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The extended zero-divisor graph of a commutative ring I

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Abstract. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The extended zero-divisor graph of R is the undirected (simple) graph $\Gamma'(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if either $Rx \cap \mathrm{Ann}(y) \neq (0)$ or $Ry \cap \mathrm{Ann}(x) \neq (0)$. It follows that the zero-divisor graph $\Gamma(R)$ is a subgraph of $\Gamma'(R)$. It is proved that $\Gamma'(R)$ is connected with diameter at most two and with girth at most four, if $\Gamma'(R)$ contains a cycle. Moreover, we characterize all rings whose extended zero-divisor graphs are complete or star. Furthermore, we study the affinity between extended zero-divisor graph and zero-divisor graph associated with a commutative ring. For instance, for a non-reduced ring R, it is proved that the extended zero-divisor graph and the zero-divisor graph of R are identical to the join of a complete graph and a null graph if and only if $ann_R(Z(R))$ is a prime ideal.

Key words: Extended zero-divisor graph, Zero-divisor graph, Complete graph.

1. Introduction

One of the interesting and active area in the last decade is using graph theoretical tools to study the algebraic structures. There are several papers devoted to the study of rings in this approach (see [1], [4], [8], [13], [2], and [16]). For most recent study of a graph that is a generalization of the classical zero-divisor graph see [9]. For a very useful survey article on the classical zero-divisor graph see [3].

Throughout this paper, R denotes a unitary commutative ring which is not an integral domain. We denote by $\mathrm{Min}(R)$, $Z(R)$, $\mathrm{Nil}(R)$ and $\mathrm{U}(R)$, the set of all minimal prime ideals of R, the set of all zero-divisor elements of R, the set of all nilpotent elements of R and the set of all invertible elements of R, respectively. An element $r \in R$ is called *regular* if $r \notin Z(R)$. We say that $\mathrm{depth}(R) = 0$, whenever every non-unit element of R is a zero-divisor. The ring R is said to be *reduced* if it has no non-zero nilpotent element. For every subset A of R, we denote the annihilator of A by $\mathrm{Ann}(A)$. Moreover, for the subset A of R we let $A^* = A \setminus \{0\}$. For any undefined notation or

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terminology in ring theory, we refer the reader to [7], [10].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. The diameter and the girth of a graph G are denoted by $\text{diam}(G)$ and $\text{girth}(G)$, respectively. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Also G is called a *null graph* if it has no edge. A complete bipartite graph of part sizes m, n is denoted by $K_{m,n}$. If $m = 1$, then the complete bipartite graph is called a *star graph*. Also, a complete graph of n vertices is denoted by K_n. For any undefined notation or terminology in graph theory, we refer the reader to [17].

In this paper, we introduce and study the notion of extended zero-divisor graph associated with a commutative ring. The *extended zero-divisor graph* of a commutative ring R is the undirected (simple) graph $\Gamma'(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if either $Rx \cap \text{Ann}(y) \neq (0)$ or $Ry \cap \text{Ann}(x) \neq (0)$. We investigate the interplay between the graph-theoretic properties of $\Gamma'(R)$ and the ring-theoretic properties of R. We study the connectedness, diameter and girth of $\Gamma'(R)$. Also we completely characterize all rings R for which $\Gamma'(R)$ is a star graph or has a vertex adjacent to every other vertex. The *zero- divisor graph* of a ring R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. In the last section of this paper, we study some relations between two graphs $\Gamma(R)$ and $\Gamma'(R)$.

2. Basic Properties of Extended Zero-Divisor Graphs

In this section, we study fundamental properties of $\Gamma'(R)$. It is shown that $\Gamma'(R)$ is always a connected graph and $\text{diam}(\Gamma'(R)) \leq 2$. Moreover, we prove that if $\Gamma'(R)$ contains a cycle, then $\text{girth}(\Gamma'(R)) \leq 4$. Finally, it is shown that if $\Gamma'(R)$ contains a cycle, then $\text{girth}(\Gamma'(R)) = 4$ if and only if R is reduced with $|\text{Min}(R)| = 2$.

The following lemma has a straightforward proof that is omitted.

Lemma 2.1 *Let R be a reduced ring, and let $x \in Z(R)^*$. Then*
(1) $\mathrm{Ann}(x) = \mathrm{Ann}(x^n)$ *for each positive integer* $n \geq 2$.
(2) $Rx \cap \mathrm{Ann}(x) = (0)$.

The following lemma has a key role in this paper.

