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The DPW method for constant mean curvature surfaces in 3-dimensional Lorentzian spaceforms, with applications to Smyth type surfaces

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Abstract. We give criteria for singularities of spacelike constant mean curvature surfaces in 3-dimensional de Sitter and anti-de Sitter spaces constructed by the DPW method, which is a generalized Weierstrass representation. We also construct some examples of spacelike CMC surfaces, including analogs of Smyth surfaces with singularities, using appropriate models to visualize them.

Key words: differential geometry, surface theory, integrable systems.

1. Introduction

We can construct CMC $H = 0$ (minimal) surfaces in the 3-dimensional Euclidean space $\mathbb{R}^3$ by using a famous integral formula involving a pair of holomorphic functions satisfying certain conditions ([32]), called the Weierstrass representation, and many examples of minimal surfaces have been constructed with it. Dorfmeister, Pedit and Wu provided a generalization of the Weierstrass representation formula ([12]), called the DPW method, for constructing CMC $H \neq 0$ surfaces in $\mathbb{R}^3$, using holomorphic data satisfying certain conditions and a matrix loop splitting called the Iwasawa splitting. As an application, they constructed generalized Smyth surfaces. In other works ([9], [10], [15], [16], [26], etc), Brander, Dorfmeister, Inoguchi, Kobayashi, Kilian, Rossman and Schmitt also constructed new examples of CMC surfaces in the 3-dimensional sphere $\mathbb{S}^3$ and hyperbolic space $\mathbb{H}^3$ (non-Euclidean positive definite spaceforms) via the DPW method. In [9], Brander, Rossman and Schmitt constructed spacelike CMC surfaces in the 3-dimensional Lorentzian space $\mathbb{R}^{2,1}$, and they classified the spacelike rotationally symmetric surfaces. (Also see [13] and [14].) In the appendix of [10] (arXiv version), spacelike CMC $|H| < 1$ surfaces in the de Sitter space $\mathbb{S}^{2,1}$ are considered.

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Here we apply this method to spacelike CMC surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and anti de-Sitter space $\mathbb{H}^{2,1}$, and we give some examples of spacelike CMC surfaces in those Lorentzian spaceforms. We can see this theory as the special case of [13], however here we also study the singularities of the resulting CMC surfaces, and look at some examples in detail (totally umbilical surfaces, round cylinders, and more interestingly, analogs of Smyth surfaces).

This paper has eight sections. Section 2 explains the DPW method for spacelike CMC surfaces in $\mathbb{R}^{2,1}$, as in [9] and [13], using Lax pairs and loop groups. In Sections 3 and 4, we describe the DPW method for spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, as in [13]. Section 5 introduces some models of $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ for visualization. We use the *hollow ball model* for $\mathbb{S}^{2,1}$, and we use the *cylindrical models* for $\mathbb{H}^{2,1}$. In Section 6, as applications, we describe the most basic examples of spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$. In Section 7, we explore singularities on CMC surfaces in Lorentzian spaceforms, and give criteria for determining certain types of singularities (cuspidal edges, swallowtails and cuspidal cross caps) in the context of Lax representations. Finally, in Section 8, we apply the methods and results in this paper to the analogs of Smyth surfaces in Lorentzian spaceforms, including results about their symmetries, singularities, and relations with the Painleve III equation.

2. The loop group method in $\mathbb{R}^{2,1}$

We consider the DPW method for constructing spacelike CMC surfaces in $\mathbb{R}^{2,1}$ as in [9], [13].

Let $\mathbb{R}^{2,1}$ be the 3-dimensional Lorentz space with Lorentz metric

$$\langle x, y\rangle_{\mathbb{R}^{2,1}} := x_1y_1+x_2y_2-x_0y_0 \quad \text{for} \quad x=(x_1,x_2,x_0), y=(y_1,y_2,y_0)\in\mathbb{R}^{2,1}.$$

We simplify the notation $\langle\cdot,\cdot\rangle$ to $\langle\cdot,\cdot\rangle_{\mathbb{R}^{2,1}}$, and use its bilinear extension to $\mathbb{C}^3$. Let Σ be a simply-connected domain in $\mathbb{C}$ with the usual complex coordinate $z=x+iy$, and let $f:\Sigma\longrightarrow\mathbb{R}^{2,1}$ be a conformally immersed spacelike surface. Since f is conformal,

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0, \quad \langle f_z, f_{\bar z}\rangle := 2e^{2u} \tag{2.1}$$

for some function $u:\Sigma\longrightarrow\mathbb{R}$. We choose the unit timelike normal vector field $N:\Sigma\longrightarrow\mathbb{H}^2$ ($\mathbb{H}^2$ is the hyperbolic 2-space in $\mathbb{R}^{2,1}$) of f, and then the mean curvature and Hopf differential are

$$H = \frac{1}{2e^{2u}}\langle f_{z\bar{z}}, N\rangle, \quad Q := \langle f_{zz}, N\rangle. \tag{2.2}$$

The Gauss-Codazzi equations are the following form in the CMC cases:

$$4u_{z\bar{z}} + Q\bar{Q}e^{-2u} - 4H^2e^{2u} = 0, \quad Q_{\bar{z}} = 0. \tag{2.3}$$

The Codazzi equation in (2.3) is equivalent to the Hopf differential Q being holomorphic, and (2.3) is invariant with respect to the transformation $Q \to \lambda^{-2}Q$ for $\lambda \in \mathbb{S}^1$. When $f(x,y)$ is a spacelike CMC in $\mathbb{R}^{2,1}$, the spectral parameter $\lambda \in \mathbb{S}^1$ allows us to create a 1-parameter family of CMC surfaces $f^\lambda = f(x,y,\lambda)$ associated to $f(x,y)$.

To describe the 2×2 matrix representation of $\mathbb{R}^{2,1}$ as in [9], [13], writing $su_{1,1}$ for the Lie algebra of the Lie group

$$\mathrm{SU}_{1,1} := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha,\ \beta \in \mathbb{C},\ \alpha\bar{\alpha} - \beta\bar{\beta} = 1 \right\}, \tag{2.4}$$

we identify $\mathbb{R}^{2,1}$ with $su_{1,1}$ via

$$\mathbb{R}^{2,1} \ni x = (x_1, x_2, x_0) \longmapsto \begin{pmatrix} ix_0 & x_1 - ix_2 \\ x_1 + ix_2 & -ix_0 \end{pmatrix} \in su_{1,1}.$$

The metric becomes, under this identification, $\langle X, Y\rangle = (1/2)\mathrm{trace}(XY)$ for $X, Y \in su_{1,1}$.

Let f be a conformal immersed spacelike surface in $\mathbb{R}^{2,1}$ with associated family f^λ, and let the identity matrix and Pauli matrices be as follows:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

Then $\{\sigma_1, \sigma_2, i\sigma_3\}$ is an orthogonal basis for $su_{1,1} \approx \mathbb{R}^{2,1}$. We can define

$$e_1 := \frac{f_x^\lambda}{|f_x^\lambda|} = \frac{f_x^\lambda}{2e^u} = \hat{F}\sigma_1\hat{F}^{-1}, \quad e_2 := \frac{f_y^\lambda}{|f_y^\lambda|} = \frac{f_y^\lambda}{2e^u} = \hat{F}\sigma_2\hat{F}^{-1},$$
$$N := \hat{F}i\sigma_3\hat{F}^{-1} \tag{2.6}$$

for $\hat{F} = \hat{F}(z, \bar{z}, \lambda) \in \mathrm{SU}_{1,1}$. For this $\hat{F}$, we get the *untwisted* 2×2 *Lax pair* in $\mathbb{R}^{2,1}$ as follows:

$$\hat{F}_z = \hat{F}\hat{U}, \ \hat{F}_{\bar{z}} = \hat{F}\hat{V}, \ \text{where}$$

$$\hat{U} = \frac{1}{2}\begin{pmatrix} -u_z & -i\lambda^{-2}Qe^{-u} \\ 2iHe^u & u_z \end{pmatrix}, \ \hat{V} = \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & -2iHe^u \\ i\lambda^2\bar{Q}e^{-u} & -u_{\bar{z}} \end{pmatrix}. \tag{2.7}$$

We change the "untwisted" setting to the "twisted" setting by the following transformation (2.8). Let F be defined by

$$\hat{F} = -\sigma_3(F^{-1})^t \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \sigma_3, \tag{2.8}$$

producing the *twisted* 2×2 *Lax pair* of f in $\mathbb{R}^{2,1}$,

$$F_z = FU, \ F_{\bar{z}} = FV, \ \text{where}$$

$$U = \frac{1}{2}\begin{pmatrix} u_z & 2i\lambda^{-1}He^u \\ -i\lambda^{-1}Qe^{-u} & -u_z \end{pmatrix}, \ V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & i\lambda\bar{Q}e^{-u} \\ -2i\lambda He^u & u_{\bar{z}} \end{pmatrix}. \tag{2.9}$$

The following Proposition 2.1 gives us a method for determining spacelike CMC $H \neq 0$ surfaces in $\mathbb{R}^{2,1}$ from given data u and Q, by choosing a solution F of (2.9) and inserting F into the Sym-Bobenko type formula (2.10).

Proposition 2.1 (Sym-Bobenko type formula for spacelike CMC surfaces in $\mathbb{R}^{2,1}$ [9] [13]) *Let Σ be a simply-connected domain in $\mathbb{C}$. Let u and Q solve* (2.3)*, and let $F = F(z, \bar{z}, \lambda)$ be a solution of the system* (2.9)*. Suppose $F \in \mathrm{SU}_{1,1}$ for all $\lambda \in \mathbb{S}^1$ and one value of z. Then $F \in \mathrm{SU}_{1,1}$ for all z. Defining the following Sym-Bobenko type formulas*

$$f = \left[\frac{1}{2H}Fi\sigma_3F^{-1} + \frac{i}{H}\lambda(\partial_\lambda F)F^{-1}\right]\Big|_{\lambda=1}, \ N = -[Fi\sigma_3F^{-1}]\big|_{\lambda=1}, \tag{2.10}$$

f is a conformally parametrized spacelike CMC $H \neq 0$ surface in $\mathbb{R}^{2,1}$ with normal N.

3. The loop group method in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$

We apply the DPW method to construct spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, as in [13].

Let $\mathbb{R}^{3,1}$, resp. $\mathbb{R}^{2,2}$, be the 4-dimensional space with metric $\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle := x_1y_1 + x_2y_2 + \epsilon \cdot x_3y_3 - x_4y_4$, where $\epsilon = 1$ for $\mathbb{R}^{3,1}$, resp. $\epsilon = -1$ for $\mathbb{R}^{2,2}$. We define the spaceform $\mathcal{S} := \{x | \langle x, x \rangle = \epsilon\}$. Thus we obtain $\mathcal{S} = \mathbb{S}^{2,1}$ (resp. $\mathcal{S} = \mathbb{H}^{2,1}$) when $\epsilon = 1$ (resp. $\epsilon = -1$).

Let Σ be a simply-connected domain in $\mathbb{C}$ with the usual complex coordinate $w = x + iy$. Let $f : \Sigma \longrightarrow \mathcal{S}$ be a conformally immersed spacelike surface. Since f is conformal,

$$\langle f_w, f_w \rangle = \langle f_{\bar{w}}, f_{\bar{w}} \rangle = 0, \quad \langle f_w, f_{\bar{w}} \rangle = 2e^{2u} \tag{3.1}$$

for some function $u : \Sigma \longrightarrow \mathbb{R}$. For the unit normal vector field N of f satisfying $\langle N, N \rangle = -1$, $\langle f_w, N \rangle = \langle f_{\bar{w}}, N \rangle = 0$, we define the mean curvature H and Hopf differential $\mathcal{A}$ as follows:

$$H := \frac{1}{2e^{2u}} \langle f_{w\bar{w}}, N \rangle, \quad \mathcal{A} := \langle f_{ww}, N \rangle. \tag{3.2}$$

The Gauss-Codazzi equations are of the following form in the CMC ($H^2 > \epsilon$) cases:

$$2u_{w\bar{w}} - 2e^{2u}(H^2 - \epsilon) + \frac{1}{2}\mathcal{A}\bar{\mathcal{A}}e^{-2u} = 0, \quad \mathcal{A}_{\bar{w}} = 0. \tag{3.3}$$

Making the change of parameter $z := 2\sqrt{H^2 - \epsilon} \cdot w$ and defining Q by $\mathcal{A} = -2\sqrt{H^2 - \epsilon} \cdot e^{-i\psi} Q$ for a real constant ψ, we have equation (2.3) with $H = \pm 1/2$:

$$4u_{z\bar{z}} + Q\bar{Q}e^{-2u} - e^{2u} = 0, \quad Q_{\bar{z}} = 0. \tag{3.4}$$

$\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ and their matrix group representations.

