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Sufficient conditions for decay estimates of the local energy and a behavior of the total energy of dissipative wave equations in exterior domains

(In memory of Professor Rentaro Agemi)

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Abstract. Decaying properties of the local energy for the dissipative wave equations with the Dirichlet boundary conditions in exterior domains are discussed. For the dissipation coefficient, natural conditions ensuring that waves trapped by obstacles may lose their energy are considered. Under this setting, two sufficient conditions for getting the decay estimates for the energy in bounded regions (i.e. the local energy) are given. These conditions bring some relaxation on classes of the dissipation coefficient which uniformly decaying estimates for the local energy hold. Further, decaying properties of the total energy are also discussed.

Key words: Dissipative wave equations, exterior problems, local energy decay, total energy decay, non-compactly supported initial data.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain of a bounded obstacle $\mathcal{O} = \mathbb{R}^n \setminus \overline{\Omega}$. Assume that the boundary $\partial\Omega$ is C^∞ and compact, and Ω is connected. Consider the mixed problem of the usual wave equation with a dissipation term and the Dirichlet condition:

$$\begin{cases} (\partial_t^2 - \triangle + a(x)\partial_t)u(t,x) = 0 & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x) & \text{on } \Omega. \end{cases} \tag{1.1}$$

Since $\mathcal{O}$ is compact, $\mathcal{O} \subset B_{R_0}$ holds for some fixed constant $R_0 > 0$, where $B_{R_0} = B_{R_0}(0)$, and $B_{R_0}(a) = \{x \in \mathbb{R}^n \mid |x-a| < R_0\}$. Throughout this paper, we always assume that $a(x) \geq 0$, which means that the term $a(x)\partial_t u(t,x)$ in (1.1) works as a dissipation.

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For initial data $\{f_1, f_2\} \in H_0^1(\Omega) \times L^2(\Omega)$ problem (1.1) has the unique solution $u(t,x) \in C([0,\infty); \dot{H}^1(\Omega))$ with $\partial_t u(t,x) \in C([0,\infty); L^2(\Omega))$ in the weak sense. For any domain $D \subset \mathbb{R}^n$, we put

$$e(t,x;u) = \frac{1}{2}\{|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2\} \quad \text{and} \quad E(u,D,t) = \int_D e(t,x;u)dx.$$

We call $E(u,D,t)$ the local energy in D of the solutions of (1.1). Note that total energy $E(u,\Omega,t)$ satisfies

$$E(u,\Omega,t) + \int_0^t \int_\Omega a(x)|\partial_t u(s,x)|^2 dxds = E(u,\Omega,0) \quad (t \geq 0). \tag{1.2}$$

This identity is given by multiplying $\overline{\partial_t u(t,x)}$ by the equation in (1.1), and using integration by parts. Since $a(x) \geq 0$, it follows that $E(u,\Omega,t) \leq E(u,\Omega,0)$ $(t \geq 0)$. Thus, the term $a(x)\partial_t u$ in (1.1) may work as a dissipation.

When the equation in (1.1) is a usual wave equation, (i.e., $a(x) \equiv 0$), many authors investigate whether the uniform decay estimates are obtained or not. For star shaped obstacles, Morawetz [21] gives the following uniform decay estimate of the local energy for solutions of (1.1) (with $a \equiv 0$):

$$E(u, \Omega \cap B_{R_0}, t) \leq C(1+t)^{-1} E(u,\Omega,0) \tag{1.3}$$

for any $t \geq 0$ and $(f_1, f_2) \in H_0^1(\Omega) \times L^2(\Omega)$ with $\operatorname{supp} f_1 \cup \operatorname{supp} f_2 \subset \overline{\Omega} \cap B_{R_0}$. Note that the constant $C > 0$ in (1.3) depends only on the obstacle $\mathcal{O}$, the radius R_0 of the supports of initial data and the dimension n. When initial data have noncompact supports, Ikehata [5] gives the same decay estimate of local energy as (1.3) even for the variable coefficients case.

To describe decay estimates of local energy, we introduce the following uniform decay rate $p_{m,R}(t)$ of the local energy of solutions $u(t,x)$ for usual wave equation (i.e. (1.1) with $a(x) \equiv 0$):

$$p_{m,R_0}(t) = \sup \left\{ \frac{E(u, \Omega \cap B_{R_0}, t)}{\|\nabla_x f_1\|^2_{H^m(\Omega)} + \|f_2\|^2_{H^m(\Omega)}} \;\middle|\; 0 \neq f_1, f_2 \in C_0^\infty(\overline{\Omega} \cap B_{R_0}) \text{ such that } u \in C^\infty(\mathbb{R} \times \overline{\Omega}) \right\},$$

where $R_0 > 0$ is a constant satisfying $\mathcal{O} \subset B_{R_0}$, and $m \geq 0$ is an integer.

Note that (1.3) given by Morawetz [21] is the same as $p_{0,R_0}(t) \leq C(1+t)^{-1}$ for some fixed constant $C > 0$ depending only on $\mathcal{O}$, R_0 and n. From now on, the above estimate is written as $p_{0,R_0}(t) = O(t^{-1})$ for brief descriptions.

The rate $p_{0,R_0}(t)$ is the uniform decay rate in the sense of Morawetz [22]. From (1.3), it follows that $p_{0,R_0}(t) = O(e^{-\alpha t})$ if $n \geq 3$ is odd, where $\alpha > 0$ is a fixed constant depending only on $\mathcal{O}$. If n is even, $p_{0,R_0}(t) = O(t^{n-1})$ holds. For odd n case, in [22] Morawetz shows that $p_{0,R_0}(t) = O(e^{-\alpha t})$ holds if only $\lim_{t\to\infty} p_{0,R_0}(t) = 0$ holds. This result is given by using Huygens' principle. For even dimensional case, we only have Huygens' principle of weak type. Hence, for even n, $\lim_{t\to\infty} p_{0,R_0}(t) = 0$ implies $p_{0,R_0}(t) = O(t^{n-1})$ only (see [8] for $n \geq 4$ and Vodev [35] for $n \geq 2$).

On the other hand, when there are trapping rays of geometrical optics, which is called trapping case, as is shown in Ralston [28], $\lim_{t\to\infty} p_{0,R_0}(t) = 0$ never occurs. For example, $\mathcal{O}$ consists of two convex bodies, the line giving the distance of two convex obstacles gives a trapped ray. This is a typical example for the trapping case. Note that there is no trapped ray in the exterior of star shaped obstacle. Thus, star shaped obstacles are one of the examples of non-trapping case.

In the case that there is no dissipation, the uniform decay estimates depend on the geometry of the obstacles. In this case, when there is no trapped ray of geometrical optics, we can get the decay estimate of $p_{0,R_0}(t)$ (see e.g. [11], [23], [29] and [34]). The energy for high frequency waves propagates along the rays of geometrical optics. As a result in Ralston [28], we can not obtain such decay estimates if there is a trapped ray. Hence, it is important to investigate propagation of singularietes for wave equations (about that, see e.g. [12] and [13] and references given there).

On the contrary, when $m > 0$, that is the case that initial data f_1 and f_2 have additional regularities than usual energy of physical sense (i.e.$(f_1, f_2) \in H^1(\Omega)\times L^2(\Omega)$), we encounter different aspects. When $m \geq 1$, Walker [36] shows that $\lim_{t\to\infty} p_{m,R_0}(t) = 0$ always holds. Hence, the problems is how fast $p_{m,R_0}(t)$ decays if $m \geq 1$. For the case that $n = 3$ and $\mathcal{O}$ consists of two convex bodies, Ikawa [2] obtains $p_{5,R_0}(t) = O(e^{-\alpha t})$ for some fixed constant $\alpha > 0$. This result is imploved by Ikawa [3] as $p_{2,R_0}(t) = O(e^{-\alpha t})$ even when $\mathcal{O}$ consists of finite many convex bodies with two additional conditions. One condition is that there is no other body inside of the convex hull of each two bodies. It is hard to say the other one correctly in here, but, this condition holds if each body is a ball and they separetes well each other.

About $p_{m,R_0}(t)$ with $m > 0$, a logarithmic upper bound, i.e. $p_{m,R_0}(t) = O((\log(1+t))^{-m})$ is given by Burq [1]. Note that to obtain this bound, there is no assumption for the obstacle $\mathcal{O}$. Hence, for arbitrary obstacle $\mathcal{O}$, energy decays at least logarithmically if initial data has additional regularities.

In this article, we discuss whether energy decay estimates can be given or not in dissipative case (i.e. $a(x) \not\equiv 0$). When the dissipative term works well, even the total energy decays uniformly. On the other hand, if the term is not so effective, the total energy remains, and scattering theory are developed. These phenomena are clarified by many authors. We can not give complete guide of it, however, let us mention some of the results before going to the main problem in this article.

In the case where $\Omega = \mathbb{R}^n$, all waves go out to far field since there is no trapped rays. Hence, decaying properties are closely connected with the asymptotic behavior of the dissipation term $a(x)$ as $|x| \to \infty$. Matsumura [14] and Mochizuki [17] gave decay estimates of the total energy when $a_0(1+|x|)^{-1} \leq a(x) \leq a_1$ and the initial data have compact support or have an appropriate decaying property as $|x| \to \infty$ respectively. Oppositely, when $0 \leq a(x) \leq a_2(1+|x|)^{-1-\delta}$ for some $\delta > 0$, Mochizuki [16, 17] showed that scattering phenomena occurs, that is, every solution of (1.1) is close to some solution of the free space problem (i.e. the case where $a(x) \equiv 0$ in $\Omega = \mathbb{R}^n$) as $t \to \infty$. Thus, in this case, the total energy never decay.

From the results introduced above, we can see that the critical order dividing the decaying or non-decaying phenomena is $a(x) = O(|x|^{-1})$. About the critical order, Mochizuki and Nakazawa [19] introduce finer order like as $a(x) \sim \{(e+|x|)\log(e+|x|)\cdot\log(\log(e+|x|))\cdots\log(\log\cdots(\log(e+|x|))\cdots))\}^{-1}$ and discuss decaying and non-decaying properties even for the case of exterior domains. All works describing in the above are discussed the case that dissipation coefficients depend also on the time variable t. Note also that Mochizuki and Nakazawa [20] treat decaying property for the case of star shaped obstacles even if the dissipation coefficient is localized near infinity. Mochizuki [18] also develops the scattering theory even in the case containing such a fine critical order.

In the case where $\Omega = \mathbb{R}^n$ in problem (1.1), Nakazawa [26] shows that non-decaying property and the scattering phenomena if $a(x)$ satisfies $0 \leq a(x) \leq \tilde{a}(|x|)$ $(x \in \mathbb{R}^n)$ with some non-increasing $\tilde{a} \in L^1((0,\infty))$ such that $\|\tilde{a}\|_{L^1((0,\infty))}$ is small enough. Only for getting non-decaying property, it suffices to assume that $\int_0^\infty a(x_0 + s\omega)ds < \infty$ for some $x_0 \in \Omega$ and

$\omega \in S^{n-1}$. Modifying the argument in Kawashita, Nakazawa and Soga [9], we can obtain this fact. Note that the half line $x_0 + s\omega$ $(s \geq 0)$ is the ray of geometrical optics, and the above condition means that the amplitude of high frequency waves propagating along this ray does not decay as $t \to \infty$. This is why the total energy does not decay. For general wave equations with dissipation terms, using microlocal defect measures, Nishiyama [27] investigates the relation between the rays of geometrical optics and the total energy, and gives a lower bound of the uniform decay rate causing by high frequency waves.