Lemma 2.2 *Let R be a ring.*

(1) *If $x - y$ is an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $\Gamma'(R)$.*
(2) *If $x - y$ is not an edge of $\Gamma'(R)$ for some distinct $x, y \in Z(R)^*$, then* $\mathrm{Ann}(x) = \mathrm{Ann}(y)$. *If R is a reduced ring, then the converse is also true.*
(3) *If* $\mathrm{Ann}(x) \nsubseteq \mathrm{Ann}(y)$ *or* $\mathrm{Ann}(y) \nsubseteq \mathrm{Ann}(x)$ *for some distinct $x, y \in Z(R)^*$, then $x - y$ is an edge of $\Gamma'(R)$.*
(4) *If $Rx \cap \mathrm{Ann}(x) \neq (0)$ for some $x \in Z(R)^*$, then x is adjacent to all other vertices in $\Gamma'(R)$. In particular, if $x \in \mathrm{Nil}(R)^*$, then x is adjacent to all other vertices.*
(5) $\Gamma'(R)[\mathrm{Nil}(R)^*]$ *is a complete subgraph of* $\Gamma'(R)$.

Proof. (1) Suppose that $x - y$ is an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Thus $xy = 0$ and clearly $x \in Rx \cap \mathrm{Ann}(y)$. Hence $x - y$ is an edge of $\Gamma'(R)$.

(2) Suppose that $x - y$ is not an edge of $\Gamma'(R)$ for some distinct $x, y \in Z(R)^*$. Then $Rx \cap \mathrm{Ann}(y) = (0)$ and $Ry \cap \mathrm{Ann}(x) = (0)$ and so $Rx\mathrm{Ann}(y) = (0)$ and $Ry\mathrm{Ann}(x) = (0)$. Hence $\mathrm{Ann}(x) = \mathrm{Ann}(y)$. If R is a reduced ring, then by Part (2) of Lemma 2.1, $Rx \cap \mathrm{Ann}(x) = (0)$, for every $x \in Z(R)^*$. This fact together with $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ imply that $x - y$ is not an edge of $\Gamma'(R)$.

(3) It is clear by Part (2).

(4) Assume that $Rx \cap \mathrm{Ann}(x) \neq (0)$ for some $x \in Z(R)^*$, and let y be another vertex of $\Gamma'(R)$. If x is not adjacent to y, then by Part (2), $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and hence $Rx \cap \mathrm{Ann}(y) \neq (0)$, a contradiction.

(5) By Part (4), it is trivial. □

By [5, Theorem 2.3], for every ring R, the zero-divisor graph $\Gamma(R)$ is a connected graph and $\mathrm{diam}(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $\mathrm{girth}(\Gamma(R)) \leq 4$ (see [15]). By using Theses facts and Lemma 2.2, we have the following result.

Theorem 2.3 *Let R be a ring. Then $\Gamma'(R)$ is connected and* $\mathrm{diam}(\Gamma'(R)) \leq 2$. *Moreover, if* $\Gamma'(R)$ *contains a cycle, then* $\mathrm{girth}(\Gamma'(R)) \leq 4$.

Proof. By Lemma 2.2 (1), $\Gamma(R)$ is a subgraph of $\Gamma'(R)$ such that $V(\Gamma'(R)) = V(\Gamma(R))$. So $\Gamma'(R)$ is connected and $\mathrm{girth}(\Gamma'(R)) \leq 4$.

Now, we show that $\mathrm{diam}(\Gamma'(R)) \leq 2$. If $\mathrm{Nil}(R) \neq (0)$, then by Lemma 2.2 (4), $\mathrm{diam}(\Gamma'(R)) \leq 2$. If $\mathrm{Nil}(R) = (0)$ and $d(x, y) \neq 1$, for some distinct $x, y \in Z(R)^*$, then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$, by Lemma 2.2 (2). Since $\mathrm{Nil}(R) = (0)$, Lemma 2.1 implies that $Ry \cap \mathrm{Ann}(y) = (0)$. Therefore, for every $0 \neq z \in \mathrm{Ann}(y)$, both of x and y are adjacent to z and hence $d(x, y) = 2$. This completes the proof. □

The next theorem shows that $\mathrm{girth}(\Gamma'(R)) = 4$ may occur.

Theorem 2.4 *Let R be a ring and $\Gamma'(R)$ contains a cycle. Then* $\mathrm{girth}(\Gamma'(R)) = 4$ *if and only if R is reduced with* $|\mathrm{Min}(R)| = 2$.