We identify $\mathbb{R}^{3,1}$, resp. $\mathbb{R}^{2,2}$, with the Hermitian symmetric group $\{X \in M_{2\times 2} \mid X = \bar{X}^t\}$, resp. another matrix group, as follows:

$$\mathbb{R}^{3,1}, \text{resp. } \mathbb{R}^{2,2} \ni x = (x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_4 + \nu \cdot x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - \nu \cdot x_3 \end{pmatrix}, \tag{3.5}$$

with $\nu = 1$ for $\mathbb{R}^{3,1}$ and $\nu = i$ for $\mathbb{R}^{2,2}$. The metric becomes, under this identification, $\langle X, Y\rangle = -(1/2)\mathrm{trace}(X\sigma_2 Y^t \sigma_2)$. In particular, $\langle X, X\rangle = -\det(X)$, and we can identify $\mathbb{S}^{2,1}$, resp. $\mathbb{H}^{2,1}$, with

$$\{X \in M_{2\times 2} \mid X = \bar{X}^t, \det(X) = -1\} = \{F\sigma_3 \bar{F}^t \mid F \in \mathrm{SL}_2(\mathbb{C})\}, \quad (3.6)$$

respectively, with $\mathrm{SU}_{1,1}$, as in (2.4) via

$$\mathbb{H}^{2,1} \ni x = (x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_4 + ix_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - ix_3 \end{pmatrix}. \quad (3.7)$$

The Sym-Bobenko type formula in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$.

Defining $\hat{F}$ and f^λ in the same way as in (2.6), once again we change the "untwisted" setting to the "twisted" setting by the transformation (2.8) defining F. We define the *twisted* 2×2 *Lax pair* of f in $\mathcal{S}$ as

$$F_z = FU, \quad F_{\bar{z}} = FV, \quad (3.8)$$

where U and V are as in (2.9) with H fixed to be $1/2$.

The following Proposition 3.1 gives us a method for determining spacelike CMC H surfaces in $\mathbb{S}^{2,1}$ with $|H| > 1$, resp. CMC H surfaces in $\mathbb{H}^{2,1}$ for any value of H, from given data u and Q.

Proposition 3.1 (Sym-Bobenko type formula for spacelike CMC surfaces in $\mathbb{S}^{2,1}$, $\mathbb{H}^{2,1}$ [13]) *Let Σ be a simply-connected domain in $\mathbb{C}$. Let u and Q solve* (3.4)*, and let $F = F(z, \bar{z}, \lambda) \in \mathrm{SL}_2(\mathbb{C})$ be a solution of the system* (3.8) *such that $F(z, \bar{z}, \lambda) \in \mathrm{SU}_{1,1}$ when $\lambda \in \mathbb{S}^1$.*

- *In the case of $\mathbb{S}^{2,1}$, set $F_0 = F|_{\lambda = e^{q/2}e^{i\psi}}$ for $q, \psi \in \mathbb{R}$, $q \neq 0$. We define the following Sym-Bobenko type formulas*

$$f = F_0 \begin{pmatrix} e^{(1/2)q} & 0 \\ 0 & -e^{-(1/2)q} \end{pmatrix} \overline{F_0}^t, \quad N = -F_0 \begin{pmatrix} e^{(1/2)q} & 0 \\ 0 & e^{-(1/2)q} \end{pmatrix} \overline{F_0}^t. \quad (3.9)$$

  Then, f is a spacelike CMC $H = -\coth(-q)$ surface in $\mathbb{S}^{2,1}$ with normal N.
- *In the case of $\mathbb{H}^{2,1}$, set $F_1 = F|_{\lambda = e^{i\gamma_1}}$ and $F_2 = F|_{\lambda = e^{i\gamma_2}}$ for $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\gamma_1 - \gamma_2 \neq n\pi$ ($n \in \mathbb{Z}$). We define the following Sym-Bobenko*

type formulas

$$\begin{aligned} f &= iF_1 \begin{pmatrix} e^{(1/2)i(\gamma_1-\gamma_2)} & 0 \\ 0 & -e^{-(1/2)i(\gamma_1-\gamma_2)} \end{pmatrix} \overline{F_2}^t, \\ N &= -F_1 \begin{pmatrix} e^{(1/2)i(\gamma_1-\gamma_2)} & 0 \\ 0 & e^{-(1/2)i(\gamma_1-\gamma_2)} \end{pmatrix} \overline{F_2}^t. \end{aligned} \tag{3.10}$$

Then, f is a spacelike CMC $H = -\cot(\gamma_1 - \gamma_2)$ surface in $\mathbb{H}^{2,1}$ with normal N.

4. Application of the DPW method to $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$

In this section, we give a description of the DPW method, and apply this method to spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ as in [9], [13]. First, we define the potential ξ:

Definition 4.1 (holomorphic potential [9], [12], [16]) Let Σ be a simply-connected domain, $z \in \Sigma$ and $\lambda \in \mathbb{C}$. A *holomorphic potential* ξ is of the form

$$\xi := A dz, \quad A = A(z,\lambda) = \sum_{j=-1}^{\infty} A_j(z)\lambda^j, \tag{4.1}$$

where each $A_j(z)$ is a 2×2 matrix that is independent of λ, is holomorphic in $z \in \Sigma$, is traceless, is a diagonal (resp. off-diagonal) matrix when j is even (resp. odd), and the upper-right entry of $A_{-1}(z)$ is never zero.

Given a holomorphic potential ξ, we then solve the equation

$$d\phi = \phi\xi, \quad \phi(z_*) = I \quad \text{for} \quad \phi \in \Lambda\mathrm{SL}_2(\mathbb{C}), \tag{4.2}$$

where $\Lambda\mathrm{SL}_2(\mathbb{C}) = \left\{\phi(\lambda) \in M_{2\times 2} \mid \phi : \mathbb{S}^1 \xrightarrow{C^\infty} \mathrm{SL}_2(\mathbb{C}), \phi(-\lambda) = \sigma_3\phi(\lambda)\sigma_3\right\}$ for some choice of initial point $z_* \in \Sigma$.

We will use the following "$\mathrm{SU}_{1,1}$-Iwasawa splitting" defined on an open dense subset $\mathcal{B}_{1,1}$, called the *Iwasawa big cell*, of this loop group $\Lambda\mathrm{SL}_2(\mathbb{C})$. The following proposition was proven in [9].

Proposition 4.1 ($\mathrm{SU}_{1,1}$-Iwasawa splitting [9]) *For all $\phi \in \mathcal{B}_{1,1}$, there exist unique loops F and B such that*

$$\phi = F \cdot B, \tag{4.3}$$

where

$$F \in \Lambda \mathrm{SU}_{1,1} \cup \left\{ \begin{pmatrix} 0 & \lambda i \\ \lambda^{-1} i & 0 \end{pmatrix} \cdot \Lambda \mathrm{SU}_{1,1} \right\}, \quad B \in \Lambda^+_{\mathbb{R}} \mathrm{SL}_2(\mathbb{C}),$$

$$\Lambda \mathrm{SU}_{1,1} = \left\{ \phi(\lambda) \in M_{2\times 2} \mid \phi : \mathbb{S}^1 \xrightarrow{C^\infty} \mathrm{SU}_{1,1}, \phi(-\lambda) = \sigma_3 \phi(\lambda) \sigma_3 \right\},$$

$$\Lambda^+_{\mathbb{R}} \mathrm{SL}_2(\mathbb{C}) = \left\{ B_+(\lambda) \in \Lambda \mathrm{SL}_2(\mathbb{C}) \;\middle|\; \begin{array}{l} B_+ \text{ extends holomorphically to } \mathbb{D}, \\ B_+(0) = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ for some } \rho > 0. \end{array} \right\}.$$

After obtaining a solution ϕ of (4.2), we restrict to $\mathcal{B}_{1,1}$ and split ϕ as in (4.3). We then input F into the "Sym-Bobenko type formula" in (2.10) (resp. (3.9) or (3.10)), and the following proposition tells us we have a conformally immersed CMC surface $f = f(z, \bar{z})$ in $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$).

Proposition 4.2 ([9]) *Let ξ be a holomorphic potential as in* (4.1) *over a simply-connected domain Σ in $\mathbb{C}$, and let $\phi : \Sigma \longrightarrow \Lambda \mathrm{SL}_2(\mathbb{C})$ be a solution of* (4.2)*. Define the open set $\Sigma^o := \phi^{-1}(\mathcal{B}_{1,1}) \subset \Sigma$, and consider the unique* $\mathrm{SU}_{1,1}$*-Iwasawa splitting on Σ^o as in* (4.3)*.*

Then, after a conformal change of parameter z and appropriate choice of definitions for u, H and Q, we have that F satisfies the Lax pair (2.9)*.*

The converse of this recipe, that any conformally immersed CMC $H \neq 0$ (resp. $H = -\coth(-q)$ or $H = -\cot(\gamma_1 - \gamma_2)$) surface in $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$) has a holomorphic potential, also holds, but we will not prove that here, see [9] [13] for details.

5. Visualization of surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$

In the next section, we introduce some examples of CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$. At that point, we wish to visualize CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, using appropriate models for $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, and we consider those models here. These models are already known, and we describe them explicitly here, in the context of our setting.

The hollow ball model of $\mathbb{S}^{2,1}$.

To visualize CMC surfaces in $\mathbb{S}^{2,1}$, we use the *hollow ball model* of $\mathbb{S}^{2,1}$, as in [15], [21], [23], [33]. We get the following identification (bijection):

$$\mathbb{S}^{2,1} \ni (x_1, x_2, x_3, x_4) \longmapsto \left(\frac{e^{\arctan(x_4)}}{\sqrt{1+x_4^2}} x_1, \frac{e^{\arctan(x_4)}}{\sqrt{1+x_4^2}} x_2, \frac{e^{\arctan(x_4)}}{\sqrt{1+x_4^2}} x_3 \right) \in \mathcal{H}. \tag{5.1}$$

In this way, $\mathbb{S}^{2,1}$ is identified with the hollow ball $\mathcal{H}$. The hollow ball model is the set $\{(y_1, y_2, y_3) \in \mathbb{R}^3 | e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}\}$.

The cylindrical model of $\mathbb{H}^{2,1}$.

To visualize CMC surfaces in $\mathbb{H}^{2,1}$, we use a model for $\mathbb{H}^{2,1}$, which we call the *cylindrical model*, like in [7] and [19]. It is actually only a model for the universal cover of $\mathbb{H}^{2,1}$, but has the advantage that certain symmetries become more apparent. We have the following homeomorphism and universal covering: $\mathbb{H}^{2,1} \approx \mathbb{H}^2 \times \mathbb{S}^1 \subset \mathbb{H}^2 \times \mathbb{R} =: \mathcal{C}$. This means that we have the following covering:

$$\begin{aligned} &\mathbb{H}^{2,1} \ni (x_1, x_2, x_3, x_4) \\ &\quad \longmapsto \left(\frac{x_1}{1+\sqrt{x_3^2+x_4^2}}, \frac{x_2}{1+\sqrt{x_3^2+x_4^2}}, \mathrm{Arg}\left(\frac{x_3 + i x_4}{\sqrt{x_3^2+x_4^2}} \right) \right) \in \mathcal{C}. \end{aligned} \tag{5.2}$$

The cylindrical model is the set $\{(y_1, y_2, y_3) \in \mathbb{R}^3 | 0 \leqq y_1^2 + y_2^2 < 1\}$.

