On cofinite-dimensional modules

Dedicated to Professor Kiiti Morita on his sixtieth birthday

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Introduction

Goldie introduced finite-dimensional modules in [4]. By dualizing the notion of finite-dimensionality, "cofinite-dimensional modules" may be defind. The object of this article is to study the properties of cofinite-dimensional modules under certain conditions. Our basic tools are coessential extensions and cocomplements in a module, and our main guides are Miyashita [9], [10] and Utumi [14].

It will be assumed throughout that R is a nonzero ring with identity and that all modules over R are unital left R-modules. Let M be a nonzero R-module and let $A\subset B$ be submodules[1)] of M. Then B is called a coessential extension of A in M iff B/A is a small submodule of M/A. This definition originates in the necessity of treating not merely small submodules of M but small submodules of factor modules of M. A set $\{A_\lambda|\lambda\in\Lambda\}$ of submodules of M is called coindependent iff $\bigcap_{i=1}^{n-1} A_{\lambda_i}+A_{\lambda_n}=M$ for any finite subset $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of Λ $(n\geqq 2)$, and M is called cofinite-dimensional iff every coindependent set of submodules of M is finite. Zelinsky proves in [17] that every linearly compact module is cofinite-dimensional. As for the coindependency, Proposition 1.3 is fundamental and Proposition 1.6 shows the relationship between coessential extensions and coindependent sets of submodules.

For a submodule A of an R-module M, a complement A' of A in M is a maximal submodule of M with respect to the property $A\cap A'=0$; dually, a cocomplement A^c of A in M is a minimal submodule of M with respect to the property $A+A^c=M$. Clearly, each direct summand of M is a complement and also a cocomplement (of some submodule) in M. Section 2 is devoted to the propositions about cocomplements in a module.

It is proved by applying Zorn's Lemma that every submodule has a

1) Henceforward, submodules, factor modules, homomorphisms, epimorphisms, etc. of left R-modules will be understood to possess the sense of "R-".

complement. But it is not always true that every submodule has a cocomplement in the module. An R-module M is called cocomplemented iff every submodule of M has a cocomplement in M, and M is called completely cocomplemented iff for any pair of submodules A, B of M with $A+B=M$, there exists a cocomplement A^c of A in M such that A^c is included in B. Every linearly compact module is completely cocomplemented (Corollary 3.7). An R-module M is called semiperfect iff every factor module of M has a projective cover. Every semiperfect module is also completely cocomplemented. The study of these modules is supplementarily shown in Section 3.

A proper submodule A of an R-module M is called couniform in M iff every proper submodule B, $A\subset B$, of M is a coessential extension of A in M, and then M is called locally couniform iff every proper submodule of M is included in a couniform submodule of M. These are of course the dual notions to uniform submodules and locally uniform modules. The uniqueness of the cardinal number of the maximal coindependent set of couniform submodules of M deduces the definition of the codimension of M. Thus, in Section 4, we obtain the following result (Proposition 4.11 and Theorem 4.13):

THEOREM. *Let M be a completely cocomplemented R-module. Then the following statements are equivalent*:

(1) *M is cofinite-dimensional.*

(2) *M satisfies the descending chain condition for cocomplements in M.*

(3) *M satisfies the ascending chain condition for cocomplements in M.*

(4) *M has a cocomplement composition series.*

(5) *M is locally couniform and the codimension of M is finite.*

(6) *M is an irredundant sum of a finite number of minimal cocomplements in M.*

It is to be noted that the verification of the above is considerably due to Theorem 3.9.

In Section 5, we mention quasi-projective modules relating to cocomplements, and also those modules which are weaker than quasi-projectives (see Conditions (I) and (II)).

Let A, A', A'' be submodules of an R-module M such that $A'\oplus A''=M$. Then a direct summand A' of M has been called a direct hull of A in M iff A' is an essential extension of A, and M has been called a direct module iff every submodule of M has a direct hull in M. Dually, a direct summand A' of M is called a codirect cover of M/A in M iff A is a coessential extension of A'' in M, and M is called a codirect module iff every factor module of M has a codirect cover in M. The direct module has been characterized as such a module M that every complement in M is a direct summand of M. Every quasi-injective module is direct. But in our dual case, the situation is complicated, as is explained in Section 6. If M is codirect, then every cocomplement in M is a direct summand of M. The converse holds under the assumption of M to be completely cocomplemented. Every codirect module is cocomplemented. Assume that M is a quasi-projective R-module. Then M is codirect if and only if M is completely cocomplemented. Furthermore, assume that M is a projective R-module. Then the following are equivalent (Corollary 6.10):

(1) M is semiperfect.
(2) M is completely cocomplemented.
(3) M is cocomplemented.
(4) M is codirect.

Therefore, for the ring R itself, ${}_RR$ is codirect if and only if R is a semiperfect ring.

In Sections 7 and 8, cofinite-dimensional codirect modules are studied by researching of their endomorphism rings. Under the assumption of ${}_RM$ to be quasi-projective and semiperfect, ${}_RM$ is finitely generated if and only if the endomorphism ring of ${}_RM$ is a semiperfect ring (Corollary 7.13).

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1. Coessential extensions and coindependent sets of submodules

The notion of a small submodule is well-known as a dual to that of an essential submodule. However, if we take notice of the essential "extension" of a submodule, the following dual is obtained:

DEFINITION 1.1. *Let M be a left R-module and let $A\subset B$ be submodules of M. Then B is called to be a coessential extension of A in M, denoted by $A\subset_{,}B\subset M$, iff $B+C=M$ implies $A+C=M$ for any submodule C of M. This is equivalent to the condition that B/A is a small submodule of the left R-module M/A.* (Cf. [2].)

Evidently, for any submodule A of M,

(1) $A \subset' A \subset M$,

(2) $A \subset' M \subset M$ implies $A = M$, and

(3) $0 \subset' A \subset M$ means that A is small in M.

The following is a fundamental result of the above definition, although easily verified:

PROPOSITION 1.2. *Let M, N be left R-modules and let A, B, C, D be submodules of M.*

(1) *Let M be a submodule of N. Then $A \subset' B \subset M$ implies that $A \subset' B \subset N$.*

(2) *Assume the inclusions $A \subset B \subset C$. Then $B \subset' C \subset M$ if and only if $B/A \subset' C/A \subset M/A$.*

(3) *Assume the inclusions $A \subset B \subset C$. Then $A \subset' C \subset M$ if and only if $A \subset' B \subset M$ and $B \subset' C \subset M$.*

(4) *If $A \subset' B \subset M$ and $C \subset' D \subset M$, then $A + C \subset' B + D \subset M$. In particular, $A \subset' B \subset M$ implies that $A + C \subset' B + C \subset M$.*

(5) *Let $\phi : M \longrightarrow N$ be a homomorphism. If $A \subset' B \subset M$, then $A\phi \subset' B\phi \subset N$*[2].

(6) *Let $\psi : N \twoheadrightarrow M$ be an epimorphism*[3]. *If $A \subset' B \subset M$, then $A\psi^{-1} \subset' B\psi^{-1} \subset N$.*

Let M be a left R-module. A set $\{A_\lambda | \lambda \in \Lambda\}$ of submodules of M is called *coindependent* (=independent in Zelinsky [17] =d-independent in Miyashita [10]) iff $\bigcap_{i=1}^{n-1} A_{\lambda_i} + A_{\lambda_n} = M$ for any finite subset $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of Λ $(n \geqq 2)$.

PROPOSITION 1.3. *For any coindependent set $\mathfrak{A} = \{A_\lambda | \lambda \in \Lambda\}$ of submodules of M, the following statements hold:*

(1) *Every subset of $\mathfrak{A}$ is coindependent.*

(2) *If $A_\lambda \subset B_\lambda (\lambda \in \Lambda)$ are submodules of M, then $\{B_\lambda | \lambda \in \Lambda\}$ is a coindependent set.*

(3) *Let B be a submodule of M such that $\bigcap_{\lambda \in \Lambda'} A_\lambda + B = M$ for any finite subset Λ' of Λ. Then $\mathfrak{A} \cup \{B\}$ is a coindependent set.*

PROOF. Both (1) and (2) are evident and so we prove only (3). Let Λ' be a finite subset of Λ and λ' an element of $\Lambda - \Lambda'$. Putting $A = \bigcap_{\lambda \in \Lambda'} A_\lambda$, we have only to deduce $(A \cap B) + A_{\lambda'} = M$ from assumption. But since

2) Homomorphisms will be written opposite to scalars.

3) The symbol "$\twoheadrightarrow$" means an epimorphism.

$B+(A\cap A_{\lambda'})=M$ and $A+A_{\lambda'}=M$, we can immediately obtain

$$\begin{aligned}(A\cap B)+A_{\lambda'} &= (A\cap B)+(A\cap A_{\lambda'})+A_{\lambda'} \\ &= A\cap\bigl(B+(A\cap A_{\lambda'})\bigr)+A_{\lambda'} \\ &= M.\end{aligned}$$

Now the following are easily seen:

COROLLARY 1.4. *If* $\{A_1, A_2, \cdots, A_n\}$ $(n\geqq 1)$ *is a coindependent set of submodules of* M *and if a submodule* B *of* M *satisfies* $\bigcap_{i=1}^{n} A_i+B=M$, *then* $\{A_1, A_2, \cdots, A_n, B\}$ *is coindependent.*

COROLLARY 1.5. *If* $\{A_1, A_2, \cdots, A_n\}$ $(n\geqq 2)$ *is a coindependent set of submodules of* M, *then* $\bigcap_{i=1}^{m} A_i+\bigcap_{i=m+1}^{n} A_i=M$ *for each* m, $1\leqq m\leqq n-1$.

The above propositions and corollaries will yield the following relationship between coessential extensions and coindependent sets of submodules:

PROPOSITION 1.6. *Let* $A\subset B$, $C\subset D$, $A_i\subset B_i$ $(i=1, 2, \cdots, n)$, $A_\lambda\subset B_\lambda$ $(\lambda\in\Lambda)$ *be submodules of* M.

(1) *Assume that* $A\subset' B\subset M$. *If* $B+C=M$, *then* $A+C=M$ *and* $A\cap C\subset' B\cap C\subset M$.

(2) *Assume that* $A\subset' B\subset M$ *and* $C\subset' D\subset M$. *If* $B+D=M$, *then* $A+C=M$ *and* $A\cap C\subset' B\cap D\subset M$.

(3) *Assume that* $A_i\subset' B_i\subset M$ *for each* i, $1\leqq i\leqq n$. *If* $\{B_i|i=1, 2, \cdots, n\}$ *is coindependent, then* $\{A_i|i=1,2,\cdots,n\}$ *is coindependent and* $\bigcap_{i=1}^{n} A_i\subset' \bigcap_{i=1}^{n} B_i\subset M$.

(4) *Assume that* $A_\lambda\subset' B_\lambda\subset M$ *for each* $\lambda\in\Lambda$. *If* $\{B_\lambda|\lambda\in\Lambda\}$ *is coindependent, then* $\{A_\lambda|\lambda\in\Lambda\}$ *is coindependent and* $\bigcap_{\lambda\in\Lambda'} A_\lambda\subset' \bigcap_{\lambda\in\Lambda'} B_\lambda\subset M$ *for any finite subset* Λ' *of* Λ.

PROOF. In order to prove (1) let D be a submodule of M such that $(B\cap C)+D=M$. Then the set $\{B, C, D\}$ is coindependent by Corollary 1.4, and so $B+(C\cap D)=M$. Hence $A+(C\cap D)=M$. It follows from this that $\{A, C, D\}$ is a coindependent set since $C+D=M$. Therefore $(A\cap C)+D=M$, as desired.

(2) follows from (1) by (3) of Proposition 1.2.

(3) holds by (2) and (4) follows from (3) easily.

REMARK. As for (1) of the above proposition, we can say a little more precisely: Assume that $A\subset{,}B\subset M$. If $B+C=M$, then $A+C=M$ and $A\cap C\subset{,}B\cap C\subset C$.

2. Cocomplements and coclosed submodules

Let M be a left R-module and let A be a submodule of M. A *cocomplement* (=d-complement in [10]) A^c of A in M is a minimal submodule of M with respect to the property $A+A^c=M$. A is called a *cocomplement* in M iff A is a cocomplement of some submodule of M in M.

The following are evident:

(1) M (resp. 0) is the only cocomplement of 0 (resp. M) in M.

(2) Every direct summand of M is a cocomplement in M.

(3) If A^c is a cocomplement of a submodule A in M, then $A\cap A^c$ is small in A^c and hence in M.

(4) If a submodule A of M has a cocomplement A^c in M, and if A^c has a cocomplement $(A^c)^c$ in M, then A^c is a cocomplement of $(A^c)^c$ in M.

A submodule A of M is called (*coessentially*) *coclosed* in M iff $B\subset{,}A\subset M$ implies $B=A$ for any submodule $B(\subset A)$ of M (see Golan [3]). Obviously, every cocomplement in M is coclosed in M. A double cocomplement A^{cc} of A in M is a cocomplement in M of some cocomplement of A in M such that $A^{cc}\subset A$. We can easily see that $A^{cc}\subset{,}A\subset M$. Actually, suppose that $A^{cc}=(A^c)^c$ for some cocomplement A^c of A in M. Since $A\cap A^c$ is small in M, i.e., $0\subset{,}A\cap A^c\subset M$, we have $A^{cc}\subset{,}(A\cap A^c)+A^{cc}=A\subset M$ by Proposition 1.2, (4).