Proof. First suppose that $\mathrm{girth}(\Gamma'(R)) = 4$. If $\mathrm{Nil}(R) \neq (0)$, then by Lemma 2.2 (4), $\mathrm{girth}(\Gamma'(R)) = 3$, a contradiction. Thus $\mathrm{Nil}(R) = (0)$. Now, let $x \in Z(R)^*$. We show that $\mathrm{Ann}(x)$ is a prime ideal of R. To see this, assume that $ab \in \mathrm{Ann}(x)$ such that $a \notin \mathrm{Ann}(x)$ and $b \notin \mathrm{Ann}(x)$. This implies that $ax \neq 0$ and $bx \neq 0$ but $axbx = 0$. So for every $0 \neq c \in \mathrm{Ann}(x)$, it is easy to see that $c - ax - bx - c$ is a triangle, a contradiction. Hence $\mathrm{Ann}(x)$ is a prime ideal. Since R is reduced, Lemma 2.1 (2) together with [12, Corollary 2.2] imply that $\mathrm{Ann}(x)$ is a minimal prime ideal. By using a similar argument, $\mathrm{Ann}(y)$ is a minimal prime ideal, for every $0 \neq y \in \mathrm{Ann}(x)$. Now, we prove that $\mathrm{Min}(R) = \{\mathrm{Ann}(x), \mathrm{Ann}(y)\}$. It is enough to show that $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) = (0)$. Assume to the contrary, $0 \neq a \in \mathrm{Ann}(x) \cap \mathrm{Ann}(y)$. Thus $a - x - y - a$ is a triangle (as $xy = 0$), a contradiction. Hence $\mathrm{Min}(R) = \{\mathrm{Ann}(x), \mathrm{Ann}(y)\}$.

Conversely, suppose that R is reduced and $|\mathrm{Min}(R)| = 2$. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be the minimal prime ideals of R. Since R is reduced, we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$, by [12, Corollary 2.4]. It is not hard to see that $\Gamma'(R) = K_{|\mathfrak{p}_1^*|, |\mathfrak{p}_2^*|}$. As $\Gamma'(R)$ contains a cycle, $\mathrm{girth}(\Gamma'(R)) = 4$. □

3. Star or Complete Extended Zero-Divisor Graphs

In this section, we classify all rings with complete or star extended zero-divisor graphs. First, we characterize all rings R for which the graph $\Gamma'(R)$ has a vertex adjacent to every other vertex. Indeed, the first result in this section is applied to characterize all rings whose extended zero-divisor graphs are star or complete.

Theorem 3.1 *Let R be a ring. Then there is a vertex of $\Gamma'(R)$ which is adjacent to every other vertex if and only if one of the following statements holds.*

(1) $R \cong R_1 \times R_2$*, where R_1 and R_2 are rings such that* $|U(R_i)| = 1$ *and* $\mathrm{depth}(R_i) = 0$*, for some* $1 \leq i \leq 2$.
(2) $\mathrm{Nil}(R) \neq (0)$

Proof. Suppose that $a \in Z(R)^*$ is adjacent to every other vertex and $\mathrm{Nil}(R) = (0)$. If $a \neq a^2$, then either $Ra \cap \mathrm{Ann}(a^2) \neq (0)$ or $Ra^2 \cap \mathrm{Ann}(a) \neq (0)$, which contradicts Lemma 2.1. Thus $a = a^2$ and so by Brauer's Lemma (see[14, 10.22]), $R \cong Ra \times R(1-a)$. We may assume that $R \cong R_1 \times R_2$ with $(1,0)$ adjacent to every other vertex. Now, for any $1 \neq u \in U(R_1)$, it is easy to see that $(1,0)$ is not adjacent to $(u,0)$, a contradiction unless $1 = u$. Hence, $|U(R_1)| = 1$. Also, if $\mathrm{depth}(R_1) \neq 0$ and $r \in R_1$ is regular, then $(1,0)$ is not adjacent to $(r,0)$, a contradiction.

Conversely, assume that one of the (1) or (2) is satisfied. Condition (2) implies that every element of $\mathrm{Nil}(R)^*$ is adjacent to every other vertex, by Lemma 2.2 (4). If (1) holds, then, with no loss of generality, one may assume that $|U(R_1)| = 1$ and $\mathrm{depth}(R_1) = 0$. It is easily seen that $(1,0)$ is adjacent to every other vertex. □

It is known that if R is a ring such that $\mathrm{depth}(R) \neq 0$, then R is infinite. Also, Ganesan ([11]) proved that if R is infinite and $Z(R) \neq (0)$, then $Z(R)$ must be infinite (in fact $|R| \leq |Z(R)|^2$ when $2 \leq |Z(R)| < \infty$). We are now in a position to characterize all rings whose extended zero-divisor graph is star.

Theorem 3.2 *Let R be a ring. Then $\Gamma'(R)$ is a star graph if and only if one of the following statements holds*:
(1) $R \cong \mathbb{Z}_2 \times D$*, where D is an integral domain.*

(2) $|Z(R)| = 3$.
(3) $\mathrm{Nil}(R)$ *is a prime ideal of* R *and* $|\mathrm{Nil}(R)| = 2$.