6. Examples

Here we introduce some examples of CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, using the DPW method.

Round cylinders.

Here we show how the DPW method makes round cylinders. Defining

$$\xi := \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz \tag{6.1}$$

for $z = x + iy \in \Sigma = \mathbb{C}$ and $\lambda \in \mathbb{S}^1$, we solve $d\phi = \phi\xi$ and determine F, obtaining

$$\phi = \begin{pmatrix} \cosh(\lambda^{-1} z) & \sinh(\lambda^{-1} z) \\ \sinh(\lambda^{-1} z) & \cosh(\lambda^{-1} z) \end{pmatrix},$$

$$F = \begin{pmatrix} \cosh(\lambda^{-1}z + \bar{z}\lambda) & \sinh(\lambda^{-1}z + \bar{z}\lambda) \\ \sinh(\lambda^{-1}z + \bar{z}\lambda) & \cosh(\lambda^{-1}z + \bar{z}\lambda) \end{pmatrix}. \tag{6.2}$$

Totally umbilical surfaces.

The DPW method produces totally umbilical surfaces via

$$\xi := \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz, \tag{6.3}$$

for $z \in \Sigma = \mathbb{C}$ and $\lambda \in \mathbb{S}^1$, we solve $d\phi = \phi\xi$ and determine F ($\phi \in \mathcal{B}_{1,1}$ when $|z| \neq 1$), obtaining

$$\phi = \exp\left(z\lambda^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & z\lambda^{-1} \\ 0 & 1 \end{pmatrix},$$

$$F = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z\lambda^{-1} \\ \bar{z}\lambda & 1 \end{pmatrix} \in \Lambda \mathrm{SU}_{1,1}.$$

Remark 6.1 These surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ are not compact, but are totally umbilic.

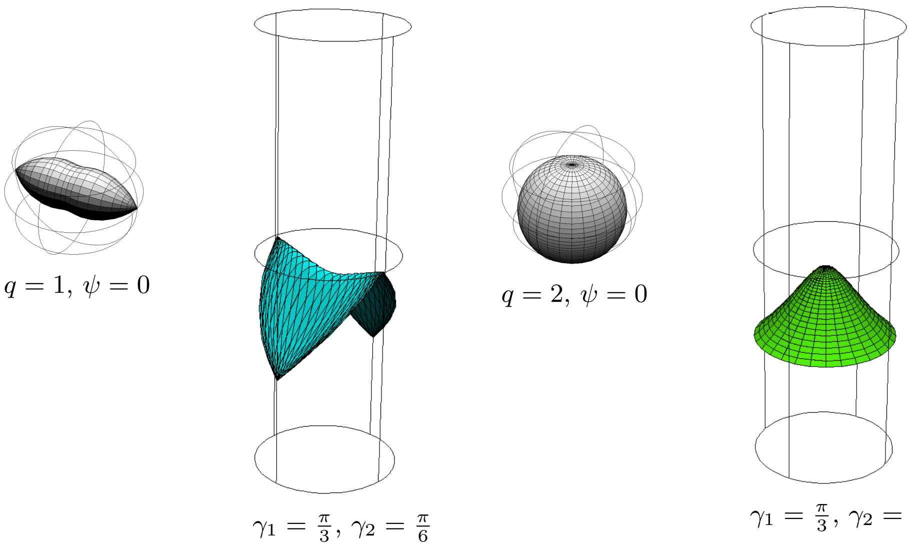

Figure 1. The left two images are round cylinders in $\mathbb{S}^{2,1}$ (resp. $\mathbb{H}^{2,1}$), and right two image are totally umbilical surfaces in $\mathbb{S}^{2,1}$ (resp. $\mathbb{H}^{2,1}$).

7. Theory for CMC surfaces with singularities

In this section, we consider the singularities of CMC surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, and we show criteria for cuspidal edges, swallowtails and cuspidal cross caps, using some geometric notions.

Relationships between Iwasawa splitting and singularities.

We first make some remarks about relationships between the DPW method and singularities. We define *small cells* as follows:

Definition 7.1 ([8], [9]) Define, for a positive integer $m \in \mathbb{Z}$,

$$\omega_m = \begin{pmatrix} 1 & 0 \\ \lambda^{-m} & 1 \end{pmatrix} : \ m \text{ odd}, \quad \omega_m = \begin{pmatrix} 1 & \lambda^{1-m} \\ 0 & 1 \end{pmatrix} : \ m \text{ even}.$$

Then the group $\Lambda\mathrm{SL}_2(\mathbb{C})$ is a disjoint union

$$\Lambda\mathrm{SL}_2(\mathbb{C}) = \mathcal{B}_{1,1} \bigsqcup_{m\in\mathbb{Z}^+} \mathcal{P}_m, \text{ where}$$

$$\mathcal{B}_{1,1} := \left(\Lambda\mathrm{SU}_{1,1} \cup \left\{ \begin{pmatrix} 0 & \lambda i \\ \lambda^{-1} i & 0 \end{pmatrix} \cdot \Lambda\mathrm{SU}_{1,1} \right\}\right) \cdot \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}),$$

is called the Iwasawa big cell, and the m-th small cell is $\mathcal{P}_m := \Lambda\mathrm{SU}_{1,1} \cdot \omega_m \cdot \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$.

Using these small cells, we have the following proposition for the case of $\mathbb{R}^{2,1}$, as in [8], [9].

Proposition 7.1 ([8], [9, Theorem 4.2]) *Let Σ be a simply connected domain, and let $\phi : \Sigma \longrightarrow \Lambda\mathrm{SL}_2(\mathbb{C})$. Define $\Sigma^0 = \phi^{-1}(\mathcal{B}_{1,1})$, $C_1 = \phi^{-1}(\mathcal{P}_1)$ and $C_2 = \phi^{-1}(\mathcal{P}_2)$. Then:*

(1) *The sets $\Sigma^0 \cup C_1$ and $\Sigma^0 \cup C_2$ are both open subsets of Σ. The sets C_i are each locally given as the zero set of a non-constant real analytic function $\mathbb{R}^2 \longrightarrow \mathbb{R}$.*
(2) *All components of the matrix F obtained by Proposition* 4.1 *on Σ^0, and evaluated at $\lambda_0 \in \mathbb{S}^1$, blow up as z approaches a point z_0 in either C_1 or C_2. In the limit, the unit normal vector N, to the corresponding surface in $\mathbb{R}^{2.1}$, becomes asymptotically lightlike, i.e. its length in the Euclidean $\mathbb{R}^3$ metric approaches infinity.*

(3) *The CMC surface $f \in \mathbb{R}^{2,1}$ given by the DPW method extends to a real analytic map $\Sigma^0 \cup C_1 \longrightarrow \mathbb{R}^{2,1}$, but is not immersed as points $z_0 \in C_1$.*
(4) *The CMC surface $f \in \mathbb{R}^{2,1}$ given by the DPW method diverges to ∞ as $z \to z_0 \in C_2$. Moreover, the induced metric on the surface blows up as such a point in the coordinate domain is approached.*

By the above proposition, at a singular point, we cannot split ϕ to $F \cdot B$ for $F \in \Lambda \mathrm{SU}_{1,1}$, and the procedure of the DPW method does not work. Thus, we consider the claims of Proposition 2.1 and 3.1 directly, without using the Iwasawa splitting. Our recipe is as follows:

(1) First we choose a real constant $H \neq 0$ (resp. $H = 1/2$) and holomorphic function Q, for the case of $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$).
(2) We define the metric function $g = e^u$ satisfying (2.3).
(3) We obtain F by solving the system (2.9).
(4) Finally, we get a conformal spacelike CMC surface f in $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$) by inputting F into the Sym-Bobenko type formula (2.10) (resp. (3.9) or (3.10)).

Criteria for singularities of spacelike CMC $H \neq 0$ surfaces in $\mathbb{R}^{2,1}$.
Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps of CMC $H \neq 0$ surfaces in $\mathbb{R}^{2,1}$, as in [30]. However, we use a different approach from [30] because we start not with harmonic maps, but with H, Q and $g = e^u$.

Let $H \neq 0$ be a real constant, and let Q be a holomorphic function. Let $g = e^u$ be a solution of (2.3). Here, in order to match [30], we use the untwisted setting. Thus we define $\hat{F}$ satisfying the system (2.7) for $\lambda = 1$. For this $\hat{F}$, we have the following untwisted version of Proposition 2.1:

Proposition 7.2 (untwisted version of Proposition 2.1) *Let Σ be a simply-connected domain in $\mathbb{C}$. Let u and Q solve* (2.3)*, and let $\hat{F} = \hat{F}(z, \bar{z}, \lambda)$ be a solution of the system* (2.7)*. Suppose $\hat{F} \in \mathrm{SU}_{1,1}$ for all $\lambda \in \mathbb{S}^1$ and one value of z. Then $\hat{F} \in \mathrm{SU}_{1,1}$ for all z where $\hat{F}$ is bounded, and $\hat{F}$ is bounded wherever u and Q are bounded. Defining the following Sym-Bobenko type formulas*

$$f = \left[-\frac{1}{H}\hat{F} i\sigma_3 \hat{F}^{-1} + \frac{i}{H}\lambda(\partial_\lambda \hat{F})\hat{F}^{-1} \right] \Big|_{\lambda=1}, \quad N = \left[\hat{F} i\sigma_3 \hat{F}^{-1}\right] \big|_{\lambda=1} . \quad (7.1)$$

f is a conformally parametrized spacelike CMC $H \neq 0$ surface in $\mathbb{R}^{2,1}$ with normal N.

We denote $\hat{F} = \hat{F}(z, \bar{z}) = e^{-u/2}\left(\begin{smallmatrix} \bar{a} & b \\ \bar{b} & a \end{smallmatrix}\right) \in \mathrm{SU}_{1,1}$, where $|a|^2 - |b|^2 = e^u = g$. So we define h and ω such that $h := -(ia/b)$ and $\omega := -2b^2$, and metric is

$$ds^2 = 4g^2 dz d\bar{z} = (1 - |h|^2)^2 |\omega|^2 dz d\bar{z}. \tag{7.2}$$

This implies that, wherever ds^2 is finite, f has a singularity if and only if $h \in \mathbb{S}^1$ or $\omega = 0$. However, by (7.1) we have

$$f_z = \frac{\omega}{2}(1 + h^2, i(1 - h^2), -2h), \quad f_{\bar{z}} = \frac{\bar{\omega}}{2}(1 + \bar{h}^2, -i(1 - \bar{h}^2), -2\bar{h}), \tag{7.3}$$

and $\omega = 0$ means that f has an isolated singular point there. Here we consider only *extended CMC surfaces* defined by the following, as in [30].

Definition 7.2 ([30]) A CMC surface f restricted to the subdomain $\mathcal{D} = \{p \in \Sigma \mid ds^2 < \infty\}$ is called an extended CMC surface if ω, resp. $h^2\omega$, is never zero on $\mathcal{D}$ when $|h| < \infty$, resp. $|h| = \infty$.

Remark 7.1 By this definition, any point $p \in \Sigma$ is singular only when $|h(p)| = 1$. (See [30].)

Remark 7.2 By [24], the normal vector N of spacelike CMC surface f satisfies that $N_{z\bar{z}}$ is parallel to N at each regular point p. This implies that $h_{z\bar{z}} + (2\bar{h}/1 - |h|^2)h_z h_{\bar{z}} = 0$ meaning that h is a harmonic map at each regular point p. Similarly, $N_z = -Hf_z - (1/2)Qe^{-2u}f_{\bar{z}}$ implies $\omega = \bar{h}_z/(1 - |h|^2)^2$ at each regular point p. However, these Equations do not necessarily hold at singular points.

Now we have the following criteria for singularities of spacelike extended CMC $H \neq 0$ surfaces, as in [30]. However, since we use different notations and a different approach, we give a sketch of the proof here. The notion of $\mathcal{A}$-equivarece used in this theorem is fundamental in singularity theory, and is explained in [17], [30], [31].

