}|x^a \pm x^b|} \right)$$

and consequently tangent cut locus of e

$$:= \bigcup_{\Sigma(x^a)^2=1} \mathrm{Ad}G \left(\Sigma \frac{\pi x^a}{\mathrm{Max}|x^a|} e_{2a-1,2a} \right).$$

REMARK 3.7. The cut point $\tilde{t}_0(X)X$ of e along γ_X is the first conjugate point if and only if there exists some $a<b$ such that $|x^a|=|x^b|=\mathrm{Max}\,|x^a|$. Note that $\pi_1(\boldsymbol{SO}(n))=\boldsymbol{Z}_2$.

Next, let F be a real n-dimensional oriented euclidean vector space with a euclidean norm $\|\ \|$, and consider $E=F\oplus F$ with the induced orientation. Let ϕ be a non-degenerate quadratic form on E defined by $\phi([x, y)]=\|x\|^2-\|y\|^2$. We denote by pr_1 (resp pr_2) the projection mapping $E\to F$ on the first factor (resp. the second factor). Now for an n-dimensional oriented subspace $P\subset E$ on which ϕ vanishes identically, it is easy to see that pr_1, pr_2 P: $E\to F$ are linear isomorhpisms. We shall say that P is positively oriented if pr_1, pr_2 are orientation preserving isomorphisms. We define $O(E:\phi):=\{P\subset E;$ positively oriented n-dimensional subspace on which ϕ vanishes identcially$\}$. Then $O(E:\phi)$ has a natural topology as a subspace of real oriented Grassmann manifold. Moreover $\boldsymbol{SO}(n)$ is homeomorphic to $O(E:\phi)$ via the map $\Phi: \boldsymbol{SO}(n)\to O(E, \phi)$ defined by $\Phi(O)=\{(x, Ox)|x\in F\}$. Especially $\Delta=\Phi(I)$ is the diagonal set. Put $\Delta^\perp=\{(x, -x)|x\in F\}$, then $\Delta^\perp\in O(E, \phi)$ if $n=2m$ but $\Delta^\perp\notin O(E, \phi)$ if $n=2m+1$. Now as in the case of $\boldsymbol{U}(n)$, we define $\Gamma_{\Delta^\perp}:=\left\{Q\in O(E, \phi);\ Q\cap\Delta^\perp\neq\{0\}\right\}$. Note that dim $(Q\cap\Delta^\perp)$ is always even for any $Q\in O(E, \phi)$. Now we have

THEOREM 3.8. *Let $O(E, \phi)$ carry the riemannian structure induced via Φ from the riemannian structure of $\boldsymbol{SO}(n)$ stated above. Then the cut locus of Δ is given by $\Gamma_{\Delta^\perp}$*

Proof. Firstly we shall shall show that for $X=\sum_{a=1}^m x^a e_{2a-1,2a}$, $(\Sigma(x^a)^2=1)$ with $\mathrm{Max}\,|x^a|=|x^{i_1}|=\cdots=|x^{i_k}|$, $\dim\Phi(\exp \tilde{t}_0(X)X)\cap\Delta^\perp=2k$ holds. In fact we may assume $|x^1|=\cdots=|x^k|>|x^{k+1}|\geqq\cdots\geqq|x^m|$. Then we see that

$$\Phi(\exp\ t_0(X)(X)) = \left\{ \left(\begin{pmatrix} u_1 \\ \vdots \\ u_{2k} \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{pmatrix}, \begin{pmatrix} -u_1 \\ \\ -u_{2k} \\ \cos\frac{\pi}{|x_1|}x_{k+1}\cdot u_{2k+1} - \sin\frac{\pi}{|x_1|}x_{k+1}\cdot u_{2k+2} \\ \sin\frac{\pi}{|x_1|}x_{k+1}\cdot u_{2k+1} + \cos\frac{\pi}{|x_1|}x_{k+1}\cdot u_{2k+2} \\ \vdots \end{pmatrix} \right) \right\},$$

has a $2k$-dimensional intersection with $\Delta^{\perp}$.

Similarly we get $\Phi(\exp tX)\cap\Delta^{\perp}=0$ if $0\leqq t<\tilde{t}_0(X)$. Since Δ and $\Delta^{\perp}$ are invariant under the action of Ad $\boldsymbol{SO}(n)$, we get

$$\dim \Phi\left(\exp(\mathrm{Ad}\, g\, \tilde{t}_0(X)\, X)\right)\cap\Delta^{\perp}>0 \text{ for every } g\in\boldsymbol{SO}(n)$$

and dim $\Phi(\exp \mathrm{Ad}\, g\, tX)\cap\Delta^{\perp}=0$ for every $g\in\boldsymbol{SO}(n)$ and $0\leqq t<\tilde{t}_0(X)$. So we know

$$\begin{aligned} O(E,\psi) &= \textit{Cut locus} \text{ of } \Delta\cup Int(\Delta)\\ &\subset \bigcup_{X\in\mathfrak{a}^*,|X|=1}\Phi(\mathrm{Ad}\,\boldsymbol{SO}(n)\exp\tilde{t}_0(X)\,X)\cup\\ &\quad\left(\bigcup_{\substack{X\in\mathfrak{a}^*\\|X|=1}}\ \bigcup_{0\leqq t<\tilde{t}_0(X)}\Phi(\mathrm{Ad}\,\boldsymbol{SO}(n)\exp tX)\right)\\ &\subset\Gamma_{\Delta^{\perp}}\cup\Gamma^0_{\Delta^{\perp}}=O(E,\psi), \end{aligned}$$

where $\Gamma^0_{\Delta^{\perp}}$ denotes the subset of $O(E,\psi)$ defined by $\Gamma^0_{\Delta^{\perp}}=\left\{P\in O(E,\psi)|P\cap\Delta^{\perp}=\{0\}\right\}$. Q. E. D.

REMARK 3.8. If we put $\Gamma^k_{\Delta^{\perp}}:=\{P\in O(E,\psi);\ \dim\ P\cap\Delta^{\perp}=2k\}=\bigcup_{X=\Sigma x^{a}e_{2a-1,2a},\,\mathrm{Max}|x^{a}|=|x^{a_1}|=\cdots=|x^{a_k}|}\Phi(\mathrm{Ad}\,\boldsymbol{SO}(n)\exp\tilde{t}_0(X)\,X)$, then Γ^k_{Δ} is a submanifold of $O(E,\psi)$ of codimension $k(2k-1)$.

§4. Examples

In this section, we shall determine the cut locus of a point in some standard non-simply connected symmetric spaces by the method of §2.