Proof. First suppose that $\Gamma'(R)$ is a star graph. By Lemma 2.2 (5), $|\mathrm{Nil}(R)| \leq 3$. We consider the following cases.

Case 1. $|\mathrm{Nil}(R)| = 1$ (i.e, R is reduced). Since $\Gamma'(R)$ is a star graph, there exists a vertex of $\Gamma'(R)$ which is adjacent to every other vertex. By Theorem 3.1 (and its proof), one may assume that $R \cong R_1 \times R_2$ with $(1, 0)$ adjacent to every other vertex, $|U(R_1)| = 1$ and $\mathrm{depth}(R_1) = 0$. If $x \in Z(R_1)^*$, it is easily seen that the induced subgraph on the vertices $(1, 0)$, $(x, 0)$ and $(0, 1)$ forms a triangle in $\Gamma'(R)$, a contradiction. So $Z(R_1) = (0)$. Similarly, $Z(R_2) = (0)$. Therefore, $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain.

Case 2. $|\mathrm{Nil}(R)| = 2$. This implies that $Z(R) \neq \mathrm{Nil}(R)$ (since $\Gamma'(R)$ is star). Let $a \in \mathrm{Nil}(R)^*$. By lemma 2.2 (4), a is adjacent to all vertices contained in $Z(R) \setminus \{0, a\}$. If $xy = 0$ for some $x, y \in Z(R) \setminus \{0, a\}$, then x is adjacent to y, a contradiction. So $\mathrm{Ann}(x) = \{0, a\}$, for every $x \in Z(R)^*$. Now, we show that $\mathrm{Nil}(R)$ is a prime ideal of R. Assume to the contrary, $xy \in \mathrm{Nil}(R)$ such that $x \notin \mathrm{Nil}(R)$, $y \notin \mathrm{Nil}(R)$. It is easy to check that $x \neq y$, $x, y \in Z(R)$, $xy \neq 0$ and $xy \in Rx \cap \mathrm{Ann}(y)$. Thus x is adjacent to y, a contradiction.

Case 3. $|\mathrm{Nil}(R)| = 3$. By Lemma 2.2 (4), $\Gamma'(R) = K_2$ and $|Z(R)| = |\mathrm{Nil}(R)| = 3$.

Conversely, assume that one of the conditions (1), (2) or (3) holds. If one of (1) and (2) holds, then the proof is obvious. If (3) holds, then there exists exactly one element $a \in \mathrm{Nil}(R)^*$. By Lemma 2.2 (4), a is adjacent to all other vertices in $\Gamma'(R)$. Since $\mathrm{Nil}(R)$ is a prime ideal of R, $\mathrm{Ann}(x) = \{0, a\}$, for every $x \in Z(R)^*$. To complete the proof, we have only to show that $Ry \cap Ra = (0)$, for every $y \in Z(R) \setminus \{0, a\}$. Suppose to the contrary, $a \in Ry$. Then $a = ry$, for some $1 \neq r \in R$. Thus $ry^2 = 0$ and so $y^2 = 0$ or $y^2 = a$ or $r = a$. Since $\mathrm{Nil}(R) = \{0, a\}$ and $a = ry$, each case of the mentioned situations leads to a contradiction. □

Theorem 3.3 *Let* R *be a non-reduced ring. Suppose that* $\Gamma'(R)$ *is a star graph. Then the following hold*:
(1) R *is indecomposable.*
(2) *Either* $|Z(R)| = 3$ *or* $|Z(R)| = \infty$.

Proof. (1) Let $R \cong R_1 \times R_2$, where R_i is a ring, for $1 \leq i \leq 2$. Then for every $a \in \mathrm{Nil}(R)^*$, the vertices of the set $\{a, (1,0), (0,1)\}$ forms a triangle, a contradiction.

(2) Let $|Z(R)| < \infty$. Then R is an Artinian (finite) ring (since $|R| \leq |Z(R)|^2$). Now, by Part (1), $Z(R) = \mathrm{Nil}(R)$ and since $\Gamma'(R)$ is a star graph, we deduce from Part (5) of Lemma 2.2 $|Z(R)| = 3$. □

The next theorem studies a complete extended zero-divisor graph associated with a reduced ring.

Theorem 3.4 *Let R be a reduced ring. Then $\Gamma'(R)$ is a complete graph if and only if $a^2 = a$ for every $a \in Z(R)$, $|U(R)| = 1$ and* $\mathrm{depth}(R) = 0$.