Let $\nu(x) = {}^t(\nu_1(x), \nu_2(x), \dots, \nu_n(x))$ be the unit outer normal vector of $\partial\Omega$ at $x \in \partial\Omega$ pointing into the outside of Ω. If the obstacle $\mathcal{O}$ is star shaped with respect to a point x_0, i.e. $\nu(x) \cdot (x - x_0) \leq 0$ $(x \in \partial\Omega = \partial\mathcal{O})$, there is no trapped ray. Thus, a part causing trapping phenomena of the boundary $\partial\Omega$ is included by the set Γ defined by

$$\Gamma = \{x \in \partial\Omega \mid \nu(x) \cdot (x - x_0) > 0\}.$$

Hence, we can expect to obtain local decay estimate (1.3) if we make the following assumptions:

(A.1) $a \in L^\infty(\Omega)$, $a(x) \geq 0$ a.e. $x \in \Omega$.
(A.2) There exist a bounded open set $\omega \subset \mathbb{R}^n$ and a constant $\varepsilon_0 > 0$ such that $\overline{\Gamma} \subset \omega$, and $a(x) \geq \varepsilon_0$ a.e. $x \in \omega \cap \overline{\Omega}$.

In what follows, we may assume that $x_0 = 0 \in \mathcal{O}$ without loss of generarity.

About this expectation, assuming (A.1) and (A.2), and $\operatorname{supp} a \subset \overline{\Omega} \cap B_R$ for some $R > 0$ as an additional assumption, Nakao [24] shows the decay estimate given by replacing $C(1+t)^{-1}$ in (1.3) by $C_\delta(1+t)^{-1+\delta}$ for any $\delta > 0$. Note that this constant C_δ may depend not only $\mathcal{O}$, R_0, n and the dissipation coefficient $a(x)$ but also $\delta > 0$ chosen arbitrary, and be large when $\delta \to 0$.

In [24], support compactness of the initial data is also assumed. This restriction is removed in Ikehata [4]. To describe this result (and a part of the main theorems of this paper), we introduce the following notations:

$$K(f_1, f_2) = \int_\Omega (1+|x|)\{|\nabla_x f_1(x)|^2 + |f_2(x)|^2\}dx$$
$$+ \|d_n(\cdot)(f_2 + a(\cdot)f_1)\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)}.$$

In the above, $d_n(x)$ are defined by $d_n(x) = |x|$ $(n \geq 3)$ and $d(x) = |x|\log(B|x|)$ $(n = 2)$, where $B > 0$ is a constant satisfying $B\inf_{x\in\Omega}|x| \geq 2$.

In [4], for any $\delta > 0$, Ikehata gives the following local energy decay estimate for solutions u of (1.1):

$$E(u, \Omega\cap B_{R_0}, t) \leq C_\delta(1+t)^{-1+\delta}K(f_1, f_2) \tag{1.4}$$

for any $t \geq 0$ and $(f_1, f_2) \in H^1_0(\Omega) \times L^2(\Omega)$ with $K(f_1, f_2) < \infty$. The constant C_δ in (1.4) depends not only on $\mathcal{O}$, n and the dissipation coefficient $a(x)$ but also $\delta > 0$ chosen arbitorary. Note that in (1.4), support compactness of the initial data is removed. For non dissipative case, serial works of Ikehata show that δ in the above estimate can be taken as $\delta = 0$. In these works of Ikehata, the argument is simpler than the original one of Morawetz [21], and also remove the restriction that the support of the initial data is compact (see e.g. Ikehata [5] and the references therein).

In this article, we give sufficient conditions for the local energy decay estimates. Throughout this paper, we put

$$J(s, x; u) = \frac{1}{2}\left|\nabla_x u(s,x) + \frac{x}{|x|}\partial_t u(s,x)\right|^2.$$

Theorem 1.1 *Assume that* (A.1), (A.2) *and there exists a functional* $L(f_1, f_2, t)$ *such that one of the following estimates holds for every solution* u *of* (1.1) *with initial data* $f_1, f_2 \in C_0^\infty(\Omega)$:

(S.1) $$\int_0^t \int_\Omega J(s, x; u)dsdx \leq L(f_1, f_2, t) \qquad (t \geq 0),$$

(S.2) $$\int_0^t \int_\Omega (a(x)|x|)^2 |\partial_t u(s,x)|^2 dsdx \leq L(f_1, f_2, t) \qquad (t \geq 0).$$

Then, there exists a constant $C > 0$ *depending only on the space dimension* n, Ω *and* $a(x)$ *such that*

$$(t-R)E(u,\Omega\cap B_R,t) \leq C\{K(f_1,f_2)+L(f_1,f_2,t)\} \qquad (t,R\geq 0, f_1,f_2\in C_0^\infty(\Omega)). \quad (1.5)$$

Remark 1.2 (1) From (1.5) and (1.2), it follows that

$$(1+t-R)E(u,\Omega\cap B_R,t) \leq C\{K(f_1,f_2)+L(f_1,f_2,t)\} \qquad (t,R\geq 0, f_1,f_2\in C_0^\infty(\Omega)).$$

Note that the constant $C>0$ in the above estimate does not depend on $R>0$. Thus, for example, we can choose $R=t/2$, and obtain

$$E(u,\Omega\cap B_{t/2},t) \leq 2C(1+t)^{-1}\{K(f_1,f_2)+L(f_1,f_2,t)\} \qquad (t\geq 0, f_1,f_2\in C_0^\infty(\Omega)).$$

(2) The local energy decay estimate in Theorem 1.1 implies that

$$tE(u,\Omega\cap B_R,t) \leq C\{K(f_1,f_2)+L(f_1,f_2,t)\}+RE(u,\Omega,0) \qquad (t\geq 0, f_1,f_2\in C_0^\infty(\Omega))$$

since it follows that $E(u,\Omega\cap B_R,t)\leq E(u,\Omega,0)$.

(3) If a functional $L(f_1,f_2,t)$ in (S.1) or (S.2) can be taken as it is less than linear order of growth in t, (1.5) gives a decay estimate of the local energy.

(4) Throughout this article, only the case of the Dirichlet problems is treated. This restriction comes from the Hardy inequality. For $n\geq 3$ or $n=2$ with the Dirichlet boundary condition, there exists a constant $C_n>0$ depending on the dimension n such that

$$\int_\Omega \frac{|f(x)|^2}{(d_n(x))^2}dx \leq C_n\int_\Omega |\nabla_x f(x)|^2 dx, \qquad (1.6)$$

holds. This is called the Hardy inequality. Hence, when $n\geq 3$, even for the case of the Cauchy problems (i.e. $\Omega=\mathbb{R}^n$ and there is no boundary condition), the same result as Theorem 1.1 can be obtained.

In what follows, two applications of Theorem 1.1 are given. The first one is to remove the restriction $\delta > 0$ and the support compactness assumptions of $a(x)$ in the decay estimate (1.4). About this problem, in the Master thesis of Suzuki [30] these restrictions are removed by introducing the following conditions on the dissipation coefficient $a(x)$: there exists a constant $C_0 > 0$ such that $a(x)|x| \leq C_0$, $|\nabla a(x)| \leq C_0 a(x)$, $\triangle a(x)|x| \leq C_0 a(x)$ $(x \in \overline{\Omega})$. In [10], these additional conditions on $a(x)$ are little bit relaxed. Using Theorem 1.1, we can extend the results in [10]. In what follows, we use the notation $\langle x\rangle = (1+|x|^2)^{1/2}$.

Theorem 1.3 *Let $n \geq 2$ and assume that* (A.1) *and* (A.2) *hold. Then there exists a constant $C > 0$ such that*

$$E(u, \Omega \cap B_{R_0}, t) \leq C(1+t)^{-1} K(f_1, f_2) \tag{1.7}$$

for any $t \geq 0$ and $(f_1, f_2) \in H_0^1(\Omega) \times L^2(\Omega)$ with $K(f_1, f_2) < \infty$ if the one of the following properties are assumed:

(A.3) *$a \in W^{1,\infty}(\Omega)$, and there exists a constant $C > 0$ such that*

$$|\nabla_x a(x)| \leq C\sqrt{a(x)}\langle x\rangle^{-1} \quad a.e. \ \ x \in \Omega.$$

(A.4) *There exist constants $C > 0$ and $R_0 > 0$ such that*

$$a(x)|x|^2 \leq C \quad a.e. \ \ x \in \Omega, \ |x| \geq R_0.$$

Examlpes satsifying (A.1), (A.2) and (A.3) are systematically made. We choose a function $F \in C^1(\mathbb{R}^n \setminus \{0\})$ satisfying $\sup_{x\in\mathbb{R}^n}(-F(x)) < \infty$ and

$$|\nabla_x F(x)| \leq C_0 e^{(|F(x)|/2)}\langle x\rangle^{-1} \qquad (x \neq 0)$$

for some fixed constant $C_0 > 0$. For this F, we put $a(x) = e^{-F(x)}$. Since

$$\sup_{x\in\mathbb{R}^n} e^{-F(x)/2} e^{(|F(x)|/2)} \leq \max\{1, e^{\sup_{x\in\mathbb{R}^n}(-F(x))}\} = C < \infty,$$

a satisfies (A.1), (A.2) and (A.3). For example, $a_1(x) = e^{-(1+|x|)^\delta}$ and $a_2(x) = (1+|x|)^{-\delta} = e^{-\delta\log(1+|x|)}$ with $\delta \geq 0$ satisfy (A.1)-(A.3). These have rather fast decaying property as $|x| \to \infty$. There are also many examples satisfying (A.1)-(A.3) with weaker decaying property, for example, $a_3(x) =$

$(\log(e+|x|))^{-\delta} = e^{-\delta \log(\log(e+|x|))}$, $a_4(x) = (\log(\log(e+|x|)))^{-\delta}$ with $\delta \geq 0$ and so forth.

In comparison with (A.3), assumption (A.4) has strong restrictions about decaying property as $|x| \to \infty$. The function a_1 with $\delta = 0$, a_2 with $\delta < 2$, and the functions a_3 and a_4 with any $\delta \geq 0$ do not satisfy (A.4). Instead of that, (A.4) does not contain any condition on regularities of $a(x)$. This is the advantage of (A.4). For example, the compact supported case is covered by (A.4). Further for any $b \in L^\infty(\Omega)$ having positive lower bound near the boundary $\partial\Omega$, the function $a_5(x) = b(x)(1+|x|)^{-2}$ satisfies (A.1), (A.2) and (A.4).

Next is an example not satisfying both (A.3) and (A.4). Consider $a_6(x) = e^{\cos(|x|)}$. It seems that $a_6(x)$ gives a strong dissipation since $a_6(x) \geq e^{-1}$ $(x \in \mathbb{R}^n)$. Despite of that, Theorem 1.3 can not conclude whether estimate (1.7) holds or not. Of course, in this case, (1.7) holds. Indeed, Nakao [25] shows that the following total energy decay:

$$E(u,\Omega,t) \leq C(1+t)^{-1}E(u,\Omega,0) \quad (t \geq 0, (f_1,f_2) \in H_0^1(\Omega) \times L^2(\Omega))$$

when (A.1), (A.2) and $a(x) \geq \varepsilon_0$ $(x \in \Omega, |x| \geq R_0)$ for some constant $\varepsilon_0 > 0$ and $R_0 > 0$ hold. Note that the above constant C depends on $\mathcal{O}$, n and $a(x)$.