The following is rather fundamental on the coessentiality:

PROPOSITION 2.1. *Let $A\subset B\subset C^c$ be submodules of M. Then $A\subset{,}B\subset M$ if and only if $A\subset{,}B\subset C^c$.*

PROOF. We have only to prove the "only if" part. Let D be a submodule of M such that $D\subset C^c$ and $B+D=C^c$. Then $B+D+C=M$ and hence $A+D+C=M$. The minimality of C^c which includes $A+D$ implies $A+D=C^c$.

PROPOSITION 2.2. *Let $A\subset C^c$ be submodules of M. Then A is coclosed (resp. a cocomplement) in M if and only if A is coclosed (resp. a cocomplement) in C^c.*

PROOF. The "coclosed" part is clear by the above.

Assume that A is a cocomplement of a submodule $A_1(\subset C^c)$ in C^c. Then $A+A_1=C^c$ and so $A+A_1+C=M$. If $B_1+A_1+C=M$ for some submodule $B_1\subset A$, then the minimality of C^c which includes B_1+A_1 yields $B_1+A_1=C^c$. The minimality of A in C^c deduces $B_1=A$. Thus A is a cocomplement of A_1+C in M.

Conversely, assume that A is a cocomplement of a submodule A_2 in M. Then $A_2\cap C^c$ is a submodule of C^c and $A+(A_2\cap C^c)=C^c$. If $B_2+(A_2\cap C^c)=C^c$ for some submodule $B_2\subset A$, then $C^c\subset B_2+A_2$ and so $B_2+A_2=M$. The minimality of A implies $B_2=A$. Thus A is a cocomplement of $A_2\cap C^c$ in C^c.

PROPOSITION 2.3. *Let $A\subset B$ and C^c be submodules of M.*

(1) *Assume that $A\subset' B\subset M$. If M/B is finitely generated, then so is M/A. In particular, if M/B is finitely generated for some small submodule B of M, then so is M.*

(2) *If M is finitely generated, then so is C^c.*

PROOF. (1) Let M/B be finitely generated: $M/B=\sum_{i=1}^{n}R(m_i+B)$ with $m_i\in M$. Then $M=\sum_{i=1}^{n}Rm_i+B$ and therefore $M=\sum_{i=1}^{n}Rm_i+A$, or $M/A=\sum_{i=1}^{n}R(m_i+A)$. This means that M/A is finitely generated.

(2) Assume that $C^c=\sum_{\lambda\in\Lambda}C_\lambda$ with submodules $C_\lambda(\lambda\in\Lambda)$ of C^c. Then $\sum_{\lambda\in\Lambda}C_\lambda+C=M$. Therefore, if M is finitely generated, $\sum_{i=1}^{n}C_{\lambda_i}+C=M$ for some C_{λ_i}. The minimality of C^c implies $\sum_{i=1}^{n}C_{\lambda_i}=C^c$. Thus C^c is finitely generated.

PROPOSITION 2.4. *Let $A\subset B$, C be submodules of M and suppose that $A\subset' C\subset M$. If B/A is coclosed (resp. a cocomplement) in M/A, then $(B+C)/C$ is coclosed (resp. a cocomplement) in M/C.*

PROOF. Let B/A be coclosed in M/A. If $D/C\subset'(B+C)/C\subset M/C$ for some submodule D, $C\subset D\subset B+C$, then $D\subset' B+C\subset M$. Since $A\subset' C\subset M$, we have $A+(B\cap D)\subset' C+(B\cap D)\subset M$, i.e., $B\cap D\subset' D\subset M$. Thus $B\cap D\subset' B+C\subset M$ and hence $B\cap D\subset' B\subset M$, or $(B\cap D)/A\subset' B/A\subset M/A$. Therefore $B\cap D=B$ by assumption. Accordingly, $B\subset D$ and $D=B+C$. This shows that $(B+C)/C$ is coclosed in M/C.

Similarly, if B/A is a cocomplement of B_1/A in M/A, then $(B+C)/C$ is a cocomplement of $(B_1+C)/C$ in M/C.

A homomorphism is called *minimal* iff its kernel is a small submodule (see Bass [1]).

COROLLARY 2.5. *Let $\phi: M \twoheadrightarrow N$ be a minimal epimorphism. If C is coclosed (resp. a cocomplement) in M, then $C\phi$ is coclosed (resp. a cocomplement) in N.*

PROOF. Let C be coclosed (resp. a cocomplement) in M. Since Ker ϕ is small in M, $(C+\text{Ker }\phi)/\text{Ker }\phi$ is coclosed (resp. a cocomplement) in $M/\text{Ker }\phi$ by the above. The isomorphism $M/\text{Ker }\phi \cong M\phi = N$ implies that $C\phi$ is coclosed (resp. a cocomplement) in N.

The following is proved easily:

PROPOSITION 2.6. *Let A be a submodule of M such that A has a double cocomplement in M. Then the following conditions are equivalent:*

(1) *A is coclosed in M.*
(2) *A is a cocomplement in M.*
(3) *$A = A^{cc}$ for some double cocomplement A^{cc} of A in M.*
(4) *$A = A^{cc}$ for every double cocomplement A^{cc} of A in M.*

3. Semiperfect and completely cocomplemented modules

Let M be a nonzero left R-module. Then we recall the following three types of modules:

(1) M is called *semiperfect* iff every factor module of M has a projective cover. This definition was given in Mares [8] under the assumption of M to be projective, but we do not add the projectivity according to the seminar note on algebra in Universität München in 1964.

(2) We should like to call M *completely cocomplemented* iff for any pair of submodules A, B of M with $A+B=M$, there exists a cocomplement A^c of A in M such that $A^c \subset B$. Such a module was defined in Miyashita [10] as a "perfect" module, but this does not coincide with a "perfect" module in Mares [8].

(3) M is called *cocomplemented* (=komplementiert in Kasch and Mares [6]) iff every submodule of M has a cocomplement in M.

A ring R is called *semiperfect* iff ${}_RR$ is a semiperfect module (see Bass [1]). Obviously, an Artinian module is completely cocomplemented and a completely cocomplemented module is cocomplemented. Moreover, a semiperfect module is completely cocomplemented. This is seen in the proof of Miyashita [10; Theorem 3.3], and indeed verified as follows:

Let M be a semiperfect module and set $A+B=M$ for submodules A, B of M. Then M/A has a projective cover $\phi: P \twoheadrightarrow M/A$. For the natural eqimorphism $\pi: B \twoheadrightarrow M/A$, where $b\pi = b+A \in M/A$ $(b \in B)$[4], there exists a homomorphism $\psi: P \longrightarrow B$ such that $\psi\pi = \phi$, by the projectivity of P. Hence $A + P\psi = M$ with $P\psi \subset B$. If $A + B' = M$ for some submodule $B' \subset P\psi$, then we have $B'\psi^{-1} + \text{Ker}\ \phi = P$. Since Ker ϕ is small in P, $B'\psi^{-1} = P$ and so $B' = P\psi$. Thus A has a cocomplement $P\psi \subset B$ in M. This shows that M is completely cocomplemented.

Now we prepare the following:

LEMMA 3.1. *Let $\psi: N \longrightarrow M$ and $\phi: M \longrightarrow N'$ be homomorphisms. Then the following are equivalent:*

(1) *ψ and ϕ are minimal epimorphisms.*

(2) *$\psi\phi$ and ϕ are minimal epimorphisms.*

(3) *ψ and $\psi\phi$ are minimal epimorphisms.*

PROOF. (1) implies (2): The minimality of $\psi\phi$ will be shown. By Proposition 1.2, (6), $0 \subset\!\!/\ \text{Ker}\ \phi \subset M$ asserts that $0\psi^{-1} \subset\!\!/\ (\text{Ker}\ \phi)\psi^{-1} \subset N$, i.e., $\text{Ker}\ \psi \subset\!\!/\ \text{Ker}\ \psi\phi \subset N$. Since $0 \subset\!\!/\ \text{Ker}\ \psi \subset N$, we obtain that $0 \subset\!\!/\ \text{Ker}\ \psi\phi \subset N$, as desired.

(2) implies (3): $M\phi = N' = N\psi\phi$ and hence $M = N\psi + \text{Ker}\ \phi$. Since Ker ϕ is small in M, $N\psi = M$; ψ is an epimorphism. Since $\text{Ker}\ \psi \subset \text{Ker}\ \psi\phi$, which is small in N, ψ is minimal.

(3) implies (1): If Ker $\psi\phi$ is small in N, then $(\text{Ker}\ \psi\phi)\psi = \text{Ker}\ \phi$ is small in M.

PROPOSITION 3.2. *Let M be a semiperfect module. Then every factor module of M and every cocomplement in M are semiperfect.*

PROOF. The first half is obvious by definition. Now let C be a cocomplement in M, let A be a submodule of C and let C^c be any cocomplement of C in M. Then $M/(A+C^c)$ has a projective cover $\phi: P \twoheadrightarrow M/(A+C^c)$, and the natural epimorphism $\pi: C/A \twoheadrightarrow M/(A+C^c)$ is minimal since $A \subset\!\!/\ A+(C^c \cap C) \subset C$. By the projectivity of P, there exists a homomorphism $\psi: P \longrightarrow C/A$ such that $\psi\pi = \phi$. Then, ψ is a projective cover of C/A by the above lemma.

4) Henceforward, the letter "π" will always be used to indicate such a natural epimorphism. Suppose the general situation that A, B, C, D are submodules of M such that $A \subset B \subset D$, $A \subset C \subset D$ and $B+C=D$. Then the natural epimorphism $\pi: C/A \twoheadrightarrow D/B$ is a mapping defined by $(c+A)\pi = c+B \in D/B$ $(c+A \in C/A)$.

PROPOSITION 3.3. *Assume that* $A \subset_{\prime} B \subset M$. *If* M/B *is semiperfect, then so is* M/A.

PROOF. Let C/A be a submodule of M/A, where $A \subset C$ are submodules of M. We shall show that M/C has a projective cover. $M/(B+C)$, which is isomorphic to a factor module of M/B, has a projective cover ϕ: $P \twoheadrightarrow M/(B+C)$, and the natural epimorphism π: $M/C \twoheadrightarrow M/(B+C)$ is minimal since $C \subset_{\prime} B+C \subset M$. By the projectivity of P, there exists a homomorphism ψ: $P \longrightarrow M/C$ such that $\psi\pi = \phi$. Then, ψ is a projective cover of M/C by Lemma 3.1.

COROLLARY 3.4. *If* M/A *is semiperfect for a small submodule* A *of* M, *then so is* M. *In particular, if* M *is semiperfect, then so is any projective cover of* M. (See [10; Proposition 3.13] and [8; Theorem 5.6].)

Let M be a completely cocomplemented module. Then the following two statements hold (see [10; pp. 89–90]):

(1) Every factor module of M is completely cocomplemented.

Let $M/A = B/A + C/A$ for submodules $A \subset B$, C of M. Since $M = B + C$, B has cocomplement $B^c \subset C$ in M. Then $(B^c + A)/A \subset C/A$ is a cocomplement of B/A in M/A.

(2) Every cocomplement C^c in M is completely cocomplemented.

Let $C^c = A + B$ for submodules A, $B \subset C^c$. Since $C + A + B = M$, $C + A$ has a cocomplement $(C+A)^c \subset B$ in M, which is a cocomplement of A in C^c.

We note here that in a completely cocomplemented module, a coclosed submodule is nothing but a cocomplement (see Proposition 2.6).

A left R-module M is called *linearly compact* iff any finitely solvable system of congruences in M:

$$\alpha \equiv a_\lambda \quad (\mathrm{mod}\ A_\lambda) \quad (\lambda \in \Lambda),$$

where $a_\lambda \in M$ and A_λ is a submodule of M for each $\lambda \in \Lambda$, is solvable (see Zelinsky [17]).

The following three results are seen substantially in Sadomierski [11; p. 335]:

LEMMA 3.5. *Let* A, $B_\lambda (\lambda \in \Lambda)$ *be submodules of* M *such that* $A + B_\lambda = M$ *for all* $\lambda \in \Lambda$. *Assume that* $\{B_\lambda | \lambda \in \Lambda\}$ *is linearly ordered by set-inclusion. If* A *is linearly compact, then* $A + \bigcap_{\lambda \in \Lambda} B_\lambda = M$.

PROOF. Let c be an arbitrary element of M. Then for each $\lambda \in \Lambda$, we have $c=a_\lambda+b_\lambda$ with $a_\lambda\in A$ and $b_\lambda\in B_\lambda$. Consider a system of congruences in A:

$$\alpha\equiv a_\lambda \qquad (\text{mod } A\cap B_\lambda) \qquad (\lambda\in\Lambda).$$

For any finite number of elements $\lambda_1, \lambda_2, \cdots, \lambda_n\in\Lambda$, there exists r, $1\leqq r\leqq n$ such that $B_{\lambda_r}=\bigcap_{i=1}^{n} B_{\lambda_i}$, by assumption. Therefore, $a_{\lambda_r}-a_{\lambda_i}=-b_{\lambda_r}+b_{\lambda_i}\in A\cap B_{\lambda_i}$ for each i, $1\leqq i\leqq n$. Accordingly, this system is finitely solvable in A, and hence this is solvable in A. Thus there exists an element a in A such that

$$a\equiv a_\lambda \qquad (\text{mod } A\cap B_\lambda) \qquad (\lambda\in\Lambda).$$

Since $c-a=b_\lambda-(a-a_\lambda)\in B_\lambda$ for any $\lambda\in\Lambda$, we can deduce that $A+\bigcap_{\lambda\in\Lambda}B_\lambda=M$.