Proof. First suppose that $\Gamma'(R)$ is a complete graph and let $a \in Z(R)^*$. Let $a^2 \neq a$. By by Lemma 2.1 (1), $\mathrm{Ann}(a^2) = Ann(a)$. Lemma 2.2 (2) implies that $a - a^2$ is not an edge of $\Gamma'(R)$, which is impossible. Thus $a^2 = a$, for every $a \in Z(R)$ and so $R \cong Ra \times R(1-a)$, for every $a \in Z(R)^*$. A similar argument to that of Theorem 3.1 completes the proof.

To prove the converse, if x is an arbitrary element of $Z(R)^*$, then it is shown that x is adjacent to every other vertex. The equality $|U(R)| = 1$ implies that $R \cong Rx \times R(1-x)$ and so we may assume that $R \cong R_1 \times R_2$. Since $|U(R)| = 1$ and $\mathrm{depth}(R) = 0$, it is easy to see that $R_1 \times (0) \cap \mathrm{Ann}(a) \neq (0)$, where $a \in Z(R)^* \setminus \{(1,0)\}$. Thus $(1,0)$ is adjacent to every other vertex and so x is also adjacent to every other vertex, as desired. □

Example 3.5 Let R be a reduced ring and let $R \cong \prod_{i \in \Lambda} R_i$, where R_i is an indecomposable ring, for every $i \in \Lambda$. If $\Gamma'(R)$ is a complete graph, then Theorem 3.4 implies that $Z(R_i) = \{0\}$ and $|U(R_i)| = 1$ and so $R_i \cong \mathbb{Z}_2$, for every $i \in \Lambda$.

The prove Theorem 3.7, the following lemma is needed.

Lemma 3.6 *Let R be a ring such that $\Gamma'(R)$ be complete. Then R is decomposable if and only if R is reduced.*

Proof. Suppose that R is decomposable and let $R \cong R_1 \times R_2$. Assume in contrary, and with no loss of generality, $|\mathrm{Nil}(R_1)| \geq 2$. Thus $2 \leq |\mathrm{Nil}(R_1)| \leq |\mathrm{J}(R_1)| \leq |U(R_1)|$ and so $|U(R_1)| \geq 2$. By a similar proof to that of Theorem 3.1, $\Gamma'(R)$ is not a complete graph, a contradiction. The other side, follows from Theorem 3.4. □

Let R be a ring and $x, y \in R$. We say that x is an Ry-regular element if $x \notin Z(Ry)$ and $RxRy \neq Ry$.

Theorem 3.7 *Let R be a non-reduced ring. Then $\Gamma'(R)$ is complete if and only if R is indecomposable and either x is not Ry-regular or y is not Rx-regular, for every distinct $x, y \in Z(R)^*$.*

Proof. If $\Gamma'(R)$ is a complete graph, then by Lemma 3.6, R is indecomposable. Moreover, it is obvious, for every two distinct elements $x, y \in Z(R)^*$ either x is not an Ry-regular element or y is not an Rx-regular element.

To prove the other side, let x, y be two distinct vertices of $V(\Gamma'(R))$. Without loss of generality, assume that x is not an Ry-regular element. If $x \in Z(Ry)$, then there is nothing to prove. If $x \notin Z(Ry)$, then $RxRy = Ry$, and so, by [7, Corollary 2.5], there exists an element $a \in Rx$ such that $(1-a)Ry = 0$. Thus $1 - a \in \mathrm{Ann}(y)$, and hence $Rx + \mathrm{Ann}(y) = R$. Now, the indecomposability of R implies that $Rx \cap \mathrm{Ann}(y) \neq (0)$. Hence $x - y$ is an edge of $\Gamma'(R)$, as desired. $\square$

4. When Extended Zero-Divisor Graphs and Zero-Divisor Graphs Are Identical?

As we have seen in the previous section, the extended zero-divisor graphs and zero-divisor graphs are close to each other. So, it may be interesting to characterize rings whose extended zero-divisor graphs are identical to zero-divisor graphs. We first study the case when R is reduced.

Theorem 4.1 *Let R be a reduced ring with $|\mathrm{Min}(R)| = n \geq 2$. Then $n = 2$ if and only if $\Gamma'(R) = \Gamma(R)$.*

Proof. First suppose that $\Gamma'(R) = \Gamma(R)$. We show that $n = 2$. Suppose to the contrary, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are three distinct minimal primes. Let $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$. Thus $\mathfrak{p}_2 \cup \mathfrak{p}_3 \nsubseteq \mathrm{Ann}(a)$ (as $\mathrm{Ann}(a) \subseteq \mathfrak{p}_2 \cap \mathfrak{p}_3$). So one may assume that $ab \neq 0$, for some $b \in \mathfrak{p}_2 \cup \mathfrak{p}_3 \setminus \mathfrak{p}_1$. With no loss of generality, assume that $b \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Obviously, $\mathrm{Ann}(b) \subseteq \mathfrak{p}_1$. Also, it follows from [12, Corollary 2.2], there exists an element $x \in \mathrm{Ann}(a)$ such that $x \notin \mathfrak{p}_1$. Therefore, $\mathrm{Ann}(a) \neq \mathrm{Ann}(b)$, and so by Lemma 2.2 (2), $a - b$ is an edge of $\Gamma'(R)$, a contradiction.

