Note on H-separable extensions

Dedicated to Professor Kiiti Morita on his 60th birthday

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It is the purpose of this note to give a (self-contained) computational proof to the principal theorem (1. 3) of [7] and a theorem concerning ring-endomorphisms of an H-separable extension. Our tool employed in this note is an H-system of an H-separable extension, which was introduced in [4].

Recently, we found that the proof of [6, Proposition 1] contained an error and the same was repeated in the proof of main part of [7, (1. 3)]. So, the present note comprehends the correction to the previous papers [6] and [7].

Throughout, A/B will represent a ring extension with common identity 1, V the centralizer $V_A(B)$ of B in A, and C the center of A.

The next will be useful occasionally in the subsequent study.

(1) *Let* $B'\subset B''$ *be intermediate rings of* A/B. *Let* $V'=V_A(B')$, *and* $V''=V_A(B'')$. *If* ${}_{B'}B'\otimes_B B''_{B''}\rightarrow {}_{B'}B''_{B''}(b'\otimes b''\mapsto b'b'')$ *splits then* ${}_{V'}V'_{V''}<\oplus_{V'}V_{V''}$.

PROOF. There exists an element $\sum_k b_k'\otimes b_k''\in(B'\otimes_B B'')^{B'}$ such that $\sum_k b_k'b_k''=1$. Then, the map $q: V\rightarrow V'$ defined by $v\mapsto\sum_k b_k'vb_k''$ is a V'-V''-homomorphism and induces the identity map on V', which means ${}_{V'}V'_{V''}<\oplus_{V'}V_{V''}$.

A/B is called an *H-separable extension* if $A\otimes_B A$ is A-A-isomorphic to an A-A-direct summand of a finite direct sum of copies of A. To be easily seen, A/B is H-separable if and only if there exist some $v_i\in V$ $(i=1,\cdots,m)$ and $\sum_j x_{ij}\otimes y_{ij}\in(A\otimes_B A)^A$ such that $\sum_{i,j}x_{ij}\otimes y_{ij}v_i=1\otimes 1$. Following [4], such a system $\{v_i;\sum_j x_{ij}\otimes y_{ij}\}_i$ will be called an *H-system* for A/B.

In what follows, we assume always A/B is an H-separable extension with an H-system $\{v_i;\sum_j x_{ij}\otimes y_{ij}\}_i$. Then the map $\eta: A\otimes_B A\rightarrow \mathrm{Hom}_C(V,A)$ $(a_1\otimes a_2\mapsto(v\mapsto a_1va_2))$ is an A-A-isomorphism, whose inverse is given by $h\mapsto\sum_{i,j}x_{ij}\otimes y_{ij}h(v_i)$ (cf. also (2. 1$'$)).

(2) *If* σ *is an arbitrary ring-endomorphism of* A *which leaves every element of* B *invariant,* $g\in\mathrm{Hom}(A_B,A_B)$ *and* $h\in\mathrm{Hom}({}_BA,{}_BA)$, *then*

(2. 1) $$\sum_{i,j} g(x_{ij}) v\sigma(y_{ij}) \sigma(a) \sigma(v_i) = g(a) v$$

and

(2. 2) $$\sum_{i,j} \sigma(v_i) \sigma(a) \sigma(x_{ij}) vh(y_{ij}) = vh(a) \qquad (a\in A, v\in V).$$

PROOF. $\sum_{i,j} x_{ij} \otimes y_{ij}av_i = a\otimes 1$ implies $\sum_{i,j} g(x_{ij})\otimes\sigma(y_{ij}av_i) = g(a)\otimes 1$. Applying η, we obtain (2. 1).

The above formulae are specialized in various ways:

(2. 3) $$\sum_{i,j} \sigma(x_{ij}) v\sigma(y_{ij}) \sigma(v_i) = \sum_{i,j} \sigma(v_i) \sigma(x_{ij}) v\sigma(y_{ij}) = v\,.$$

(2. 1′) $$\sum_{i,j} g(x_{ij}) vy_{ij}av_i = g(a) v\,.$$

(2. 2′) $$\sum_{i,j} v_i ax_{ij} vh(y_{ij}) = vh(a)\,.$$

In particular, we have $\sum_{i,j} v_i x_{ij} v y_{ij} = v$, which means:

(3) *V_C is f.g. (finitely generated) projective* ([3, p. 112]).