PROPOSITION 3.6. *Assume that $A+B=M$ for submodules A, B of M. If A is linearly compact, then there exists a cocomplement A^c of A in M such that $A^c\subset B$.*

PROOF. Consider the set $\mathfrak{B}$ of all submodules B' of M such that $B'\subset B$ and $A+B'=M$, with the order opposite to the set-inclusion. Let $\mathfrak{B}'=\{B_\lambda|\lambda\in\Lambda\}$ be a nonempty chain in $\mathfrak{B}$. Then, by the above lemma, we have $A+\bigcap_{\lambda\in\Lambda}B_\lambda=M$ since A is linearly compact. This means that $\mathfrak{B}'$ contains an upper bound in $\mathfrak{B}$. Thus by Zorn's Lemma $\mathfrak{B}$ has a maximal element, which is a required cocomplement of A in M.

Now the following is evident, noting that every submodule of a lineary compact module is linearly compact:

COROLLARY 3.7. *If M is a linearly compact module, then M is completely cocomplemented.*

Let M be a cocomplemented module. Then every factor module of M is cocomplemented. Actually, let B/A be a submodule of M/A with submodules $A\subset B$ of M. If B^c is a cocomplement of B in M, then $(B^c+A)/A$ is a cocomplement of B/A in M/A.

The (Jacobson) *radical* of M (i.e., the sum of all small submodules of M) will be denoted by $\mathrm{J}(M)$.

PROPOSITION 3.8. *Let M be cocomplemented and C a cocomplement in M. Then $C/\mathrm{J}(C)$ is semisimple. In particular, if M is cocomplemented, then $M/\mathrm{J}(M)$ is semisimple.* (Cf. [7; p. 13].)

PROOF. Let $A/\mathrm{J}(C)$ be a submodule of $C/\mathrm{J}(C)$, where A, $\mathrm{J}(C)\subset A\subset C$, is a submodule of M. By assumption, there exists a cocomplement A^c of A in M. Then,

$$A+\big((A^c\cap C)+\mathrm{J}(C)\big)=(A+A^c)\cap C=C.$$

And next,

$$A\cap\big((A^c\cap C)+\mathrm{J}(C)\big)=(A\cap A^c)+\mathrm{J}(C)=\mathrm{J}(C).$$

Because, $A\cap A^c$ is small in M and hence in C (Proposition 2.1). Thus, $C/\mathrm{J}(C)$ is the direct sum of the submodules $A/\mathrm{J}(C)$ and $((A^c\cap C)+\mathrm{J}(C))/\mathrm{J}(C)$. This shows that $C/\mathrm{J}(C)$ is semisimple.

THEOREM 3.9[5)]. *Let M be completely cocomplemented and let A, B, C be submodules of M such that $A\subset B$ and $A+C=M$. Then for any cocomplement B^c of B in M with $B^c\subset C$, there exists a cocomplement A^c of A in M such that $B^c\subset A^c\subset C$.*

PROOF. Let $B^c\subset C$ be a cocomplement of B in M. Since $M/B^c=(A+B^c)/B^c+C/B^c$ is completely cocomplemented, there exists a cocomplement $D/B^c\subset C/B^c$ of $(A+B^c)/B^c$ in M/B^c, where D, $B^c\subset D\subset C$, is a submodule of M. Hence $A+D=M$, so that there exists a cocomplement $A^c\subset D$ of A in M. On the other hand $(A+B^c)/B^c\cap D/B^c$ is small in D/B^c, i.e., $B^c\subset,(A+B^c)\cap D\subset D$. Since $B^c+(B\cap D)=D$, we have $B^c\cap(B\cap D)\subset,(A+B^c)\cap(B\cap D)\subset D$. Hence $0\subset,(A+B^c)\cap B\cap D\subset D$. Therefore it follows from $A^c+((A+B^c)\cap B\cap D)=(A^c+((A+B^c)\cap B))\cap D=D$ that $A^c=D$. Thus we obtain $B^c\subset A^c\subset C$, as required.

Now the following are easy by the above:

COROLLARY 3.10. *Let M be completely cocomplemented and let $A\subset B$ be submodules of M. Then for any cocomplement A^c of A (resp. B^c of B) in M, there exists a cocomplement B^c of B (resp. A^c of A) in M such that $B^c\subset A^c$.*

COROLLARY 3.11. *Let M be completely cocomplemented and let A be a submodule of M, $A\subsetneqq B$ a cocomplement in M. Then for any cocomplement A^c of A (resp. B^c of B) in M, there exists a cocomplement B^c of B (resp. A^c of A) in M such that $B^c\subsetneqq A^c$.*

5) The dual statements of 3.9–12 will hold. Of course, we need not assume that M is "(completely) complemented".

PROPOSITION 3.12. *Let M be completely cocomplemented and let $A \subset B$ be submodules of M. Then for any double cocomplement A^{cc} of A in M, there exists a double cocomplement B^{cc} of B in M such that $A^{cc} \subset B^{cc}$.*

PROOF. Let $A^{cc} = (A^c)^c$ be a double cocomplement of A in M. For the cocomplement A^c, there exists a cocomplement B^c of B in M such that $B^c \subset A^c$. Since $A^{cc} \subset B$, there exists a cocomplement $(B^c)^c$ of B^c in M such that $A^{cc} \subset (B^c)^c \subset B$, by the above theorem. Hence this cocomplement $(B^c)^c$ is a double cocomplement of B in M, completing the proof.

PROPOSITION 3.13. *Let M be completely cocomplemented and C a cocomplement in M. Then there exists a one-to-one inclusion preserving correspondence between the set of all cocomplements in M that include C and the set of all cocomplements in M/C.*

PROOF. Let $A = A^{cc}$ be a cocomplement in M including C. Then $A/C + (A^c + C)/C = M/C$. Suppose that $A'/C + (A^c + C)/C = M/C$ for a submodule A', $C \subset A' \subset A$, of M. Since $A' + A^c = M$, the minimality of $A = A^{cc}$ implies $A' = A$. Therefore A/C is a cocomplement of $(A^c + C)/C$ in M/C.

Conversely, let B/C be a cocomplement in M/C. By Proposition 3.12, there exists a double cocomplement B^{cc} of B in M such that $C \subset B^{cc}$. Thus $B^{cc}/C \subset_{\prime} B/C \subset M/C$, so that $B^{cc}/C = B/C$, since B/C is coclosed in M/C. Hence $B = B^{cc}$ is a cocomplement in M.

Therefore our proposition holds.

4. Cofinite-dimensional modules

Let M be a nonzero left R-module. Then M is called *sum-irreducible* iff for any proper submodules A, B of M, $A + B$ is a proper submodule of M. This is equivalent to the condition that every proper submodule of M is small in M. Let A be a proper submodule of M. Then A is called a *couniform* (=d-uniform in [10]) submodule of M, or couniform in M iff M/A is a sum-irreducible module. This is equivalent to the condition that $A \subset_{\prime} B \subset M$ for any proper submodule B, $A \subset B$, of M. Evidently, a simple module is sum-irreducible and a sum-irreducible module is indecomposable. For a ring R, ${}_R R$ is sum-irreducible if and only if R is a local ring. Every maximal submodule is a couniform submodule.

PROPOSITION 4.1. *Let $A \subset B$ be proper submodules of M.*

(1) *If A is couniform in M, then B is couniform in M.*

(2) *Assume that* $A\subset, B\subset M$. *If* B *is couniform in* M, *then* A *is couniform in* M.

(3) B *is couniform in* M *if and only if* B/A *is couniform in* M/A.

(4) *Assume that* A *is proper in* B. *If* A *is couniform in* B, *then* $A+B^c=M$ *or* $A+B^c$ *is couniform in* M *for any cocomplement* B^c *of* B *in* M.

(5) *Let* $A_\lambda \subsetneqq B$ $(\lambda\in\Lambda)$ *be submodules of* M, B *a cocomplement in* M, *and* B^c *a cocomplement of* B *in* M. *If* $\{A_\lambda|\lambda\in\Lambda\}$ *is a coindependent set of couniform submodules of* B, *then* $\{A_\lambda+B^c|\lambda\in\Lambda\}$ *is a coindependent set of couniform submodules of* M.

Proof. (1) and (3) are obvious by Proposition 1.2, (3) and (2), respectively.

(2) If C is a submodule with $A\subset C\subsetneqq M$, then $C\subset, B+C\subset M$. This shows that $B+C$ is proper in M. Therefore $B\subset, B+C\subset M$ and hence $A\subset, C\subset M$.

(4) If C is a submodule with $A+B^c\subset C\subsetneqq M$, then $B\cap C\subsetneqq B$ and so $A\subset, B\cap C\subset B$. Therefore $A+B^c\subset, (B\cap C)+B^c=C\subset M$.

(5) is obvious by (4), since each $A_\lambda+B^c$ is proper in M.

Proposition 4.2. *Let* M *be completely cocomplemented and* A *a nonzero submodule of* M. *Then the following conditions are equivalent*:

(1) A *is a minimal cocomplement in* M (*i. e., minimal as a nonzero cocomplement in* M).

(2) A *is a cocomplement in* M *and any cocomplement* A^c *of* A *in* M *is a couniform submodule of* M.

(3) A *is a cocomplement in* M *of some couniform submodule of* M.

(4) A *is sum-irreducible and not small in* M.

Proof. (1) implies (2): Let A^c be a cocomplement of A in M. The assumption $A^c=M$ would imply $(A^c)^c=0$ and so $A=A^{cc}=0$, a contradiction. Hence A^c is proper in M. If B is a submodule with $A^c\subset B\subsetneqq M$, B has a nonzero cocomplement $B^c\subset A$ in M. The minimality of A deduces $B^c=A$. Therefore A^c is a double cocomplement of B in M, and consequently $A^c\subset, B\subset M$.

(2) implies (3) obviously.

(3) implies (4): Assume that $A=B^c$, where B is a counifrom submodule of M. Then $M=A+B$ with $B\subsetneqq M$, which means that A is not small in M. Next, let C, $D\subsetneqq A$ be submodules of M. Then $B+C$, $B+D\subsetneqq M$ and $B\subset, B+C\subset M$, $B\subset, B+D\subset M$ since B is couniform in M. Hence

$B\subset' B+C+D\subset M$, so that $B+C+D\subsetneqq M$. Thus $C+D\subsetneqq B^c=A$, showing that A is sum-irreducible.

(4) implies (1): Let A be not small in M. Then there exists a submodule B of M such that $A+B=M$ and $B\subsetneqq M$. Then A includes a cocomplement B^c of B in M. Assume that $B^c\neq A$. If A is sum-irreducible, B^c is small in A and hence in M. This contradicts the fact that B is proper in M. Thus, $A=B^c$ is a cocomplement in M. If $C\subsetneqq A$ is a cocomplement in M, then $C^c=M$, so that $C=0$. Therefore A is a minimal cocomplement in M.

PROPOSITION 4.3. *Let M be completely cocomplemented and A a proper submodule of M. Then the following conditions are equivalent*:

(1) *A is a couniform cocomplement in M.*

(2) *A is minimally couniform in M.*

(3) *A is a maximal cocomplement in M (i. e., maximal as a proper cocomplement in M).*

PROOF. (1) implies (2): Let $B\subset A$ be a couniform submodule of M. Since A is proper in M, $B\subset' A\subset M$, so that $B=A$. Because, A is cocolosed in M.

(2) implies (3): By Proposition 4.1, (2), any double cocomplement A^{cc} of A in M is couniform in M. The minimality of A yields $A^{cc}=A$, showing that A is a cocomplement in M. Let B be a cocomplement in M such that $A\subset B\subsetneqq M$. Since A is couniform in M, $A\subset' B\subset M$ and we have $A=B$. Because, B is coclosed in M.

(3) implies (1): Let B be a submodule of M such that $A\subset B\subsetneqq M$. Then by Proposition 3.12, there exists a double cocomplement B^{cc} of B in M such that $A\subset B^{cc}$. Since B^{cc} is proper in M, the maximality of A asserts $A=B^{cc}$. Therefore $A\subset' B\subset M$, as requested.

Let M be a nonzero left R-module. Then M is called *locally couniform* iff every proper submodule of M is included in a couniform submodule of M. (Cf. [9; p. 167].)

PROPOSITION 4.4. *Let M be completely cocomplemented. Then the following statements are equivalent*:

(1) *M is locally couniform.*

(2) *Every proper cocomplement in M is included in a maximal cocomplement in M.*

(3) *Every nonzero cocomplement in M includes a minimal cocomplement in M.*

Proof. (1) implies (2): Let A be a proper cocomplement in M. Then A is included in a couniform submodule B of M. By Proposition 3.12, there exists a double cocomplement B^{cc} of B in M which includes A. Since B^{cc} is couniform in M, B^{cc} is a maximal cocomplement in M by the above proposition.

(2) implies (3): Let A be a nonzero cocomplement in M. Any cocomplement A^c of A in M is proper in M and hence included in a maximal cocomplement B in M. Then A includes a cocomplement B^c of B in M which is nonzero. By Propositions 4.2 and 4.3, B^c is a minimal cocomplement in M.

(3) implies (1): Let A be a proper submodule of M. If $A^{cc}=(A^c)^c$ is a double cocomplement of A in M, $A^c \neq 0$ includes a minimal cocomplement B in M. Then for A^{cc}, there exists a cocomplement B^c of B in M such that $A^{cc} \subset B^c$. Since $B=B^{cc}\neq 0$, B^c is proper in M, so that $A+B^c$ is proper in M. Because B^c is couniform in M by Proposition 4.2, so is $A+B^c$ in M, which includes A.