Theorem 7.1 ([30]) *Let Σ be a simply connected domain, and let $f : \Sigma \longrightarrow \mathbb{R}^{2,1}$ be a spacelike extended CMC $H \neq 0$ surface. Then*:

(1) *f is a front at a singular point $p \in \Sigma$ (i.e. $h(p) \in \mathbb{S}^1$) if and only if*

$\mathrm{Re}(h_z/h^2\omega)|_p \neq 0$. *If this is the case, p is a non-degenerate singular point.*

(2) *f is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)\bigg|_p \neq 0 \quad \textit{and} \quad \mathrm{Im}\left(\frac{h_z}{h^2\omega}\right)\bigg|_p \neq 0.$$

(3) *f is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)\bigg|_p \neq 0,\ \mathrm{Im}\left(\frac{h_z}{h^2\omega}\right)\bigg|_p = 0 \text{ and}$$
$$\mathrm{Re}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\}\bigg|_p \neq \mathrm{Re}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\}\bigg|_p.$$

(4) *f is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)\bigg|_p = 0,\ \mathrm{Im}\left(\frac{h_z}{h^2\omega}\right)\bigg|_p \neq 0 \text{ and}$$
$$\mathrm{Im}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\}\bigg|_p \neq \mathrm{Im}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\}\bigg|_p.$$

Proof. The essential idea behind proving this is to input h, ω into the Kenmotsu type representation as in [3] and to compute the same way as in [30]. □

Here we give equivalent conditions for Theorem 7.1, as follows (in this corollary $\langle f_z, f_z\rangle_{\mathbb{C}^3} := (1/4)(|f_x|^2_{\mathbb{R}^3} - |f_y|^2_{\mathbb{R}^3}) - (i/2)\langle f_x, f_y\rangle_{\mathbb{R}^3}$ is the bilinear extension of the $\mathbb{R}^3$ inner product):

Corollary 7.1 *Let Σ be a simply connected domain, and let $f : \Sigma \longrightarrow \mathbb{R}^{2,1}$ be a spacelike extended CMC $H \neq 0$ surface, given by a real constant H, a holomorphic function Q and a metric function g. Then*:

(1) *f is a front at a singular point $p \in \Sigma$ if and only if* $\mathrm{Re}(Q/\langle f_z, f_z\rangle_{\mathbb{C}^3})|_p \neq$

0. *If this is the case, p is a non-degenerate singular point.*

(2) *f is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\Bigg|_p \neq 0 \quad \text{and} \quad \mathrm{Im}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\Bigg|_p \neq 0.$$

(3) *f is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\Bigg|_p \neq 0, \quad \mathrm{Im}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\Bigg|_p = 0 \quad \text{and}$$

$$\mathrm{Re}\left[\frac{Q_z\langle f_z, f_z\rangle_{\mathbb{C}^3} - 2Q\langle f_{zz}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{\overline{Q}}{\langle f_{\bar z}, f_{\bar z}\rangle_{\mathbb{C}^3}}\right]\Bigg|_p$$

$$\neq \mathrm{Re}\left[\frac{-2Q\langle f_{z\bar z}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right]\Bigg|_p .$$

(4) *f is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\Bigg|_p = 0, \quad \mathrm{Im}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\Bigg|_p \neq 0 \quad \text{and}$$

$$\mathrm{Im}\left[\frac{Q_z\langle f_z, f_z\rangle_{\mathbb{C}^3} - 2Q\langle f_{zz}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{\overline{Q}}{\langle f_{\bar z}, f_{\bar z}\rangle_{\mathbb{C}^3}}\right]\Bigg|_p$$

$$\neq \mathrm{Im}\left[\frac{-2Q\langle f_{z\bar z}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right]\Bigg|_p .$$

Proof. Using the Hopf differential $Q = \langle f_{zz}, N\rangle_{\mathbb{R}^{2,1}} = \omega h_z$ and $\langle f_z, f_z\rangle_{\mathbb{C}^3} = 2\omega^2 h^2$, we draw the conclusions from Theorem 7.1. □

Criteria for singularities of spacelike CMC $H^2 > 1$ surfaces in $\mathbb{S}^{2,1}$. In this section, we study singularities of spacelike CMC H ($H^2 > 1$) surfaces f in $\mathbb{S}^{2,1}$, similarly to Theorem 7.1 for spacelike CMC H ($H \neq 0$) surfaces in $\mathbb{R}^{2,1}$. However, in $\mathbb{R}^{2,1}$ we used $\lambda \in \mathbb{S}^1$ in the Sym-Bobenko type formula and thus the solution $\hat{F}$ in (2.7) is in $\mathrm{SU}_{1,1}$ for that λ, while in $\mathbb{S}^{2,1}$ this will not be the case. In the case of $\mathbb{S}^{2,1}$, the λ we use in the Sym-Bobenko type formula is not in $\mathbb{S}^1$, and so $\hat{F}|_\lambda \notin \mathrm{SU}_{1,1}$, and this creates complications for attempting to imitate Theorem 7.1. We remedy this problem by using the s-spectral deformation to shift to a new CMC surface $\hat{f}$ in $\mathbb{S}^{2,1}$ of the same type where arguments like those proving Theorem 7.1 can be used. Noting that, by Lemma 7.1, as we have not restricted the full class of surfaces being considered, we are still proving a result (Theorem 7.5) that applies to all spacelike CMC surfaces in $\mathbb{S}^{2,1}$ with constant mean curvature greater than 1 in absolute value. Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps on CMC $H^2 > 1$ surfaces in $\mathbb{S}^{2,1}$.

Let $f : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a spacelike CMC surface for a simply-connected domain $\Sigma \subset \mathbb{C}$, with the metric $ds^2 = 4g^2 dw d\bar{w} = 4e^{2u} dw d\bar{w}$ and unit normal vector N. First, we consider the *moving frame* $\mathfrak{F}$ such that

$$f = \mathfrak{F}\sigma_3\overline{\mathfrak{F}}^t, \ \frac{f_x}{2e^u} = \mathfrak{F}\sigma_1\overline{\mathfrak{F}}^t, \ \frac{f_y}{2e^u} = \mathfrak{F}\sigma_2\overline{\mathfrak{F}}^t, \ N = \mathfrak{F}\overline{\mathfrak{F}}^t.$$

Then, we have

$$\mathfrak{F}_w = \mathfrak{F}\boldsymbol{A}, \ \mathfrak{F}_{\bar{w}} = \mathfrak{F}\boldsymbol{B}, \ \text{where}$$

$$\boldsymbol{A} = \frac{1}{2}\begin{pmatrix} -u_w & -e^{-u}\mathcal{A} \\ 2e^u(1-H) & u_w \end{pmatrix}, \ \boldsymbol{B} = \frac{1}{2}\begin{pmatrix} u_{\bar{w}} & -2e^u(1+H) \\ -e^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix}. \quad (7.4)$$

For this $\mathfrak{F}$, the compatibility condition implies the Gauss and Codazzi equations (3.3). We define *s-spectral deformations* as follows:

Definition 7.3 ([29]) The *s-spectral deformation* of the CMC surface f in $\mathbb{S}^{2,1}$ is the deformation defined by $(1+H) \to s(1+H)$, $(1-H) \to s^{-1}(1-H)$ in Equations (7.4) for the parameter $s > 0$.

The s-spectral deformation maps CMC surfaces to other CMC surfaces conformally, as follows (the analogous result in the case of $\mathbb{H}^3$ was proven in [29]):

Theorem 7.2 *For all $s \in \mathbb{R}_{>0}$, the s-spectral deformation deforms a surface f in $\mathbb{S}^{2,1}$, with mean curvature H, metric $ds^2 = 4e^{2u} dw d\bar{w}$ and Hopf differential $\mathcal{A}$, into a surface f^s with mean curvature $H^s = (s(1+H) - s^{-1}(1-H))/(s(1+H) + s^{-1}(1-H))$, metric $4e^{2u^s} dw d\bar{w} = 4k^2 e^{2u} dw d\bar{w}$ and Hopf differential $\mathcal{A}^s = k\mathcal{A}$ for $k = (s(1+H) + s^{-1}(1-H))/2$.*

Proof. We notice that the s-spectral deformation implies that

$$\begin{cases} k(1+H^s) = s(1+H) \\ k(1-H^s) = s^{-1}(1-H) \end{cases}, \text{ equivalently } \begin{cases} k = \dfrac{s(1+H)+s^{-1}(1-H)}{2} \\ H^s = \dfrac{s(1+H)-s^{-1}(1-H)}{s(1+H)+s^{-1}(1-H)}. \end{cases}$$

Then (3.3) holds with $\epsilon = 1$, and u, $\mathcal{A}$ replaced by $u^s = u + \log|k|$, $\mathcal{A}^s = k\mathcal{A}$. Thus the deformation family of surfaces exists. $\square$

Lemma 7.1 $(f^s)^{1/s} = f$.

Proof. The $1/s$-spectral deformation of f^s is

$$\begin{cases} k^{1/s}\left(1+(H^s)^{1/s}\right) = \dfrac{1}{s}(1+H^s) \\ k^{1/s}\left(1-(H^s)^{1/s}\right) = s(1-H^s) \end{cases},$$

$$\text{equivalently } \begin{cases} k^{1/s} = \dfrac{s^{-1}(1+H^s)+s(1-H^s)}{2} = \dfrac{1}{k} \\ (H^s)^{1/s} = \dfrac{s^{-1}(1+H^s)-s(1-H^s)}{s^{-1}(1+H^s)+s(1-H^s)} = H. \end{cases}$$

Similarly we have $(\mathcal{A}^s)^{1/s} = \mathcal{A}$, $(u^s)^{1/s} = u$. $\square$

We define the (twisted) s-spectral Lax pair.

Definition 7.4 (s-spectral Lax pair) We define $\mathfrak{F}^s$ as a solution of the following system:

$$\mathfrak{F}^s_w = \mathfrak{F}^s \boldsymbol{A}^s, \quad \mathfrak{F}^s_{\bar{w}} = \mathfrak{F}^s \boldsymbol{B}^s,$$

where

$$\begin{aligned} \boldsymbol{A}^s &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -u_w & -e^{-u}\mathcal{A} \\ 2e^u s^{-1}(1-H) & u_w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} u_w & 2e^u s^{-1}(1-H) \\ -e^{-u}\mathcal{A} & -u_w \end{pmatrix}, \end{aligned}$$

$$\boldsymbol{B}^s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} u_{\bar{w}} & -2e^u s(1+H) \\ -e^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathcal{A}} \\ -2e^u s(1+H) & u_{\bar{w}} \end{pmatrix}.$$

Further, we define the form $\Omega^s := (\mathfrak{F}^s)^{-1} d\mathfrak{F}^s$.

Theorem 7.3 *For f given by the frame $\mathfrak{F}$, and mean curvature $H^2 > 1$, there exists a member of the s-spectral deformation, for the special value $s = s_0 := \sqrt{(H-1)/(H+1)}$, that generates a frame $\tilde{\mathfrak{F}} = \mathfrak{F}^{s_0} \in \mathrm{SU}_{1,1}$. This frame $\tilde{\mathfrak{F}}$ represents the lift of a harmonic map in $\mathbb{H}^2$.*

Proof. It is easy to see that choosing $s = s_0 := \sqrt{(H-1)/(H+1)}$ gives the only deformation that makes the Maurer-Cartan form become an $\mathrm{su}_{1,1}$-valued form. □

As s approaches s_0, the mean curvature goes to infinity, and $\tilde{f} := \tilde{\mathfrak{F}}\sigma_3\overline{\tilde{\mathfrak{F}}}^t$ degenerates to a point, but there still exists a map $\tilde{\mathfrak{F}}$ from Σ to $\mathrm{SU}_{1,1}$ such that $\tilde{\mathfrak{F}}^{-1}d\tilde{\mathfrak{F}} = \tilde{\Omega}$ defined by the following (7.5). The harmonic map is the natural projection of the adjusted frame $\tilde{\mathfrak{F}}$ (defined just below) to $\mathbb{H}^2$.