1°. **Real Grassmann manifold** $\mathrm{M}=\boldsymbol{SO}(n+m)/\{(\boldsymbol{O}(n)\times\boldsymbol{O}(m))\cap\boldsymbol{SO}(n+m)\}$(dim $M=mn\geqq 2, m\geqq n$). In this case $\pi_1(M)\cong\boldsymbol{Z}_2$. Now put $\mathfrak{g}=\mathfrak{so}(n+m)=\left\{\begin{pmatrix}A & B\\ -{}^tB & D\end{pmatrix}\middle|\begin{matrix}{}^tA+A=0\\ {}^tD+D=0\end{matrix}\right\}$, $\mathfrak{k}=\mathfrak{o}(n)\times\mathfrak{o}(m)=\left\{\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\middle|\begin{matrix}{}^tA+A=0\\ {}^tD+D=0\end{matrix}\right\}$, $\mathfrak{m}=\left\{\begin{pmatrix}0 & B\\ -{}^tB & 0\end{pmatrix}\right\}$ where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of $\boldsymbol{SO}(n+m)$ (resp. $\boldsymbol{SO}(n+m)\cap(\boldsymbol{O}(n)\times\boldsymbol{O}(m))$) and $\mathfrak{k}$ (resp. $\mathfrak{m}$) is the $+1$ (resp. -1) -eigenspace of the involutive automorphism $d\theta$ in $\mathfrak{so}(n+m)$ which is induced from $\theta\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}A & -B\\ -C & D\end{pmatrix}$ on $\boldsymbol{SO}(n+m)$. Then $(\boldsymbol{SO}(n+m), \boldsymbol{SO}(n+m)\cap(\boldsymbol{O}(n)\times\boldsymbol{O}(m)), \theta)$ is a compact riemannian symmetric pair with the invariant metric $\langle\ ,\ \rangle$ defined by $Q(X, Y):=-\frac{1}{2}\mathrm{tr}\ XY,\ X, Y\in\mathfrak{m}$. Now

$$e_{ij}:=\begin{array}{cc} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \\ \\ \end{array} & \left(\begin{array}{cc|c} & 1 & \\ -1 & & 0 \\ \hline 0 & & 0 \end{array}\right) \end{array},\quad f_{\alpha\beta}:=\begin{array}{cc} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \\ \alpha \\ \beta \end{array} & \left(\begin{array}{c|cc} 0 & & 0 \\ \hline 0 & & 1 \\ & -1 & \end{array}\right) \end{array},\quad g_{i\alpha}:=\begin{array}{cc} & \begin{array}{cc} i & \alpha \end{array} \\ \begin{array}{c} i \\ \alpha \end{array} & \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \end{array}$$

$$(1\leqq i<j\leqq n)\quad (n+1\leqq\alpha<\beta\leqq n+m)\quad (1\leqq i\leqq n,\ n+1\leqq\alpha\leqq n+m)$$

forms an orthonormal basis with respect to $\langle\ ,\ \rangle$, and $\{g_{i\alpha}\}$ forms an orthonormal basis for $\mathfrak{m}$. Now from Lie multiplication table

$$\begin{aligned}[][g_{i\alpha},\ g_{j\beta}] &= -\delta_{\alpha\beta}\,e_{ij}-\delta_{ij}\,f_{\alpha\beta}\,,\\ [g_{i\alpha},\ e_{jk}] &= -\delta_{ik}\,g_{j\alpha}+\delta_{ij}\,g_{k\alpha}\,,\\ [g_{i\alpha},\ f_{\delta\alpha}] &= \delta_{\alpha\gamma}\,g_{i\delta}-\delta_{\alpha\delta}\,g_{i\gamma}\,,\end{aligned}$$

we know that $\{g_{1,n+1},\cdots,g_{n,2n}\}$ forms a Cartan subalgebra $\mathfrak{a}$ of $(\mathfrak{g},\mathfrak{k},d\theta)$ and for $X=\sum_{i=1}^{n}x_i\,g_{i\ n+i}$ we have

$$R(X,g_{j\beta})\,X=\begin{cases}(x_j^2+x_{\beta-n}^2)\,g_{j\beta}-2x_j\,x_{\beta-n}\,g_{\beta-n\ n+j}\,, & \text{if } \beta\leqq 2n\\ x_j^2g_{j\beta}\,, & \text{if } m>n \text{ and } \beta>2n.\end{cases}$$

So the eigenvalues of the symmetric linear transformation $V\to R(X,V)\,X$ on $\mathfrak{m}$ are given by 0 (with eigenvectors $g_{j,n+j}\,(j=1,\cdots,n)$), $(x_j-x_k)^2$ (with eigenvector $1/\sqrt{2}\,\{g_{j,n+k}+g_{k,n+j}\}\ (j<k)$), $(x_j+x_k)^2$ (with eigenvector $1/\sqrt{2}\,\{g_{j,n+k}-g_{k,n+j}\}\ (j<k)$) and x_j^2 (with eigenvectors $g_{j\beta}\,(n+m\geqq\beta>2n)$, where x_j^2 occurs only when $m>n$. Thus the tangent conjugate locus of o is given by $\bigcup\left\{\mathrm{ad}h\left(\sum_{i=1}^{n}\frac{rx^i\pi}{|x_p\pm x_q|}\,g_{i,n+i}\right);\ h\in K=\{\boldsymbol{O}(n)\times\boldsymbol{O}(m)\}\cap\boldsymbol{SO}(n+m),\ \Sigma(x_i)^2=1,\right.$ $\left.p<q,\ r\in\boldsymbol{Z}-\{0\}\text{ (multiplicity 1)}\right\}$ $(n>1)$, and $\bigcup\left\{\mathrm{ad}h\left(\sum_{i=1}^{n}\frac{rx_i\pi}{|x_p|}\,g_{i\ n+i}\right);\ h\in K,\right.$ $\left.\Sigma x_i^2=1,\ p=1,\cdots,n,\ r\in\boldsymbol{Z}-\{0\}\text{ (multiplicity } m-n)\right\}$ if $m>n$. Especially the first tangent conjugate point $t_0(X)\,X$ of o along a geodesic γ_X, $X=\Sigma x_i\,g_{i\ n+i}$ $(\Sigma x_i^2=1)$ is given by

$$t_0(X)=\operatorname*{Min}_{p<q}\frac{\pi}{|x_p\pm x_q|}\ \text{ if } n<1 \text{ (and } =\pi \text{ if } n=1).$$

On the orther hand we have $\Gamma(G,K)=\left\{A\in\mathfrak{a}\,|\exp\ A\in(\boldsymbol{O}(n)\times\boldsymbol{O}(m))\cap\boldsymbol{SO}\right.$ $\left.(n+m)\right\}=\left\{\sum_{i=1}^{n}m_i\,\pi g_{i\ n+i}\right\}$ and thus we get

$$\tilde{t}_0(X):=\operatorname*{Min}_{A\in\Gamma(G,K)-\{0\}}\frac{\langle A,A\rangle}{2\langle X,A\rangle}=\frac{\pi}{2\,\mathrm{Max}|x_p|}\ \left(\leqq\frac{\pi}{\operatorname*{Max}_{p<q}|x_p\pm x_q|}\right).$$

So we know that the tangent cut locus of o is given by $\left\{\mathrm{ad}h\left(\sum_{i=1}^{n}\frac{\pi x_i}{2\,\mathrm{Max}|x_p|}g_{i\,n+i}\right);\ h\in K,\ \Sigma x_i^2=1\right\}$. Now $\boldsymbol{SO}(n+m)/(\boldsymbol{O}(n)\times\boldsymbol{O}(m))\cap\boldsymbol{SO}(n+m)$ may be identified with the manifold $G_{n,m}(\boldsymbol{R})$ of all n-dimensional subspaces P of $\boldsymbol{R}^{n+m}$. Put

$$o(=\boldsymbol{R}^n):=\left\{\begin{pmatrix}u_1\\ \vdots\\ u_n\\ 0\\ \vdots\\ 0\end{pmatrix};\ u_i\in\boldsymbol{R}\right\},\quad o^{\perp}(=\boldsymbol{R}^m):=\left\{\begin{pmatrix}0\\ \vdots\\ 0\\ u_{n+1}\\ \vdots\\ u_{n+m}\end{pmatrix};\ u_i\in\boldsymbol{R}\right\}\ \text{and}$$

$W_l:=\{Z\in G_{n,m};\ \dim Z\cap o^{\perp}=l\}$. Note that W_l is a submanifold of codimension l^2. Then we get a result which is obtained previously by Y-C. Wong by a different more geometric argument.