The second application is for decay property of total energy decay if the dissipation satisfies the following condition:

(A.5) There exist constants $a_0 > 0$ and $R_0 > 0$ such that

$$a(x)|x| \geq a_0 \ \text{ a.e. } \ x \in \Omega \cap (B_{R_0})^c.$$

Theorem 1.4 *Let $n \geq 2$ and assume that* (A.1), (A.2) *and* (A.5) *hold. Then there exists a constant $C > 0$ such that*

$$E(u,\Omega,t) \leq C(1+t)^{-\mu}K(f_1,f_2) \tag{1.8}$$

for any $t \geq 0$ and $(f_1,f_2) \in H_0^1(\Omega) \times L^2(\Omega)$ with $K(f_1,f_2) < \infty$, where $\mu = \min\{1, a_0\}$.

Assumption (A.5) is weaker than that "$a(x) \geq \varepsilon_0 > 0$ a.e. in Ω" in Nakao [25], however, we need additional condition $K(f_1,f_2) < \infty$ on the initial data. Note also that Mochizuki and Nakazawa [19] investigate the similar case to (A.5) with time dependent dissipation. They show the estimate

$$E(u,\Omega,t) \leq O(t^{-\min\{1,\,\beta_0/2\}})$$

if the dissipation coefficient $b(t,x)$ depending also in time satisfies

$$\beta_0(e+r+t)^{-1} \leq b(t,x) \leq b_1 \qquad (t \geq 0, x \in \Omega). \tag{1.9}$$

Note that decay rate in (1.8) is better than that in [19]. Instead of that advantage, we need $K(f_1,f_2) < \infty$, which is stronger than that in [19]. When $a_0 \leq 1$, the decay rate $O(t^{-a_0})$ is the same as in the lower bound estimates given by Nishiyama [27]. Thus, in this case, estimate (1.8) is optimal. Note also that Matsumura [15] and Uesaka [33] give estimates corresponding to $E(u,\Omega,t) = O(t^{-\min\{1,\beta_0\}})$ for compact supported initial data if (1.9) holds.

If $a_0 > 1$, the situation is different. For the case of the Cauchy problem (i.e. $\Omega = \mathbb{R}^n$ and there is no boundary condition), Ikehata, Todorova and Yordanov [7] give the optimal decay estimates of the total energy for compactly supported initial data. They assume that the dissipation coefficient $a \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and there exist constants $0 < a_0 \leqq a_1$ such that

$$\frac{a_0}{(1+|x|^2)^{1/2}} \leq a(x) \leq \frac{a_1}{(1+|x|^2)^{1/2}} \qquad (x \in \mathbb{R}^n).$$

In the above setting, they show that for any fixed $\delta > 0$, $E(u,\Omega,t) = O(t^{-\min\{a_0,n-\delta\}})$ $(n \geq 3)$ and $E(u,\Omega,t) = O(t^{-\min\{a_0-\delta,2\}})$ $(n = 2)$ hold. Thus, they find another threshold $a_0 = n$, which comes from the aspect corresponding to the heat equations (i.e. the low frequency parts of solutions). On the other hand, optimality of estimate (1.8) reflects on high frequency parts of waves. Thus, there are differences between (1.8) and the estimates of Ikehata, Todorova and Yordanov [7]. Note that in [7], the upper bound estimate of $a(x)$ in the above assumption is needed. If a decays more slowly like $a(x) \sim (1+|x|)^{-\alpha}$ with $0 \leq \alpha < 1$, as is in Todorova and Yordanov [32], for any fixed $\varepsilon > 0$, $E(u,\Omega,t) = O(t^{\varepsilon-(n-\alpha)/(2-\alpha)-1})$ holds. Thus, decay rates of the total energy change although this is also caused by the low frequency parts.

2. Preliminaries

We begin by introducing basic identities and estimates to obtain theorems describing in introduction. In what follows, only for the solutions u of problem (1.1) with $f_1, f_2 \in C_0^\infty(\Omega)$ are treated. Every solution u of (1.1) in the space $\bigcap_{j=0}^{1} C^{1-j}([0,\infty); H_0^j(\Omega))$ can be approximated by a sequence of solutions u_j $(j = 1, 2, \ldots)$ of (1.1) with $u_j(0,\cdot)$, $\partial_t u_j(0,\cdot) \in C_0^\infty(\Omega)$. Thus, we can always assume that f_1, $f_2 \in C_0^\infty(\Omega)$. From usual existence theorems and the finite propagation property of the solutions of wave equation, we can see that $u \in \bigcap_{j=0}^{1} C^{2-j}([0,\infty); H^j(\Omega))$ and $\operatorname{supp} u$ is compact in $[0,T] \times \overline{\Omega}$ for any $T > 0$.

We choose any $\eta \in C^1(\overline{\Omega})$ of real-valued. Multiplying $t\partial_t\overline{u}$, $\eta\partial_t\overline{u}$, $\eta\overline{u}$, $x \cdot \partial_x \overline{u}$ by the equation in (1.1) respectively, and integrating by parts, we obtain the following identities:

$$tE(u,\Omega,t) + \int_0^t \int_\Omega sa(x)|\partial_t u|^2 dxds = \int_0^t E(u,\Omega,s)ds, \tag{2.1}$$

$$\begin{aligned}\int_\Omega \eta(x)e(t,x;u)dx &+ \int_0^t \int_\Omega a(x)\eta(x)\left|\partial_t u\right|^2 dxds \\ &+ \operatorname{Re}\int_0^t \int_\Omega (\nabla_x \eta \cdot \nabla_x u)\overline{\partial_t u} dxds = \int_\Omega \eta(x)e(0,x;u)dx,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\int_0^t \int_\Omega \eta\big(|\nabla_x u|^2 - |\partial_t u|^2\big)dxds &+ \operatorname{Re}\left[\int_\Omega \eta\overline{u}\partial_t u dx\right]_0^t \\ &= -\frac{1}{2}\left[\int_\Omega a(x)\eta|u|^2dx\right]_0^t - \operatorname{Re}\int_0^t\int_\Omega (\nabla_x\eta\cdot\nabla_x u)\overline{u}dxds,\end{aligned} \tag{2.3}$$

$$\begin{aligned}\frac{n}{2}\int_0^t\int_\Omega \big(|\partial_t u|^2 - |\nabla_x u|^2\big)dxds &+ \int_0^t\int_\Omega |\nabla_x u|^2 dxds \\ &+ \operatorname{Re}\left[\int_\Omega \partial_t u(x\cdot\nabla_x\overline{u})dx\right]_0^t \\ &= -\operatorname{Re}\int_0^t\int_\Omega a(x)\partial_t u(x\cdot\nabla_x\overline{u})dxds + \frac{1}{2}\int_0^t\int_{\partial\Omega} x\cdot\nu(x)|\partial_\nu u|^2 dS_x ds,\end{aligned} \tag{2.4}$$

where $[f]_0^t = f(t) - f(0)$. Note that identities (2.2) and (2.3) hold even for $\eta \in W^{1,\infty}_{loc}(\Omega)$. For any f_1 and $f_2 \in C_0^\infty(\Omega)$ and any $t \geq 0$, there exists

$R > 0$ such that $\operatorname{supp} u \subset [0,t] \times (\overline{\Omega} \cap B_R)$ since the propagation speed is less than 1. Noting $W^{1,\infty}_{loc}(\Omega) \subset W^{1,1}_{loc}(\Omega)$, we can choose a seqence $\{\eta_j\}$ in $C^1(\overline{\Omega})$ satisfying $\eta_j \to \eta$ in $W^{1,1}(\Omega \cap B_R)$ as $j \to \infty$. Using this sequence $\{\eta_j\}$, we can show the above mentioned facts.

Adding (2.4) to the equality obtained by multiplying $(n-1)/2$ by (2.3) with $\eta = 1$, we have

$$\begin{aligned}
&\int_0^t E(u,\Omega,s)ds \\
&\quad = -\frac{n-1}{2}\operatorname{Re}\left[\int_\Omega \overline{u}\partial_t u dx\right]_0^t - \frac{n-1}{4}\left[\int_\Omega a(x)|u|^2 dx\right]_0^t \\
&\quad\quad - \operatorname{Re}\left[\int_\Omega \partial_t u(x\cdot\nabla_x\overline{u})dx\right]_0^t - \operatorname{Re}\int_0^t\int_\Omega a(x)\partial_t u(x\cdot\nabla_x\overline{u})dxds \\
&\quad\quad + \frac{1}{2}\int_0^t\int_{\partial\Omega} x\cdot\nu(x)|\partial_\nu u|^2 dS_x ds. \qquad (2.5)
\end{aligned}$$

To handle the boundary integral in (2.5), we need assumption (A.2). As is in Ikehata [4] and Nakao [24], (A.2) implies the following estimate:

$$\begin{aligned}
\int_0^t\int_\Gamma x\cdot\nu(x)\left|\frac{\partial u}{\partial\nu}\right|^2 dS_x ds &\le CE(u,\Omega,0) + C\int_0^t\int_\omega a(x)|u|^2 dxds \\
&\quad + C\|u(t,\cdot)\|^2_{L^2(\Omega)} + C\,\|u(0,\cdot)\|^2_{L^2(\Omega)}. \qquad (2.6)
\end{aligned}$$

Thus we need to control $\|u(t,\cdot)\|^2_{L^2(\Omega)}$. Here, we use a variation of Lemma 2.6 in Ikehata [4] originated in Ikehata and Matsuyama [6].

Lemma 2.1 *There exists a constant $C > 0$ such that every solution $u(t,x)$ of* (1.1) *with the inital data $f_1, f_2 \in C_0^\infty(\Omega)$ satisfies the following estimate*:

$$\begin{aligned}
&\|u(t,\cdot)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega a(x)\,|u(s,x)|^2\,dxds \\
&\quad \le C\big(\|d_n(\cdot)(f_2 + a(\cdot)f_1)\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)}\big) \qquad (t \ge 0).
\end{aligned}$$

Note that in this article, the Hardy inequality (1.6) is needed only for getting Lemma 2.1. This is the reason why only the Dirichlet problem (1.1) is treated and the case of the Cauchy problems are excluded for $n = 2$.

Identity (2.5), estimate (2.6) and Lemma 2.1 imply that there exists a constant $C > 0$ depending only on the space dimension n, Ω and $a(x)$ satisfying

$$\int_0^t E(u, \Omega, s)ds + \frac{n-1}{4}\int_\Omega a(x)|u(t,x)|^2 dx \\ \le CK(f_1, f_2) + \int_\Omega |x| e(t, x; u)dx + I(t; u) \quad (t \ge 0), \tag{2.7}$$

where

$$I(t; u) = -\operatorname{Re}\left[\int_0^t\int_\Omega a(x)\partial_t u(s,x) x \cdot \overline{\nabla_x u(s,x)} dxds\right].$$

We also need the following estimates for $|x| \ge t$, i.e. the outside of the propagation cone:

$$\int_\Omega (|x| - t)_+ e(t, x; u)dx \le \int_\Omega |x| e(0, x; u)dx, \tag{2.8}$$

$$\int_0^t \int_\Omega a(x)(|x| - s)_+ |\partial_t u(s,x)|^2 dxds \le \int_\Omega |x| e(0, x; u)dx, \tag{2.9}$$

where $(|x| - t)_+ = \max\{|x| - t, 0\}$. As is in Ikehata [4], (2.8) is obtained by using the idea showing weighted estimates given in Todorova and Yordanov [31]. In [30] and [10], we also need estimate (2.9), which is given by the same manner as for (2.8).