(4) *A/B is a separable extension* ([3, Theorem 2. 2]).

PROOF. Since V_C is f. g. projective by (3), there exists a C-epimorphism q: $V\to C$ which induces the identity map on C. Obviously, $\sum_{i,j} x_{ij}\otimes y_{ij} q(v_i)$ is in $(A\otimes_B A)^A$ and $\sum_{i,j} x_{ij} y_{ij} q(v_i) = q(\sum_{i,j} x_{ij} y_{ij} v_i) = 1$, which means that A/B is separable.

Next, by a brief computation with (2. 1′) and (2. 2′), we see that the map ξ: $V\otimes_C V\to \mathrm{Hom}({}_BA_B, {}_BA_B)$ $(u_1\otimes u_2\mapsto(a\mapsto u_1 a u_2))$ is a V-V-isomorphism, whose inverse is given by $h\mapsto\sum_i \sum_j h(x_{ij}) y_{ij}\otimes v_i = \sum_i v_i \otimes \sum_j x_{ij} h(y_{ij})$.

(5) *If ${}_BB<\oplus_BA$ or $B_B<\oplus A_B$ then $V_A(V)=B$* ([5, Proposition 1. 2]).

PROOF. Let p: $A\to B$ be a left B-epimorphism which induces the identity map on B. Then, for $a\in V_A(V)$ we have $p(a) = \sum_{i,j} v_i a x_{ij} p(y_{ij}) = a\sum_{i,j} v_i x_{ij} p(y_{ij}) = a$ by (2. 2′). Hence, $V_A(V)=B$.

Let $\mathfrak{B}_l$ be the set of all intermediate rings B' of A/B such that ${}_{B'}B'_B<\oplus_{B'}A_B$ and ${}_{B'}B'\otimes_B A_A\to{}_{B'}A_A$ $(b'\otimes a\mapsto b'a)$ splits, and $\mathfrak{V}_l$ the set of all intermediate rings V' of V/C such that ${}_{V'}V'<\oplus_{V'}V$ and ${}_{V'}V'\otimes_C V_V\to{}_{V'}V_V$ $(v'\otimes v\mapsto v'v)$ splits. Similarly, we can consider the sets $\mathfrak{B}_r$ and $\mathfrak{V}_r$. Finally, let $\mathfrak{B}$ be the set of all intermediate rings B' of A/B such that B'/B is separable and ${}_{B'}B'_{B'}<\oplus_{B'}A_{B'}$, and $\mathfrak{V}$ the set of all intermediate rings V' of V/C such that V'/C is separable. Needless to say, $\mathfrak{B}$ is a subset of $\mathfrak{B}_l$. (In [7], $\mathfrak{B}$ was denoted as $\mathfrak{D}$.)

(6) *Let B' be an intermediate ring of A/B with $V'=V_A(B')$. If ${}_{B'}B'_B<\oplus_{B'}A_B$ or ${}_BB'_{B'}<\oplus_BA_{B'}$ then $V_A(V')=B'$. Especially, if B' is in $\mathfrak{B}_l$* (resp. $\mathfrak{B}_r$) *then V' is in $\mathfrak{V}_l$* (resp. $\mathfrak{V}_r$) *and $V_A(V')=B'$.*

PROOF. Let p: $A\to B'$ be a B'-B-epimorphism which induces the identity map on B'. Then, by $\sum_j p(x_{ij})\, y_{ij} \in V'$ and (2. 1′), we obtain for every $b'' \in V_A(V')$, $p(b'') = \sum_{i,j} p(x_{ij})\, y_{ij} b'' v_i = b'' \sum_{i,j} p(x_{ij})\, y_{ij} v_i = b''$. This proves $V_A(V') = B'$. Henceforth, we assume further that ${}_{B'}B' \otimes_B A_A \to {}_{B'}A_A$ $(b' \otimes a \mapsto b'a)$ splits. Then, ${}_{V'}V' < \oplus\, {}_{V'}V$ by (1). We define a V'-V-homomorphism $V \to V' \otimes_C V$ by $v \mapsto \sum_i \sum_j p(x_{ij})\, v y_{ij} \otimes v_i = \sum_i \sum_j p(x_{ij})\, y_{ij} \otimes v_i v$ (cf. the definition of ξ and (2. 1′)). Then, $\sum_{i,j} p(x_{ij})\, v y_{ij} v_i = v$ means that ${}_{V'}V' \otimes_C V_V \to {}_{V'}V_V$ $(v' \otimes v \mapsto v'v)$ splits. Hence, $V' \in \mathfrak{B}_l$.