THEOREM 4.1. (*Y-C. Wong* [12]). *Cut locus of o in $G_{n,m}(\boldsymbol{R})$ is given by $V_1=W_1\cup\cdots\cup W_n$.*

Proof. First, note that every geodesic in $G_{n,m}(\boldsymbol{R})$ emanating from o may be expressed in the form $t\to\exp t\ \mathrm{Ad}h(\Sigma x_i g_{i\,n+i})\cdot o$, where $h\in K$, $\Sigma x_i^2=1$. Now for $X=\sum_{i=1}^{n}x_i g_{i\,n+i}\ (\Sigma(x_i)^2=1)$ with $\operatorname*{Max}_i|x_i|=|x_{i_1}|=\cdots=|x_{i_k}|$, we shall show that $\exp\tilde{t}_0(X)\cdot o\in W_k$. In fact we may assume $\mathrm{Max}\,|x_i|=|x_1|=\cdots=|x_k|>|x_{k+1}|\geqq\cdots\geqq|x_n|$. Then we have

$$\exp\tilde{t}_0(X)\cdot o=\Big\{{}^t\Big(\underbrace{0,\cdots,0}_{k},\ \cos\frac{x_{k+1}\pi}{2|x_1|}u_{k+1},\cdots,\cos\frac{x_n\pi}{2|x_1|}u_n,$$
$$-u_1,\cdots,-u_k,-\sin\frac{x_{k+1}\pi}{2|x_1|}u_{k+1},\cdots,-\sin\frac{x_n\pi}{2|x_1|}u_n,\underbrace{0,\cdots,0}_{m-n}\Big);$$
$$(u_1,\cdots,u_n)\in\boldsymbol{R}^n\Big\},$$

from which our assertion is obvious. Similarly we know that

$$\exp tX\cdot o\in W_0\ \text{for}\ 0\leqq t<\tilde{t}_0(X).$$

Since $\mathrm{Ad}\big((\boldsymbol{O}(n)\times\boldsymbol{O}(m))\cap\boldsymbol{SO}(n+m)\big)$ leaves o and $o^{\perp}$ invariant, we get

$$G_{n,m}(\boldsymbol{R})=\mathrm{Int}(o)\cup\text{Cut locus of }o\subset W_0\cup\bigcup_{k=1}^{n}W_k=G_{n,m}(\boldsymbol{R}).$$

Q. E. D.

REMARK 4.2. If $m>n=1$, then $G_{1,m}(\boldsymbol{R})$ is the m-dimensional real projective space. In this case the tangent conjugate locus of o is $\cup\,\mathrm{Ad}\,k(m\pi)\,g_{1,2}$ and the conjugate locus of o is $\{o\}$.

REMARK 4.3. Put $V_l:=W_l\cup\cdots\cup W_n$, $\widetilde{W}_k:=\{Z\in G_{n,m}(\boldsymbol{R});\ \dim(Z\cap o)$

$=k\}$, and $\widetilde{V}_l=\widetilde{W}_l \cup \cdots \cup \widetilde{W}_n$. Then for conjugate locus of o, Wong [13] has announced a result which asserts that conjugate locus of o is given by $V_2 \cup \widetilde{V}_1$ if $n<m$ and $V_2 \cup \widetilde{V}_2$ if $n=m$. But his result seems not to be correct. In fact in general, $\left(\exp \Sigma \frac{x_i \pi}{|x_p \pm x_q|} g_{i\,n+i}\right) \cdot o$ has zero intersection with o or $o^\perp$. Since $G_{1,m}(\boldsymbol{R})$ is the n-dimensional real projective space for $n=1$, we shall consider the first conjugate locus $\bigcup_{\substack{h \in K \\ X \in \mathfrak{a}, \langle X, X\rangle=1}} (\exp \mathrm{Ad}\, h t_0(X)\, X) \cdot o$ of o for $n \geqq 2$. Then the first tangent conjugate point $t_0(X)\, X$ of o is the cut point if and only if $\mathrm{Max}\,|x_p|$ is taken at least for two x_p's. We shall define for $X=\Sigma x_i\, g_{i,n+i}\, (\Sigma x_i^2=1)$, $\sharp X=k$ when $\mathrm{Max}\,|x_p|=|x_{i_1}|=\cdots=|x_{i_k}|$. Then as the above theorem shows we have for $l \geqq 2$, $\cup \{\exp \mathrm{Ad}\, h t_0(X)\, X \cdot o\,;\ h \in K,\ X \in \mathfrak{a},\ \langle X, X\rangle=1,\ \sharp X=l\}=W_l$. On the otherhand for X with $\sharp X=1$ $(\mathrm{Max}\,|x_p|=|x_i|)$ we have

$$1 \geqq \frac{|x_i|}{\mathrm{Max}|x_p \pm x_q|}>1/2, \qquad 0 \leqq \frac{|x_j|}{\mathrm{Max}|x_p \pm x_q|}<1/2 \quad (j \neq i),$$

where $\frac{|x_i|}{\mathrm{Max}|x_p \pm x_q|}=1$ if and only if $|x_i|=1$ and other x_j's are equal to zero. So if $\sharp X=1$, $\cup \{(\exp \mathrm{Ad}\, h t_0(X)\, X) \cdot o\} \subset W_0$. On the other hand for $\sharp X=1$, $(\exp \mathrm{Ad}\, h t_0(X)\, X) \cdot o \in \widetilde{V}_1$ if and only if $x_j=0$ for at least one j and $(\exp (\mathrm{Ad}\, h t_0(X)\, X)) \cdot o=o$ if and only if $x_j=1$ holds for some j. But the whole conjugate locus seems to be fairly complicated and we shall only mention the following:

$$\cup \left\{\left(\exp \mathrm{Ad}\, h \left(\sum_i \frac{r x_i}{|x_p|} \pi g_{i,n+i}\right)\right) o\,;\ h \in K,\ \Sigma x_i^2=1\right\}=\widetilde{V}_1 \text{ for } r \in \boldsymbol{Z}-\{0\}.$$

$$\cup \left\{\left(\exp \mathrm{Ad}\, h \left(\sum_i \frac{r x_i}{|x^p \underset{(+)}{-} x^q|} \pi g_{i,n+i}\right)\right) o\,;\ h \in K,\ \Sigma x_i^2=1,\ x^p \underset{(-)}{+} x^q=0\right\} \text{ is equal to}$$

V_2 if r is odd and is equal to $\widetilde{V}_2$ is r is even.

REMARK 4.4. In case of complex or quaternionic Grassmann manifolds $G_{n,m}(\boldsymbol{C})$, $G_{n,m}(\boldsymbol{H})$ the cut locus and the first conjugate locus coincides because they are simply connected. In these cases Wong [13] has also investigated the conjugate loci, which seems to be incorrect. So we shall give the tangent conjugate loci of $G_{n,m}(\boldsymbol{C})$ and $G_{n,m}(\boldsymbol{H})$.