Noting $\langle x\rangle \le 1 + |x|$ and $|x| \le t + (|x| - t)_+$ for $|x| \ge R$ and $|x| \le s + (|x| - s)_+$, we can obtain the following estimates:

$$\int_\Omega \langle x\rangle e(t, x; u)dx \le CK(f_1, f_2) + tE(u, \Omega, t) - (t - R)E(u, \Omega \cap B_R, t), \tag{2.10}$$

$$\int_0^t \int_\Omega a(x)\langle x\rangle |\partial_t u|^2 dxds \le CK(f_1, f_2) + \int_0^t \int_\Omega a(x)s |\partial_t u|^2 dxds. \tag{2.11}$$

Concluding this section, we introduce another identity which is also useful to show Theorems 1.1, 1.3 and 1.4. Choosing $\eta(x) = (\langle x\rangle_\delta)^\beta (= (\delta + |x|^2)^{\beta/2})$ $(\delta > 0)$ in identity (2.2), we have

$$\int_\Omega (\langle x\rangle_\delta)^\beta e(t,x;u)dx + \int_0^t \int_\Omega a(x)(\langle x\rangle_\delta)^\beta |\partial_t u|^2\, dxds \\ + \mathrm{Re}\int_0^t \int_\Omega (\beta(\langle x\rangle_\delta)^{\beta-2} x\cdot\nabla_x u)\overline{\partial_t u} dxds = \int_\Omega (\langle x\rangle_\delta)^\beta e(0,x;u)dx.$$

We add $\int_0^t \int_\Omega \beta(\langle x\rangle_\delta)^{\beta-2}|x|e(s,x;u)dxds$ to the both side of the above identity. Since

$$e(t,x;u) + \mathrm{Re}\Big(\frac{x}{|x|}\cdot\nabla_x u\Big)\overline{\partial_t u} = \frac{1}{2}\Big|\nabla_x u + \frac{x}{|x|}\partial_t u\Big|^2,$$

it follows that

$$\int_\Omega (\langle x\rangle_\delta)^\beta e(t,x;u)dx + \int_0^t \int_\Omega a(x)(\langle x\rangle_\delta)^\beta |\partial_t u|^2\, dxds \\ + \beta\int_0^t \int_\Omega (\langle x\rangle_\delta)^{\beta-2}|x|J(s,x;u)dsdx \\ = \int_\Omega (\langle x\rangle_\delta)^\beta e(0,x;u)dx + \beta\int_0^t \int_\Omega (\langle x\rangle_\delta)^{\beta-2}|x|e(s,x;u)dxds. \quad (2.12)$$

Taking the limit as $\delta \to +0$ in (2.12), we obtain

$$\int_\Omega |x|^\beta e(t,x;u)dx + \int_0^t \int_\Omega a(x)|x|^\beta |\partial_t u|^2\, dxds \\ + \beta\int_0^t \int_\Omega |x|^{\beta-1}J(s,x;u)dsdx \\ = \int_\Omega |x|^\beta e(0,x;u)dx + \beta\int_0^t \int_\Omega |x|^{\beta-1}e(s,x;u)dxds. \quad (2.13)$$

Note that even for the case of the Cauchy problems (i.e. $\Omega = \mathbb{R}^n$), (2.13) is vaild for $\beta > -n+1$. Since $|\langle x\rangle^{\beta-1} - \langle x\rangle^{\beta-2}|x|| \le \langle x\rangle^{\beta-2}/(\langle x\rangle + |x|) \le \langle x\rangle^{\beta-3}$, from (2.12) with $\delta = 1$, it follows that

$$
\begin{aligned}
&\int_\Omega \langle x\rangle^\beta e(t,x;u)dx + \int_0^t \int_\Omega a(x)\langle x\rangle^\beta \left|\partial_t u\right|^2 dxds \\
&\quad + \beta \int_0^t \int_\Omega \langle x\rangle^{\beta-1} J(s,x;u)dsdx \\
&\qquad \le \int_\Omega \langle x\rangle^\beta e(0,x;u)dx + \beta \int_0^t \int_\Omega \langle x\rangle^{\beta-1} e(s,x;u)dxds \\
&\qquad\quad + 2\beta \int_0^t \int_\Omega \langle x\rangle^{\beta-3} e(s,x;u)dxds \qquad (\beta \ge 0). \qquad (2.14)
\end{aligned}
$$

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. First we show (1.5) assuming (S.1). Choosing $\beta = 1$ in (2.13), we have

$$
\begin{aligned}
&\int_\Omega |x| e(t,x;u)dx + \int_0^t \int_\Omega a(x)|x| \left|\partial_t u\right|^2 dxds + \int_0^t \int_\Omega J(s,x;u)dsdx \\
&\quad = \int_\Omega |x| e(0,x;u)dx + \int_0^t E(u,\Omega,s)ds. \qquad (3.1)
\end{aligned}
$$

From assumption (S.1) and (3.1), it follows that

$$
\int_0^t E(u,\Omega,s)ds \le L(f_1,f_2,t) + \int_\Omega |x| e(t,x;u)dx + \int_0^t \int_\Omega a(x)|x| \left|\partial_t u\right|^2 dxds.
$$

Combining this estimate with (2.10), (2.11) and (2.1) we obtain

$$
\begin{aligned}
&\int_0^t E(u,\Omega,s)ds \\
&\quad \le L(f_1,f_2,t) + CK(f_1,f_2) \\
&\qquad + tE(u,\Omega,t) - (t-R)E(u,\Omega\cap B_R,t) + \int_0^t \int_\Omega a(x)s \left|\partial_t u\right|^2 dxds \\
&\quad = L(f_1,f_2,t) + CK(f_1,f_2) + \int_0^t E(u,\Omega,s)ds - (t-R)E(u,\Omega\cap B_R,t),
\end{aligned}
$$

where $C > 0$ is a constant in (2.10) and (2.11). The above estimate implies that

$$(t-R)E(u,\Omega\cap B_R,t) \le CK(f_1,f_2) + L(f_1,f_2,t) \qquad (t,R\ge 0, f_1,f_2\in C_0^\infty(\Omega)).$$

Hence we obtain estimate (1.5) in Theorem 1.1 if we assume (S.1).

Next we show (1.5) assuming (S.2). We begin by estimating $I(t;u)$ in (2.7). Since it follows that

$$\begin{aligned}
&\left|a(x)\partial_t u(s,x)x\cdot\nabla_x\overline{u(s,x)}\right| \\
&\quad = \left|a(x)|x|\partial_t u\frac{x}{|x|}\cdot\nabla_x\overline{u}\right| \\
&\quad = a(x)|x|\left|\left(\nabla_x\overline{u}+\frac{x}{|x|}\partial_t\overline{u}\right)\cdot\frac{x}{|x|}\partial_t u - \frac{x}{|x|}\cdot\frac{x}{|x|}|\partial_t u|^2\right| \\
&\quad \le \frac{1}{4}\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right|^2 + \big((a(x)|x|)^2 + a(x)|x|\big)|\partial_t u|^2,
\end{aligned}$$

assumption (S.2) and (1.2) implies that

$$\begin{aligned}
I(t;u) &\le \frac{1}{2}\int_0^t\int_\Omega J(s,x;u)dxds + \int_0^t\int_\Omega (a(x)|x| + (a(x)|x|)^2)|\partial_t u|^2 dxds \\
&\le \frac{1}{2}A(t) + B(t) + L(f_1,f_2,t),
\end{aligned}$$

where we put

$$A(t) = \int_0^t\int_\Omega J(s,x;u)dxds \quad\text{and}\quad B(t) = \int_0^t\int_\Omega a(x)|x||\partial_t u|^2 dxds. \tag{3.2}$$

From the above estimate, (2.7) and (3.1), it follows that

$$\begin{aligned}
&\int_\Omega |x|e(t,x;u)dx + B(t) + A(t) \\
&\quad \le \int_\Omega |x|e(0,x;u)dx + \int_0^t E(u,\Omega,s)ds
\end{aligned}$$

$$\le CK(f_1,f_2) + \int_\Omega |x|e(t,x;u)dx + I(t;u)$$
$$\le \int_\Omega |x|e(t,x;u)dx + \frac{1}{2}A(t) + B(t) + CK(f_1,f_2) + L(f_1,f_2,t),$$

which implies that $A(t) \le 2CK(f_1,f_2)+2L(f_1,f_2,t)$. This means that (S.1) is satisfied if we take $L(f_1,f_2,t)$ in (S.1) as $2CK(f_1,f_2)+2L(f_1,f_2,t)$. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

We give a proof of Theorem 1.3. First assuming (A.1), (A.2) and (A.4), we show the local decay estimate (1.7). In this case, since it follows that

$$a(x)|x|^2 \le a(x)\{(|x|^2 - R_0^2)_+ + R_0^2\} \le C + R_0^2\|a\|_{L^\infty(\Omega)} \quad (x \in \Omega),$$

from (1.2), we obatin

$$\int_0^t \int_\Omega (a(x)|x|)^2 \left|\partial_t u(s,x)\right|^2 dsdx$$
$$\le (C + R_0^2\|a\|_{L^\infty(\Omega)}) \int_0^t \int_\Omega a(x) \left|\partial_t u(s,x)\right|^2 dsdx$$
$$\le (C + R_0^2\|a\|_{L^\infty(\Omega)})E(u,\Omega,0).$$

Since $E(u,\Omega,0) \le K(f_1,f_2)$, it follows that (S.2) in Theorem 1.1 holds for $L(f_1,f_2,t) = (C + R_0^2\|a\|_{L^\infty(\Omega)})K(f_1,f_2)$. Thus, using Theorem 1.1 we get (1.7).

Next assume that (A.1), (A.2) and (A.3) hold. From (3.1) and (2.7), it follows that

$$\int_\Omega |x|e(t,x;u)dx + \int_0^t \int_\Omega a(x)|x| \left|\partial_t u\right|^2 dxds + \int_0^t \int_\Omega J(s,x;u)dsdx$$
$$\le CK(f_1,f_2) + \int_\Omega |x|e(t,x;u)dx + \frac{1}{2}\int_0^t\int_\Omega a(x)|x|\big(|\partial_t u|^2 + |\nabla_x u|^2\big)dxds,$$

which implies that

$$\frac{1}{2}B(t) + A(t) \leq CK(f_1, f_2) + \frac{1}{2}\int_0^t \int_\Omega a(x)|x| \left|\nabla_x u\right|^2 dxds, \tag{4.1}$$

where and in what follows, we use the notations given in (3.2).