Definition 7.5 We call $\tilde{\mathfrak{F}} : \Sigma \longrightarrow \mathrm{SU}_{1,1}$ the *adjusted frame* of $\mathfrak{F}$ and the form $\tilde{\Omega} = \tilde{\mathfrak{F}}^{-1}d\tilde{\mathfrak{F}}$ the *adjusted Maurer-Cartan form*, where

$$\tilde{\Omega} = \frac{1}{2} \begin{pmatrix} u_w & -2e^u\sqrt{H^2-1} \\ -e^{-u}\mathcal{A} & -u_w \end{pmatrix} dw + \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathcal{A}} \\ -2e^u\sqrt{H^2-1} & u_{\bar{w}} \end{pmatrix} d\bar{w}$$
$$=: \tilde{\boldsymbol{A}}dw + \tilde{\boldsymbol{B}}d\bar{w}. \tag{7.5}$$

Theorem 7.4 *Let Σ be a simply-connected domain. Let $a > 0$ $(a \neq 1)$ be an arbitrary real constant, and let*

$$\beta_1(a) := \frac{a^{-1}-1}{2} \begin{pmatrix} 0 & -2e^u\sqrt{H^2-1} \\ 0 & 0 \end{pmatrix} dw,$$

$$\beta_2(a) := \frac{a-1}{2} \begin{pmatrix} 0 & 0 \\ -2e^u\sqrt{H^2-1} & 0 \end{pmatrix} d\bar{w}.$$

Define $\hat{\Omega} := \tilde{\Omega} + \beta_1(a) + \beta_2(a)$. Then we have the following:

(1) $d\hat{\Omega} + \frac{1}{2}[\hat{\Omega} \wedge \hat{\Omega}] = 0$.

(2) *If $\hat{\mathfrak{F}}$ is a $\mathrm{SL}_2(\mathbb{C})$-valued solution of $\hat{\Omega} = \hat{\mathfrak{F}}^{-1} d\hat{\mathfrak{F}}$, then $\hat{f} = \hat{\mathfrak{F}} \sigma_3 \overline{\hat{\mathfrak{F}}}^t$ is a conformal spacelike CMC surface with $\hat{H} = (a^2+1)/(a^2-1)$.*

Proof. For $s = \sqrt{(H-1)/(H+1)}a \in \mathbb{R}_{>0}$, we have $\Omega^s = \hat{\Omega}$, by direct computation. Thus we have existence of $\hat{f}$, and $\hat{H} = H^s = (a^2+1)/(a^2-1)$. □

Remark 7.3 Defining $G := \hat{\mathfrak{F}} \cdot \tilde{\mathfrak{F}}^{-1}$, we have $G\sigma_3 \bar{G}^t = \hat{\mathfrak{F}} \sigma_3 \overline{\hat{\mathfrak{F}}}^t = \hat{f}$.

As noted previously, we will consider the criteria for singularities of $\hat{f}$ instead of f.

We denote $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(w, \bar{w}) = e^{-u/2}\left(\begin{smallmatrix} u_1 & u_2 \\ \overline{u_2} & \overline{u_1} \end{smallmatrix}\right) \in \mathrm{SU}_{1,1}$, where $g = e^u$ is the metric function of f, and $|u_1|^2 - |u_2|^2 = g$. So we define $h := \overline{u_2}/u_1$ and $\omega := u_1^2$. By Remark 7.3, we have $\hat{f} = \hat{\mathfrak{F}} \sigma_3 \overline{\hat{\mathfrak{F}}}^t = G\sigma_3 \bar{G}^t$. Setting $\hat{k} = ((a - a^{-1})\sqrt{H^2-1})/2$, we get that $\hat{f}$ has metric

$$ds^2 := 4\hat{g}^2 dw d\bar{w} = 4\hat{k}^2 (1 - |h|^2)^2 |\omega|^2 dw d\bar{w}.$$

Thus this implies that, wherever ds^2 is finite, $\hat{f}$ has a singularity if and only if $h \in \mathbb{S}^1$ or $\omega = 0$. However we consider only extended CMC surfaces $\hat{f}$ defined in the same way as Definition 7.2.

We have the following criteria for singularities of spacelike extended CMC $\hat{H}^2 > 1$ surfaces in $\mathbb{S}^{2,1}$. The proof of Theorem 7.5 is parallel to the proof of Theorem 3.1 in [17]. (Also see [30], [31].)

Theorem 7.5 *Let Σ be a simply connected domain, and let $\hat{f} : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a spacelike extended CMC $\hat{H}^2 > 1$ surface, given by Theorem 7.4. Then:*

(1) *A point $p \in \Sigma$ is a singular point if and only if $h \in \mathbb{S}^1$.*

(2) *$\hat{f}$ is a front at a singular point $p \in \Sigma$ if and only if $\mathrm{Re}(h_w/h^2\omega)|_p \neq 0$. If this is the case, p is non-degenerate singular point.*

(3) *$\hat{f}$ is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p \neq 0 \quad \text{and} \quad \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p \neq 0.$$

(4) $\hat{f}$ *is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0, \quad \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Big|_p = 0 \text{ and}$$

$$\mathrm{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\Big|_p \neq \mathrm{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\Big|_p .$$

(5) $\hat{f}$ *is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Big|_p = 0, \quad \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0 \text{ and}$$

$$\mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\Big|_p \neq \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\Big|_p .$$

Proof. (1) This is clear (like in Remark 7.1).
(2) First we define

$$\nu := G\begin{pmatrix} 1+|h|^2 & 2\bar{h} \\ 2h & 1+|h|^2 \end{pmatrix}\bar{G}^t, \tag{7.6}$$

and this ν is the Lorentz normal vector field of $\hat{f}$ on the regular set of $\hat{f}$. This is not a unit vector, but extends smoothly across the singular set. By Lemma 1.6 of [17], $\hat{f}$ is a front at a singular point p if and only if ν is not proportional to $D_\eta^{\mathbb{R}^{3,1}}\nu$ at p, for the null direction η of $\hat{f}$ and the canonical connection $D^{\mathbb{R}^{3,1}}$. Since $d\hat{f} = G\{2\hat{k}\left(\begin{smallmatrix} -h & -1 \\ -h^2 & -h \end{smallmatrix}\right)\omega dw - 2\hat{k}\left(\begin{smallmatrix} \bar{h} & \bar{h}^2 \\ 1 & \bar{h} \end{smallmatrix}\right)\bar{\omega}d\bar{w}\}\bar{G}^t$, the null direction is $\eta = (i/h\omega)\partial_w - (i/\bar{h}\bar{\omega})\partial_{\bar{w}}$ at a singular point p. We also have, at p,

$$D_\eta^{\mathbb{R}^{3,1}}\nu = G\left\{\frac{i}{h\omega}\begin{pmatrix} h_w\bar{h} & 0 \\ 2h_w & h_w\bar{h} \end{pmatrix} - \frac{i}{\bar{h}\bar{\omega}}\begin{pmatrix} h\bar{h}_{\bar{w}} & 2\bar{h}_{\bar{w}} \\ 0 & h\bar{h}_{\bar{w}} \end{pmatrix}\right\}\bar{G}^t.$$

On the other hand, we have $\langle\nu,\nu\rangle = \langle D_\eta^{\mathbb{R}^{3,1}}\nu,\nu\rangle = 0$ at p, thus $D_\eta^{\mathbb{R}^{3,1}}\nu$ is proportional to ν if and only if $D_\eta^{\mathbb{R}^{3,1}}\nu$ is a null vector, which is equivalent to

$$0 = \det\left(D_\eta^{\mathbb{R}^{3,1}}\nu\right) = 4\left(\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\right)^2. \tag{7.7}$$

By the identification between $\mathbb{R}^{3,1}$ and the Hermitian symmetric group (3.5), we have

$$G^{-1}\hat{f}(\bar{G}^t)^{-1} = (0,0,1,0), \quad G^{-1}\hat{f}_w(\bar{G}^t)^{-1} = -\hat{k}\omega(h^2+1, -i(h^2-1), 0, 2h),$$
$$G^{-1}\hat{f}_{\bar{w}}(\bar{G}^t)^{-1} = -\hat{k}\bar{\omega}(\bar{h}^2+1, -i(1-\bar{h}^2), 0, 2\bar{h}),$$

and we define $\mathbf{n} := GI\bar{G}^t$. This implies that $G^{-1}\mathbf{n}(\bar{G}^t)^{-1} = (0,0,0,1)$. By Proposition 1.2 of [17], we can define

$$\boldsymbol{\lambda} := \det(\hat{f}, \hat{f}_x, \hat{f}_y, \mathbf{n}) = 4\hat{k}^2|\omega|^2(1+|h|^2)(1-|h|^2).$$

Then, at a singular point p, $d\boldsymbol{\lambda} = -8\hat{k}^2|\omega|^2(dh\cdot\bar{h} + h\cdot d\bar{h})$, and thus p is a non-degenerate point if and only if $h_w \neq 0$.
(3)&(4) These cases are proven analogously to the proofs of (3)&(4) in Theorem 7.1.
(5) We define the limiting tangent bundle, as in [17], as $\{X \in T\mathbb{S}^{2,1}|_{\hat{f}(\Sigma)}; \langle X, \nu\rangle = 0\}$. By direct computation, we notice any section X of the limiting tangent bundle is parametrized by

$$X = G\begin{pmatrix} \bar{\zeta}\bar{h} + \zeta h & \zeta(|h|^2+1) \\ \bar{\zeta}(|h|^2+1) & \bar{\zeta}\bar{h} + \zeta h \end{pmatrix}\bar{G}^t$$

for some $\zeta : \Sigma \longrightarrow \mathbb{C}$, and $X \nparallel \nu$ exactly when $\mathrm{Im}(\zeta h)|_p \neq 0$ at a singular point p. For such X such that $\mathrm{Im}(\zeta h)|_p \neq 0$, we define $\boldsymbol{\psi} := \langle D_\eta^{\mathbb{R}^{3,1}} X, \nu\rangle = -4i\mathrm{Re}(h_w/h^2\omega)\mathrm{Im}(\zeta h)$. We can apply Theorem 1.4 of [17], and the conditions to have a cuspidal cross cap are

$$\begin{aligned}
&\det(\gamma_t, \eta) \neq 0,\ \boldsymbol{\psi} = 0 \text{ and } \partial_t\boldsymbol{\psi} \neq 0 \\
&\iff \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right) \neq 0,\ \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right) = 0 \text{ and} \\
&\qquad \mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\} \neq \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}.
\end{aligned}$$

□

Criteria for singularities of spacelike CMC surfaces in $\mathbb{H}^{2,1}$.

Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps of CMC surfaces in $\mathbb{H}^{2,1}$. In this case as well, like in the previous case of $\mathbb{S}^{2,1}$, we will shift from one surface f to another surface $\hat{f}$, again without causing any restriction on the class of surfaces involved. However, in the case of $\mathbb{H}^{2,1}$, the reason for doing this is different, because now we will indeed have the frames $\mathfrak{F}_1$, $\mathfrak{F}_2 \in \mathrm{SU}_{1,1}$. But the fact that two frames $\mathfrak{F}_1$, $\mathfrak{F}_2$ are now involved again causes us to switch from f to $\hat{f}$.