Lemma 4.5. *If M has a strictly descending chain of an infinite number of cocomplements in M, then there exists a coindependent set of an infinite number of proper submodules of M*

Proof. Let $M=C_0^c \supsetneqq C_1^c \supsetneqq C_2^c \supsetneqq \cdots$ be a strictly descending chain of cocomplements in M. Then each $C_i+C_{i+1}^c$ is proper in M $(i=0, 1, 2, \cdots)$. Noting $C_n^c \subset \bigcap_{i=0}^{n-1} (C_i+C_{i+1}^c)$ $(n=1, 2, \cdots)$, it is easily seen that $\{C_i+C_{i+1}^c | i=0, 1, 2, \cdots\}$ is a coindependent set of proper submodules of M, by Corollary 1.4.

Lemma 4.6. *Let M be completely cocomplemented. If there exists a coindependent set of an infinite number of proper submodules of M, then M has a strictly ascending chain of an infinite number of cocomplements in M.*

Proof. Let $\{A_i | i=0, 1, 2, \cdots\}$ be a coindependent set of proper submodules of M. Consider the submodules of M: $B_i=\bigcap_{j=1}^{i} A_j$ $(i=1, 2, \cdots)$. Then $M \supset B_1 \supset B_2 \supset \cdots$ gives an ascending chain $B_1^c \subset B_2^c \subset \cdots \subset M$, which is strict. Because, assume $B_i^c=B_{i+1}^c$. Since $B_i^c+B_{i+1}=M$, there exists a cocomplement $(B_i^c)^c \subset B_{i+1}$ of B_i^c in M. But this is a double cocomplement of

B_i in M, so that $B_{i+1} \subset' B_i \subset M$. Since $A_{i+1} + B_i = M$, we have $A_{i+1} = A_{i+1} + B_{i+1} = M$, a contradiction.

DEFINITION 4.7. *A nonzero left R-module M is called to be cofinite-dimensional iff every coindependent set of proper submodules of M is finite.*

PROPOSITION 4.8. *Let M be cofinite-dimensional. Then every factor module of M and every cocomplement in M are cofinite-dimensional.*

PROOF. The first half is obvious. Now let C^c be a cocomplement in M and assume that $\{A_i | i = 1, 2, \cdots, n\}$ is a coindependent set of proper submodules of C^c. Since $\bigcap_{i=1}^{n-1} (A_i + C) + (A_n + C) \supset \left(\bigcap_{i=1}^{n-1} A_i + A_n \right) + C = M$, we deduce that $\{A_i + C | i = 1, 2, \cdots, n\}$ is a coindependent set of proper submodules of M. Thus it follows that C^c is cofinite-dimensional.

Under the assumption that $A \subset' B \subset M$, if $\{A_i/A | i = 1, 2, \cdots, n\}$ is a coindependent set of proper submodules of M/A, then $\{(A_i + B)/B | i = 1, 2, \cdots, n\}$ is a coindependent set of proper submodules of M/B. Thus we obtain:

PROPOSITION 4.9. *Assume that $A \subset' B \subset M$. If M/B is cofinite-dimensional, then so is M/A. In particular, if M/B is cofinite-dimensional for a small submodule B of M, then so is M.*

The following was given in Zelinsky [17; Proposition 6], but we shall prove by making use of Lemma 3.5.

PROPOSITION 4.10. *If M is a linearly compact module, then M is cofinite-dimensional.*

PROOF. Assume that $\{A_i | i = 1, 2, \cdots\}$ is a coindependent set of proper submodules of M and put $A_0 = \bigcup_{j \geqq 1} \bigcap_{i \geqq j} A_i$. Fix the elements a_i of M arbitrarily such that $a_0 = 0$ and a_i is not contained in A_i for each $i \geqq 1$. Now consider a system of congruences in M:

$$\alpha \equiv a_i \qquad (\text{mod } A_i) \qquad (i = 0, 1, 2, \cdots).$$

Treating only $A_1, A_2, \cdots, A_n$ $(n \geqq 2)$, $A_i + \bigcap_{j \neq i} A_j = M$ implies that we can set $a_i = a_i' + b_i$ $(i = 1, 2, \cdots, n)$, where $a_i' \in A_i$ and $b_i \in \bigcap_{j \neq i} A_j$. If we put $b = \sum_{i=1}^{n} b_i$, then

$$b = b_i + \sum_{j \neq i} b_j \equiv b_i \equiv a_i \qquad (\text{mod } A_i) \qquad (i = 1, 2, \cdots, n).$$

On the other hand, since

$$\bigcap_{i=1}^{n} A_i + \bigcap_{i=1}^{j} A_{n+i} = M \qquad (j=1, 2, \cdots),$$

we can conclude by Lemma 3.5 that

$$\bigcap_{i=1}^{n} A_i + \bigcap_{i\geqq 1} A_{n+i} = M .$$

Thus there holds $b=b'+b''$ for some $b' \in \bigcap_{i=1}^{n} A_i$, $b'' \in \bigcap_{i\geqq 1} A_{n+i}$. Hence,

$$b'' \equiv b \equiv a_i \qquad (\mathrm{mod}\ A_i) \qquad (i=1, 2, \cdots, n),$$

$$b'' \equiv 0 = a_0 \qquad (\mathrm{mod}\ A_0).$$

This shows that our present system is finitely solvable. Therefore it has a solution c in M;

$$c \equiv a_i \qquad (\mathrm{mod}\ A_i) \qquad (i=0, 1, 2, \cdots).$$

Let c, contained in A_0, be in A_m with $m \geqq 1$. Then $c - a_m \in A_m$ yields $a_m \in A_m$, a contradiction. Thus we deduce that M is cofinite-dimensional.

Lemma 4.5, Corollary 3.11 and Lemma 4.6 assert the following:

PROPOSITION 4.11. *Let M be completely cocomplemented. Then the following statements are equivalent*:

(1) *M is cofinite-dimensional.*
(2) *M satisfies the descending chain condition for cocomplements in M.*
(3) *M satisfies the ascending chain condition for cocomplements in M.*

A finite chain of submodules of M:

$$M = C_0 \supsetneqq C_1 \supsetneqq C_2 \supsetneqq \cdots \supsetneqq C_{n-1} \supsetneqq C_n = 0$$

is called a *cocomplement composition series* of M iff each C_{i+1} is a maximal cocomplement in C_i $(i=0, 1, \cdots, n-1)$. This is equivalent to the condition that each C_i is a cocomplement in M $(i=0, 1, \cdots, n)$ and there exists no cocomplement in M which is strictly intermediate between C_i and C_{i+1} $(i=0, 1, \cdots, n-1)$.

Let $M \supset A$, $N \supset B$ be left R-modules and submodules of them. Then we shall say that A is *cosimilar* to B in (M, N): $A \sim B\,(M, N)$, iff there exist coessential extensions $A \subset' A_1 \subset M$ and $B \subset' B_1 \subset N$ such that M/A_1 is isomorphic to N/B_1. (Cf. [10; p. 106].) This cosimilarity is an "equivalence relation". To show the transitivity, assume that $A \sim B$ (M, N) and $B \sim C$

(N, L). Then $M/A_1 \cong N/B_1$ and $N/B_2 \cong L/C_1$ for some coessential extensions $A\subset' A_1 \subset M$, $B \subset' B_1 \subset N$; $B\subset' B_2 \subset N$, $C \subset' C_1 \subset L$. These isomorphisms imply $A_2/A_1 \cong (B_1+B_2)/B_1$ and $(B_1+B_2)/B_2 \cong C_2/C_1$ for some $A_1 \subset' A_2 \subset M$ and $C_1 \subset' C_2 \subset L$, since B_1, $B_2 \subset' B_1+B_2 \subset N$. Hence $M/A_2 \cong N/(B_1+B_2) \cong L/C_2$, where $A \subset' A_2 \subset M$ and $C \subset' C_2 \subset L$. Thus $A \sim C$ (M, L).

Assume that $A \subset' A_1 \subset M$ and $B \subset' B_1 \subset N$. If $A_1 \sim B_1$ (M, N), then $A \sim B$ (M, N). Therefore, if A is a small submodule of M, then $A \sim 0$[6].

Now the following is easily verified:

PROPOSITION 4.12. *Let $M_i \supset A_i$ $(i=1, 2)$ be left R-modules and submodules of them, and let P_i be projective covers of M_i/A_i $(i=1, 2)$. Then $A_1 \sim A_2$ (M_1, M_2) if and only if P_1 is isomorphic to P_2.*

A set $\{A_\lambda | \lambda \in \Lambda\}$ of submodules of M is called *homogeneous* if $A_\lambda \sim A_{\lambda'}$ for all λ, $\lambda' \in \Lambda$.

By [10; § 5], we know the following results:

(1) If M has a couniform submodule, then there exists a maximal coindependent set of couniform submodules of M.

(2) Let $\{A_\lambda | \lambda \in \Lambda\}$ and $\{B_\gamma | \gamma \in \Gamma\}$ be maximal coindependent homogeneous sets of couniform submodules of M such that $A_\lambda \sim B_\gamma$ for $\lambda \in \Lambda$ and $\gamma \in \Gamma$. Then $\sharp\Lambda = \sharp\Gamma$.

(3) Let $\{A_\lambda | \lambda \in \Lambda\}$ be a maximal coindependent set of couniform submodules of M. Then for any $\lambda_0 \in \Lambda$, $\{A_\lambda | A_\lambda \sim A_{\lambda_0} (\lambda \in \Lambda)\}$ is a maximal coindependent homogeneous set of couniform submodules of M.

(4) Let $\{A_\lambda | \lambda \in \Lambda\}$ and $\{B_\gamma | \gamma \in \Gamma\}$ be maximal coindependent sets of couniform submodules of M. Then there exists a one-to-one correspondence $\chi: \Lambda \longrightarrow \Gamma$ such that $A_\lambda \sim B_{\chi(\lambda)}$ for all $\lambda \in \Lambda$.

(5) Assume that M has a couniform submodule. Then we can define the *codimension* of M as the cardinal number of Λ: codim $M = \sharp\Lambda$, where Λ is denoted in the condition that $\{A_\lambda | \lambda \in \Lambda\}$ is a maximal coindependent set of couniform submodules of M.

THEOREM 4.13. *Let M be completely cocomplemented. Then the following are equivalent:*

(1) *M is cofinite-dimensional.*

(2) *M has a cocomplement composition series.*

(3) *M is locally couniform and* codim *M is finite.*

6) In case of $N=M$, "(M, M)" will be omitted.

(4) *M is an irredundant sum of a finite number of minimal cocomplements in M.*

If one of these equivalent conditions is satisfied, then the following hold:

(5) *The length of any cocomplement composition series of M is equal to* codim M.

(6) *If M has two cocomplement composition series:*

$$M=C_0\supsetneqq C_1\supsetneqq C_2\supsetneqq\cdots\supsetneqq C_{n-1}\supsetneqq C_n=0,$$
$$M=D_0\supsetneqq D_1\supsetneqq D_2\supsetneqq\cdots\supsetneqq D_{n-1}\supsetneqq D_n=0,$$

then there exist cocomplements $(C_i/C_{i+1})^c$ of C_i/C_{i+1} in M/C_{i+1} and $(D_j/D_{j+1})^c$ of D_j/D_{j+1} in M/D_{j+1} $(0\leqq i,\ j\leqq n-1)$, and a permutation χ of the numbers $0, 1, 2, \cdots, n-1$ *such that*

$$(C_i/C_{i+1})^c\sim(D_j/D_{j+1})^c\quad(M/C_{i+1},\ M/D_{j+1}),$$

where $j=\chi(i)$, $0\leqq i\leqq n-1$.

PROOF. By Proposition 4.11, (1) implies (2).

(2) implies (3), (5) and (6): Assume that

$$M=C_0\supsetneqq C_1\supsetneqq C_2\supsetneqq\cdots\supsetneqq C_{n-1}\supsetneqq C_n=0$$

is a cocomplement composition series of M.

First, let A be a proper submodule of M. Then there exist cocomplements $C_i''\subset A$ of C_i in M $(i=0, 1, 2, \cdots, r)$ such that $A+C_{r+1}$ is proper in M for some r, $0\leqq r\leqq n-1$. Since $C_r''+C_{r+1}$ is couniform in M (Propositions 4.1 and 4.3), $A+C_{r+1}$ is couniform in M. Thus M is a locally couniform module.

Next, let C_i^c be any cocomplements of C_i in M $(i=0, 1, 2, \cdots, n)$. Then $\{C_i^c+C_{i+1}\,|\,i=0, 1, 2, \cdots, n-1\}$ is a coindependent set of couniform submodules of M (see Proposition 4.5). Moreover, this is a maximal coindependent set. Because,

$$C_i\subset'(C_{i-1}\cap C_{i-1}^c)+C_i=C_{i-1}\cap(C_{i-1}^c+C_i)\subset M,$$
$$C_{i+1}\subset'(C_i\cap C_i^c)+C_{i+1}\subset'(C_{i-1}\cap(C_{i-1}^c+C_i)\cap C_i^c)+C_{i+1}\subset M,$$

and so we have

$$0\subset'\bigcap_{i=0}^{n-1}(C_i^c+C_{i+1})\subset M.$$

Thus codim M is finite and the length of the cocomplement composition series of M is equal to codim M.