To give estimate for the integral in the right hand side of (4.1), we use identity (2.3) with $\eta(x) = a(x)|x|$. Since identity (2.3) still valids even for $\eta \in W^{1,\infty}_{loc}(\Omega)$, we obtain

$$\begin{aligned}
&\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds + \mathrm{Re}\left[\int_\Omega a(x)|x|u\overline{\partial_t u}dx\right]_0^t \\
&\quad = -\frac{1}{2}\left[\int_\Omega (a(x))^2|x||u|^2dx\right]_0^t \\
&\qquad - \mathrm{Re}\int_0^t \int_\Omega \left\{a(x)\left(\frac{x}{|x|}\cdot\nabla_x u\right)\overline{u} + |x|(\nabla_x a\cdot\nabla_x u)\overline{u}\right\}dxds.
\end{aligned}$$

Since

$$\begin{aligned}
\left|a(x)|x|f_1(x)\overline{f_2(x)}\right| &\leq |x| \left|f_2(x) + a(x)f_1(x)\right| \left|f_2(x)\right| + |x| \left|f_2(x)\right|^2 \\
&\leq d_n(x)^2 \left|f_2(x) + a(x)f_1(x)\right|^2 + (1 + |x|) \left|f_2(x)\right|^2,
\end{aligned}$$

and

$$\begin{aligned}
(a(x))^2|x| \left|f_1(x)\right|^2 &\leq 2|x|\{\left|f_2(x) + a(x)f_1(x)\right|^2 + \left|f_2(x)\right|^2\} \\
&\leq 2d_n(x)^2 \left|f_2(x) + a(x)f_1(x)\right|^2 \\
&\quad + 4\|a\|^2_{L^\infty(\Omega)} \left|f_1(x)\right|^2 + 4(1 + |x|) \left|f_2(x)\right|^2,
\end{aligned}$$

from the definition of $K(f_1, f_2)$, it follows that

$$\begin{aligned}
&\left|\int_\Omega a(x)|x|u(0,x)\overline{\partial_t u(0,x)}dx\right| + \left|\int_\Omega (a(x))^2|x||u(0,x)|^2dx\right| \\
&\quad \leq 5(1 + \|a\|^2_{L^\infty(\Omega)})K(f_1, f_2).
\end{aligned}$$

The above equality and estimate imply that

$$
\begin{aligned}
&\int_0^t \int_\Omega a(x)|x|\,|\nabla_x u|^2\,dxds \\
&\quad \le B(t) + q(t) + CK(f_1, f_2) \\
&\qquad - \operatorname{Re} \int_0^t \int_\Omega \left\{ a(x)\left(\frac{x}{|x|}\cdot\nabla_x u\right)\overline{u} + |x|(\nabla_x a\cdot\nabla_x u)\overline{u}\right\} dxds, \qquad (4.2)
\end{aligned}
$$

where

$$
q(t) = -\operatorname{Re}\int_\Omega a(x)|x|u(t,x)\overline{\partial_t u(t,x)}dx.
$$

Next, we treat the last integral of the right hand side of (4.2). Noting that

$$
a(x)\left(\frac{x}{|x|}\cdot\nabla_x u\right)\overline{u} = a(x)\frac{x}{|x|}\cdot\left(\nabla_x u + \frac{x}{|x|}\partial_t u\right)\overline{u} - a(x)\left(\frac{x}{|x|}\cdot\frac{x}{|x|}\partial_t u\right)\overline{u},
$$

we get

$$
-\operatorname{Re}\left[a(x)\left(\frac{x}{|x|}\cdot\nabla_x u\right)\overline{u}\right] \le a(x)\left|\nabla_x u + \frac{x}{|x|}\partial_t u\right||u| + \frac{a(x)\partial_t|u|^2}{2}.
$$

For the term $|x|(\nabla_x a\cdot\nabla_x u)\overline{u}$, we use assumption (A.3) and get

$$
\begin{aligned}
&-\operatorname{Re}\left[|x|(\nabla_x a\cdot\nabla_x u)\overline{u}\right] \\
&\quad = -\operatorname{Re}\left[|x|\nabla_x a\cdot\left(\nabla_x u + \frac{x}{|x|}\partial_t u\right)\overline{u} - |x|\nabla_x a\cdot\frac{x}{|x|}(\partial_t u)\overline{u}\right], \\
&\quad \le C\sqrt{a}\left|\nabla_x u + \frac{x}{|x|}\partial_t u\right||u| + \frac{x\cdot\nabla_x a}{2}\partial_t|u|^2.
\end{aligned}
$$

Adding them, we obtain

$$
\begin{aligned}
&-\operatorname{Re}\left[a(x)\left(\frac{x}{|x|}\cdot\nabla_x u\right)\overline{u} + |x|(\nabla_x a\cdot\nabla_x u)\overline{u}\right] \\
&\quad \le \frac{1}{2}\left|\nabla_x u + \frac{x}{|x|}\partial_t u\right|^2 + (\|a\|_{L^\infty(\Omega)} + C^2)a(x)|u|^2 + \frac{a(x) + x\cdot\nabla_x a}{2}\partial_t(|u|^2).
\end{aligned}
$$

Hence noting (A.3) and $a \in L^\infty(\Omega)$ again, and using Lemma 2.1 we obtain

$$\begin{aligned}
&- \operatorname{Re} \int_0^t \int_\Omega \left\{ a(x) \left(\frac{x}{|x|} \cdot \nabla_x u \right) \overline{u} + |x| (\nabla_x a \cdot \nabla_x u) \overline{u} \right\} dx ds \\
&\leq \int_0^t \int_\Omega J(s, x; u) dx ds + C \int_0^t \int_\Omega a(x) |u|^2 dx ds \\
&\quad + \left[\int_\Omega \frac{a(x) + x \cdot \nabla_x a}{2} |u|^2 dx \right]_0^t \\
&\leq A(t) + CK(f_1, f_2)
\end{aligned}$$

with different constants $C > 0$ from that in (A.3). This estimate and (4.2) imply that

$$\int_0^t \int_\Omega a(x) |x| \left| \nabla_x u \right|^2 dx ds \leq B(t) + CK(f_1, f_2) + q(t) + A(t).$$

Hence (4.1) and the above estimate yield

$$\frac{1}{2} B(t) + A(t) \leq CK(f_1, f_2) + \frac{1}{2} \big\{ q(t) + A(t) + B(t) + CK(f_1, f_2) \big\},$$

which implies that

$$A(t) \leq 3CK(f_1, f_2) + q(t) \qquad (t \geq 0). \tag{4.3}$$

From the definition of q, it follows that

$$\begin{aligned}
\int_0^t q(s) ds &= - \int_0^t \int_\Omega \frac{a(x)|x|}{2} \partial_t |u|^2 ds dx \\
&= \int_\Omega \frac{a(x)|x|}{2} \{ |f_1(x)|^2 - |\partial_t u(t, x)|^2 \} dx \\
&\leq \int_\Omega \frac{a(x)|x|}{2} |f_1(x)|^2 dx < \infty \quad (t \geq 0),
\end{aligned}$$

since $f_1 \in C_0^\infty(\Omega)$. Choose $\tau > 0$ arbitrary. Since $A(t)$ is non-negative and increasing for $t \geq 0$, (4.3) and the above estimate imply

$$A(\tau) \le \frac{1}{t-\tau}\int_\tau^t A(s)ds \le 3CK(f_1, f_2) + \frac{1}{t-\tau}\int_\Omega \frac{a(x)|x|}{2}|f_1(x)|^2 dx$$

for any $t > \tau$. Taking $t \to \infty$, we obtain $A(\tau) \le 3CK(f_1, f_2)$. Thus, (S.1) in Theorem 1.1 is satisfied if $L(f_1, f_2, t)$ in (S.1) is chosen by $L(f_1, f_2, t) = 3CK(f_1, f_2)$. Thus, Theorem 1.1 and (2) of Remark 1.2 imply Theorem 1.3. □

5. Proof of Theorem 1.4

In the beginning, we introduce the following lemma which implies decay estimates for the total energy:

Lemma 5.1 *Assume that there exist constants* $0 < \mu \le 1$ *and* $C_0 > 0$ *such that*

$$\int_\Omega \langle x\rangle^\mu e(t,x;u)dx + \int_0^t \int_\Omega \langle x\rangle^{\mu-1} J(s,x;u)dsdx \le C_0 K(f_1, f_2) \qquad (t \ge 0), \quad (5.1)$$

then there exists a constant $C_1 > 0$ *such that*

$$E(u,\Omega,t) \le C_1(1+t)^{-\mu}K(f_1, f_2) \quad (t \ge 0, f_1, f_2 \in C_0^\infty(\Omega)).$$

Proof. Assuming (5.1), we show (S.1) in Theorem 1.1 holds if we take $L(f_1, f_2, t) = C'K(f_1, f_2)(1+t)^{1-\mu}$ for some constant $C' > 0$ independent of f_1, f_2 and t. Note that $\langle t\rangle^{\mu-1} \le \langle x\rangle^{\mu-1}$ for $|x| \le t$. This and (5.1) imply that

$$\begin{aligned}\langle t\rangle^{\mu-1}\int_0^t \int_{\Omega\cap B_s} J(s,x;u)dxds &\le \int_0^t \int_{\Omega\cap B_t} \langle t\rangle^{\mu-1} J(s,x;u)dxds \\ &\le \int_0^t \int_\Omega \langle x\rangle^{\mu-1} J(s,x;u)dsdx \le C_0 K(f_1, f_2).\end{aligned}$$

On the othe hand, since $\mu > 0$, from (5.1), it follows that

$$\langle t\rangle^\mu \int_{\Omega\cap(B_t)^c} e(t,x;u)dx \le \int_{\Omega\cap(B_t)^c} \langle x\rangle^\mu e(s,x;u)dx \le C_0 K(f_1, f_2), \quad (5.2)$$

which yields

$$\int_0^t \int_{\Omega\cap(B_s)^c} J(s,x;u)dxds \le 2\int_0^t \int_{\Omega\cap(B_s)^c} e(s,x;u)dxds$$
$$\le \int_0^t C_0K(f_1,f_2)\langle s\rangle^{-\mu}ds$$
$$\le \frac{2C_0}{1-\mu}K(f_1,f_2)(1+t)^{1-\mu}$$

if $0<\mu<1$. Combining these estimates, for $0<\mu<1$, we obtain

$$\int_0^t \int_{\Omega} J(s,x;u)dxds \le CK(f_1,f_2)(1+t)^{1-\mu}$$

for some constant $C>0$. Note that for $\mu=1$, the above estimate is given by assumption (5.1) itself.

Thus, when $0<\mu\le 1$, assumption (S.1) in Theorem 1.1 is satisfied for $L(f_1,f_2,t)=CK(f_1,f_2)(1+t)^{1-\mu}$ for some constant $C>0$. Theorem 1.1 and (1) of Remark 1.2 imply that there exists a constant $C>0$ independent of t such that

$$E(u,\Omega\cap B_{t/2},t) \le CK(f_1,f_2)(1+t)^{-\mu} \qquad (t\ge 0, f_1,f_2\in C_0^\infty(\Omega)).$$

Similarly to (5.2), we also have

$$\int_{\Omega\cap(B_{t/2})^c} e(t,x;u)dx \le CK(f_1,f_2)(1+t)^{-\mu}$$

for any $t\ge 0$ and f_1, $f_2\in C_0^\infty(\Omega)$. Combining the above two estimates with

$$E(u,\Omega,t)=E(u,\Omega\cap B_{t/2},t)+\int_{\Omega\cap(B_{t/2})^c} e(t,x;u)dx,$$

we obtain

$$E(u,\Omega,t) \le CK(f_1,f_2)(1+t)^{-\mu} \qquad (t\ge 0, f_1,f_2\in C_0^\infty(\Omega)),$$

which completes the proof of Lemma 5.1. □

From Lemma 5.1, to obtain Theorem 1.4, it suffices to show (5.1). We need the following lemma converting energy estimates for space derivatives to ones for time derivative:

Lemma 5.2 *Assume that the dissipation coefficient $a(x)$ satisfies* (A.5). *For R_0 in assumption* (A.5)*, we take arbitrary $r_0 > R_0$ and function $\eta \in C^\infty(\mathbb{R}^n)$ with $0 \le \eta \le 1$, $\eta = 1$ $(|x| \ge r_0 + 2)$ and $\eta = 0$ $(|x| \le r_0 + 1)$. We also choose a non-positive $\alpha \le 0$ arbitrarily. For these fixed r_0, η and $\alpha \le 0$, there exists a constant $C_{\eta,\alpha} > 0$ such that*

$$\left| \int_0^t \int_\Omega \eta(x) \langle x \rangle^\alpha \big(|\nabla_x u|^2 - |\partial_t u|^2 \big) dxds \right| \le \varepsilon \int_0^t \int_{\Omega \cap B^c_{r_0+1}} \langle x \rangle^{\alpha-1} \left| \nabla_x u + \frac{x}{|x|} \partial_t u \right|^2 dxds + \frac{C_{\eta,\alpha}}{\varepsilon} K(f_1, f_2)$$

holds for any $t \ge 0$ and $0 < \varepsilon \le 1$.