2°. **Complex Grassmann menifold $G_{n,m}(\boldsymbol{C})=\boldsymbol{U}(n+m)/\boldsymbol{U}(n) \times \boldsymbol{U}(m)$ $(n \leqq m)$.** Put $G=\boldsymbol{U}(n+m)$, $K=\boldsymbol{U}(n) \times \boldsymbol{U}(m)$ and let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). If we define the involutive automorphism θ of G by $\theta \begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$, then $\mathfrak{k}$ $\left(\text{resp. } \mathfrak{m}=\left\{\begin{pmatrix} 0 & B \\ -{}^t\bar{B} & 0 \end{pmatrix}\right\}\right)$ is the $+1$ (resp.

-1) -eigenspace of $d\theta$. Now if we put $D_{i\alpha} := \begin{array}{c} \\ i \\ \alpha \end{array}\!\!\begin{array}{c} \begin{array}{cc} i & \alpha \end{array} \\ \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \end{array}$ $(1 \leqq i \leqq n,\ n+1 \leqq \alpha \leqq n+m)$, $E_{i\alpha} := \begin{array}{c} \\ i \\ \alpha \end{array}\!\!\begin{array}{c} \begin{array}{cc} i & \alpha \end{array} \\ \left(\begin{array}{c|c} 0 & \sqrt{-1} \\ \hline \sqrt{-1} & 0 \end{array}\right) \end{array}$ $(1 \leqq i \leqq n,\ n+1 \leqq \alpha \leqq n+m)$, then $\{D_{i\alpha}, E_{i\alpha}\}$ forms an orthonormal basis of $\mathfrak{m}$ $(=T_o G_{n,m}(\boldsymbol{C}))$ with respect to the invariant riemnnian structure defined by $Q(X, Y) := -\frac{1}{2}\operatorname{tr} XY$, $X, Y \in \mathfrak{g}$. Note that $\{D_{i,n+i}\}_{i=1,\cdots,n}$ forms a Cartan subalgebra $\mathfrak{g}$ of $(\mathfrak{a}, \mathfrak{k}, d\theta)$. Then for $X := \sum_{i=1}^{n} x^i D_{i\ n+i}$ $(\Sigma x_i^2 = 1)$ we get

$$R(X, D_{j\ n+j})\ X = 0\ (j=1, \cdots, n),$$
$$R\left(X, \frac{1}{\sqrt{2}}(D_{j\alpha} + D_{\alpha-n\ n+j})\right) X = (x^j - x^{\alpha-n})^2 \frac{1}{\sqrt{2}}(D_{j\alpha} + D_{\alpha-n\ n+j}) \quad (n+1 \leqq n+j < \alpha \leqq 2n),$$
$$R\left(X, \frac{1}{\sqrt{2}}(D_{j\alpha} - D_{\alpha-n\ n+j})\right) X = (x^j + x^{\alpha-n})^2 \frac{1}{\sqrt{2}}(D_{j\alpha} - D_{\alpha-n\ n+j}) \quad (n+1 \leqq n+j < \alpha \leqq 2n),$$
$$R\left(X, \frac{1}{\sqrt{2}}(E_{j\alpha} - E_{\alpha-n\ n+j})\right) X = (x^j - x^{\alpha-n})^2 \frac{1}{\sqrt{2}}(E_{j\alpha} - E_{\alpha-n\ n+j}) \quad (n+1 \leqq n+j < \alpha \leqq 2n),$$
$$R\left(X, \frac{1}{\sqrt{2}}(E_{j\alpha} + E_{\alpha-n\ n+j})\right) X = (x^j + x^{\alpha-n})^2 \frac{1}{\sqrt{2}}(E_{j\alpha} + E_{\alpha-n\ n+j}) \quad (n+1 \leqq n+j < \alpha \leqq 2n),$$
$$R(X, E_{j\ n+j})\ X = (2x^j)^2 E_{j\ n+j}(j=1, \cdots, n),$$
$$R(X, D_{j\alpha})\ X = (x^j)^2 D_{j\alpha}(\alpha > 2n \text{ possible only when } m > n),$$
$$R(X, E_{j\alpha})\ X = (x^j)^2 E_{jn}(\alpha > 2n \text{ possible only when } m > n).$$

Thus the tangent conjugate locus is given by

$$\left\{\operatorname{ad}h\left(\sum_{i=1}^{\alpha} \frac{r\pi x_i}{|x^p \pm x^q|} D_{i,n+i}\right);\ h \in K,\ \Sigma x_i^2 = 1,\ p < q,\ r \in \boldsymbol{Z} - \{0\} \quad (\text{multiplicity } 2)\ (n > 1)\right\},$$

$$\left\{\operatorname{ad}h\left(\sum_{i=1}^{n} \frac{r\pi x_i}{2|x^p|} D_{i,n+i}\right);\ h \in K,\ \Sigma x_i^2 = 1,\ p = 1, \cdots, n,\ r \in \boldsymbol{Z} - \{0\} \quad (\text{multiplicity } 1)\right\}.$$

and if $m > n$

$$\left\{\mathrm{ad}h\left(\sum_{i=1}^{n}\frac{r\pi}{|x_p|}x_i D_{i\,n+i}\right);\ h\in K,\ \Sigma x_i^2=1,\ p=1,\ \cdots,\ n,\ r\in \boldsymbol{Z}-\{0\}\right.$$
$$\left.(\text{multiplicity } 2(m-n))\right\}.$$

On the other hand in this case $\Gamma(G, K):=\{A\in\mathfrak{a}|\exp A\in K\}=\{\Sigma m_i\,\pi D_{i,n+i}|(m_i)\in\boldsymbol{Z}^n\}$ and $\tilde{t}_0(X):=\underset{A\in\Gamma(G,K)-\{0\}}{\mathrm{Min}}\dfrac{\langle A, A\rangle}{2|\langle X, A\rangle|}=\underset{p}{\mathrm{Min}}\dfrac{\pi}{2|x_p|}$ for $X=\Sigma x_i D_{i\,n+i}$. Thus the (tangent) cut locus coincides with the first (tangent) conjugate locus. Now $G_{n,m}(\boldsymbol{C})$ is the space of all complex n-dimensional subspaces of $\boldsymbol{C}^{n+m}$.