Finally, let

$$M=D_0\supsetneqq D_1\supsetneqq D_2\supsetneqq\cdots\supsetneqq D_{n-1}\supsetneqq D_n=0$$

be another cocomplement composition series of M. Then we have two maximal coindependent sets

$$\{C_i^c+C_{i+1}|i=0,1,2,\cdots,n-1\},\qquad \{D_j^c+D_{j+1}|j=0,1,2,\cdots,n-1\}$$

of couniform submodules of M. Accordingly, there exists a permutation χ of the numbers $0,1,2,\cdots,n-1$ such that

$$C_i^c+C_{i+1}\sim D_j^c+D_{j+1},\ j=\chi(i)\qquad (0\leqq i\leqq n-1).$$

Hence there exist coessential extensions

$$C_i^c+C_{i+1}\subset' C_i'\subset M,\qquad D_j^c+D_{j+1}\subset' D_j'\subset M$$

such that M/C_i' is isomorphic to M/D_j'. Since

$$(C_i^c+C_{i+1})/C_{i+1}\subset' C_i'/C_{i+1}\subset M/C_{i+1},$$

$$(D_j^c+D_{j+1})/D_{j+1}\subset' D_j'/D_{j+1}\subset M/D_{j+1},$$

we have

$$(C_i^c+C_{i+1})/C_{i+1}\sim(D_j^c+D_{j+1})/D_{j+1}\qquad (M/C_{i+1},\ M/D_{j+1}).$$

As is easily seen, $(C_i^c+C_{i+1})/C_{i+1}$ is a cocomplement of C_i/C_{i+1} in M/C_{i+1} and $(D_j^c+D_{j+1})/D_{j+1}$ is a cocomplement of D_j/D_{j+1} in M/D_{j+1}. Thus (6) has been deduced.

(3) implies (4): Assume codim $M=n$ and let $\{C_1, C_2, \cdots, C_n\}$ be a maximal coindependent set of couniform submodules of M. Then $\bigcap_{i=1}^{n} C_i$ is a small submodule of M since M is locally couniform. By $C_i+\bigcap_{j\neq i} C_j=M$, there exist cocomplements $C_i^c\subset\bigcap_{j\neq i} C_j$ of C_i in M $(1\leqq i\leqq n)$, which are minimal cocomplements in M. Now,

$$\sum_{i=1}^{n} C_i^c+\bigcap_{i=1}^{n} C_i=\sum_{i=2}^{n} C_i^c+(C_1^c+C_1)\cap\left(\bigcap_{i=2}^{n} C_i\right)$$

$$=\sum_{i=2}^{n} C_i^c+\bigcap_{i=2}^{n} C_i=\cdots=M,$$

so that $\sum_{i=1}^{n} C_i^c=M$. The irredundancy of the sum follows from $\sum_{j\neq i} C_j^c\subset C_i\subsetneqq M$.

(4) implies (3): Assume that M is an irredundant sum of a finite number of minimal cocomplements C_i in M: $M=\sum_{i=1}^{n} C_i$.

First, let A be a proper submodule of M. Then there exists r, $1\leqq r\leqq n$, such that

$$A+C_1+\cdots+C_{r-1}+C_r=M,$$
$$A+C_1+\cdots+C_{r-1}\subsetneqq M.$$

Therefore C_r has a cocomplement $C'_r\subset A+C_1+\cdots+C_{r-1}$ in M, which is couniform in M by Proposition 4.2. Thus A is included in a couniform submodule $A+C_1+\cdots+C_{r-1}$ of M. This shows that M is locally couniform.

Next, put $D_i=\sum_{j\neq i}C_j$ (being proper in M) for each i, $1\leqq i\leqq n$. Since $C_i+D_i=M$, C_i has a cocomplement $C_i^c\subset D_i$ in M. Since C_i^c are couniform in M, so are D_i in M $(1\leqq i\leqq n)$. Hence $\{D_1, D_2, \cdots, D_n\}$ is a coindependent set of couniform submodules of M, because $\bigcap_{j=1}^{i}D_j+D_{i+1}$ includes $C_{i+1}+\sum_{j\neq i+1}C_j=M$ $(1\leqq i\leqq n-1)$. On the other hand, $C_i^c\subset' D_i\subset M$ and so $0\subset' C_i\cap C_i^c\subset' C_i\cap D_i\subset M$ $(1\leqq i\leqq n)$. Hence $\sum_{i=1}^{n}(C_i\cap D_i)=\bigcap_{i=1}^{n}D_i$ is a small submodule of M. This yields that the above coindependent set $\{D_1, D_2, \cdots, D_n\}$ is maximal. Thus codim $M=n$.

(3) implies (1): Suppose that $\{A_1, A_2, \cdots, A_n\}$ is a coindependent set of proper submodules of M. Then each A_i is included in a couniform submodule B_i of M, since M is locally couniform. Hence $\{B_1, B_2, \cdots, B_n\}$ is a coindependent set of couniform submodules of M, so that $n\leqq$ codim M. This completes the proof.

By Propositions 4.8. and 3.13, the theorem gives the following:

COROLLARY 4.14. *Let M be completely cocomplemented and C a cocomplement in M. Then M is cofinite-dimensional if and only if M/C and C are cofinite-dimensional. In this case,* codim $M=$ codim $M/C+$ codim C.

5. Quasi-projective and pseudo-projective modules

Henceforth, we shall adopt the following notations: M is a nonzero left R-module and S is the (R-)endomorphism ring of M, acting on the right side of M. Therefore $M={}_RM_S$ is an (R, S)-bimodule. The (Jacobson) radical of M is denoted by $\mathrm{J}(M)$.

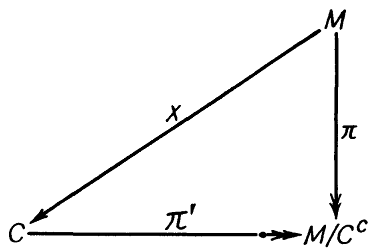
M is called *quasi-projective* iff for any submodule A of M and for any homomorphism ϕ: $M\longrightarrow M/A$, there exists an endomorphism $x\in S$ such that $\phi=x\pi(M\twoheadrightarrow M/A)$, where $\pi(M\twoheadrightarrow M/A)$ means the natural epimorphism of M onto M/A (see the footnote 4)). Evidently, every direct summand of a quasi-projective module is also quasi-projective. Among some characterizations, we use the following one (see [10; Proposition 2.1]):

M is quasi-projective if and only if for any submodules A, B of M and for any epimorphism $\phi: B \twoheadrightarrow M/A$, there exists a homomorphism $\psi: M \longrightarrow B$ with $\pi(M \twoheadrightarrow M/A) = \psi\phi$.

The following is seen in Miyashita [10; Theorem 2.3]:

PROPOSITION 5.1. *Let M be quasi-projective and C a cocomplement in M such that C has a cocomplement C^c in M. Then C is a direct summand of M.*

PROOF. For the natural epimorphisms $\pi(M \twoheadrightarrow M/C^c)$ and $\pi'(C \twoheadrightarrow M/C^c)$, the quasi-projectivity of M yields the existence of an endomorphism $x \in S$ such that $Mx \subset C$ and $x\pi' = \pi$.



Since $C^c x$ is included in $C \cap C^c$ and small in M, it follows from

$$M = M(1-x) + Mx = M(1-x) + Cx + C^c x$$

that

$$M = M(1-x) + Cx = M(1-x) + C.$$

The minimality of C^c which includes $M(1-x)$ deduces $M(1-x) = C^c$, and so $C^c + Cx = M$. Noting that C (a cocomplement in M) is a cocomplement of C^c in M, the minimality of C which includes Cx implies $Cx = C$, and so $Cx = Mx$. Hence $C + \mathrm{Ker}\, x = M$ and the minimality of $C^c = M(1-x)$ which includes $\mathrm{Ker}\, x$ asserts $\mathrm{Ker}\, x = C^c$, and so $M(1-x)x = 0$ or $x = x^2$. Therefore we conclude that $C = Mx$ is a direct summand of M, as desired.

The following is the result of Kasch and Mares [6]:

COROLLARY 5.2. *If M is projective and cocomplemented, then M is semiperfect.*

PROOF. Let A be a submodule of M. By the above, a cocomplement A^c of A in M is a direct summand of M and hence projective. Thus the natural epimorphism $\pi: A^c \twoheadrightarrow M/A$ is a projective cover of M/A.

Thus, the following are equivalent if M is projective:

(1) M is semiperfect.

(2) M is completely cocomplemented.

(3) M is cocomplemented.

Dualizing the notion of Singh and Jain [12], we shall say that M is *pseudo-projective* iff for any submodule A of M and for any epimorphism $\phi: M \twoheadrightarrow M/A$, there exists an endomorphism $x \in S$ such that $\phi = x\pi(M \twoheadrightarrow M/A)$.

The following are analogous characterizations of pseudo-projectives and verified easily:

(1) For any submodule A of M and for any epimorphism ϕ: $M \twoheadrightarrow M/A$, there exists an endomorphism $x \in S$ such that $\pi(M \twoheadrightarrow M/A) = x\phi$.

(2) For any left R-module N with epimorphisms ϕ, ψ: $M \twoheadrightarrow N$, there exists an endomorphism $x \in S$ such that $\phi = x\psi$.

(3) For any submodule A of M and for any epimorphisms ϕ, ψ: $M \twoheadrightarrow M/A$, there exists an endomorphism $x \in S$ such that $\phi = x\psi$.

Clearly quasi-projectivity implies pseudo-projectivity but we do not know how weak the latter is comparing with the former. Every direct summand of a pseudo-projective module is also pseudo-projective just as in the case of quasi-projectives.

Now we state the conditions concerning a left R-module M. See Utumi [14].

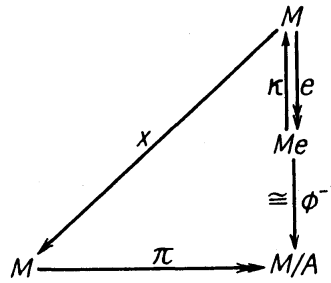
CONDITION (I): Let A be an arbitrary submodule of M. If M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

CONDITION (II): Let e, f be arbitrary idempotents of S. If $Me + Mf = M$, then $Me \cap Mf$ is a direct summand of M.

CONDITION (II'): Let e, f be arbitrary idempotents of S. If $Me + Mf = M$, then there exists an idempotent g of S such that $Mg = Me$ and $M(1-g) \subset Mf$.

LEMMA 5.3. *If M is pseudo-projective, then M satisfies Condition* (I). *Furthermore Condition* (I) *implies Condition* (II), *which is equivalent to Condition* (II').

PROOF. Pseudo-projectivity implies Condition (I): Let A be a submodule of M and let ϕ be an isomorphism of M/A to Me, $e = e^2 \in S$, a direct summand of M. Let κ be the canonical injection of Me into M. Then, since M is pseudo-projective, we have an endomorphism $x \in S$ such that $e\phi^{-1} = x\pi(M \twoheadrightarrow M/A)$.



Hence $\phi\kappa\chi\pi$ is the identity mapping of M/A, so that A is a direct summand of M.

Condition (I) implies Condition (II): Assume that $Me+Mf=M$ with $e=e^2$, $f=f^2\in S$. Then $M/\mathrm{Ker}\,(e-ef)$ is isomorphic to $M(e-ef)=M(1-f)$. By assumption there exists $g=g^2\in S$ such that $\mathrm{Ker}\,(e-ef)=Mg$. Since $M(1-e)\subset\mathrm{Ker}\,(e-ef)$, we have $M(1-e)(1-g)=0$ and so $ge=geg$. Further $Mg(e-ef)=0$ deduces $ge=gef$. Thus it follows that $Me\cap Mf=Mge$, where ge is an idempotent of S.

Condition (II) implies Condition (II'): Assume that $Me+Mf=M$, $Me\cap Mf=Mh$; $e=e^2$, $f=f^2$, $h=h^2\in S$. Then we have $Me\oplus(Mf\cap M(1-h))=M$, since $Mfh\subset Me$ and $Mf(1-h)=M(1-fh)f$.

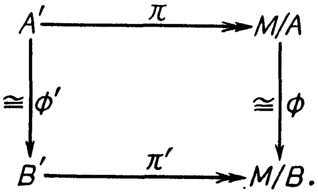
Condition (II') implies Condition (II): Assume that $Me+Mf=M$, $Mh=Me$, $M(1-h)\subset Mf$; $e=e^2$, $f=f^2$, $h=h^2\in S$. Then we have $Me\cap Mf=Mfh$, where fh is an idempotent of S.

6. Codirect modules

The following are the dual notions of direct hulls and (uniquely) direct modules in [13].

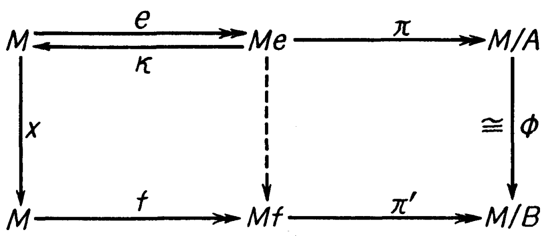
DEFINITION 6.1. *Let A be a submodule of M and Me, $e=e^2\in S$, a direct summand of M with $M(1-e)\subset A$. Then Me is called to be a codirect cover of M/A in M iff $M(1-e)\subset' A\subset M$, or equivalently iff $Ae=A\cap Me$ is a small submodule of M.*

DEFINITION 6.2. *M is called to be a codirect module iff every factor module of M has a codirect cover in M. Moreover, a codirect module M is called to be uniquely codirect iff for any submodules A, B of M, every isomorphism ϕ between M/A and M/B is induced by an isomorphism ϕ' between any codirect covers A' and B' of M/A and M/B in M respectively, in the sense that the following diagram is commutative:*



PROPOSITION 6.3. *M is uniquely codirect if and only if M is pseudo-projective and codirect.*

PROOF. Suppose that M is pseudo-projective and codirect. Let ϕ: $M/A \longrightarrow M/B$ be an isomorphism for submodules A, B of M, and let Me and $Mf(e=e^2,\ f=f^2 \in S)$ be codirect covers of M/A and M/B in M respectively.