Proof. Changing η in identity (2.3) with $\eta(x)\langle x \rangle^\alpha$, we obtain

$$\int_0^t \int_\Omega \eta(x) \langle x \rangle^\alpha \big(|\nabla_x u|^2 - |\partial_t u|^2 \big) dxds + \mathrm{Re} \left[\int_\Omega \eta(x) \langle x \rangle^\alpha \overline{u} \partial_t u dx \right]_0^t = -\frac{1}{2} \left[\int_\Omega a(x) \eta(x) \langle x \rangle^\alpha |u|^2 dx \right]_0^t - \mathrm{Re} \int_0^t \int_\Omega \big(\nabla_x (\eta(x) \langle x \rangle^\alpha) \cdot \nabla_x u \big) \overline{u} dxds.$$

Since $\eta(x)\langle x \rangle^\alpha$ is bounded for $\alpha \le 0$, from (1.2) and Lemma 2.1, it follows that

$$\left| \int_\Omega \eta(x) \langle x \rangle^\alpha \overline{u} \partial_t u dx \right| \le C \|u(t,\cdot)\|_{L^2(\Omega)} \|\partial_t u(t,\cdot)\|_{L^2(\Omega)} \le CK(f_1, f_2) \qquad (t \ge 0),$$

$$\left| \int_\Omega \eta(x) \langle x \rangle^\alpha a(x) u^2 dx \right| \le C \|a\|_{L^\infty(\Omega)} \|u(t,\cdot)\|^2_{L^2(\Omega)} \le CK(f_1, f_2) \qquad (t \ge 0),$$

which implies that

$$\begin{aligned}&\left|\int_0^t\int_\Omega \eta(x)\langle x\rangle^\alpha\big(|\nabla_x u|^2-|\partial_t u|^2\big)dxds\right|\\&\qquad\le CK(f_1,f_2)+\left|\int_0^t\int_\Omega(\nabla_x(\eta(x)\langle x\rangle^\alpha)\cdot\nabla_x u)\overline{u}dxds\right|.\end{aligned}\tag{5.3}$$

Noting that

$$\begin{aligned}&\big(\nabla_x(\eta(x)\langle x\rangle^\alpha)\cdot\nabla_x u\big)\overline{u}\\&\quad=\left((\nabla_x\eta)\langle x\rangle^\alpha+\alpha\eta\langle x\rangle^{\alpha-1}\frac{x}{\langle x\rangle}\right)\cdot(\nabla_x u)\overline{u}\\&\quad=\left((\nabla_x\eta)\langle x\rangle^\alpha+\alpha\eta\langle x\rangle^{\alpha-1}\frac{x}{\langle x\rangle}\right)\cdot\left\{\left(\nabla_x u+\frac{x}{|x|}\partial_t u\right)\overline{u}-\frac{x}{|x|}\overline{u}\partial_t u\right\}\end{aligned}$$

and $2\mathrm{Re}(\partial_t u\overline{u})=\partial_t\big(|u|^2\big)$, we have

$$\begin{aligned}&\int_0^t\int_\Omega(\nabla_x(\eta\langle x\rangle^\alpha)\cdot\nabla_x u)\overline{u}dxds\\&\quad=\int_0^t\int_\Omega\left((\nabla_x\eta)\langle x\rangle^\alpha+\alpha\eta\langle x\rangle^{\alpha-1}\frac{x}{\langle x\rangle}\right)\cdot\left(\nabla_x u+\frac{x}{|x|}\partial_t u\right)\overline{u}dxds\\&\qquad-\left[\int_\Omega\frac{\langle x\rangle|x|^{-1}x\cdot\nabla_x\eta+\alpha\eta|x|\langle x\rangle^{-1}}{2}\langle x\rangle^{\alpha-1}|u|^2dx\right]_0^t.\end{aligned}$$

From Lemma 2.1 it follows that

$$\left|\left[\int_\Omega\big(\langle x\rangle|x|^{-1}x\cdot\nabla_x\eta+\alpha\eta|x|\langle x\rangle^{-1}\big)\langle x\rangle^{\alpha-1}|u|^2dx\right]_0^t\right|\le C_\alpha K(f_1,f_2)$$

since $\alpha\le 0$ yields $\sup_{x\in\mathbb{R}^n}\left|\big(\langle x\rangle|x|^{-1}x\cdot\nabla_x\eta+\alpha\eta|x|\langle x\rangle^{-1}\big)\langle x\rangle^{\alpha-1}\right|\le C_\alpha<\infty$. Noting $\operatorname{supp}\eta\subset\mathbb{R}^n\setminus B_{r_0+1}$, we get

$$\begin{aligned}&\left|\int_0^t\int_\Omega\left((\nabla_x\eta)\langle x\rangle^\alpha+\alpha\eta\langle x\rangle^{\alpha-1}\frac{x}{\langle x\rangle}\right)\cdot\left(\nabla_x u+\frac{x}{|x|}\partial_t u\right)\overline{u}dxds\right|\\&\quad\le\varepsilon\int_0^t\int_{\Omega\cap B_{r_0+1}^c}\langle x\rangle^{\alpha-1}\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right|^2dxds\end{aligned}$$

$$+\frac{C}{\varepsilon}\int_0^t\int_{\Omega\cap B^c_{r_0+1}}\langle x\rangle^{\alpha-1}\big\{((1+|x|)|\nabla_x\eta(x)|)^2|u|^2+|\alpha|^2|\eta(x)|^2|u|^2\big\}dxds.$$

Since $\operatorname{supp}\nabla_x\eta\subset B_{r_0+2}\setminus B_{r_0+1}$ and $a(x)|x|a_0^{-1}\geq 1$ on $\operatorname{supp}\eta$ which is given by assumption (A.5), we obtain

$$\begin{aligned}
&\int_0^t\int_\Omega\langle x\rangle^{\alpha-1}\big\{(\langle x\rangle|\nabla_x\eta(x)|)^2|u|^2+|\alpha|^2|\eta(x)|^2|u|^2\big\}dxds\\
&\quad\leq a_0^{-1}\int_0^t\int_\Omega a(x)|x|\langle x\rangle^{\alpha-1}\big\{(\langle x\rangle|\nabla_x\eta(x)|)^2+|\alpha|^2|\eta(x)|^2\big\}|u|^2dxds\\
&\quad\leq a_0^{-1}\sup_{x\in\mathbb{R}^n}\langle x\rangle^{\alpha}\{|\nabla_x\eta(x)|^2\langle x\rangle^2+|\alpha|^2|\eta(x)|^2\}\int_0^t\int_\Omega a(x)|u|^2dxds.
\end{aligned}$$

From $\alpha\leq 0$ and Lemma 2.1, it follows that

$$\int_0^t\int_\Omega\langle x\rangle^{\alpha-1}\big\{(\langle x\rangle|\nabla_x\eta(x)|)^2|u|^2+|\alpha|^2|\eta(x)|^2|u|^2\big\}dxds\leq C_\alpha K(f_1,f_2).$$

Summarizing the above estimates, we obtain

$$\begin{aligned}
&\left|\int_0^t\int_\Omega(\nabla_x(\eta(x)\langle x\rangle^\alpha)\cdot\nabla_x u)\overline{u}dxds\right|\\
&\quad\leq\varepsilon\int_0^t\int_{\Omega\cap B^c_{r_0+1}}\langle x\rangle^{\alpha-1}\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right|^2dxds+\frac{C_{\alpha,\eta}}{\varepsilon}K(f_1,f_2).
\end{aligned}$$

Combining this estimate with (5.3), we obtain Lemma 5.2. $\square$

To show Theorem 1.4, we need more precise estimate than (2.7). Recall the Morawetz identity originally given by Morawetz [21].

Proposition 5.3 *For any $v\in H^2_{loc}(\mathbb{R}\times\Omega)$ and a real valued function $F\in C^2(\mathbb{R}^n\setminus\{0\})$, we have the following identity*:

$$\begin{aligned}
&\operatorname{Re}\left[F\left(x\cdot\nabla_x\overline{v}+\frac{n-1}{2}\overline{v}\right)(\partial_t^2-\triangle)v\right]\\
&\quad=\partial_t(X(t,x;v))+\operatorname{div}(Y(t,x;v))+Z(t,x;v)
\end{aligned}$$

$$+\left(2^{-1}\mathrm{div}(Fx)-\frac{n-1}{2}F\right)\left(|\partial_t v|^2-|\nabla_x v|^2\right)+F|\nabla_x v|^2$$
$$+\mathrm{Re}[(\nabla_x F\cdot\nabla_x v)x\cdot\nabla_x\overline{v}],$$

where

$$X(t,x;v)=\mathrm{Re}\left[F(x)\partial_t v(t,x)\left(x\cdot\nabla_x\overline{v(t,x)}+\frac{n-1}{2}\overline{v(t,x)}\right)\right],$$
$$Y(t,x;v)=2^{-1}\left(|\nabla_x v|^2-|\partial_t v|^2\right)Fx$$
$$-\mathrm{Re}\left[F(x\cdot\nabla_x\overline{v})\nabla_x v+\frac{n-1}{2}Fv\nabla_x\overline{v}-\frac{n-1}{4}|v|^2\nabla_x F\right],$$
$$Z(t,x;v)=-\frac{n-1}{4}(\triangle F)(x)|v(t,x)|^2.$$

The case of $F=1$ in Proposition 5.3 is just (2.5).

Now, let us choose $F(x)$ in Proposition 5.3 as

$$F(x)=\chi(r)=\frac{1}{r}\int_0^r(g(s)-\tilde{g}(s))ds,\quad (r=|x|),$$

where g and $\tilde{g}\in C_0^\infty([0,\infty))$ satisfy the following properties:

(i) $0\le g\le 1$, g is non-increace in $[0,\infty)$, $g(r)=1$ $(r\le R_0+3)$ and $g(r)=0$ $(r\ge R_0+4)$.

(ii) $0\le\tilde{g}(r)\le(1+r)^{-1}$, $\mathrm{supp}\,\tilde{g}\subset(R_0+5,\infty)$ and $\int_0^\infty g(s)ds=\int_0^\infty\tilde{g}(s)ds$.

We can take g, $\tilde{g}$ as follows: First we make g satisfying (i). Noting that

$$\int_{R_0+5}^r(1+s)^{-1}ds=\log(1+r)-\log(R_0+6)\to\infty\quad(r\to\infty),$$

for g, we can choose a function $\tilde{h}\in C_0^\infty((R_0+5,\infty))$ satisfying $0\le\tilde{h}(r)\le(1+r)^{-1}$ and $\int_0^\infty g(s)ds<\int_0^\infty\tilde{h}(s)ds$. If we put $\tilde{g}(x)=\left(\int_0^\infty g(s)ds\right)\left(\int_0^\infty\tilde{h}(s)ds\right)^{-1}\tilde{h}(r)$, then $\tilde{g}$ satisfies property (ii) in the above.

Lemma 5.4 *The function χ satisfies the following properties*:

(i) $\chi \in C_0^\infty([0,\infty))$, $\chi(r) = 1$ $(r \leq R_0 + 3)$, $\chi'(r) \leq 0$ $(r \geq 0)$.
(ii) $r\chi'(r) + \chi(r) = (r\chi(r))' = g(r) - \tilde{g}(r)$ $(r \geq 0)$.