Put $o:=\{{}^t(x_1, \cdots, x_n, \underbrace{0, \cdots, 0}_{m})\in\boldsymbol{C}^{n+m}\}$ and $o^\perp:=\{{}^t(\underbrace{0, \cdots, 0}_{n}, x_{n+1}, \cdots, x_{n+m})\in\boldsymbol{C}^{n+m}\}$ and define $V_l:=\{Z\in G^{n,m}(\boldsymbol{C});\ \mathrm{dim}Z\cap o^\perp\geqq l\}$, $\widetilde{V}_l:=\{Z\in G_{n,m}(\boldsymbol{C})|\mathrm{dim}\,Z\cap o\geqq l\}$. Now the same argument as before tells us that the first conjugate locus is equal to V_1. Now it is easy to see $\cup\left\{\exp\left(\mathrm{Ad}\,h\sum_{i=1}^{n}\dfrac{r\pi}{2|x_p|}x_i D_{i\,n+i}\right)\cdot o;\ h\in K,\ \Sigma x_i^2=1,\ 1\leqq p\leqq n\right\}$ is equal to V_1 if r is odd and is equal to $\widetilde{V}_1$ if r is even. Similarly $\cup\left\{\exp\left(\mathrm{Ad}\,h\sum_{i=1}^{n}\dfrac{r\pi}{|x_p|}x_i D_{i\,n+i}\right)o;\ h\in K,\ \Sigma x_i^2=1,\ 1\leqq p<n\right\}$ is equal to $\widetilde{V}_1$ for $r\in\boldsymbol{Z}$. Thus if $n=1$ (i.e. complex projective case) the conjugate locus is given by by $V_1\cup\widetilde{V}_1$. Now Wong ([13]) has announced that the conjugate locus of o is given by $V_1\cup\widetilde{V}_1$ for $G_{n,m}(\boldsymbol{C})$. But $\exp\left(\mathrm{Ad}\,h\,\Sigma\dfrac{r\pi x_i}{|x_p\pm x_q|}D_{i\,n+i}\right)\cdot o$ has generally zero intersection with o or $o^\perp$ and for $n>1$ the conjugate locus seems to be fairly complicated. We shall remark only $\cup\left\{\exp\left(\mathrm{Ad}\,h\sum_{i=1}^{n}\dfrac{r\pi}{x_p+x_q}x_i D_{i\,n+i}\right)o;\ h\in K,\ \Sigma x_i^2=1,\ x_p=x_q\right\}$ is equal to V_2 if r is odd and is equal to $\widetilde{V}_2$ if r is even, and $\cup\left\{\exp\left(\mathrm{Ad}\,h\sum_{i=1}^{n}\dfrac{r\pi}{x_p-x_q}x_i D_{i\,n+i}\right)o;\ h\in K,\ \Sigma x_i^2=1,\ x_p+x_q=0\right\}$ is equal to V_2 if r is odd and is equal to $\widetilde{V}_2$ if r is even.

3°. **Quaternion Grassmann manifolds $G_{n,m}(\boldsymbol{H})=\boldsymbol{Sp}(n+m)/\boldsymbol{Sp}(n)\times\boldsymbol{Sp}(m)$ $(n\leqq m)$.** Put $G=\boldsymbol{Sp}(n+m)$, $K=\boldsymbol{Sp}(n)\times\boldsymbol{Sp}(m)$ and let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebra of G, K. We may consider $\mathfrak{sp}(n)=\left\{\begin{pmatrix}\alpha & \beta\\ -{}^t\bar{\beta} & \bar{\alpha}\end{pmatrix}\in\boldsymbol{U}(2n);\ \beta \text{ symmetric}\right\}$ and then $\mathfrak{g}$ may be identified with $\left\{\begin{array}{c|cc} & 2n & 2m\\ \hline 2n & A & B\\ 2m & -{}^t\bar{B} & D\end{array};\ A\in\mathfrak{sp}(n),\ D\in\mathfrak{sp}(m),\right.$

$B=\begin{array}{cc} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ n \end{array} & \left(\begin{array}{c|c} \alpha & \beta \\ \hline -{}^t\bar{\beta} & \bar{\alpha} \end{array}\right) \end{array} \in \mathfrak{M}_{m+n}(\boldsymbol{C})\Big\}$, If we define an involutive automorphism θ on $\boldsymbol{Sp}(n+m)$ by $\theta\begin{array}{c} \begin{array}{cc} 2n & 2m \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$, then $\mathfrak{k}$ (resp. $\mathfrak{m}=\Big\{\begin{pmatrix} 0 & B \\ -{}^t\bar{B} & 0 \end{pmatrix}$; $B=\begin{pmatrix} \alpha & \beta \\ -{}^t\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathfrak{M}_{m+n}(\boldsymbol{C})\Big\}$) is the $+1$ (resp. -1)-eigenspace of $d\theta$. Now $\mathfrak{m}$ has the following orthonormal basis $\{H_{ij}, K_{ij}, L_{i\alpha}, M_{i\alpha}\}$ $(1\leqq i\leqq n,\ 1\leqq j\leqq m,\ n+1\leqq\alpha\leqq n+m)$ with respect to the inner product $Q(X, Y)=-\frac{1}{4}\operatorname{tr}XY$, where we put

$$H_{ij} := \left(\begin{array}{c|c} 0 & B_{ij} \\ \hline -{}^t\bar{B}_{ij} & 0 \end{array}\right) \text{ with } B_{ij} = \begin{array}{cc} & \begin{array}{cc} j & m+j \end{array} \\ \begin{array}{c} i \\ n+j \end{array} & \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) \end{array},$$

$$K_{ij} := \left(\begin{array}{c|c} 0 & C_{ij} \\ \hline -{}^t\bar{C}_{ij} & 0 \end{array}\right) \text{ with } C_{ij} = \begin{array}{cc} & \begin{array}{cc} j & m+j \end{array} \\ \begin{array}{c} i \\ n+j \end{array} & \left(\begin{array}{c|c} \sqrt{-1} & 0 \\ \hline 0 & -\sqrt{-1} \end{array}\right) \end{array},$$

$$L_{i\alpha} := \left(\begin{array}{c|c} 0 & D_{i\alpha} \\ \hline -{}^t\bar{D}_{i\alpha} & 0 \end{array}\right) \text{ with } D_{i\alpha} = \begin{array}{cc} & \begin{array}{cc} \alpha-n & \alpha \end{array} \\ \begin{array}{c} i \\ n+i \end{array} & \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right) \end{array} \text{ and}$$

$$M_{i\alpha} := \left(\begin{array}{c|c} 0 & E_{i\alpha} \\ \hline -{}^t E_{i\alpha} & 0 \end{array}\right) \text{ with } E_{i\alpha} = \begin{array}{cc} & \begin{array}{cc} \alpha-n & \alpha \end{array} \\ \begin{array}{c} i \\ n+i \end{array} & \left(\begin{array}{c|c} 0 & \sqrt{-1} \\ \hline \sqrt{-1} & 0 \end{array}\right) \end{array}.$$

Now $\{H_{ii};\ 1\leqq i\leqq n\}$ forms a Cartan subalgebra of $(\mathfrak{g}, \mathfrak{k}, d\theta)$. Then we get for $X=\sum_{i=1}^{n} x_i H_{ii}$, $\Sigma(x^i)^2=1$,

$$R\Big(X, \frac{1}{\sqrt{2}}(H_{ij}+H_{ij})\Big)\, X = (x_i-x_j)^2\frac{1}{\sqrt{2}}(H_{ij}+H_{ji}) \quad 1\leqq i<j\leqq n,$$

$$R\Big(X, \frac{1}{\sqrt{2}}(H_{ij}-H_{ji})\Big)\, X = (x_i+x_j)^2\frac{1}{\sqrt{2}}(H_{ij}-H_{ji}) \quad 1\leqq i<j\leqq n,$$

$$R(X, H_{ii})\, X = 0, \quad i=1, \cdots, n,$$

$$R(X, H_{ij})\, X = x_i^2 H_{ij} \text{ for } j>n \text{ (possible only when } m>n),$$

$$R\Big(X, \frac{1}{\sqrt{2}}(K_{ij}+K_{ji})\Big)\, X = (x_i+x_j)^2\frac{1}{\sqrt{2}}(K_{ij}+K_{ji}) \quad 1\leqq i<j\leqq n,$$