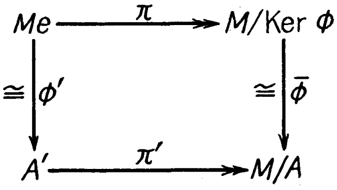


Since M is pseudo-projective, there exists an endomorphism $x \in S$ such that $e\pi\phi = xf\pi'$. Then κxf is a homomorphism of Me into Mf with $\kappa xf\pi' = \pi\phi$, where κ is the canonical injection of Me into M. Further, κxf is a minimal epimorphism by Lemma 3.1, since $\pi\phi$ and π' are minimal epimorphisms. Since

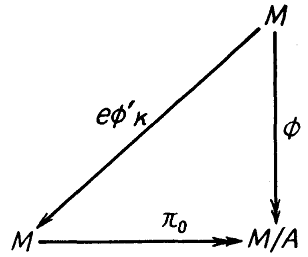
$$M(1-e) \subset \text{Ker } e\kappa xf \subset \text{Ker } e\kappa xf\pi' = \text{Ker } e\pi\phi = A\,,$$

it follows that $M(1-e) \subset'$ Ker $e\kappa xf \subset M$. But $Me\kappa xf = Mf$ implies that Ker $e\kappa xf$ is a direct summand of M, because of Condition (I) for M (which is deduced by the pseudo-projectivity). Hence Ker $e\kappa xf = M(1-e)$ and so Ker $\kappa xf = 0$. Thus κxf is an isomorphism which induces ϕ. Therefore M is now uniquely codirect.

Conversely, suppose that M is uniquely codirect. Let A be a submodule of M and let ϕ be an epimorphism of M onto M/A. Then ϕ induces an isomorphism $\bar{\phi}$: $M/\text{Ker } \phi \cong M/A$. Now, let Me and A' be codirect covers of $M/\text{Ker } \phi$ and M/A in M respectively. Then, by assumption, there exists an isomorphism ϕ' such that the diagram



is commutative. Let κ be the canonical injection of the direct summand A' into M. Then $e\phi'\kappa$ is an endomorphism of M and the diagram



is commutative, since $\kappa\pi_0=\pi'$ and $(1-e)\phi=0$. Thus M is pseudo-projective.

PROPOSITION 6.4. *If M is codirect, then every cocomplement in M is a direct summand of M. Conversely, let M be completely cocomplemented. If every cocomplement in M is a direct summand of M, then M is codirect.*

PROOF. Suppose that M is codirect and let C be a cocomplement in M. Then M/C has a codirect cover Me, $e=e^2\in S$, in M, i.e., $M(1-e)\subset\!,C\subset M$. Since C is coclosed in M, $C=M(1-e)$, as desired.

Conversely, suppose that every cocomplenet in M is a direct summand of M. If M is completely cocomplemented, then $A^{cc}\subset\!,A\subset M$ for any submodule A of M. Since A^{cc} is a direct summand of M: (say) $A^{cc}\oplus B=M$, M/A has a codirect cover B in M. Thus M is codirect.

The following is immediate by Proposition 5.1 and the above:

COROLLARY 6.5. *If M is quasi-projective and completely cocomplemented, then M is codirect.*

PROPOSITION 6.6. *Every direct summand of a codirect module is codirect.*

PROOF. Let Me $(e=e^2\in S)$ be a direct summand of a codirect module M. If $A\subset Me$ is a submodule of M, then M/A has a codirect cover Mf $(f=f^2\in S)$ in M, i.e., $M(1-f)\subset\!,A\subset M$. Hence $M(1-f)\subset\!,A\subset Me$ by Proposition 2.1. It follows from $M(1-f)\oplus(Me\cap Mf)=Me$ that $Me\cap Mf$ is a codirect cover of Me/A in Me. This shows that Me is codirect.

PROPOSITION 6.7. *If M is codirect, then M is cocomplemented.*

PROOF. Let A be a submodule of a codirect module M. Then M/A has a codirect cover $Me\,(e=e^2\in S)$ in M, i.e., $M(1-e)\subset\!,A\subset M$, and so

$A+Me=M$. If $A+B=M$ for a submodule $B \subset Me$, then $M(1-e)+B=M$ and therefore $B=Me$. This means that Me is a cocomplement of A in M. Thus M is cocomplemented.

PROPOSITION 6.8. *If M is codirect with Condition* (II), *then M is completely cocomplemented.*

PROOF. Let A, B be submodules of M satisfying $A+B=M$. Since M is codirect, there exist idempotents e, $f \in S$ such that $M(1-e) \subset' A \subset M$, $M(1-f) \subset' B \subset M$. Then $M(1-e)+M(1-f)=M$ implies that $M(1-e) \cap M(1-f)=Mg$ for some $g=g^2 \in S$, by Condition (II). Accordingly $Mg \oplus Me' = M(1-e)$, $Mg \oplus Mf' = M(1-f)$ with some idempotents e', $f' \in S$. Since $M(1-e) \cap Mf'=0$ and $M(1-e) \oplus Mf'=M$, there exists an idempotent $h \in S$ such that $M(1-h)=M(1-e)$ and $Mh=Mf'$. Evidently $A+Mh=M$. Moreover, if a submodule $C \subset Mh$ satisfies $A+C=M$, then $M(1-e)+C=M$ and so $Mh=Ch=C$. Thus $Mh \subset B$ is a cocomplement of A in M, showing that M is completely cocomplemented.

Now the following corollaries are obvious:

COROLLARY 6.9. *Let M be quasi-projective. Then the following are equivalent*:

(1) *M is codirect.*
(2) *M is uniquely codirect.*
(3) *M is completely cocomplemented.*

COROLLARY 6.10. *Let M be projective. Then the following are equivalent*:

(1) *M is semiperfect.*
(2) *M is completely cocomplemented.*
(3) *M is cocomplemented.*
(4) *M is codirect.*
(5) *M is uniquely codirect.*

COROLLARY 6.11. *For a ring R, ${}_RR$ is codirect if and only if R is a semiperfect ring.*

Now we prepare a result on the (Jacobson) radical $J(M)$ of a semiperfect module M.

(1) If $M \neq 0$ is projective, then $J(M) \neq M$. (See [1] and [8].)

(2) If M is projective and semiperfect, then $J(M)$ is a small submodule of M.

This is Theorem 3.3 of Mares [8], but we can replace "semiperfect" by "codirect" in the assumption. Thus, if $M(1-e)\subset' J(M)\subset M$ for $e=e^2\in S$, we have

$$\begin{aligned} M(1-e) &= M(1-e)\cap J(M) \\ &= M(1-e)\cap\big(J(Me)\oplus J\big(M(1-e)\big)\big) \\ &= \big(M(1-e)\cap J(Me)\big)\oplus J\big(M(1-e)\big) = J\big(M(1-e)\big). \end{aligned}$$

Hence $M(1-e)=0$, i.e., $J(M)$ is small in M.

(3) If M is semiperfect, then $J(M)$ is a small submodule of M.

This is a known result. Let $\phi: P \twoheadrightarrow M$ be a projective cover of M. Since M is semiperfect, so is P by Corollary 3.4. Then $J(P)$ is small in P by the above. Hence $J(P)\phi$ is small in M and included in $J(M)$. Conversely, if A is a small submodule of M, then $\operatorname{Ker}\phi = 0\phi^{-1}\subset' A\phi^{-1}\subset P$ by Proposition 1.2, (6). But $0\subset' \operatorname{Ker}\phi\subset P$, and so $0\subset' A\phi^{-1}\subset P$. Thus $A\phi^{-1}\subset J(P)$, so that $A=(A\phi^{-1})\phi\subset J(P)\phi$. Consequently $J(M)=J(P)\phi$ is small in M. Thus we can set up:

PROPOSITION 6.12. *If M is semiperfect, then* $J(M)$ *is a small submodule of M.*

7. Codirect modules with Condition (I)

In this section we shall investigate the endomorphism ring of a codirect module with Condition (I).

Henceforward, we shall understand the following: $M={}_RM_S$ is a nonzero (R, S)-bimodule, where S is the endomorphism ring of ${}_RM$. We put

$$Y(S)=\{x\in S \mid Mx \text{ is small in } M\}.$$

This is an ideal of S containing no nonzero idempotent, and we have $MY(S)\subset J(M)$, where $J(M)$ is the (Jacobson) radical of ${}_RM$. By $\bar{S}$ we denote the residue class ring of S modulo $Y(S)$: $\bar{S}=S/Y(S)$, and $\bar{x}$ is the residue class of $x\in S$ modulo $Y(S)$.

The following is rather tight and verified easily:

PROPOSITION 7.1. *M is sum-irreducible if and only if M is codirect and indecomposable.*

LEMMA 7.2. *Suppose that M satisfies Condition* (I). *Then* $\mathrm{Y}(S)\subset \mathrm{J}(S)$.

PROOF. Let $x\in S$. Then $M(1-x)=M$ since Mx is small in M. By Condition (I), Ker $(1-x)$ is a direct summand of M. But Ker $(1-x)\subset Mx$ and so Ker $(1-x)$ is small in M. Thus we have Ker $(1-x)=0$. Hence $1-x$ is a unit of S, showing that $\mathrm{Y}(S)\subset \mathrm{J}(S)$.

The following may be compared with [10; Theorem 3.6] or [15; Theorem 4.2].

PROPOSITION 7.3. *Let M be codirect with Condition* (I). *Then the following are equivalent*:

(1) *M is indecomposable.*
(2) *M is sum-irreducible.*
(3) *S is a local ring.*

PROOF. (1) implies (2) because of Proposition 7.1.

(2) implies (3): Let $x\in S$. Then $Mx+M(1-x)=M$ deduces $Mx=M$ or $M(1-x)=M$, since M is sum-irreducible. If $Mx=M$, Ker x is a direct summand of M by Condition (I). Since $M\neq 0$ is indecomposable, Ker $x=0$ and so x is a unit of S. Similarly, if $M(1-x)=M$, then $1-x$ is a unit of S. This shows that S is a local ring.

(3) implies (1), since in a local ring, 0 and 1 are the only idempotents.

Mares [8; Theorem 2.4] or Miyashita [10; Theorem 2.12] shows the following in essence:

(1) If M is pseudo-projective, then $\mathrm{Y}(S)=\mathrm{J}(S)$.

(2) If M is pseudo-projective and cocomplemented, then $\bar{S}=S/\mathrm{Y}(S)$ is a (von Neumann) regular ring.

But we shall maintain here under slightly different assumptions.

PROPOSITION 7.4. *Assume that M is codirect with Condition* (I). *Then* $\mathrm{Y}(S)=\mathrm{J}(S)$ *and $\bar{S}$ is a regular ring.*

PROOF. The inclusion $\mathrm{Y}(S)\subset \mathrm{J}(S)$ is deduced by Lemma 7.2. Now let $y\in S$. Then, by Proposition 6.7, My has a cocomplement in M and it is a direct summand of M by Proposition 6.4: say $(My)^c=M(1-e)$, $e=e^2\in S$. Since $My+M(1-e)=M$, $Mye=Me$ yields that Ker ye is a direct summand $Mf\,(f=f^2\in S)$ of M, by Condition (I). Therefore, for any element $m\in M$ we can find a unique $m'\in M(1-f)$ such that $me=m'ye$. This implies the existence of an endomorphism $z \in S$ such that $e = zye$. Since $M(y-yzy) \subset My \cap M(1-e)$, which is small in M, we deduce $y-yzy \in \mathrm{Y}(S)$. Thus $\bar{S}$ is a regular ring.

If in particular $y \in \mathrm{J}(S)$, then $y \in \mathrm{Y}(S)$ since $1-yz$ is a unit of S. This completes the proof.

On lifting idempotents modulo $\mathrm{Y}(S)$ we have the following:

PROPOSITION 7.5. *Assume that M is codirect with Condition* (II), *and let x, $e=e^2 \in S$. If $\bar{x}=\bar{x}\bar{e}=\bar{x}^2$, then there exists an endomorphism $f=fe=f^2 \in S$ such that $\bar{x}=\bar{f}$.*

PROOF. It follows from $\bar{x}=\bar{x}\bar{e}$ that $M(x-xe)$ is small in M. Hence $M(1-x)+Mxe=M$. Let g, h be idempotents in S such that

$$M(1-g) \subset' M(1-x) \subset M, \qquad M(1-h) \subset' Mxe \subset M.$$

Then $M(1-g)+M(1-h)=M$ and there exists, by Condition (II′), an endomorphism $f=f^2 \in S$ such that $M(1-f)=M(1-g)$, $Mf \subset M(1-h)$. Since $M(1-x)x$ including $M(1-f)x$ and $Mxe(1-x)$ including $Mf(1-x)$ are both small in M, $(1-f)x$ and $f(1-x)$, and hence $x-f$ are contained in $\mathrm{Y}(S)$. As $Mf \subset Mxe$, we have $f=fe$, completing the proof.

Now we shall mention some results concerning coindependent sets of direct summands of M.