Proof. From definition of χ, we have $\chi \in C^\infty([0,\infty))$ and

$$\chi(r) = \frac{1}{r}\int_0^r (g(s) - \tilde{g}(s))ds = \frac{1}{r}\int_0^r 1ds = 1 \qquad (r \leq R_0 + 3).$$

For any $R_1 > 0$ with $\operatorname{supp}\tilde{g} \subset (R_0+5, R_1)$, it follows that $\chi(r) = 0$ $(r \geq R_1)$ since

$$\chi(r) = r^{-1}\int_0^r (g(s) - \tilde{g}(s))ds = r^{-1}\int_0^\infty (g(s) - \tilde{g}(s))ds = 0 \qquad (r > R_1).$$

Thus we obtain $\chi \in C_0^\infty([0,\infty))$. Note that we also have

$$r\chi'(r) + \chi(r) = (r\chi(r))' = g(r) - \tilde{g}(r) \qquad (r \geq 0).$$

Last, we show $\chi'(r) \leq 0$ $(r \geq 0)$. For $r < R_0 + 5$, we have

$$\chi'(r) = \frac{-1}{r^2}\int_0^r g(s)ds + \frac{g(r)}{r} = \frac{-1}{r^2}\left(\int_0^r g(s)ds - rg(r)\right) \leq 0 \quad (r < R_0 + 5)$$

since g is non-increasing. Since $\tilde{g} \geq 0$ and $g = 0$ for $r \geq R_0 + 5$, it follows that

$$\int_0^r g(s)ds - \int_0^r \tilde{g}(s)ds \geq \int_0^\infty g(s)ds - \int_{R_0+5}^\infty \tilde{g}(s)ds = 0 \quad (r \geq R_0 + 5).$$

This implies that

$$\begin{aligned}\chi'(r) &= \frac{-1}{r^2}\int_0^r (g(s) - \tilde{g}(s))ds + \frac{g(r) - \tilde{g}(r)}{r} \\ &= \frac{-1}{r^2}\left(\int_0^\infty g(s)ds - \int_{R_0+5}^r \tilde{g}(s)ds\right) - \frac{\tilde{g}(r)}{r} \leq 0 \quad (r \geq R_0 + 5).\end{aligned}$$

This completes the proof of Lemma 5.4. $\square$

From $\nabla_x F = \chi'(|x|)(x/|x|)$, it follows that

$$2^{-1}\mathrm{div}(Fx) - \frac{n-1}{2}F = \frac{1}{2}\big(|x|\chi'(|x|) + \chi(|x|)\big)$$

$$\mathrm{Re}[(\nabla_x F \cdot \nabla_x u)x \cdot \nabla_x \overline{u}] = \left(\chi'(|x|)\frac{x}{|x|} \cdot \nabla_x u\right)x \cdot \nabla_x \overline{u} = |x|\chi'(|x|)\left|\frac{x}{|x|} \cdot \nabla_x u\right|^2$$

$$\triangle F = \mathrm{div}\left(\chi'(|x|)\frac{x}{|x|}\right) = \chi''(|x|) + \frac{n-1}{|x|}\chi'(|x|).$$

Hence Proposition 5.3, equation (1.1) and (ii) of Lemma 5.4 imply

$$\begin{aligned}
&\mathrm{Re}\left[-\chi(|x|)a(x)\left(x \cdot \nabla_x \overline{u} + \frac{n-1}{2}\overline{u}\right)\partial_t u\right] \\
&\quad = \partial_t(X(t,x;u)) + \mathrm{div}(Y(t,x;u)) + (g(|x|) - \tilde{g}(|x|))e(t,x;u) \\
&\qquad - |x|\chi'(|x|)\left(|\nabla_x u|^2 - \left|\frac{x}{|x|} \cdot \nabla_x u\right|^2\right) \\
&\qquad - \frac{n-1}{4}\left(\chi''(|x|) + \frac{n-1}{|x|}\chi'(|x|)\right)|u|^2 .
\end{aligned} \tag{5.4}$$

We integrate (5.4) in $[0,t] \times \Omega$. As is in Morawetz [21], integration by parts implies

$$\int_0^t \int_\Omega \mathrm{div} Y dxds = \int_0^t \int_{\partial\Omega} \nu(x) \cdot Y dS_x ds = -\frac{1}{2}\int_0^t \int_{\partial\Omega} F\nu \cdot x|\partial_\nu u|^2 dS_x ds$$

since from the Dirichlet condition, it follows that $\nabla_x u(t,x) = \nu(x)\partial_\nu u(t,x)$ on $\mathbb{R} \times \partial\Omega$. Noting Lemma 5.4 and $|\nabla_x u|^2 - |x/|x| \cdot \nabla_x u|^2 \geq 0$, we obtain

$$\begin{aligned}
&\int_0^t \int_\Omega g(|x|)e(s,x;u)dxds - \int_0^t \int_\Omega \tilde{g}(|x|)e(s,x;u)dxds \\
&\quad \leq -\mathrm{Re}\int_0^t \int_\Omega \chi(|x|)a(x)\left(x \cdot \nabla_x \overline{u} + \frac{n-1}{2}\overline{u}\right)\partial_t u dxds \\
&\qquad - \mathrm{Re}\left[\int_\Omega \chi(|x|)\partial_t u\left(x \cdot \nabla_x \overline{u} + \frac{n-1}{2}\overline{u}\right)dx\right]_0^t \\
&\qquad + \frac{1}{2}\int_0^t\int_{\partial\Omega} x \cdot \nu(x)\,|\partial_\nu u|^2\, dS_x ds
\end{aligned}$$

$$+\frac{n-1}{4}\int_0^t\int_\Omega\left\{\chi''(|x|)+\frac{n-1}{|x|}\chi'(|x|)\right\}|u|^2dxds$$

From (2.6) and Lemma 2.1, the boundary integral in the above are estimated as

$$\int_0^t\int_{\partial\Omega}x\cdot\nu(x)\,|\partial_\nu u|^2\,dS_xds\le CK(f_1,f_2)\qquad(t\ge 0).$$

Since $\chi'(r)\le 0$ $(r\ge 0)$, for any $r\ge 0$, it follows that $1=\chi(0)\ge\chi(r)\ge\lim_{r\to\infty}\chi(r)=0$. Thus Lemma 2.1 implies that

$$\begin{aligned}\left|2\,\mathrm{Re}\int_0^t\int_\Omega\chi(|x|)a(x)\overline{u}\partial_t udxds\right|&=\left|\int_0^t\int_\Omega\chi(|x|)a(x)\partial_t|u|^2dxds\right|\\&=\left|\left[\int_\Omega\chi(|x|)a(x)|u|^2dx\right]_0^t\right|\le\|a\|_{L^\infty(\Omega)}K(f_1,f_2)\qquad(t\ge 0).\end{aligned}$$

From (1.2) and Lemma 2.1 and boundness of $|x|\chi(|x|)$ in $\mathbb{R}^n$, it also follows that

$$\begin{aligned}\left|\int_\Omega\chi(|x|)\partial_t u\Big(x\cdot\nabla_x\overline{u}+\frac{n-1}{2}\overline{u}\Big)dx\right|&\le C\int_\Omega\big(|\partial_t u|^2+|\nabla_x u|^2+|u|^2\big)dx\\&\le CK(f_1,f_2)\qquad(t\ge 0).\end{aligned}$$

For the integral of $\mathrm{Re}\,\chi(|x|)a(x)(x\cdot\nabla_x\overline{u})\partial_t u$ in $[0,t]\times\Omega$, noting

$$\begin{aligned}&\chi(|x|)a(x)(x\cdot\nabla_x\overline{u})\partial_t u\\&\quad=\chi(|x|)a(x)|x|\left(\overline{\nabla_x u+\frac{x}{|x|}\partial_t u}\right)\cdot\frac{x}{|x|}\partial_t u-\chi(|x|)a(x)|x||\partial_t u|^2\end{aligned}$$

and $|x|\chi(|x|)=0$ $(|x|\ge R_1)$ for some $R_1>R_0+5$, for any $\beta\le 1$, we obtain

$$\begin{aligned}&\left|\chi(|x|)a(x)(x\cdot\nabla_x\overline{u})\partial_t u\right|\\&\quad\le C\left\{\langle R_1\rangle^{(1-\beta)/2}\langle x\rangle^{(\beta-1)/2}a(x)\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right||\partial_t u|+a(x)|\partial_t u|^2\right\}\end{aligned}$$

$$\leq \varepsilon\langle x\rangle^{\beta-1}\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right|^2+C_\varepsilon\{1+\langle R_1\rangle^{1-\beta}\|a\|_{L^\infty(\Omega)}\}a(x)|\partial_t u|^2.$$

This and (1.2) imply

$$\begin{aligned}&\left|\mathrm{Re}\int_0^t\int_\Omega \chi(|x|)a(x)(x\cdot\nabla_x\overline{u})\partial_t u dxds\right|\\&\quad\leq \varepsilon\int_0^t\int_\Omega\langle x\rangle^{\beta-1}\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right|^2 dxds+C_{\varepsilon,\beta}K(f_1,f_2)\quad(t\geq 0,\varepsilon>0).\end{aligned}$$

Summarizing the above estimates, we obtain for any $\beta\leq 1$, $0<\varepsilon$,

$$\begin{aligned}&\int_0^t\int_\Omega g(|x|)e(s,x;u)dxds\\&\quad\leq \varepsilon\int_0^t\int_\Omega\langle x\rangle^{\beta-1}\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right|^2 dxds+C_{\varepsilon,\beta}K(f_1,f_2)\\&\qquad+\int_0^t\int_\Omega\tilde{g}(|x|)e(s,x;u)dxds\\&\qquad+\frac{n-1}{4}\int_0^t\int_\Omega\left\{\chi''(|x|)+\frac{n-1}{|x|}\chi'(|x|)\right\}|u|^2dxds.\end{aligned}\tag{5.5}$$

Using (5.5), we show the following lemma:

Lemma 5.5 *Assume that the dissipation coefficient $a(x)$ satisfies* (A.5)*. Then for any $\varepsilon>0$ and $\beta\leq 1$ there exists a constant $C_{\beta,\varepsilon}>0$ such that*

$$\begin{aligned}&\int_0^t\int_\Omega g(|x|)e(s,x;u)dxds\\&\quad\leq \varepsilon\int_0^t\int_\Omega\langle x\rangle^{\beta-1}\left|\nabla_x u+\frac{x}{|x|}\partial_t u\right|^2 dxds+C_{\beta,\varepsilon}K(f_1,f_2)\\&\qquad\qquad\qquad(t\geq 0,0<\varepsilon\leq 1,\beta\leq 1).\end{aligned}$$

Proof. Since $\chi'(|x|)\neq 0$ implies $R_0+3\leq|x|$, from (A.5), it follows that $a(x)|x|a_0^{-1}\geq 1$ for $x\in\Omega$ with $\chi'(|x|)\neq 0$. Combining this with Lemma 2.1, we get

$$\int_0^t \int_\Omega \left\{\chi''(|x|) + \frac{n-1}{|x|}\chi'(|x|)\right\}|u|^2 dxds$$
$$\leq \sup_{r\geq 0} \left|r\chi''(r) + (n-1)\chi'(r)\right| \int_0^t \int_{\Omega\cap(B_{R_0+3})^c} \frac{1}{|x|} a_0^{-1} a(x)|x||u|^2 dxds$$
$$\leq C \int_0^t \int_\Omega a(x)|u|^2 dxds \leq CK(f_1, f_2).$$