$$R\left(X, \frac{1}{\sqrt{2}}(K_{ij}-K_{ji})\right) X=(x_i-x_j)^2 \frac{1}{\sqrt{2}}(K_{ij}-K_{ji}), \quad 1 \leqq i<j \leqq n,$$
$$R(X, K_{ii}) X=(2x_i)^2 K_{ii}, \quad i=1, \cdots, n,$$
$$R(X, K_{ij}) X=(x_i)^2 K_{ij} \text{ for } j>n \text{ (possible only when } m>n),$$
$$R\left(X, \frac{1}{\sqrt{2}}(L_{i\alpha}+L_{\alpha-n,n+i})\right) X=(x_i+x_{\alpha-n})^2 \frac{1}{\sqrt{2}}(L_{i\alpha}+L_{\alpha-n\; n+i}) \quad n+i<\alpha \leqq 2n,$$
$$R\left(X, \frac{1}{\sqrt{2}}(L_{i\alpha}-L_{\alpha-n,n+i})\right) X=(x_i-x_{\alpha-n})^2 \frac{1}{\sqrt{2}}(L_{i\alpha}-L_{\alpha-n\; n+i}) \quad n+i<\alpha \leqq 2n,$$
$$R(X, L_{i\; n+i}) X=(2x_i)^2 L_{i\; n+i}, \quad i=1, \cdots, n,$$
$$R(X, L_{i\alpha}) X=x_i^2 L_{i\alpha} \text{ for } \alpha>2n \text{ (possible onlywhen } m>n),$$
$$R\left(X, \frac{1}{\sqrt{2}}(M_{i\alpha}+M_{\alpha-n\; n+i})\right) X=(x_i+x_{\alpha-n}) \frac{1}{\sqrt{2}}(M_{i\alpha}+M_{\alpha-n\; n+i}) X \quad n+j<\alpha \leqq 2n,$$
$$R\left(X, \frac{1}{\sqrt{2}}(M_{i\alpha}-M_{\alpha-n\; n+i})\right) X=(x_i-x_{\alpha-n}) \frac{1}{\sqrt{2}}(M_{i\alpha}-M_{\alpha-n\; n+i}) X \quad n+j<\alpha \leqq 2n,$$
$$R(X, M_{i\; n+j}) X=(2x_i)^2 M_{i\; n+i}, \quad i=1, \cdots, n,$$
$$R(X, M_{i\alpha}) X=x_i^2 M_{i\alpha} \text{ if } \alpha>2n \text{ (possible only when } m>n).$$

Thus the tangent conjugate locus of $M=G_{n,m}(\boldsymbol{H})$ is given by $\bigcup_{p<q}\left\{\mathrm{Ad}\, h \sum_{i=1}^{n} \frac{r\pi x_i}{|x^p \pm x^q|} H_{ii} ;\, h \in K,\, \sum_{i=1}^{n} x_i^2=1,\, r \in \boldsymbol{Z}-\{0\} \text{ (with mutiplicity 4)}\right\}$, (possible only when $n>1$), $\bigcup_{p}\left\{\mathrm{Ad}\, h \sum_{i=1}^{n} \frac{r\pi x_i}{2|x_p|} H_{ii} ;\, h \in K,\, \Sigma x_i^2=1,\, r \in \boldsymbol{Z}-\{0\} \text{ (with multiplicity 3)}\right\}$, and $\bigcup_{p}\left\{\mathrm{Ad}\, h \sum_{i=1}^{n} \frac{r\pi x_i}{|x_p|} H_{ii} ;\, h \in K,\, \Sigma x_i^2=1,\, r \in \boldsymbol{Z}-\{0\} \text{ (with multiplicity } 4(m-n)\right\}$ (possible only when $m>n$). On the other hand, in this case $\Gamma(G, K)=\{A \in \mathfrak{a} \mid \exp A \in K\}=\{\Sigma m_i \pi H_{ii} ;\, m_i \in \boldsymbol{Z}\}$. So we have $\tilde{t}_0(X) ;=\underset{A \in \Gamma(G,K)-\{0\}}{\mathrm{Min}} \frac{\langle A, A\rangle}{2|\langle X, A\rangle|}=\mathrm{Max} \frac{\pi}{2|x_p|}$ for a unit $X=\Sigma x_i H_{ii} \in \mathfrak{a}$, from which we know that the first conjugate locus coincides with the cut locus of o. Now $G_{n,m}(\boldsymbol{H})$ may be identified with manifold of all quaternionic n dimensional subspaces of $\boldsymbol{H}^{n+m}$. Put $o:=\{{}^t(x_1, \cdots, x_n, 0, \cdots, 0) \in \boldsymbol{H}^{n+m} ;\, x_i \in \boldsymbol{H}\}$ and $o^{\perp}:=\{{}^t(0, \cdots, 0, x_{n+1}, \cdots, x_{n+m}) \in \boldsymbol{H}^{n+m} ;\, x_i \in \boldsymbol{H}\}$ and $V_l:=\{Z \in G_{n,m}(\boldsymbol{H}) ;\, \dim(Z \cap o^{\perp}) \geqq l\}$ and $\tilde{V}_l:=\{Z \in G_{n,m}(\boldsymbol{H}) ;\, \dim(Z \cap o) \geqq l\}$. Then the argument as before shows that the cut locus of o is given by V_1. Now it is easy to see $\bigcup\left\{\left(\exp \mathrm{Ad}\, h \sum_{i=1}^{n} \frac{r\pi x_i}{2|x_p|} H_{ii}\right) \cdot o ;\, h \in K,\, \Sigma x_i^2=1\right\}$ is equal to V_1 (resp. $\tilde{V}_1$) if r

is odd (resp. if r is even), and $\cup\left\{\left(\exp \operatorname{Ad} h \sum_{i=1}^{n} \frac{r\pi x_i}{|x_p|} H_{ii}\right)\cdot o\,;\ h\in K,\ \Sigma x_i^2=1\right\}$ is equal to $\widetilde{V}_1$ for every $r\in \boldsymbol{Z}-\{0\}$. Thus if $n=1$ (i.e. quaternion projective space), then the whole conjugate locus of o is given by $V_1\cup\widetilde{V}_1$. On the other hand Wong [13] has announced that the conjugate locus of o is equal to $V_1\cup\widetilde{V}_1$ for every n, which seems to be incorrect. In fact, generally $\left(\exp \operatorname{Ad} h \Sigma \frac{r\pi}{|x_p\pm x_q|} x_i H_{ii}\right)\cdot o$ has zero intersection with $o^{\perp}$, o, and for $n>1$ the conjugate locus seems to be fairly complicated. We shall mention only the following: $\cup\left\{\exp\left(\operatorname{Ad} h \sum_{i=1}^{n} \frac{r\pi}{x_p+x_q} x_i H_{ii}\right)\cdot o\,;\ h\in K,\ \Sigma x_i^2=1,\ x_p=x_q\right\}$ is equal to V_2 (resp. $\widetilde{V}_2$) if r is odd (resp. even), and $\cup\left\{\exp\left(\operatorname{Ad} h \sum_{i=1}^{n} \frac{r\pi}{x_p-x_q} x_i H_{ii}\right)\cdot o\,;\ h\in K,\ \Sigma x_i^2=1,\ x_p+x_q=0\right\}$ is equal to V_2 (resp. $\widetilde{V}_2$) if r is odd (resp. even).

4°. $\boldsymbol{U}(n)/\boldsymbol{O}(n)$. see [10].