Conversely, suppose that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two distinct minimal prime ideals of R. It is not hard to check that $\Gamma(R) = \Gamma'(R) = K_{|\mathfrak{p}_1^*|,|\mathfrak{p}_2^*|}$. $\square$

The following corollaries follow from Theorem 4.1.

Corollary 4.2 *Let R be a reduced ring that is not an integral domain. Then the following statements are equivalent*:

(1) $\mathrm{girth}(\Gamma'(R)) = \infty$.
(2) $\Gamma'(R) = \Gamma(R)$ *and* $\mathrm{girth}(\Gamma(R)) = \infty$.
(3) $\mathrm{girth}(\Gamma(R)) = \infty$.
(4) $|\mathrm{Min}(R)| = 2$ *and at least one minimal prime ideal of* R *has exactly two distinct elements.*
(5) $\Gamma(R) = K_{1,n}$ *for some* $n \geq 1$.
(6) $\Gamma'(R) = K_{1,n}$ *for some* $n \geq 1$.

Proof. (1) ⇒ (2) By proof of Theorem 4.1, $|\mathrm{Min}(R)| = 2$ and so $\Gamma'(R) = \Gamma(R)$, $\mathrm{girth}(\Gamma(R)) = \infty$. (2) ⇒ (3) is clear. (3) ⇔ (4) ⇔ (5) follow from [6, Theorem 2.4]. (5) ⇒ (6) By [6, Theorem 2.4], we have $|\mathrm{Min}(R)| = 2$. Now, Theorem 4.1 implies that $\Gamma'(R) = \Gamma(R)$. (6) ⇒ (1) is clear. □

Corollary 4.3 *Let R be a reduced ring that is not an integral domain. Then the following statements are equivalent*:

(1) $\mathrm{girth}(\Gamma'(R)) = 4$.
(2) $\Gamma'(R) = \Gamma(R)$ *and* $\mathrm{girth}(\Gamma(R)) = 4$.
(3) $\mathrm{girth}(\Gamma(R)) = 4$.
(4) $|\mathrm{Min}(R)| = 2$ *and each minimal prime ideal of* R *has at least three distinct elements.*
(5) $\Gamma(R) = K_{m,n}$ *with* $m, n \geq 2$.
(6) $\Gamma'(R) = K_{m,n}$ *with* $m, n \geq 2$.

Proof. (1) ⇒ (2) By Theorems 2.4 and 4.1, $\Gamma'(R) = \Gamma(R)$, and so $\mathrm{girth}(\Gamma(R)) = 4$. (2) ⇒ (3) is clear. (3) ⇔ (4) ⇔ (5) are obtained by [6, Theorem 2.2]. (5) ⇒ (6) By [6, Theorem 2.2], $|\mathrm{Min}(R)| = 2$. Now, Theorem 4.1 implies that $\Gamma'(R) = \Gamma(R)$, and thus (6) ⇒ (1) is clear. □

In view of Corollaries 4.2 and 4.3, we have the following corollary.

Corollary 4.4 *Let R be a reduced ring with $|\mathrm{Min}(R)| = n \geq 2$. Then the following statements are equivalent*:
(1) $n = 2$.
(2) $\Gamma'(R) = \Gamma(R)$.
(3) $\mathrm{girth}(\Gamma'(R)) = \mathrm{girth}(\Gamma(R)) = \{4, \infty\}$.

In the rest of this section, we focus on non-reduced rings for which $\Gamma(R)$ and $\Gamma'(R)$ are identical.

Theorem 4.5 *Let R be a non-reduced ring. Then the following statements are equivalent*:

(1) $\Gamma(R) = \Gamma'(R)$.
(2) *If $xy \neq 0$ for some $x, y \in Z(R)$, then* $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ *and* $\mathrm{Ann}(x)$ *is a prime ideal of R.*

Proof. (1) $\Rightarrow$ (2) Suppose that $xy \neq 0$, for some $x, y \in Z(R)$. Since $\Gamma(R) = \Gamma'(R)$, we deduce that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$, by Lemma 2.2 (2). We now show that $\mathrm{Ann}(x)$ is a prime ideal of R. Let $ab \in \mathrm{Ann}(x)$, $a \notin \mathrm{Ann}(x)$ and $b \notin \mathrm{Ann}(x)$. So $xa \neq 0$, $xb \neq 0$, $a, b \in Z(R)$. By Lemma 2.2 (4), $x, y \notin \mathrm{Nil}(R)$ and hence either $x \neq a$ or $x \neq b$. With no loss of generality, one may assume that $x \neq b$. Thus $ax \in Rx \cap \mathrm{Ann}(b)$ and so $xb = 0$, a contradiction. Therefore, $\mathrm{Ann}(x)$ is a prime ideal of R, as desired.