Since $\tilde{g}$ is chosen satisfying $0 \leq \tilde{g}(r)$ $(r \geq 0)$ and $\operatorname{supp} \tilde{g} \subset (R_0 + 5, R_1)$ for some $R_0 + 5 < R_1$, it follows that

$$\int_0^t \int_\Omega \tilde{g}(|x|)e(s, x; u)dxds \leq \langle R_1\rangle^{2-\beta} \int_0^t \int_{\Omega\cap(B_{R_0+5})^c} \langle x\rangle^{\beta-2} e(s, x; u)dxds$$

for $\beta \leq 1$. From these estimates and (5.5), to obtain Lemma 5.5, it suffices to show for any $0 < \beta \leq 1$, $0 < \varepsilon$,

$$\int_0^t \int_{\Omega\cap(B_{R_0+5})^c} \langle x\rangle^{\beta-2} e(s, x; u)dxds$$
$$\leq \varepsilon \int_0^t \int_\Omega \langle x\rangle^{\beta-3} \left|\nabla_x u + \frac{x}{|x|}\partial_t u\right|^2 dxds + C_{\beta,\varepsilon} K(f_1, f_2). \tag{5.6}$$

We use Lemma 5.2 with $r_0 = R_0 + 3$ and $\alpha = \beta - 2 \leq -1$. Since η in Lemma 5.2 satisfies $\operatorname{supp} \eta \subset (B_{R_0+4})^c$, $0 \leq \eta \leq 1$ and $\eta(x) = 1$ for $|x| \geq R_0 + 5$, we obtain

$$\int_0^t \int_{\Omega\cap(B_{R_0+5})^c} \langle x\rangle^{\beta-2} |\nabla_x u|^2 dxds$$
$$\leq \int_0^t \int_\Omega \eta(x)\langle x\rangle^{\beta-2} |\nabla_x u|^2 dxds$$
$$\leq \int_0^t \int_{\Omega\cap(B_{R_0+4})^c} \langle x\rangle^{\beta-2} |\partial_t u|^2 dxds + C_\varepsilon K(f_1, f_2)$$
$$+ \varepsilon \int_0^t \int_{\Omega\cap(B_{R_0+4})^c} \langle x\rangle^{\beta-3} \left|\nabla_x u + \frac{x}{|x|}\partial_t u\right|^2 dxds.$$

From this estimate, it follows that for any $\beta \le 1$

$$
\begin{aligned}
&\int_0^t \int_{\Omega\cap(B_{R_0+5})^c} \langle x\rangle^{\beta-2} e(s,x;u)dxds \\
&\quad \le \int_0^t \int_{\Omega\cap(B_{R_0+4})^c} \langle x\rangle^{\beta-2} |\partial_t u|^2 dxds \\
&\qquad + \varepsilon \int_0^t \int_{\Omega\cap(B_{R_0+4})^c} \langle x\rangle^{\beta-3} \left|\nabla_x u + \frac{x}{|x|}\partial_t u\right|^2 dxds + C_{\beta,\varepsilon} K(f_1,f_2).
\end{aligned}
$$

From (A.5), it follows that $a(x)|x|a_0^{-1} \ge 1$ $(|x| \ge R_0)$. Hence (1.2) yields

$$
\begin{aligned}
&\int_0^t \int_{\Omega\cap(B_{R_0+4})^c} \langle x\rangle^{\beta-2} \left|\partial_t u\right|^2 dxds \\
&\quad \le \int_0^t \int_{\Omega\cap(B_{R_0+4})^c} a(x)|x|a_0^{-1} \langle x\rangle^{\beta-2} |\partial_t u|^2 dxds \\
&\quad \le a_0^{-1} \int_0^t \int_\Omega a(x)|\partial_t u|^2 dxds \le a_0^{-1} E(u,\Omega,0) \quad (\beta \le 1).
\end{aligned}
$$

These estimates give (5.6), which completes the proof of Lemma 5.5. $\square$

Next, we consider $\int_0^t \int_\Omega \langle x\rangle^{\beta-1}(1-g(|x|))e(s,x;u)dxds$ for $\beta \le 1$. We put $\eta(x) = 1 - g(|x|)$, $r_0 = R_0 + 2$ and $\alpha = \beta - 1$ in Lemma 5.2. Then it follows that

$$
\begin{aligned}
&\int_0^t \int_\Omega \langle x\rangle^{\beta-1}(1-g(|x|))e(s,x;u)dxds \\
&\quad = \int_0^t \int_\Omega \langle x\rangle^{\beta-1}(1-g(|x|))\left|\partial_t u\right|^2 dxds \\
&\qquad + \frac{1}{2}\int_0^t \int_\Omega \langle x\rangle^{\beta-1}(1-g(|x|))\left(\left|\nabla_x u\right|^2 - \left|\partial_t u\right|^2\right)dxds \\
&\quad \le \int_0^t \int_\Omega \langle x\rangle^{\beta-1}(1-g(|x|))\left|\partial_t u\right|^2 dxds \\
&\qquad + \varepsilon \int_0^t \int_\Omega \langle x\rangle^{\beta-2} \left|\nabla_x u + \frac{x}{|x|}\partial_t u\right|^2 dxds + C_{\beta,\varepsilon} K(f_1,f_2).
\end{aligned}
$$

Note that we also have

$$\begin{aligned}
&\int_0^t \int_\Omega \langle x\rangle^{\beta-1}(1-g(|x|))\,|\partial_t u|^2\,dxds \\
&\qquad \le a_0^{-1}\int_0^t \int_\Omega (1-g(|x|))a(x)\langle x\rangle^{\beta}\,|\partial_t u|^2\,dxds
\end{aligned}$$

since $0 \le g \le 1$, $a_0^{-1}a(x)|x| \ge 1$ for $|x| \ge R_0$ and $1-g(|x|) \ne 0$ implies that $R_0+3 \le |x|$. These estimates yield

$$\begin{aligned}
&\int_0^t \int_\Omega \langle x\rangle^{\beta-1}(1-g(|x|))e(s,x;u)dxds \\
&\quad \le a_0^{-1}\int_0^t \int_\Omega (1-g(|x|))a(x)\langle x\rangle^{\beta}\,|\partial_t u|^2\,dxds \\
&\qquad + \varepsilon\int_0^t \int_\Omega \langle x\rangle^{\beta-2}\left|\nabla_x u + \frac{x}{|x|}\partial_t u\right|^2 dxds + C_{\beta,\varepsilon}K(f_1,f_2) \\
&\hspace{20em} (t \ge 0, 0 < \varepsilon \le 1, \beta \le 1).
\end{aligned}$$

Combining this estimate with Lemma 5.5, we arrive at the following estimates:

Lemma 5.6 *Assume that the dissipation coefficient $a(x)$ satisfies* (A.5). *Then for any $1 \ge \varepsilon > 0$ and $1 \ge \beta$, there exists a constan $C_{\beta,\varepsilon} > 0$ such that*

$$\begin{aligned}
&\int_0^t \int_\Omega \langle x\rangle^{\beta-1}e(s,x;u)dxds \\
&\quad \le a_0^{-1}\int_0^t \int_\Omega (1-g(|x|))a(x)\langle x\rangle^{\beta}\,|\partial_t u|^2\,dxds \\
&\qquad + \varepsilon\int_0^t \int_\Omega \langle x\rangle^{\beta-1}J(s,x;u)dxds + C_{\beta,\varepsilon}K(f_1,f_2) \quad (t \ge 0).
\end{aligned}$$

From Lemma 5.6, it follows that

$$\int_0^t \int_\Omega \langle x\rangle^{\beta-2}e(s,x;u)dxds \le C_\beta K_1(f_1,f_2) \qquad (t \ge 0, \beta \le 1). \tag{5.7}$$

Indeed, for $\beta \le 1$, taking β and ε in Lemma 5.6 as $\beta - 1 (\le 0)$ and $\varepsilon = 1/4$, and using (1.2), we get

$$\begin{aligned}\int_0^t \int_\Omega \langle x\rangle^{\beta-2} e(s,x;u)dxds &\le a_0^{-1} \int_0^t \int_\Omega (1-g(|x|))a(x)\langle x\rangle^{\beta-1} |\partial_t u|^2 \, dxds \\ &\quad + \frac{1}{2}\int_0^t \int_\Omega \langle x\rangle^{\beta-2} e(s,x;u)dxds + C_\beta K(f_1,f_2) \\ &\le \frac{1}{2}\int_0^t \int_\Omega \langle x\rangle^{\beta-2} e(s,x;u)dxds + C_\beta K(f_1,f_2) \\ &\qquad (t \ge 0).\end{aligned}$$

This implies (5.7).

We continue the proof of Theorem 1.4. From Lemma 5.6, (5.7) and (2.14), it follows that

$$\begin{aligned}&\int_\Omega \langle x\rangle^{\beta} e(t,x;u)dx + \int_0^t \int_\Omega a(x)\langle x\rangle^{\beta} |\partial_t u|^2 \, dxds \\ &\quad + \beta \int_0^t \int_\Omega \langle x\rangle^{\beta-1} J(s,x;u)dsdx \\ &\quad \le \int_\Omega \langle x\rangle^{\beta} e(0,x;u)dx + \beta \int_0^t \int_\Omega \langle x\rangle^{\beta-1} e(s,x;u)dxds \\ &\qquad + 2\beta \int_0^t \int_\Omega \langle x\rangle^{\beta-3} e(s,x;u)dxds \\ &\quad \le \beta a_0^{-1} \int_0^t \int_\Omega (1-g(|x|))a(x)\langle x\rangle^{\beta} |\partial_t u|^2 \, dxds \\ &\qquad + \beta\varepsilon \int_0^t \int_\Omega \langle x\rangle^{\beta-1} J(s,x;u)dxds + C_{\beta,\varepsilon} K(f_1,f_2)\end{aligned}$$

since $\langle x\rangle^{\beta-3} \le \langle x\rangle^{\beta-2}$ $(0 < \beta \le 1)$. This implies that

$$\begin{aligned}&\int_\Omega \langle x\rangle^{\beta} e(t,x;u)dx + \int_0^t \int_\Omega (1-\beta a_0^{-1} + \beta a_0^{-1} g(|x|))a(x)\langle x\rangle^{\beta} |\partial_t u|^2 \, dxds \\ &\quad + \beta(1-\varepsilon)\int_0^t \int_\Omega \langle x\rangle^{\beta-1} J(s,x;u)dsdx \le C_{\beta,\varepsilon} K(f_1,f_2) \quad (t \ge 0)\end{aligned}$$

for any fixed $1 \geq \varepsilon > 0$ and $0 < \beta \leq 1$. Choosing $\varepsilon = 1/2$ in the above, we obtain the following key estimate:

$$\int_\Omega \langle x\rangle^\beta e(t,x;u)dx + \int_0^t \int_\Omega (1 - \beta a_0^{-1} + \beta a_0^{-1} g(|x|))a(x)\langle x\rangle^\beta |\partial_t u|^2 \, dxds$$
$$+ \frac{\beta}{2}\int_0^t \int_\Omega \langle x\rangle^{\beta-1} J(s,x;u)dsdx \leq C_\beta K(f_1,f_2) \quad (t \geq 0, 0 < \beta \leq 1). \tag{5.8}$$

Now, we are in the position to show (5.1). For $\mu = \min\{1, a_0\}$, we take $\beta = \mu$ in (5.8). Since $0 < \mu \leq 1$ and

$$1 - \mu a_0^{-1} + \mu a_0^{-1} g(|x|) \geq 1 - \mu a_0^{-1} \geq 0,$$

(5.8) implies that

$$\int_\Omega \langle x\rangle^\mu e(t,x;u)dx + \mu \int_0^t \int_\Omega \langle x\rangle^{\mu-1} J(s,x;u)dsdx \leq CK(f_1,f_2),$$

which means that (5.1) holds for $\mu = \min\{1, a_0\}$. Hence, from Lemma 5.1, we obtain Theorem 1.4.

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