(7) *If* V' *is in* $\mathfrak{B}_l$ (resp. $\mathfrak{B}_r$) *then* $B' = V_A(V')$ *is in* $\mathfrak{B}_l$ (resp. $\mathfrak{B}_r$) *and* $V_A(B') = V'$.

PROOF. Since ${}_{V'}V' \otimes_C V_V \to {}_{V'}V_V$ $(v' \otimes v \mapsto v'v)$ splits, there exists an element $\sum_k v'_k \otimes u_k \in (V' \otimes_C V)^{V'}$ such that $\sum_k v'_k u_k = 1$. Obviously, $\sum_k v'_k x_{ij} u_k \in B'$ and ${}_{B'}B'_B < \oplus\, {}_{B'}A_B$ by (1). Next, we consider an arbitrary left V'-epimor phism $q: V \to V'$ which induces the identity map on V', and define the map $\iota: A \to B' \otimes_B A$ by $a \mapsto \sum_{i,j} \sum_k v'_k x_{ij} u_k \otimes y_{ij} q(v_i)\, a$. By (2. 1′), we have $\sum_{i,j,k} v'_k x_{ij} u_k v y_{ij} q(v_i)\, b'a = \sum_k v'_k q(u_k v)\, b'a = b' \sum_{i,j,k} v'_k x_{ij} u_k v y_{ij} q(v_i)\, a$ $(b' \in B',\ v \in V)$. Then, regarding $B' \otimes_B A$ as a submodule of $A \otimes_B A$, we see that ι is a B'-A-homomorphism and $\sum_{i,j,k} v'_k x_{ij} u_k y_{ij} q(v_i)\, a = \sum_k v'_k q(u_k)\, a = a$. Hence, $B' \in \mathfrak{B}_l$. Moreover, if $v'' \in V_A(B')$ then $v'' = v''1 =$ $v'' \sum_{i,j,k} v'_k x_{ij} u_k y_{ij}\, q(v_i) = \sum_k v'_k q(u_k v'') \in V'$, which means $V_A(B') = V'$.

Now, as a combination of (6) and (7), we readily obtain the main part of [7, (1. 3)]:

THEOREM 1. *Let* A/B *be an H-separable extension.*

(a) $B' \mapsto V_A(B')$ *and* $V' \mapsto V_A(V')$ *are mutually converse* $1-1$ *correspondences between* $\mathfrak{B}_l$ (resp. $\mathfrak{B}_r$) *and* $\mathfrak{B}_l$ (resp. $\mathfrak{B}_r$).

(b) $B' \mapsto V_A(B')$ *and* $V' \mapsto V_A(V')$ *are mutually converse* $1-1$ *correspondences between* $\mathfrak{B}$ *and* $\mathfrak{B}$.

PROOF. It remains only to prove (b). First, we claim $\mathfrak{B} \subset \mathfrak{B}_l$. Given $V' \in \mathfrak{B}$, we put $C^* = V_{V'}(V')$, $V'' = V_V(V')$, and $U = V_V(C^*)$. Since V'/C is separable, we have ${}_{V''}V''_{V''} < \oplus\, {}_{V''}V_{V''}$ by (1). Recalling that V_C is f.g. projective by (3), we see that V''_C is f.g. projective. Combining this with the separability of C^*/C (cf. [1, Theorem 2. 3]), one will readily see that V''_{C^*} is f.g. projective, so that $C^*_{C^*} < \otimes\, V''_{C^*}$. On the other hand, since V'/C^* is central separable by [1, Theorem 2. 3], we have $V' \otimes_{C^*} V'' \cong V' \cdot V'' = U$ by [1, Theorem 3. 1] and ${}_U U_U < \oplus\, {}_U V_U$ by (1). Hence, ${}_{V'}V' < \oplus\, V_{V'}$ and $V' \in \mathfrak{B}_l$. Moreover, if we set $B' = V_A(V')$ then ${}_{B'}B'_{B'} < \oplus\, {}_{B'}A_{B'}$ by (1) and ${}_{B'}B' \otimes_B A_A \to {}_{B'}A_A$ $(b' \otimes a \mapsto b'a)$ splits by (a). Hence B'/B is separable and $B' \in \mathfrak{B}$. Similarly, we can prove that $V_A(B')$ is in $\mathfrak{B}$. Now, the rest of part of the proof is immediate by (a).