LEMMA 7.6. *Assume that M satisfies Condition* (I). *Let e_1, e_2 be idempotents of S such that $\bar{S}\bar{e}_1+\bar{S}\bar{e}_2=\bar{S}$. Then $Se_1+Se_2=S$ and $Me_1+Me_2=M$. Furthermore, there exists an idempotent f of S such that $Me_1 \cap Me_2 = Mf$. It follows that $Se_1 \cap Se_2=Sf$ and $\bar{S}\bar{e}_1 \cap \bar{S}\bar{e}_2=\bar{S}\bar{f}$.*

PROOF. If $\bar{S}\bar{e}_1+\bar{S}\bar{e}_2=\bar{S}$, then there exist $x_1=x_1e_1$, $x_2=x_2e_2$ in S such that $\bar{x}_1+\bar{x}_2=\bar{1}$. Hence $1-(x_1+x_2) \in \mathrm{Y}(S) \subset \mathrm{J}(S)$ (Lemma 7.2), and so x_1+x_2 is a unit of S. Thus there exist $y_1=y_1e_1$, $y_2=y_2e_2$ in S such that $y_1+y_2=1$. Hence $Se_1+Se_2=S$ and $Me_1+Me_2=M$. By Condition (II) (implied by Condition (I)), there exists an idempotent f in S such that $Me_1 \cap Me_2=Mf$. This yields $Se_1 \cap Se_2=Sf$ evidently, and this implies $\bar{S}\bar{f} \subset \bar{S}\bar{e}_1 \cap \bar{S}\bar{e}_2$. Since $e_1-e_1y_1=e_1y_2$, $e_2-e_2y_2=e_2y_1 \in Se_1 \cap Se_2=Sf$, it holds for any element $\bar{z}$ of $\bar{S}\bar{e}_1 \cap \bar{S}\bar{e}_2$ that

$$\bar{z}=\overline{z(y_1+y_2)}=\overline{ze_2y_1}+\overline{ze_1y_2} \in \bar{S}\bar{f}\,.$$

Thus $\bar{S}\bar{e}_1 \cap \bar{S}\bar{e}_2=\bar{S}\bar{f}$, completing the proof.

The following are easily deduced by the above:

PROPOSITION 7.7. *Assume that M satisfies Condition* (I). *Let $e_\lambda (\lambda \in \Lambda)$ be idempotents of S. If $\{{}_{\bar{S}}\bar{S}\bar{e}_\lambda | \lambda \in \Lambda\}$ is a coindependent set, then so is $\{{}_R Me_\lambda | \lambda \in \Lambda\}$.*

PROPOSITION 7.8. *Assume that M is codirect with Condition* (I). *Let $x_\lambda \in S$ $(\lambda \in \Lambda)$ and let $\{{}_{\bar{S}}\bar{S}\bar{x}_\lambda | \lambda \in \Lambda\}$ be a coindependent set of principal left ideals of $\bar{S}$. Then thre exist idempotents $e_\lambda (\lambda \in \Lambda)$ of S such that $\bar{S}\bar{x}_\lambda = \bar{S}\bar{e}_\lambda$ for each $\lambda \in \Lambda$, and it follows that $\{{}_R Me_\lambda | \lambda \in \Lambda\}$ is a coindependent set of direct summands of M.*

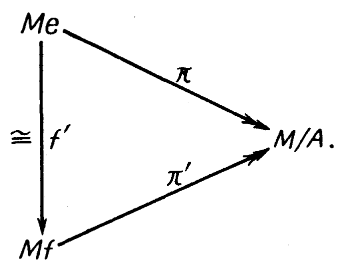
The next statement gives a kind of uniqueness of codirect covers.

PROPOSITION 7.9. *Assume that M is codirect with Condition* (II). *Let A be a submodule of M. Then codirect covers of M/A in M are isomorphic to one another.*

PROOF. Let Me and $Mf (e = e^2, f = f^2 \in S)$ be both codirect covers of M/A in M. Then

$$M(1-e) \subset' A \subset M, \qquad M(1-f) \subset' A \subset M$$

and hence $Me + M(1-f) = M$. By Condition (II) for M, there exists an idempotent g of S such that $Me \cap M(1-f) = Mg$. But $Mg = 0$ since $Mg = Mge$ is included in Ae which is small in M. This means that the contraction mapping f' of f to Me is an isomorphism: $Me \cong Mef = Mf$. Since $Me(1-f) \subset A$, we have the following commutative diagram:



Now assume that M is codirect with Condition (I) and let e be a nonzero idempotent of S. Then, by Proposition 6.8, M is deduced to be completely cocomplemented, and Propositions 4.2 and 7.3 imply that the following conditions are equivalent:

(1) Me is a minimal cocomplement in M.

(2) $M(1-e)$ is a couniform submodule of M.

(3) Me is sum-irreducible.
(4) Me is indecomposable (i. e., e is a primitive idempotent).
(5) e is a local idempotent (i. e., eSe is a local ring)[7].
Thus the preparations have been complete to prove the next:

THEOREM 7.10. *Assume that M is codirect with Condition* (I). *Then the following are equivalent*:
(1) *M is cofinite-dimensional.*
(2) *M is a direct sum of a finite number of indecomposable submodules.*
(3) *S is a semiperfect ring.*

PROOF. (1) implies (2): By assumption, there exists a maximal coindependent set $\{C_1, C_2, \cdots, C_n\}$ of couniform direct summands of M, where $n=\text{codim } M$. If $n=1$, then $C_1=0$ and so M is an indecomposable module. Let $n>1$. Then $C_i+\bigcap_{j\neq i} C_j=M$ implies that there exist cocomplements $C_i^c\subset \bigcap_{j\neq i} C_j$ of C_i in $M(1\leqq i\leqq n)$. Each direct summand C_i^c is a minimal cocomplement in M (by Proposition 4.2) and so it is an indecomposable submodule of M. Next, it follows that $M=\sum_{i=1}^{n} C_i^c$, as in the proof of Theorem 4.13. Actually this is a direct sum. Because, $C_i^c\cap\sum_{j\neq i} C_j^c$ is included in $C_i^c\cap C_i$, a small direct summand of M by Condition (II), which is zero. Therefore (2) is implied.

Obviously (2) implies (1) by Theorem 4.13.

The statement (2) says that S has a finite orthogonal set of local idempotents whose sum is 1. As is well-known, this is equivalent to the condition that S is a semiperfect ring. Thus the proof is complete.

By Corollaries 3.7, 6.9 and Proposition 4.10, our theorem deduces the following:

COROLLARY 7.11. *If M is linearly compact and quasi-projective, then S is a semiperfect ring.*

COROLLARY 7.12. *For a ring R, if ${}_RR$ is linearly compact, then R is a semiperfect ring.*

Evidently, a semisimple module ${}_RN$ is quasi-projective, completely cocomplemented and codirect. Hence by the above theorem, N is cofinite-dimensional if and only if N is a direct sum of a finite number of inde-

7) Thus, we can deduce that every primitive idempotent in a semiperfect ring is local.

composable (i. e., simple) submodules. Namely, N is cofinite-dimensional if and only if N is finitely generated.

Now assume that M is semiperfect. Then, because $J(M)$ is small in M (Proposition 6.12) and $M/J(M)$ is semisimple (Proposition 3.8), the following conditions are equivalent:

(1) M is cofinite-dimensional.

(2) $M/J(M)$ is cofinite-dimensional. (Propositions 4.8 and 4.9)

(3) $M/J(M)$ is finitely generated.

(4) M is finitely generated. (Proposition 2.3)

Thus we reach the following (cf. [8; Theorem 6.1]):

COROLLARY 7.13. *Assume that M is quasi-projective and semiperfect. Then M is finitely generated if and only if S is a semiperfect ring.*

8. Uniquely codirect modules

In this section we shall obtain some results on uniquely codirect modules in view of their automorphisms which induce the isomorphisms between codirect covers.

LEMMA 8.1. *Assume that M is uniquely codirect. Let A_i $(i=1,2)$ be submodules of M, and let $Me_i (e_i=e_i^2\in S)$ be codirect covers of M/A_i in M $(i=1,2)$. Then A_1 is cosimilar to A_2 if and only if Me_1 is isomorphic to Me_2.*

PROOF. Suppose $A_1\sim A_2$. Then there exist coessential extensions $A_i\subset\prime A_i'\subset M$ $(i=1,2)$ such that $M/A_1'\cong M/A_2'$. Since $M(1-e_i)\subset\prime A_i'\subset M$, each Me_i is a codirect cover of M/A_i' in M. Thus we have $Me_1\cong Me_2$, because M is uniquely codirect.

Conversely, suppose $Me_1\cong Me_2$. Then, trivially, $0\sim 0$ (Me_1, Me_2). On the other hand, since each A_ie_i is small in Me_i, $Me_i/A_ie_i\cong M/A_i$ implies $0\sim A_i$ (Me_i, M). Thus $A_1\sim A_2$, as required.

LEMMA 8.2. *Let $\{A_\lambda|\lambda\in\Lambda\}$ and $\{B_\gamma|\gamma\in\Gamma\}$ be maximal coindependent sets of couniform submodules of M, and C a proper submodule of M. Then $\{A_\lambda|\lambda\in\Lambda\}\cup\{C\}$ is coindependent if and only if $\{B_\gamma|\gamma\in\Gamma\}\cup\{C\}$ is coindependent.*

PROOF. Suppose that $\{A_\lambda|\lambda\in\Lambda\}\cup\{C\}$ is not coindependent. Then $\bigcap_{i=1}^{n} A_{\lambda_i}+C$ is proper in M for some $\lambda_1, \lambda_2, \cdots, \lambda_n\in\Lambda$. We may assume here the positive integer n minimal. Then we shall show that $C_0=\bigcap_{i=1}^{n} A_{\lambda_i}+C$

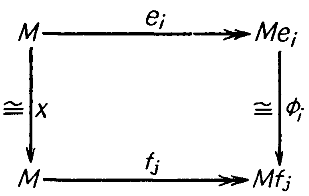
is a couniform submodule of M. If $n=1$, C_0 is couniform by $A_1 \subset C_0$. Let $n>1$. Noting that $\bigcap_{i=2}^{n} A_{\lambda_i} + A_{\lambda_1} = M$ and $\bigcap_{i=2}^{n} A_{\lambda_i} + C = M$, we have an epimorphism

$$\phi: M/A_{\lambda_1} \twoheadrightarrow M/C_0$$

defined by $(m + A_{\lambda_1})\phi = m + C_0 \left(m \in \bigcap_{i=2}^{n} A_{\lambda_i}\right)$. Let $C_0 \subset D$ be a proper submodule of M. Then $(D/C_0)\phi^{-1}$ is proper in M/A_{λ_1}, and hence small in M/A_{λ_1}, since A_{λ_1} is couniform in M. Therefore $(D/C_0)\phi^{-1}\phi = D/C_0$ is small in M/C_0. This implies that C_0 is couniform in M.

If $C_0 \in \{B_\gamma | \gamma \in \Gamma\}$, say $C_0 = B_{\gamma_0}$ $(\gamma_0 \in \Gamma)$, then $C + B_{\gamma_0} = B_{\gamma_0}$ is proper in M, so that $\{B_\gamma | \gamma \in \Gamma\} \cup \{C\}$ is not coindependent. If $C_0 \notin \{B_\gamma | \gamma \in \Gamma\}$, then $\{B_\gamma | \gamma \in \Gamma\} \cup \{C_0\}$ is not coindependent by the maximality of $\{B_\gamma | \gamma \in \Gamma\}$, so that $\{B_\gamma | \gamma \in \Gamma\} \cup \{C\}$ is not coindependent. Thus the proof completes.

PROPOSITION 8.3. *Assume that M is uniquely codirect. Let $\{A_i | i=1, 2, \cdots, n\}$, $\{B_j | j=1, 2, \cdots, n\}$ be maximal coindependent sets of couniform submodules of M, and let Me_i, Mf_j $(e_i = e_i^2,\ f_j = f_j^2 \in S)$ be codirect covers of M/A_i, M/B_j in M respectively $(1 \leqq i, j \leqq n)$. Then there exist a permutation χ of the numbers $1, 2, \cdots, n$ and isomorphisms ϕ_i of Me_i to Mf_j, where $j = \chi(i)$ for any i, $1 \leqq i \leqq n$. Furthermore, there exists an automorphism $x \in S$ such that x induces each isomorphism ϕ_i, i.e., the diagram*



is commutative with $j = \chi(i)$, $1 \leqq i \leqq n$.

PROOF. We have already known that there exists a permutation χ of $1, 2, \cdots, n$ such that $A_i \sim B_j$ with $j = \chi(i)$, $1 \leqq i \leqq n$. By Lemma 8.1, then, we have isomorphisms ϕ_i: $Me_i \cong Mf_j$, where $j = \chi(i)$, $1 \leqq i \leqq n$. The coessential extensions

$$M(1-e_i) \subset' A_i \subset M, \qquad M(1-f_j) \subset' B_j \subset M$$

assert that $\{M(1-e_i) | i=1, 2, \cdots, n\}$, $\{M(1-f_j) | j=1, 2, \cdots, n\}$ are maximal coindependent sets of couniform direct summands of M. Accordingly there exist idempotents e, $f \in S$ such that

$$\bigcap_{i=1}^{n} M(1-e_i) = M(1-e), \qquad \bigcap_{j=1}^{n} M(1-f_j) = M(1-f),$$

by Condition (II). Then $M(1-f)+Mf=M$ implies $M(1-e)+Mf=M$ by Lemma 8.2. Actually Mf is a cocomplement of $M(1-e)$ in M, which is shown by using Lemma 8.2 again, and thus we have $M(1-e)\oplus Mf=M$. Therefore, the contraction mapping of $1-f$ to $M(1-e)$ induces an isomorphism ϕ': $M(1-e)\cong M(1-e)(1-f)=M(1-f)$.