Let $f : \Sigma \longrightarrow \mathbb{H}^{2,1}$ be a spacelike CMC surface for a simply-connected domain $\Sigma \subset \mathbb{C}$, with the metric $ds^2 = 4g^2 dw d\bar{w} = 4e^{2u} dw d\bar{w}$ and unit normal vector N. First, we consider frames $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathrm{SU}_{1,1}$ such that

$$f = \mathfrak{F}_1 \overline{\mathfrak{F}_2}^t, \ \frac{f_x}{2e^u} = \mathfrak{F}_1 \sigma_1 \overline{\mathfrak{F}_2}^t, \ \frac{f_y}{2e^u} = \mathfrak{F}_1 \sigma_2 \overline{\mathfrak{F}_2}^t, \ N = \mathfrak{F}_1 i\sigma_3 \overline{\mathfrak{F}_2}^t.$$

Then, we have

$$\begin{aligned}
\Omega_1 &:= (\mathfrak{F}_1)^{-1} d\mathfrak{F}_1 \\
&= \frac{1}{2}\begin{pmatrix} -u_w & -ie^{-u}\mathcal{A} \\ 2e^u(1+iH) & u_w \end{pmatrix} dw + \frac{1}{2}\begin{pmatrix} u_{\bar{w}} & 2e^u(1-iH) \\ ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} d\bar{w} \\
&=: \boldsymbol{A}_1 dw + \boldsymbol{B}_1 d\bar{w}, \qquad (7.8) \\
\Omega_2 &:= (\mathfrak{F}_2)^{-1} d\mathfrak{F}_2 \\
&= \frac{1}{2}\begin{pmatrix} -u_w & ie^{-u}\mathcal{A} \\ 2e^u(1-iH) & u_w \end{pmatrix} dw + \frac{1}{2}\begin{pmatrix} u_{\bar{w}} & 2e^u(1+iH) \\ -ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} d\bar{w} \\
&=: \boldsymbol{A}_2 dw + \boldsymbol{B}_2 d\bar{w}, \qquad (7.9)
\end{aligned}$$

For these $\mathfrak{F}_1$ and $\mathfrak{F}_2$, the compatibility condition implies the Gauss and Codazzi equations (3.3). We define *s-spectral deformations* as follows, now using $s \in \mathbb{S}^1$ and the terms $1 \pm iH$ (in Definition 7.3 we used $s > 0$ and the terms $1 \pm H$):

Definition 7.6 The *s-spectral deformation* of the CMC surface f in $\mathbb{H}^{2,1}$ is the deformation defined by $(1+iH) \longrightarrow s(1+iH)$ and $(1-iH) \longrightarrow \bar{s}(1-iH)$ in the equations (7.8) and (7.9) for the complex parameter $s \in \mathbb{S}^1$.

The s-spectral deformation maps CMC surfaces to other CMC ones conformally, as in the following Theorem 7.6. The proof is analogous to the one of Theorem 7.2, but s^{-1} becomes $\bar{s}$.

Theorem 7.6 *For all $s \in \mathbb{S}^1$, the s-spectral deformation deforms a surface f in $\mathbb{H}^{2,1}$, with mean curvature H, metric $ds^2 = 4e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}$, into a surface f^s with mean curvature $H^s = (s(1+iH) - \bar{s}(1-iH))/(i\{s(1+iH) + \bar{s}(1-iH)\})$, metric $4e^{2u^s}dwd\bar{w} = 4k^2e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}^s = k\mathcal{A}$ for $k = (s(1+iH) + \bar{s}(1-iH))/2$.*

The next lemma is proven like for Lemma 7.1:

Lemma 7.2 $(f^s)^{\bar{s}} = f$.

We define the (twisted) s-spectral Lax pair.

Definition 7.7 (s-spectral Lax pair) We define $\mathfrak{F}_1^s$ and $\mathfrak{F}_2^s$ as solutions of the following systems:

$$(\mathfrak{F}_j^s)_w = \mathfrak{F}_j^s \boldsymbol{A}_j^s, \quad (\mathfrak{F}_j^s)_{\bar{w}} = \mathfrak{F}_j^s \boldsymbol{B}_j^s \quad (j = 1, 2),$$

where

$$\boldsymbol{A}_1^s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -u_w & -ie^{-u}\mathcal{A} \\ 2e^u s(1+iH) & u_w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} u_w & 2e^u s(1+iH) \\ -ie^{-u}\mathcal{A} & -u_w \end{pmatrix},$$
$$\boldsymbol{B}_1^s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} u_{\bar{w}} & 2e^u \bar{s}(1-iH) \\ ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & ie^{-u}\bar{\mathcal{A}} \\ 2e^u \bar{s}(1-iH) & u_{\bar{w}} \end{pmatrix},$$
$$\boldsymbol{A}_2^s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -u_w & ie^{-u}\mathcal{A} \\ 2e^u \bar{s}(1-iH) & u_w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} u_w & 2e^u \bar{s}(1-iH) \\ ie^{-u}\mathcal{A} & -u_w \end{pmatrix},$$

$$\boldsymbol{B}_2^s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} u_{\bar{w}} & 2e^u s(1+iH) \\ -ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & -ie^{-u}\bar{\mathcal{A}} \\ 2e^u s(1+iH) & u_{\bar{w}} \end{pmatrix}.$$

Further, we define the forms $\Omega_1^s := (\mathfrak{F}_1^s)^{-1} d\mathfrak{F}_1^s$ and $\Omega_2^s := (\mathfrak{F}_2^s)^{-1} d\mathfrak{F}_2^s$.

Definition 7.8 For all f given by $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and mean curvature H, there exists a member of the s-spectral deformation, for the special value $s = s_0 := i\sqrt{(1-iH)/(1+iH)}$, that generates frames $\tilde{\mathfrak{F}}_1 = \mathfrak{F}_1^{s_0}$, $\tilde{\mathfrak{F}}_2 = \mathfrak{F}_2^{s_0} \in \mathrm{SU}_{1,1}$. We call $\tilde{\mathfrak{F}}_1$ and $\tilde{\mathfrak{F}}_2$ the *adjusted frames* of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and the forms $\tilde{\Omega}_1 = \tilde{\mathfrak{F}}_1^{-1} d\tilde{\mathfrak{F}}_1$ and $\tilde{\Omega}_2 = \tilde{\mathfrak{F}}_2^{-1} d\tilde{\mathfrak{F}}_2$ the *adjusted Maurer-Cartan forms*, where

$$\tilde{\Omega}_1 = \frac{1}{2} \begin{pmatrix} u_w & 2e^u i\sqrt{H^2+1} \\ -ie^{-u}\mathcal{A} & -u_w \end{pmatrix} dw + \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & ie^{-u}\bar{\mathcal{A}} \\ -2e^u i\sqrt{H^2+1} & u_{\bar{w}} \end{pmatrix} d\bar{w}$$
$$=: \tilde{\boldsymbol{A}}_1 dw + \tilde{\boldsymbol{B}}_1 d\bar{w}, \tag{7.10}$$
$$\tilde{\Omega}_2 = \frac{1}{2} \begin{pmatrix} u_w & -2e^u i\sqrt{H^2+1} \\ ie^{-u}\mathcal{A} & -u_w \end{pmatrix} dw + \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & -ie^{-u}\bar{\mathcal{A}} \\ 2e^u i\sqrt{H^2+1} & u_{\bar{w}} \end{pmatrix} d\bar{w}$$
$$=: \tilde{\boldsymbol{A}}_2 dw + \tilde{\boldsymbol{B}}_2 d\bar{w}. \tag{7.11}$$

These forms satisfy $\tilde{\Omega}_1 = \sigma_3 \tilde{\Omega}_2 \sigma_3$.

Theorem 7.7 *Let $\tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2 : \Sigma \longrightarrow \mathrm{SU}_{1,1}$, where Σ is a simply-connected domain. Let $a \in \mathbb{S}^1$ $(a \neq 1)$ be an arbitrary constant, and let*

$$\beta_1 := \begin{pmatrix} 0 & 2e^u i\sqrt{H^2+1} \\ 0 & 0 \end{pmatrix} dw, \quad \beta_2 := \begin{pmatrix} 0 & 0 \\ -2e^u i\sqrt{H^2+1} & 0 \end{pmatrix} d\bar{w}.$$

Define $\hat{\Omega}_1 := \tilde{\Omega}_1 + ((a-1)/2)\beta_1 + ((\bar{a}-1)/2)\beta_2$ and $\hat{\Omega}_2 := \tilde{\Omega}_2 + ((-\bar{a}+1)/2)\beta_1 + ((-a+1)/2)\beta_2$. Then we have the following:

(1) $d\hat{\Omega}_1 + (1/2)[\hat{\Omega}_1 \wedge \hat{\Omega}_1] = 0$, $d\hat{\Omega}_2 + (1/2)[\hat{\Omega}_2 \wedge \hat{\Omega}_2] = 0$.
(2) *If $\hat{\mathfrak{F}}_1$ and $\hat{\mathfrak{F}}_2$ are $\mathrm{SU}_{1,1}$-valued solutions of $\hat{\Omega}_1 = \hat{\mathfrak{F}}_1^{-1} d\hat{\mathfrak{F}}_1$ and $\hat{\Omega}_2 = \hat{\mathfrak{F}}_2^{-1} d\hat{\mathfrak{F}}_2$, then $\hat{f} = \hat{\mathfrak{F}}_1 \overline{\hat{\mathfrak{F}}_2}^t$ is a conformal spacelike CMC surface with $\hat{H} = (a+\bar{a})/(i(a-\bar{a}))$.*

Proof. For $s = ia\sqrt{(1-iH)/(1+iH)} \in \mathbb{S}^1$, we have $\Omega_1^s = \hat{\Omega}_1$ and $\Omega_2^s = \hat{\Omega}_2$, by direct computations. Thus we have existence of $\hat{f}$, and $\hat{H} = H^s = (a+\bar{a})/(i(a-\bar{a}))$. $\square$

Remark 7.4 Defining $G_1 := \hat{\mathfrak{F}}_1 \cdot \tilde{\mathfrak{F}}_1^{-1}$ and $G_2 := \hat{\mathfrak{F}}_2 \cdot \tilde{\mathfrak{F}}_2^{-1}$, then $G_1 \overline{G_2}^t = \hat{\mathfrak{F}}_1 \overline{\hat{\mathfrak{F}}_2}^t = \hat{f}$.

As noted previously, again we will consider the criteria for singularities of $\hat{f}$ instead of f.

We denote $\tilde{\mathfrak{F}}_1 = \tilde{\mathfrak{F}}_1(w, \bar{w}) = e^{-u/2}\left(\begin{smallmatrix} u_1 & u_2 \\ \overline{u_2} & \overline{u_1} \end{smallmatrix}\right) \in \mathrm{SU}_{1,1}$, where $g = e^u$ is the metric function of f, and $|u_1|^2 - |u_2|^2 = g$. So we define $h := \overline{u_2}/u_1$ and $\omega := u_1^2$. By Remark 7.2, we have $\hat{f} = \hat{\mathfrak{F}}_1 \overline{\hat{\mathfrak{F}}_2}^t = G_1 \overline{G_2}^t$. By the definition of G_1 and G_2, we have

$$
\begin{aligned}
G_1^{-1} dG_1 &= (a-1) i \sqrt{H^2+1} \begin{pmatrix} -h & 1 \\ -h^2 & h \end{pmatrix} \omega dw \\
&\quad - (\bar{a}-1) i \sqrt{H^2+1} \begin{pmatrix} \bar{h} & -\bar{h}^2 \\ 1 & -\bar{h} \end{pmatrix} \bar{\omega} d\bar{w}, \\
G_2^{-1} dG_2 &= (-\bar{a}+1) i \sqrt{H^2+1} \begin{pmatrix} h & 1 \\ -h^2 & -h \end{pmatrix} \omega dw \\
&\quad - (-a+1) i \sqrt{H^2+1} \begin{pmatrix} -\bar{h} & -\bar{h}^2 \\ 1 & \bar{h} \end{pmatrix} \bar{\omega} d\bar{w}.
\end{aligned}
$$

Setting $\hat{k} = ((a-\bar{a}) i \sqrt{H^2+1})/2$, $\hat{f}$ has metric $ds^2 := 4\hat{g}^2 dw d\bar{w} = 4\hat{k}^2(1-|h|^2)^2 |\omega|^2 dw d\bar{w}$. Thus this implies that, wherever ds^2 is finite, $\hat{f}$ has a singularity if and only if $h \in \mathbb{S}^1$ or $\omega = 0$. However we consider only extended CMC surfaces $\hat{f}$ defined in the same way as Definition 7.2.

Now we have the following criteria for singularities of spacelike extended CMC surfaces in $\mathbb{H}^{2,1}$. The proof of Theorem 7.8 is parallel to the proof of Theorem 7.5.

Theorem 7.8 *Let Σ be a simply connected domain, and let $\hat{f} : \Sigma \longrightarrow \mathbb{H}^{2,1}$ be a spacelike extended CMC surface, given by Theorem 7.7. Then:*

(1) *A point $p \in \Sigma$ is a singular point if and only if $h \in \mathbb{S}^1$.*
(2) *$\hat{f}$ is a front at a singular point $p \in \Sigma$ if and only if* $\mathrm{Im}(h_w/h^2\omega)|_p \neq 0$.