COROLLARY. *Let A/B be an H-separable extension with $V_A(V)=B$, and B' is in $\mathfrak{B}$. If the center Z' of B' is contained in the center of B then $V_{B'}(V_{B'}(B))=B$ (and conversely).*

PROOF. By Theorem 1 (b), $V'=V_A(B')$ is separable over C and $V_A(V')\cap V'=B'\cap V'=Z'$. Hence, V' is a central separable algebra over Z' by [1, Theorem 2.3]. Since Z' is contained in $V_B(B,)$ we have $V=V'\otimes_{Z'}V_V(V')=V'\otimes_{Z'}V_{B'}(B)$ by [1, Theorem 3.1]. It follows then $V_{B'}(V_{B'}(B))=B'\cap V_A(V_{B'}(B))=V_A(V')\cap V_A(V_{B'}(B))=V_A(V)=B$.

(8) *Every ring-homomorphism σ of A leaving every element of B invariant is a monomorphism.*

PROOF. In fact, if $\sigma(a)=0$ then $0=\sum_{i,j}x_{ij}\sigma(y_{ij})\sigma(a)\sigma(v_i)=a$ by (2.1).

We shall conclude this note with the following theorem.

THEOREM 2. *Let A/B be an H-separable extension, and σ a ring-endomorphism of A which leaves every element of B invariant.*

(a) *If $V_A(\sigma(A))=C$ then σ is an automorphism.*

(b) *If σ leaves every element of C invariant, then σ is an automorphism. Especially, if $C\subset B$ then σ is an automorphism.*

(c) *If ${}_BB<\oplus_BA$ or $B_B<\oplus A_B$ then σ is an automorphism.*

PROOF. Let $A'=\sigma(A)$, and $C'=V_A(A')$.

(a) To be easily seen, $\sum_{s,j}\sigma(x_{rs})y_{rs}x_{ij}\sigma(y_{ij})$ is in $C'=C$. Hence, if a is an arbitrary element of A then by (2.1′) we have $\sigma(\sum_{i,j}x_{ij}\sigma(y_{ij})a\sigma(v_i))$ $=\sum_{r,s}\sigma(x_{rs})y_{rs}(\sum_{i,j}x_{ij}\sigma(y_{ij})a\sigma(v_i))v_r=\sum_{r,i}(\sum_{s,j}\sigma(x_{rs})y_{rs}x_{ij}\sigma(y_{ij}))a\sigma(v_i)v_r$ $=a\sum_{r,i}\sum_{s,j}\sigma(x_{rs})y_{rs}x_{ij}\sigma(y_{ij})\sigma(v_i)v_r=a\sum_{r,s}\sigma(x_{rs})y_{rs}\sum_{i,j}x_{ij}\sigma(y_{ij}v_i)v_r=$ $a\sum_{r,s}\sigma(x_{rs})y_{rs}v_r=a$. This together with (8) implies that σ is an automorphism.

(b) Obviously, A' is an H-separable extension of B, and hence a separable extension of B by (4). By (a), our proof will be complete if we can prove that C' coincides with C. We consider here the C-homomorphisms f: $C'\otimes_C V\to V$ defined by $c'\otimes v\mapsto c'\sigma(v)$ and g: $V\to C'\otimes_C V$ defined by $v\mapsto \sum_i\sum_j\sigma(x_{ij})v\sigma(y_{ij})\otimes v_i$. Then, by (2.3) $fg(v)=\sum_{i,j}\sigma(x_{ij})v\sigma(y_{ij}v_i)=v$. On the other hand, $gf(c'\otimes v)=\sum_i\sum_j\sigma(x_{ij})c'\sigma(v)\sigma(y_{ij})\otimes v_i=c'\sum_i\sigma(\sum_j x_{ij}vy_{ij})$ $\otimes v_i=c'\sum_{i,j}x_{ij}vy_{ij}\otimes v_i=c'\otimes v$. Hence, $C'\otimes_C V\cong V$. Since, V_C is f.g. projective by (3) and ${}_{C'}C'_{C'}<\oplus_{C'}V_{C'}$ by (1), C'_C is an f.g. projective module of rank 1. Hence, by [2, Corollaire du Théorème 1], it follows $C'=C$.

(c) This is immediate by (5) and (b).

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