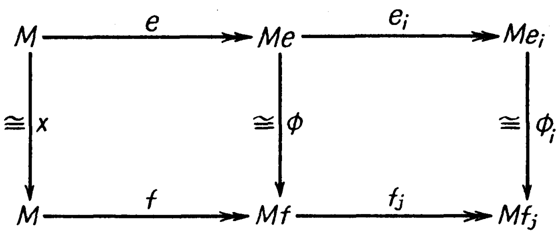
On the other hand, the compositions of the canonical isomorphisms

$$Me \cong M/M(1-e) \cong \prod_{i=1}^{n} M/M(1-e_i) \cong \prod_{i=1}^{n} Me_i,$$

$$Mf \cong M/M(1-f) \cong \prod_{j=1}^{n} M/M(1-f_j) \cong \prod_{j=1}^{n} Mf_j,$$

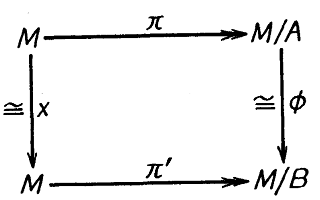
and the isomorphisms ϕ_i: $Me_i \cong Mf_j$ $(j=\chi(i))$ give an isomorphism ϕ: $Me\cong Mf$. Namely, ϕ is induced as $e_i\phi_i = e\phi f_j$, $j=\chi(i)$, for any i, $1\leqq i\leqq n$ (by noting $e_i=ee_i$, $f_j=ff_j$).

Consequently, the pair (ϕ, ϕ') of isomorphisms yields an automorphism x of M, and the commutativity of the diagram



(where $j=\chi(i)$, $1\leqq i\leqq n$), is now obvious. This completes the proof.

PROPOSITION 8.4. *Assume that M is uniquely codirect and cofinite-dimensional. Let A, B be submodules of M such that there exists an isomorphism ϕ of M/A to M/B. Then ϕ is induced by an automorphism $x\in S$, i.e., the diagram*



(with the natural epimorphisms π, π') is commutative.

PROOF. First, ϕ extends to an isomorphism ϕ' between codirect covers

Me and Mf $(e=e^2,\ f=f^2\in S)$ of M/A and M/B in M respectively, or the diagram

$$\begin{array}{ccc} Me & \xrightarrow{\pi_0} & M/A \\ \cong\downarrow\phi' & & \cong\downarrow\phi \\ Mf & \xrightarrow{\pi_0'} & M/B \end{array}$$

is commutative. Let codim $Me=$ codim $Mf=n$. Then there exist idempotents $e_i=ee_ie=e_i^2\in S$ $(1\leqq i\leqq n)$ such that $\{M(e-e_i)|i=1,2,\cdots,n\}$ is a maximal coindependent set of couniform direct summands of Me. The contraction mapping ϕ_i of the isomorphism ϕ' to Me_i gives a direct summand Mf_i, $f_i=ff_if=f_i^2\in S$, of Mf for each i, $1\leqq i\leqq n$. Namely we have ϕ_i: $Me_i\cong Mf_i$ and $M(e-e_i)\phi'=M(f-f_i)$ for each i, $1\leqq i\leqq n$. Therefore $\{M(f-f_i)|i=1,2,\cdots,n\}$ is a maximal coindependent set of couniform direct summands of Mf. Hence from the contraction mappings e_i' of e_i to Me and f_i' of f_i to Mf, the following commutative diagram follows:

$$\begin{array}{ccc} Me & \xrightarrow{e_i'} & Me_i \\ \cong\downarrow\phi' & & \cong\downarrow\phi_i \\ Mf & \xrightarrow{f_i'} & Mf_i\,. \end{array}$$

Hence the diagram

$$\begin{array}{ccc} Me & \xrightarrow{(e_i')} & \prod_{i=1}^{n} Me_i \\ \cong\downarrow\phi' & & \cong\downarrow(\phi_i) \\ Mf & \xrightarrow{(f_i')} & \prod_{i=1}^{n} Mf_i \end{array}$$

is commutative. However, the kernel of (e_i') is $\bigcap_{i=1}^{n} M(e-e_i)=0$ (being a small direct summand of Me), and (e_i') is surjective since $\{M(e-e_i)|i=1,2,\cdots,n\}$ is a coindependent set. Therefore (e_i') and similarly (f_i') are isomorphisms.

By Proposition 4.1, $\{M(e-e_i)\oplus M(1-e)|i=1,2,\cdots,n\}$ and $\{M(f-f_i)\oplus M(1-f)|i=1,2,\cdots,n\}$ are coindependent sets of couniform direct summands of M. But these can be extended to maximal coindependent sets of couniform direct summands of M, since M is (of course, completely cocomplemented and) cofinite-dimensional. Then by the above proposition, there exists an automorphism $x \in S$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{e_i} & Me_i \\ \cong \downarrow x & & \cong \downarrow \phi_i \\ M & \xrightarrow{f_i} & Mf_i \end{array}$$

is commuative for any i, $1 \leqq i \leqq n$, because each Me_i is trivially a codirect cover of $M/(M(e-e_i) \oplus M(1-e))$ in M. Accordingly, the diagram

$$\begin{array}{ccc} M & \xrightarrow{(e_i)} & \prod_{i=1}^{n} Me_i \\ \cong \downarrow x & & \cong \downarrow (\phi_i) \\ M & \xrightarrow{(f_i)} & \prod_{i=1}^{n} Mf_i \end{array}$$

is commutative, where (e_i) and (f_i) are surjective since $\{M(e-e_i) \oplus M(1-e) | i=1, 2, \cdots, n\}$ and $\{M(f-f_i) \oplus M(1-f) | i=1, 2, \cdots, n\}$ are coindependent. Consequently, attending to $(e_i)(e_i')^{-1}=e$ and $(f_i)(f_i')^{-1}=f$, we obtain the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{e} & Me \\ \cong \downarrow x & & \cong \downarrow \phi' \\ M & \xrightarrow{f} & Mf. \end{array}$$

Thus x is a desired automorphism, completing the proof.

The following is a well-known characterization of quasi-projective modules (see Miyashita [10; Theorem 2.7] or Wu and Jans [16]):

Let P be a projective left R-module and K a small submodule of P. If T is the endomorphism ring of ${}_RP$, then the following are equivalent:

(1) P/K is quasi-projective.

(2) $K=KT$.

(3) K is the sum of all submodules N such that P/N is quasi-projective.

As an analogous statement we can obtain the following last result:

PROPOSITION 8.5. *Let P be a finitely generated*[8] *projective semiperfect left R-module and K a small submodule of P. Moreover let T be the*

8) See the remark preceding to Corollary 7.13.

endomorphism ring of ${}_RP$, acting on the right, and T' the set of all surjective endomorphisms in T. Then the following conditions are equivalent:

(1) *P/K is pseudo-projective.*

(2) *$K=KT'$.*

(3) *K is the sum of all submodules N such that P/N is pseudo-projective.*

PROOF. (1) implies (3) obviously.

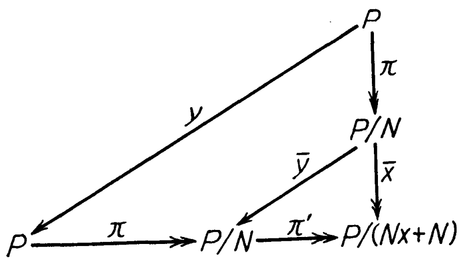
(3) implies (2): Suppose $x\in T'$ and let $N\subset K$ be a submodule of P such that P/N is pseudo-projective. Then we shall show $Nx\subset N$, which yields (2). We consider the natural epimorphisms $\pi: P\twoheadrightarrow P/N$ and $\pi': P/N\twoheadrightarrow P/(Nx+N)$. Since x induces an epimorphism

$$\bar{x}: \mathrm{P}/N\twoheadrightarrow P/(Nx+N)$$

by

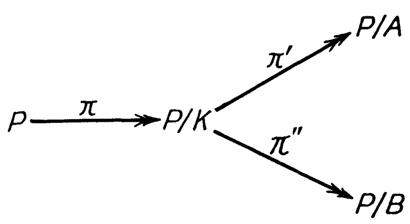
$$(p+N)\bar{x}=px+(Nx+N) \qquad (p\in P),$$

the pseudo-projectivity of P/N implies the existence of an endomorphism $\bar{y}$ of P/N with $\bar{x}=\bar{y}\pi'$.



Since P is projective, we have an endomorphism $y\in T$ such that $\pi\bar{y}=y\pi$. Hence $Ny\subset N$ and $P(y-x)\subset Nx+N$. Let $N'\subset P$ be the inverse image of N under $y-x$. Then $N+N'=P$, where N is small in P, so that we obtain $N'=P$. Therefore, $P(y-x)\subset N$ and so $N(y-x)\subset N$. Thus $Nx\subset N$ follows from $Ny\subset N$, as required.

(2) implies (1): Suppose that M is a left R-module such that there exist epimorphisms ϕ, ψ: $P/K\twoheadrightarrow M$ with Ker $\phi=A/K$ and Ker $\psi=B/K$, where $K\subset A$, B are submodules of P. We consider the natural epimorphisms



and the isomorphisms

$$\bar{\phi}: P/A \cong M, \qquad \bar{\psi}: P/B \cong M$$

by

$$(p+A)\bar{\phi} = (p+K)\phi, \qquad (p+B)\bar{\psi} = (p+K)\psi \qquad (p \in P),$$

so that $\pi'\bar{\phi} = \phi$ and $\pi''\bar{\psi} = \psi$. Then by Proposition 8.4, the isomorphism $\bar{\phi}\bar{\psi}^{-1}$ is induced by an automorphism $x \in T'$, i.e., the diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi\pi'} & P/A \\ \cong\downarrow x & & \cong\downarrow \bar{\phi}\bar{\psi}^{-1} \\ P & \xrightarrow{\pi\pi''} & P/B \end{array}$$

is commutative. By assumption, $Kx \subset K$ and thus x can induce an endomorphism $\bar{x}$ of P/K such that $\phi = \bar{x}\psi$. This shows that P/K is pseudo-projective.

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[19] H. ZÖSCHINGER: Komplemente als direkte Summanden, Algebra-Berichte 6, Mathematisches Institut der Universität München, 1973.

Supplementary notes (June 30, 1975).

1. We add the following as references:

[18]′ H. ZÖSCHINGER: Komplementierte Moduln über Dedekindringen, "J. Algebra, 29 (1974), 42-56".

[19]′ H. ZÖSCHINGER: Komplemente als direkte Summanden, "Arch. Math., 25 (1974), 241-253".

[20] H. ZÖSCHINGER: Moduln, die in jeder Erweiterung ein Komplement haben, Math. Scand., 35 (1974), 267-287.

We remark here on terminologies; our *cocomplement*=*Komplement* in [18]′ and *completely cocomplemented* = *supplemented* in [3] = *supplementiert* in [19]′.

2. On linearly compact modules, we refer to the following:

[21] B. J. MÜLLER: Linear compactness and Morita duality, J. Algebra, 16 (1970), 60-66.

[22] T. ONODERA: Linearly compact modules and cogenerators, J. Fac. Sci. Hokkaido Univ., 22 (1972), 116-125.

[23] T. ONODERA: Linearly compact modules and cogenerators II, Hokkaido Math. J., 2 (1973), 243-251.

The result that every linearly compact module is cocomplemented has been given in [11; Proposition 2.6] and also in [22; Theorem 5] (their *complemented*=our *cocomplemented*). See also [20].

Lemma 3.5 arouses our interest; we may say that the dual of this lemma holds trivially without the assumption of the submodule A to be linearly compact. As another proposition of such a form, we can recall

[21 ; Lemma 2]. See also [23 ; §3].

3. The following is a corollary to Theorem 4.13:

COROLLARY 4.15. *Let $M(\neq 0)$ be completely cocomplemented. Then M is cofinite-dimensional if and only if there exists a coindependent set of a finite number of couniform submodules of M such that the intersection of them is a small submodule of M.*

4. Let A be a submodule of a nonzero R-module M. The (Jacobson) radical $J(A, M)$ of A in M is defined to be the sum of all coessential extensions of A in M. Then $J(A, M)$ coincides with the intersection of all maximal submodules of M that include A in case A is included in a maximal submodule of M and $J(A, M)=M$ otherwise. Thus, the (Jacobson) radical $J(M)$ of M is nothing but $J(0, M)$. The following two conditions for M are equivalent:

($*$) Every proper submodule of M is included in a maximal submodule of M.

($**$) For every submodule A of M, $J(A, M)$ is a coessential extension of A in M.

Thus, if M satisfies ($*$), then $J(M)$ is small in M. Conversely, if M is cocomplemented and if $J(M)$ is small in M, then M satisfies ($*$). Because, every submodule A of M which has a cocomplement in M yields $A+J(0, M)=J(A, M)$. This phenomenon will be compared with the fact that every nonzero submodule of M includes a minimal submodule if and only if the socle of M is an essential submodule.

5. An R-module M is called cosemisimple iff M satisfies the following equivalent statements (Fuller [24]):

(1) Every simple R-module is M-injective.

(2) Every proper submodule of M is an intersection of maximal submodules.

(3) Every finitely cogenerated factor module of M is semisimple.

Now we can paraphrase (2) by using our words:

(4) The radical of every factor module of M is zero, i.e., $A=J(A, M)$ for every submodule A of M.

(4)′ Every submodule of M is (coessentially) coclosed in M.

We can also check the dual conditions to (2), (4)′ and the next:

(2)′ Every proper submodule of M is an intersection of coindependent maximal submodules.

The cosemisimple cocomplemented module is nothing but a semisimple module.

[24] K. R. FULLER: Relative projectivity and injectivity classes determined by simple modules, J. London Math. Soc. (2), 5 (1972), 423–431.

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