If this is the case, p is non-degenerate singular point.

(3) $\hat{f}$ *is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0.$$

(4) $\hat{f}$ *is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0,\ \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Big|_p = 0 \text{ and}$$
$$\mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\Big|_p \neq \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\Big|_p.$$

(5) $\hat{f}$ *is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Big|_p = 0,\ \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0 \text{ and}$$
$$\mathrm{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\Big|_p \neq \mathrm{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\Big|_p.$$

Proof. We can prove this theorem by computing the same way as in the proof of Theorem 7.5. □

8. Analogues of Smyth surfaces in Lorentzian spaceforms and their singularities

B. Smyth studied a generalization of Delaunay surfaces in $\mathbb{R}^3$, which are CMC surfaces with rotationally invariant metrics, in [27]. These surfaces are called *Smyth surfaces*, and there are numerous studies about them. For example, in [28], Timmreck et al. showed properness of Smyth surfaces in $\mathbb{R}^3$, and A. I. Bobenko and A. Its studied relationships between Smyth surfaces and Painleve III equations in [6]. The DPW method was applied to Smyth surfaces in Riemannian spaceforms, in [6], [12], [16] for example.

(See Figure 2.) Recently, in [9], D. Brander et al. constructed the analogue of Smyth surfaces in $\mathbb{R}^{2,1}$.

Here we will construct the analogues of Smyth surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, in addition to $\mathbb{R}^{2,1}$, and show that there are different kinds of Smyth surfaces in semi-Riemannian spaceforms, some which have singularities before reaching an end, and some which do not. We also identify the types of singularities on Smyth surfaces, using the criteria in Section 7.

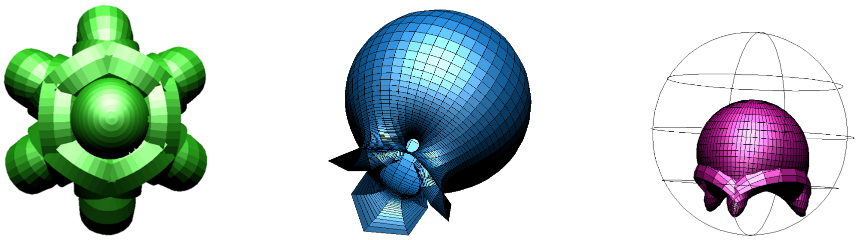

Figure 2. The left image is a 6-legged Smyth surface in $\mathbb{R}^3$, the middle is a 3-legged Smyth surface in $\mathbb{S}^3$, and the right is a 3-legged Smyth surface in $\mathbb{H}^3$.

Reflective symmetry of Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$.

Define

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^k & 0 \end{pmatrix} dz, \quad c \in \mathbb{C}, \quad z \in \Sigma = \mathbb{C}, \tag{8.1}$$

and take a solution ϕ such that $d\phi = \phi\xi$ and $\phi_{z=0} = I$. If $k = 0$ and $c \in \mathbb{S}^1$, then we have a round cylinder, as in Section 6. However, when $k \neq 0$ or $c \notin \mathbb{S}^1 \cup \{0\}$, Iwasawa splitting of ϕ is not so simple, and the surface f has singularities in some cases where the $\Lambda\mathrm{SU}_{1,1}$-Iwasawa splitting of ϕ approaches small cells.

Now we can assume $c \in \mathbb{R}_{>0}$ using a reparametrization of z and a rigid motion of f, as in [9].

Theorem 8.1 ([9]) *The surface $f : \Sigma^0 = \phi^{-1}(\mathcal{B}_{1,1}) \longrightarrow \mathbb{R}^{2,1}$, produced via the DPW method, from ξ in* (8.1), *with $\phi|_{z=0} = I$ and $\lambda = 1$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

Proof. Consider the reflections $R_l(z) = e^{2\pi i l/(k+2)}\bar{z}$ of the domain $\Sigma = \mathbb{C}$, for $l \in \{0, 1, \ldots, k+1\}$. Note that $\xi(R_l(z), \lambda) = A_l \cdot \xi(\bar{z}, \lambda) \cdot A_l^{-1}$, where

$$A_l := \begin{pmatrix} e^{\pi i l/(k+2)} & 0 \\ 0 & e^{-\pi i l/(k+2)} \end{pmatrix} : \text{ constant in } z, \lambda.$$

Since $d\left(\phi(R_l(z),\lambda)\cdot A_l\right) = d\phi(R_l(z),\lambda)\cdot A_l\cdot\xi(\bar{z},\lambda)$, and since any solutions of this equation differ by a factor that is constant in z, we have $\phi(R_l(z),\lambda)\cdot A_l = \mathfrak{A}\cdot\phi(\bar{z},\lambda)$. However the initial condition $\phi(z,\lambda)|_{z=0} = I$ implies that $\mathfrak{A} = A_l$, so $\phi(R_l(z),\lambda) = A_l\cdot\phi(\bar{z},\lambda)\cdot A_l^{-1}$. It is easy to see that this relation extends to the factors F and B in the Iwasawa splitting $\phi = F\cdot B$, and so we have a frame F which satisfies $F(R_l(z),\overline{R_l(z)},\lambda) = A_l\cdot F(\bar{z},z,\lambda)\cdot A_l^{-1}$. Note that $c\in\mathbb{R}_{>0}$ implies $\xi(\bar{z},\lambda) = \overline{\xi(z,\bar{\lambda})}$, $\phi(\bar{z},\lambda) = \overline{\phi(z,\bar{\lambda})}$ and $F(\bar{z},\lambda) = \overline{F(z,\bar{\lambda})}$, thus we have $F(R_l(z),\overline{R_l(z)},\lambda) = A_l\cdot\overline{F(z,\bar{z},\bar{\lambda})}\cdot A_l^{-1}$. Inserting this into the Sym-Bobenko type formula, we have

$$f(R_l(z),\overline{R_l(z)}) = -A_l\cdot\overline{f(z,\bar{z})}\cdot A_l^{-1}. \tag{8.2}$$

The transformation $f(z,\bar{z}) \longrightarrow -\overline{f(z,\bar{z})}$ represents a reflection across the plane $\{x_1 = 0\}$ of $\mathbb{R}^{2,1}$, and conjugation by A_l represents a rotation by angle $2\pi l/(k+2)$ about the x_0-axis, proving the result. □

Theorem 8.2 *The surfaces $f:\phi^{-1}(\mathcal{B}_{1,1}) \longrightarrow \mathbb{S}^{2,1}$, produced via the DPW method, from ξ in* (8.1)*, with $\phi|_{z=0} = I$ and $F_0 = F|_{\lambda=e^{\frac{q}{2}}}$ for $q\in\mathbb{R}$, $q\neq 0$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

Proof. As the proof of Theorem 8.1, we have $F(R_l(z),\overline{R_l(z)},\lambda) = A_l\cdot\overline{F(z,\bar{z},\bar{\lambda})}\cdot A_l^{-1}$. Inserting this into the Sym-Bobenko type formula, we have $f(R_l(z),\overline{R_l(z)}) = A_l\cdot\overline{f(z,\bar{z})}\cdot\overline{A_l}^t$. The transformation $f(z,\bar{z}) \longrightarrow \overline{f(z,\bar{z})}$ represents a reflection across the plane $\{x_2 = 0\}$ of $\mathbb{R}^{3,1}$, and conjugation by A_l represents a rotation by angle $2\pi l/(k+2)$ about the x_4-axis. □

Similarly, we can prove:

Theorem 8.3 *The surfaces $f:\phi^{-1}(\mathcal{B}_{1,1}) \to \mathbb{H}^{2,1}$, produced via the DPW method, from ξ in* (8.1)*, with $\phi|_{z=0} = I$, $F_1 = F|_{\lambda=e^{i\gamma_1}}$ and $F_2 = F|_{\lambda=e^{i\gamma_2}}$ for $\gamma_1,\gamma_2\in\mathbb{R}\backslash\{0\}$ and $\gamma_1 = -\gamma_2$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

Remark 8.1 The surfaces f in Theorems 8.1, 8.2 and 8.3 extend to $\phi^{-1}(C_1)$ at singularities (see Proposition 7.1, (3)).

The Gauss equation of Smyth surfaces.

Here we assume the mean curvature is $H = 1/2$, as in [6], [9], [28], and then we show that the metric of Smyth surfaces is rotational invariant. Before giving the theorems, we create the notation $g := e^u$ for the metric function, and we have the following Lax pair

$$F_z = FU, \quad F_{\bar{z}} = FV, \text{ where}$$

$$U = \frac{1}{2}\begin{pmatrix} \dfrac{g_z}{g} & i\lambda^{-1}g \\ -i\lambda^{-1}Qg^{-1} & -\dfrac{g_z}{g} \end{pmatrix}, \; V = \frac{1}{2}\begin{pmatrix} -\dfrac{g_{\bar{z}}}{g} & i\lambda\bar{Q}g^{-1} \\ -i\lambda g & \dfrac{g_{\bar{z}}}{g} \end{pmatrix}, \tag{8.3}$$

and Gauss equation

$$4(g_{z\bar{z}} \cdot g - g_z \cdot g_{\bar{z}}) + Q\bar{Q} - g^4 = 0. \tag{8.4}$$

Theorem 8.4 ([9]) *The Gauss equation* (8.4) *for a surface in* $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ *or* $\mathbb{H}^{2,1}$ *generated by* ξ *in* (8.1)*, with* $\phi|_{z=0} = I$*, is equivalent to a special case of the Painleve III equations, and the metric function* g*, which is a solution of* (8.4)*, is rotational invariant.*

Using polar coordinates $z = re^{i\theta}$, (8.4) and the suitable choice of the initial conditions are

$$g\left(g_{rr} + \frac{g_r}{r}\right) - (g_r)^2 + c^2r^{2k} - g^4 = 0, \; g|_{r=0} = 1, \; g_r|_{r=0} = 0. \tag{8.5}$$

We can assume $c = 1$ in (8.5) by a change of coordinate:

Lemma 8.1 *After an appropriate change of coordinate* z*, the Gauss equation* (8.5) *for* $(k+2)$*-legged Smyth surfaces in* $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ *or* $\mathbb{H}^{2,1}$*, becomes*

$$g\left(g_{rr} + \frac{g_r}{r}\right) - (g_r)^2 + r^{2k} - g^4 = 0, \quad g|_{r=0} = q^2, \quad g_r|_{r=0} = 0 \tag{8.6}$$

for a real constant $q > 0$.

Proof. Let ϕ be a solution of $d\phi = \phi\xi$ for $\xi = \left(\begin{smallmatrix} 0 & \lambda^{-1} \\ \lambda^{-1}cz^k & 0 \end{smallmatrix}\right)dz$ and $\phi|_{z=0} = I$ with $\Lambda\mathrm{SU}_{1,1}$-Iwasawa splitting $\phi = F \cdot B$. We define $\breve{\phi} = \phi\left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$. By the uniqueness of the Iwasawa splitting $\breve{\phi} = \breve{F} \cdot \breve{B}$ of $\breve{\phi}$, we have $\breve{F} = F$ and

$\breve{B} = B\left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$. Furthermore, we notice that $\breve{B}|_{\lambda=0,z=0} = \left(\begin{smallmatrix} q & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$, since $\breve{F}|_{z=0} = I$, and this implies that $\breve{g}|_{r=0} = q^2$. On the other hand, we have

$$\breve{\xi} = \breve{\phi}^{-1} d\breve{\phi} = \begin{pmatrix} 0 & q^{-2}\lambda^{-1} \\ q^2\lambda^{-1}c^2z^k & 0 \end{pmatrix} dz = \begin{pmatrix} 0 & \lambda^{-1} \\ q^{2k+4}\lambda^{-1}c^2\breve{z}^k & 0 \end{pmatrix} d\breve{z}$$

for $\breve{z} := q^{-2}z$, and we can let $\breve{c} = q^{2k+4}c$. In this way, we can change c to 1. $\square$

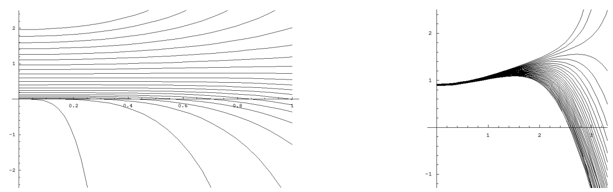

Figure 3. Solutions of a special case of Painleve III (near the origin).