5°. $\boldsymbol{U}(2n)/\boldsymbol{Sp}(n)$. Put $G=\boldsymbol{U}(2n)$, $K=\boldsymbol{Sp}(n)$ and let $\mathfrak{g}=\left\{\left(\begin{array}{c|c} A & B \\ \hline -{}^t\bar{B} & D\end{array}\right);\ A,\ D\in\mathfrak{u}(n)\right\}$ be the Lie algebra of G, $\mathfrak{k}=\left\{\left(\begin{array}{c|c} A & B \\ \hline -\bar{B} & \bar{A}\end{array}\right);\ B \text{ symmetric},\ A\in\mathfrak{u}(n)\right\}$ be the Lie algebra of K. If we define the involutive automorphism θ of $\boldsymbol{U}(2n)$ by $\theta\begin{pmatrix} A & B \\ C & D\end{pmatrix}=\begin{pmatrix} \bar{D} & -\bar{C} \\ -\bar{B} & \bar{A}\end{pmatrix}$, then $\mathfrak{k}$ is the $+1$-eigenspace of $d\theta$ and the -1-eigenspace $\mathfrak{m}$ of $d\theta$ has an orthonormal basis

$$\left\{\mathfrak{A}_i:=\begin{array}{c} \\ i \\ n+i \end{array}\!\!\begin{array}{c} \begin{array}{cc} i & n+i \end{array} \\ \left(\begin{array}{c|c} \sqrt{-2} & 0 \\ \hline 0 & \sqrt{-2}\end{array}\right)\end{array}\ (1\leqq i\leqq n),\quad \mathfrak{D}_{ik}:=\begin{array}{c|cccc} & i & k & n+i & n+k \\ \hline i & & 1 & & \\ k & -1 & & & \\ n+i & & & & -1 \\ n+k & & & 1 & \end{array}\ (1\leqq i<k\leqq n),\right.$$

$$\mathfrak{Z}_{ik}:=\begin{array}{c|cccc} & i & k & n+i & n+k \\ \hline i & & \sqrt{-1} & & \\ k & \sqrt{-1} & & & \\ n+i & & & & \sqrt{-1} \\ n+k & & & \sqrt{-1} & \end{array}\ (1\leqq i<k\leqq n),\quad \mathfrak{F}_{ik}:=\begin{array}{c|cccc} & i & k & n+i & n+k \\ \hline i & & & & 1 \\ k & & & -1 & \\ n+i & & 1 & & \\ n+k & -1 & & & \end{array}$$

$$(1 \leqq i < k \leqq n), \quad \mathfrak{S}_{ik} := \begin{array}{c} \\ i \\ k \\ \\ n+i \\ n+k \end{array} \left(\begin{array}{cc|cc} & & n+i & n+k \\ & & & \sqrt{-1} \\ & 0 & -\sqrt{-1} & \\ \hline & -\sqrt{-1} & & \\ \sqrt{-1} & & & 0 \end{array} \right) \left.(1 \leqq i < k \leqq n)\right\} \text{ with}$$

respect to an invariant riemannian structure defined by $Q(X, Y) = -\frac{1}{4}$ tr XY, X, $Y \in \mathfrak{g}$. Now $\mathfrak{a} := \{\mathfrak{A}_i\,;\ 1 \leqq i \leqq n\}$ is a Cartan subalgebra of $(\mathfrak{g}, \mathfrak{k}, d\theta)$ and the curvature tensors at o are given as follows. For a unit $X \in \Sigma x_i \mathfrak{A}_i \in \mathfrak{a}$,

$$R(X, \mathfrak{D}_{jk})\ X = 2(x_j - x_k)^2\, \mathfrak{D}_{jk}\,,$$
$$R(X, \mathfrak{Z}_{jk})\ X = 2(x_j - x_k)^2\, \mathfrak{Z}_{jk}\,,$$
$$R(X, \mathfrak{F}_{jk})\ X = 2(x_j - x_k)^2\, \mathfrak{F}_{jk}\,,$$
$$R(X, \mathfrak{S}_{jk})\ X = 2(x_j - x_k)^2\, \mathfrak{S}_{jk}\,.$$

Thus the tangent conjugate locus of o is given by $\cup \left\{ \mathrm{Ad}\, h \sum_{i=1}^{n} \frac{m\pi x_i}{\sqrt{2}\,|x_p - x_q|} \mathfrak{A}_i\,;\ h \in K,\ \Sigma x_i^2 = 1,\ p < q,\ m \in \boldsymbol{Z} - \{0\} \right\}$ and especially the first tangent conjugate point of o along a geodesic γ_X: $t \to \mathrm{Exp}\ tX$, $X = \sum_{i=1}^{n} x_i \mathfrak{A}_i \in \mathfrak{a}$ $(\Sigma x_i^2 = 1)$ is given by $\mathrm{Min}_{p<q} \frac{\pi}{\sqrt{2}\,|x_p - x_q|}$. On the other hand, since $\Gamma(G, K) := \{A \in \mathfrak{a}$ $|\exp A \in K\}$ is given by $\left\{ \sum_{i=1}^{n} \frac{m_i \pi}{\sqrt{2}} \mathfrak{A}_i\,;\ m_i \in \boldsymbol{Z} \right\}$, we get $\tilde{t}_0(X) ; = \frac{\pi}{2\sqrt{2}} \mathrm{Min}_{p} \frac{1}{|x_p|}$ $\left(\leqq \mathrm{Min}_{p<q} \frac{\pi}{\sqrt{2}\,|x_p - x_q|} \right)$. Thus the tangent cut locus is given by $\cup \left\{ \mathrm{Ad}\, h \sum_{i=1}^{n} \frac{\pi x_i}{2\sqrt{2}\,\mathrm{Max}|x_p|} \mathfrak{A}_i\,;\ h \in K,\ \Sigma x_i^2 = 1 \right\}$. Now let $\boldsymbol{H}^{2n} = \{(a, b) | a, b \in \boldsymbol{H}^n\}$ be $2n$-dimensional quaternionic right vector space. We shall define a hermitian skew symmetric forms α in $\boldsymbol{H}^{2n}$ by

$$\alpha\Big((a, b), (c, d)\Big) = \langle a, d \rangle - \langle b, c \rangle\,,$$

where $\langle\ ,\ \rangle$ denotes the symplectic product of $\boldsymbol{H}^n$. Let $M = \{P = \{(a, b)\} \subset \boldsymbol{H}^{2n}$: n-diemensional subspace on which $\alpha|_P \equiv 0\}$. Note that $P_0 = \{(a, 0)\}$, $P_0^{\perp} = \{(0, b)\} \in M$. Now we have