(2) $\Rightarrow$ (1) If $xy = 0$ for all $x, y \in Z(R)$, then $\Gamma(R)$ is complete and so by Lemma 2.2 (1), $\Gamma'(R)$ is complete, i.e, $\Gamma(R) = \Gamma'(R)$. To complete the proof, we show that if $xy \neq 0$, then $Rx \cap \mathrm{Ann}(y) = (0)$ and $Ry \cap \mathrm{Ann}(x) = (0)$. Since $\mathrm{Ann}(x) = \mathrm{Ann}(y)$, it suffices to show that $Rx \cap \mathrm{Ann}(x) = (0)$ and $Ry \cap \mathrm{Ann}(y) = (0)$. If $x \in \mathrm{Ann}(x)$, then $x \in \mathrm{Ann}(y)$ and so $xy = 0$, a contradiction. Thus $x \notin \mathrm{Ann}(x)$. Also, if $Rx \cap \mathrm{Ann}(x) \neq (0)$, then there exists $rx \neq 0$ such that $(rx)x = rx^2 = 0$, for some $r \neq 1$. Since $x^2 \notin \mathrm{Ann}(x)$ ($\mathrm{Ann}(x)$ is a prime ideal), $r \in \mathrm{Ann}(x)$, a contradiction. Hence $Rx \cap \mathrm{Ann}(x) = (0)$. Similarly, $Ry \cap \mathrm{Ann}(y) = (0)$. □

In light of Theorem 4.5 and [5, Theorem 2.5], we state the following corollary.

Corollary 4.6 *Let R be a non-reduced ring and suppose that $\Gamma'(R) = \Gamma(R)$. Then the following hold*:
(1) *$Z(R)$ is an ideal of R.*
(2) $\mathrm{Nil}(R)^2 = 0$.
(3) $\mathrm{Ann}(Z(R)) = \mathrm{Nil}(R)$.

Suppose that R is a non-reduced ring. The proof of [5, Theorem 2.5] shows that if there exists a vertex of $\Gamma(R)$ which is adjacent to every other vertex say a, then $a \in \mathrm{Ann}(Z(R))$. By using this fact the following corollary is proved.

Corollary 4.7 *Let R be a non-reduced ring. Then $\Gamma(R) = \Gamma'(R) = K_n \vee \overline{K}_m$ if and only if $\mathrm{Ann}(Z(R))$ is a prime ideal of R.*

Proof. First suppose that $\Gamma(R) = \Gamma'(R) = K_n \vee \overline{K}_m$. Since $\Gamma(R) = K_n \vee \overline{K}_m$, every vertex of K_n is adjacent to all other vertices but there is no adjacency between two arbitrary vertices of $\overline{K}_m$. This implies that $\mathrm{Ann}(Z(R)) = V(K_n) \cup \{0\}$. Thus $xy \neq 0$, for every $x, y \in V(\overline{K}_m)$, and so $\mathrm{Ann}(x) = \mathrm{Ann}(y) = \mathrm{Ann}(Z(R))$. Theorem 4.5 implies that $\mathrm{Ann}(Z(R))$ is a prime ideal of R.

Conversely, assume that $\mathrm{Ann}(Z(R))$ is a prime ideal of R. This implies that $xy = 0$, for all $x, y \in \mathrm{Ann}(Z(R))$ and $xy \neq 0$, for all $x, y \in Z(R) \setminus \mathrm{Ann}(Z(R))$. Now, it is easy to see that $\Gamma(R)[\mathrm{Ann}(Z(R))^*]$ and $\Gamma(R)[Z(R) \setminus \mathrm{Ann}(Z(R))]$ are two subgraphs of $\Gamma(R)$ such that $\Gamma(R)[\mathrm{Ann}(Z(R))^*]$ is complete, $\Gamma(R)[Z(R) \setminus \mathrm{Ann}(Z(R))]$ is null and $\Gamma(R) = \Gamma(R)[\mathrm{Ann}(Z(R))^*] \vee \Gamma(R)[Z(R) \setminus \mathrm{Ann}(Z(R))]$. We finally show that $\Gamma(R) = \Gamma'(R)$. Obviously, $xy \neq 0$, if and only if $x, y \in Z(R) \setminus \mathrm{Ann}(Z(R))$. This together with $\mathrm{Ann}(Z(R))$ is prime imply that if $xy \neq 0$, then $\mathrm{Ann}(x) = \mathrm{Ann}(y) = \mathrm{Ann}(Z(R))$. So $\mathrm{Ann}(x)$ is a prime ideal of R. Now, by Theorem 4.5, $\Gamma(R) = \Gamma'(R)$. □