Some examples of the metric function g are seen in Figure 3. By the previous studies [18] and [20], there are at least three kinds of solutions g of this special case of Painleve III equations (8.6). This implies that (spacelike) Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, near the origin $\mathbf{0} \in \mathbf{\Sigma}$, are classified into the following three cases (See Figures 3, 4, 6 and 8.):

- The first kind does not have singularities and g diverges to ∞.
- The second kind does not have singularities and g does not diverge to $\pm\infty$. This case is unique, and given by g with the following initial condition:

$$g|_{r=0} = q_0^2, \quad g_r|_{r=0} = 0 \quad \text{for}$$
$$q_0 = \left(1 + \frac{k}{2}\right)^{k/(4+2k)} 2^{k/(2+k)} \sqrt{\frac{\Gamma\left(\frac{1}{2} + \frac{k}{4+2k}\right)}{\Gamma\left(\frac{1}{2} - \frac{k}{4+2k}\right)}}.$$

- The third kind has singularities before g diverges to $-\infty$.

Smyth surfaces with singularities.

Here we only consider Smyth surfaces that have singularities before g diverges to $-\infty$. By numerical calculation, we know that these Smyth surfaces have cuspidal edges, swallowtails and cuspidal cross caps, using

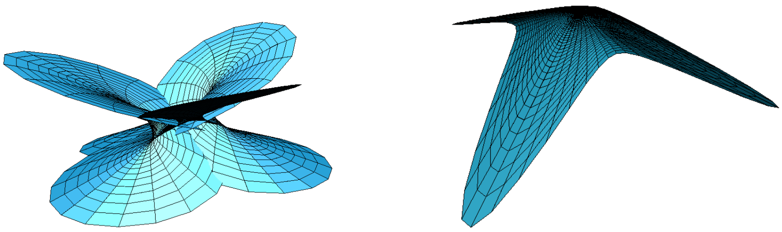

Figure 4. The left image is a 3-legged Smyth surface with singularities in $\mathbb{R}^{2,1}$, and the right image is one with no singularities in $\mathbb{R}^{2,1}$.

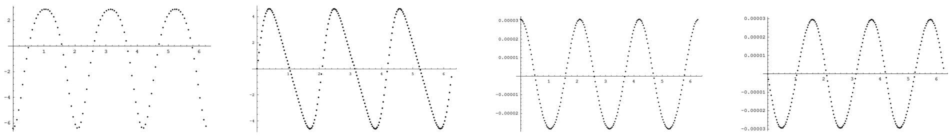

Figure 5. The values of The values of $\mathrm{Re}(h_z/h^2\omega)$, $\mathrm{Im}(h_z/h^2\omega)$, $\mathrm{Re}\{\overline{(h_z/h)}(h_z/h^2\omega)_z\} - \mathrm{Re}\{(h_z/h)(h_z/h^2\omega)_{\bar z}\}$ and $\mathrm{Im}\{\overline{(h_z/h)}(h_z/h^2\omega)_z\} - \mathrm{Im}\{(h_z/h)(h_z/h^2\omega)_{\bar z}\}$ for a 3-legged Smyth surface in $\mathbb{R}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \leq \theta \leq 2\pi$. (left to right).

criteria as in Section 7, see Figures 4∼9.

Fact 8.1 *There exist Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ which have singularities before g diverges to $-\infty$, and which have cuspidal edges, swallowtails and cuspidal cross caps.* (*See Figures* 5, 7 *and* 9.)

Here we show, for the surfaces in Fact 8.1, that there are at least $2(k+2)$-swallowtails for the case of $\mathbb{R}^{2,1}$, without relying on numerical calculation, and using only geometric properties. Before doing that, we have some lemmas.

Lemma 8.2 *Let $\hat{F} = \hat{F}(z, \bar{z}, \lambda)$ be the solution of the untwisted Lax pair* (2.7) *with $\hat{F}|_{z=0} = I$ for the case of a Smyth surface. Then $\hat{F}(z) = \sigma_3 \overline{\hat{F}(\bar{z})} \sigma_3$ for $\lambda = 1$.*

Corollary 8.1 (1) *$h(z) = -\overline{h(\bar{z})}$ and $\omega(z) = \overline{\omega(\bar{z})}$.*
(2) *At $(r, \theta) = (r_0, 0)$ for r_0 such that $g(r_0) = 0$, we have $h(r_0, 0) = \pm i$ (i.e. $h(r_0, 0) \in i\mathbb{R} \cap \mathbb{S}^1$), $\omega(r_0, 0) \in \mathbb{R} \setminus \{0\}$ and $\omega_z(r_0, 0) = \overline{\omega_{\bar{z}}(r_0, 0)}$.*

Proof. By direct computation, we have $\hat{U}(z) = -\hat{V}(\bar{z})^t$ and $\hat{V}(z) = -\hat{U}(\bar{z})^t$. By this equation and $\hat{F}(z)|_{z=0} = I$, we get the conclusion. □

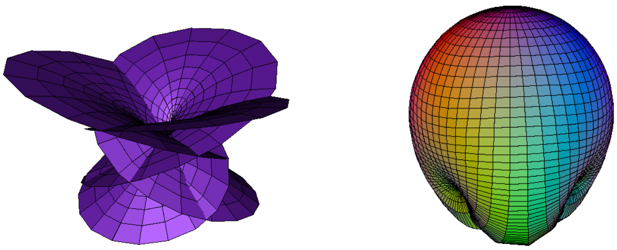

Figure 6. The left image is a 3-legged Smyth surface with singularities in $\mathbb{S}^{2,1}$, and the right image is one with no singularities in $\mathbb{S}^{2,1}$.

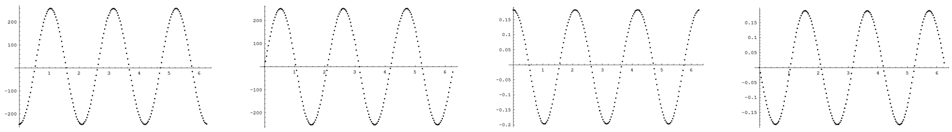

Figure 7. The values of $\mathrm{Re}(h_w/h^2\omega)$, $\mathrm{Im}(h_w/h^2\omega)$, $\mathrm{Re}\{\overline{(h_w/h)}(h_w/h^2\omega)_w\} - \mathrm{Re}\{(h_w/h)(h_w/h^2\omega)_{\bar{w}}\}$ and $\mathrm{Im}\{\overline{(h_w/h)}(h_w/h^2\omega)_w\} - \mathrm{Im}\{(h_w/h)(h_w/h^2\omega)_{\bar{w}}\}$ for a 3-legged Smyth surface in $\mathbb{S}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \le \theta \le 2\pi$. (left to right).

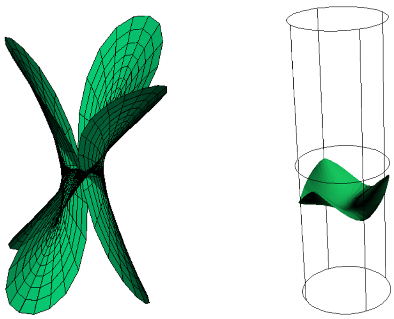

Figure 8. The left image is a 3-legged Smyth surface with singularities in $\mathbb{H}^{2,1}$, and the right image is one with no singularities in $\mathbb{H}^{2,1}$.

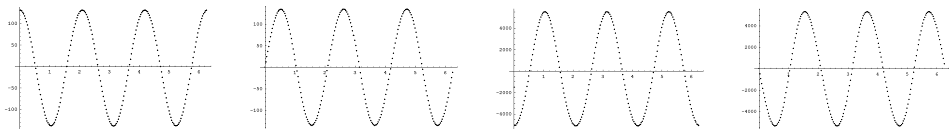

Figure 9. The values of $\mathrm{Im}(h_w/h^2\omega)$, $\mathrm{Re}(h_w/h^2\omega)$, $\mathrm{Im}\{\overline{(h_w/h)}(h_w/h^2\omega)_w\} - \mathrm{Im}\{(h_w/h)(h_w/h^2\omega)_{\bar{w}}\}$ and $\mathrm{Re}\{\overline{(h_w/h)}(h_w/h^2\omega)_w\} - \mathrm{Re}\{(h_w/h)(h_w/h^2\omega)_{\bar{w}}\}$ for a 3-legged Smyth surface in $\mathbb{H}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \le \theta \le 2\pi$ (left to right).

Theorem 8.5 *Let $\hat{f}(z) = \hat{f}(r,\theta)$ be a $(k+2)$-legged Smyth surface in $\mathbb{R}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$. Then $\hat{f}$ has a swallowtail at $(r_0, 0)$.*

Proof. We will use the criteria of Theorem 7.1. First we check that

$$\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right) = \mathrm{Re}\left(\frac{Q}{h^2\omega^2}\right) \neq 0 \tag{8.7}$$

at $(r_0, 0)$. By using $Q(r,0) = -r^k \in \mathbb{R}_{<0}$ and Corollary 8.1, we notice that (8.7) holds. Similarly, we also have

$$\mathrm{Im}\left(\frac{h_z}{h^2\omega}\right) = \mathrm{Im}\left(\frac{Q}{h^2\omega^2}\right) = 0 \tag{8.8}$$

at $(r_0, 0)$. Lastly, we check that

$$\mathrm{Re}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\} \neq \mathrm{Re}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\} \tag{8.9}$$

at $(r_0, 0)$. By direct computation, this is equivalent to

$$\begin{aligned}\mathrm{Re}&\left\{\overline{\left(\frac{Q}{h\omega}\right)}\left(\frac{Q_z h^2\omega^2 - Q(2Qh\omega + 2h^2\omega\omega_z)}{h^4\omega^4}\right)\right\}\\ &\neq \mathrm{Re}\left\{\left(\frac{Q}{h\omega}\right)\left(\frac{-2Qh^2\omega\omega_{\bar{z}}}{h^4\omega^4}\right)\right\}.\end{aligned} \tag{8.10}$$

Applying Corollary 8.1 to (8.10), we have $h^4\omega^4 \in \mathbb{R} \setminus \{0\}$ and $Q/h\omega = -\overline{(Q/h\omega)} \in i\mathbb{R} \setminus \{0\}$. Thus, (8.10) is equivalent to

$$-\mathrm{Im}\{Q_z h^2\omega - 2Q^2h - 2Qh^2\omega_z\} \neq \mathrm{Im}\{-2Qh^2\omega_{\bar{z}}\}. \tag{8.11}$$

Using $Q(r,0) = -r^k \in \mathbb{R}_{<0}$, $Q_z(r,0) = -kr^{k-1} \in \mathbb{R}_{<0}$ and Corollary 8.1, (8.11) becomes

$$\pm 2r_0^{2k} \neq 0. \tag{8.12}$$

As this is clear, we concluded that (8.9) holds. $\square$

Similarly, we have the same conclusion when $\theta = \pi/(k+2)$.

Theorem 8.6 *Let $\hat{f}(z) = \hat{f}(r,\theta)$ be a $(k+2)$-legged Smyth surface in $\mathbb{R}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$ and $g_r(r_0) = -r_0^k$. Then $\hat{f}$ has a swallowtail at $(r_0, \pi/(k+2))$.*

By above two theorems and reflective symmetry, we get the following main result:

Theorem 8.7 *If a $(k+2)$-legged Smyth surface in $\mathbb{R}^{2,1}$ has singularities before g diverges to $-\infty$, then it has at least $2(k+2)$ swallowtails.*

Remark 8.2 We have checked numerically that there are cuspidal cross caps along the cuspidal edges between each adjacent pair of swallowtails, using item (4) of Corollary 7.1, item (5) of Theorem 7.5 and item (5) of Theorem 7.8. Thus the surface as in Theorem 8.7 will also have at least $2(k+2)$ cuspidal cross caps.

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