LEMMA 4.5. *M is diffeomorphic to $\boldsymbol{U}(2n)/\boldsymbol{Sp}(n)$.*

Proof. Put $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \boldsymbol{Sp}(2n)$. Then $\tilde{A} \in \boldsymbol{Sp}(2n)$ leaves α invariant if and only if $\tilde{A}$ commutes with J. Now let $H=\{A\in \boldsymbol{Sp}(2n)\,;\ \tilde{A}J=J\tilde{A}\}=\left\{\tilde{A}=\begin{pmatrix} A & B \\ -B & A\end{pmatrix}\in \boldsymbol{Sp}(2n)\right\}$, then H acts transitively on M. In fact if P, $Q\in M$, then P, JP (resp. Q, JQ) are orthogonal and so there exists an $\tilde{A}\in \boldsymbol{Sp}(2n)$ which commutes with J and maps P onto Q. Now it is easy to see that the isotropy subgroup of H at P_0 is given by $L:=\left\{\begin{pmatrix} A & 0 \\ 0 & A\end{pmatrix};\ A\in \boldsymbol{Sp}(n)\right\}$, which may be identified with $\boldsymbol{Sp}(n)$. Next we shall show that H is isomorphic to $\boldsymbol{U}(2n)$. In fact, for $\tilde{A}=\begin{pmatrix} A & B \\ -B & A\end{pmatrix}\in \boldsymbol{Sp}(2n)$ set $A=a_1+ja_2$, $B=b_1+jb_2$, where a_i, b_i $(i=1, 2)$ are $n\times n$-complex matrices, and define $\varphi(A):=\begin{pmatrix} a_1+ib_1 & \bar{a}_2+i\bar{b}_2 \\ -a_2-ib_2 & \bar{a}_1+i\bar{b}_1\end{pmatrix}$, then we can show that φ is an isomorphism between H and $\boldsymbol{U}(2n)$ and maps L onto $\left(\left\{\begin{matrix} a_1 & \bar{a}_2 \\ -a_2 & \bar{a}_1\end{matrix}\right)\in \boldsymbol{U}(2n)\,;\ {}^t\bar{a}_1 a_1+{}^t\bar{a}_2 a_2=1,\ {}^t a_1 a_2={}^t a_2 a_1\right\}$. Thus we have a diffeomorphism $\Phi: M=H/L\cong G/K$. Q.E.D.

Now we get

THEOREM 4. 6. *Let M carry the riemannian structure which is induced from that of $\boldsymbol{U}(2n)/\boldsymbol{Sp}(n)$. Then the cut locus of P_0 is given by* $\Gamma_{P_0^\perp}:=\left\{Q\in M\,|\,Q\cap P_0^\perp \neq \{0\}\right\}$.

Proof. We may identify M and $H/\boldsymbol{Sp}(n)$. Let $\Phi: H/\boldsymbol{Sp}(n)\to \boldsymbol{U}(2n)/\boldsymbol{Sp}(n)$ be the diffeomorphism given in the Lemma. Firstly we shall show that $\cup\left\{\varphi^{-1}\left(\exp\left(\mathrm{Ad}h\left(\sum_{i=1}^{n}\frac{\pi x_i}{2\sqrt{2}\,\mathrm{Max}|x_i|}\,\mathfrak{A}_i\right)\right)\right)P_0\,;\ h\in K,\ \Sigma x_i^2=1,\ \mathrm{Max}\,|x_i|=|x_{i_1}|=\cdots=|x_{i_k}|\right\}$ is contained in $\{Q\in M\,|\,\dim\,(Q\cap P_0^\perp)=k\}$, and $\cup\left\{\varphi^{-1}\left(\exp\left(\mathrm{Ad}h\left(\sum_{i=0}^{n} tx_i\,\mathfrak{A}_i\right)\right)\right)P_0;\ h\in K,\ \Sigma x_i^2=1,\ 0\leqq t<\tilde{t}_0(X)\right\}$ is contained in $\{Q\in M\,|\,\dim\,(Q\cap P_0^\perp)=0\}$. In fact, let $\mathrm{Max}\,|x_i|=|x_{i_1}|=\cdots=|x_{i_k}|$. We may assume $\mathrm{Max}\,|x_i|=|x_1|=\cdots=|x_k|>|x_{k+1}|\geqq\cdots\geqq|x_n|$. Then

$$\left(\varphi^{-1}\left(\exp \sum_i \frac{\pi x_i}{2\sqrt{2}\,\mathrm{Max}|x_i|}\,\mathfrak{A}_i\right)\right)(P_0)=$$

$$\left\{\begin{pmatrix} a_1 \cos\left(\frac{\pi}{2}\,\frac{\sqrt{2}\,x_1}{\sqrt{2}\,\mathrm{Max}|x_i|}\right) \\ \vdots \\ a_n \cos\left(\frac{\pi}{2}\,\frac{\sqrt{2}\,x_n}{\sqrt{2}\,\mathrm{Max}|x_i|}\right) \\ -a_1 \sin\left(\frac{\pi}{2}\,\frac{\sqrt{2}\,x_1}{\sqrt{2}\,\mathrm{Max}|x_i|}\right) \\ \vdots \\ -a_n \sin\left(\frac{\pi}{2}\,\frac{\sqrt{2}\,x_n}{\sqrt{2}\,\mathrm{Max}|x_i|}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{k+1} \cos \frac{\pi x_{k+1}}{2\mathrm{Max}|x_i|} \\ a_n \cos \frac{\pi x_n}{2\mathrm{Max}|x_i|} \\ -a_1 \\ \vdots \\ -a_k \\ -a_{k+1} \sin \frac{\pi x_{k+1}}{2\mathrm{Max}|x_i|} \\ \vdots \end{pmatrix}\right\}$$

$$\subset \{Q \in M;\ \dim\ Q \cap P_0^{\perp} = k\}$$

Since $\mathrm{Ad}\,K$ leaves P_0, $P_0^{\perp}$ invariant we get our assertion. Thus we get

$$\begin{aligned} M &= \text{Cut locus of } P_0 \cup \mathrm{Int}\,(P_0) \\ &= \left(\bigcup_{h \in K, \Sigma x_i^2 = 1} \varphi^{-1}\left(\exp\left(\mathrm{Ad}\,h\left(\sum_i \frac{\pi x_i}{2\sqrt{2}\,\mathrm{Max}|x_i|}\,\mathfrak{A}_i\right)\right)\right) P_0\right) \\ &\quad \cup \left(\bigcup_{h \in K, \Sigma x_i^2 = 1} \varphi^{-1}\left(\exp\left(\mathrm{Ad}\,h \bigcup_{0 \leq t < \frac{\pi}{2\sqrt{2}\,\mathrm{Max}|x_i|}} \left(\sum_i t x_i\, \mathfrak{A}_i\right)\right)\right) P_0\right) \\ &\subset \left\{Q \in M \mid \dim Q \cap P_0^{\perp} > 0\right\} \cup \left\{Q \in M \mid \dim\ Q \cap P_0^{\perp} = (0)\right\} = M. \end{aligned}$$

Q. E. D.

REMARK 4.7. Note that $\pi_1(M) \cong \boldsymbol{Z}$ and $\Gamma_{P_0^{\perp}}^{k} := \left\{Q \in M;\ \dim\ Q \cap P_0^{\perp} = k\right\}$ $= \bigcup \varphi^{-1}\left(\exp\left(\mathrm{Ad}h\left(\sum_{i=1}^{n} t x_i\, \mathfrak{A}_i\right)\right)\right) P_0;\ h \in K,\ \Sigma x_i^2 = 1,\ \mathrm{Max}\ |x_i| = |x_{i_1}| = \cdots = |x_{i_k}|\Big\}$ defines a submanifold of codimension $k(2k-1)$ in M. Thus in these examples, cut loci has a natural stratification. It seems to be interesting to know whether this holds for all compact symmetric spaces.

REMARK 4.8. After the preparation of the present note, Prof. S. Murakami informed me that H. Naito has studied the tangent cut loci of compact symmetric R-spaces.

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