Theorem 4.8 *Let R be a non-reduced ring. Then the following statements are equivalent*:

(1) *$\Gamma'(R)$ is a star graph.*
(2) $\mathrm{girth}(\Gamma'(R)) = \infty$.
(3) $\Gamma'(R) = \Gamma(R)$ *and* $\mathrm{girth}(\Gamma(R)) = \infty$.
(4) $\mathrm{Ann}(Z(R))$ *is a prime ideal of R and either* $|Z(R)| = |\mathrm{Ann}(Z(R))| = 3$ *or* $= |\mathrm{Ann}(Z(R))| = 2$ *and* $|Z(R)| = \infty$.
(5) $\Gamma'(R) = K_{1,1}$ *or* $\Gamma'(R) = K_{1,\infty}$.
(6) $\Gamma(R) = K_{1,1}$ *or* $\Gamma(R) = K_{1,\infty}$.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) If $a \in \mathrm{Nil}(R)^*$, then a is adjacent to every other vertex in $\Gamma'(R)$. Since $\mathrm{girth}(\Gamma'(R)) = \infty$ and $\Gamma(R)$ is a connected subgraph of $\Gamma'(R)$, we conclude that $\Gamma'(R) = \Gamma(R)$, and so $\mathrm{girth}(\Gamma(R)) = \infty$.

(3) $\Rightarrow$ (4) Since R is a non-reduced ring, one may easily see that $\Gamma'(R)$ is a star graph and hence by Corollary 4.7, the result holds. (4) $\Rightarrow$ (5) is clear by Corollary 4.7.

(5) $\Rightarrow$ (6) is clear, as $\Gamma(R)$ is a connected subgraph of $\Gamma'(R)$.

(6) $\Rightarrow$ (1) If $\Gamma(R) = K_{1,1}$, then there is nothing to prove. Let $\Gamma(R) = K_{1,\infty}$. By [5, Theorem 2.5], $Z(R) = \mathrm{Ann}(a)$ for a non-zero element $a \in R$, and $\mathrm{Ann}(x) = \mathrm{Ann}(y) = \{0, a\} = \mathrm{Nil}(R)$, for every $x, y \in Z(R) \setminus \{0, a\}$. If $x - y$ is an edge of $\Gamma'(R)$ which is not an edge of $\Gamma(R)$, then we can suppose that $xy \neq 0$ and $Rx \cap \mathrm{Ann}(y) \neq (0)$. This implies that $rxy = 0$, for some $0 \neq rx \in Rx$. Therefore, $rx = a$ and so $rx^2 = 0$. Since $x^2 \notin \mathrm{Nil}(R)$, we deduce that $r = a$, a contradiction. This completes the proof. $\square$

We close this paper with the following example in which we investigate the relation between two graphs $\Gamma(\mathbb{Z}_n)$ and $\Gamma'(\mathbb{Z}_n)$.

Example 4.9 Let $R = \mathbb{Z}_n$. Since $\mathbb{Z}_n$ is an Artinian ring, Corollary 4.4 implies that if $\mathbb{Z}_n$ is not local, then $\Gamma(\mathbb{Z}_n) = \Gamma'(\mathbb{Z}_n)$ if and only if $n = pq$, for distinct prime numbers p, q. Moreover, in this case, $\Gamma(\mathbb{Z}_n) = K_{p-1,q-1}$. If $\mathbb{Z}_n$ is local, then $\Gamma(\mathbb{Z}_n) = \Gamma'(\mathbb{Z}_n)$ if and only if $n = p^2$, where p is a prime number. Moreover, in this case, $\Gamma(\mathbb{Z}_n) = K_{p-1}$. For instance it is easy to see that $\Gamma(\mathbb{Z}_{10}) = K_{1,4} = \Gamma'(\mathbb{Z}_{10})$. Also, for local rings $\mathbb{Z}_{25}$ and $\mathbb{Z}_8$, we can easily check that $\Gamma(\mathbb{Z}_{25}) = K_4 = \Gamma'(\mathbb{Z}_{25})$, but $\Gamma(\mathbb{Z}_8) = K_{1,2} \neq K_3 = \Gamma'(\mathbb{Z}_8)$.

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