Multipliers of Lorentz spaces

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1. Introduction

Let G and Γ be locally compact abelian groups in Pontrjagin duality and respectively with Haar measures λ and η such that the Plancherel theorem holds. In this paper we investigate the multipliers on Lorentz spaces $L(p, q)(G)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. For convenience, we review briefy in section 2 what we need on the fundamental theory of $L(p, q)$ spaces. Many propositions concerning $L(p, q)$ spaces one can refer to Hunt [2], Blozinsky [1], Yap [8] and [9].

In this paper, it is essentially to investigate the multipliers of $L(p, q)$ in which the identity $\mathfrak{M}(L^1(G), F(G)) \cong F(G)$ will be true for the cases of $F(G) = L(p, q)(G)$ or $F(G) = A(p, q)(G)$ defined in Yap [8]. It follows that the identity $\mathfrak{M}(L^1(G), A^p(G)) \cong A^p(G)$ for $1 \leq p \leq 2$ in Lai [4; Proposition 5.2] will be a consequence in this paper. Further we would give an answer for the question risen in [4].

2. Preliminaries on $L(p, q)$ spaces

DEFINITIONS. Let f be a measurable function defined on a measure space (X, μ). We assume that the functions f are finite valued almost everywhere and for $y > 0$,

$$\mu\{x \in X ; |f(x)| > y\} < \infty .$$

The *distribution function* of f is defined by

$$\lambda_f(y) = \mu\{x \in X ; |f(x)| > y\}, \quad y > 0 .$$

The (nonnegative) *rearrangement* of f is defined by

$$f^*(t) = \inf\{y > 0 ; \lambda_f(y) \leq t\} = \sup\{y > 0 ; \lambda_f(y) > t\}, \quad t > 0 .$$

The *average function* of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds, \quad t > 0 .$$

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Note that $\lambda_f(\cdot)$, $f^*(\cdot)$, $f^{**}(\cdot)$ are nonincreasing and right continuous functions on $(0, \infty)$ (cf. Hunt [2]).

The *Lorentz space* denoted by $L(p, q)(X, \mu)$, for brevity by $L(p, q)$, is defined to be the collection of all f such that $\|f\|^*_{(p,q)}<\infty$ where

$$\|f\|^*_{(p,q)} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [t^{1/p}f^*(t)]^q \frac{dt}{t}\right)^{1/q}, & 0<p<\infty,\ 0<q<\infty \\ \sup_{t>0} t^{1/p} f^*(t), & 0<p\leq\infty,\ q=\infty\,. \end{cases}$$

It follows from Hunt [2; p. 253] that $\|f\|^*_{(p,p)}=\|f\|_p$ and if $0<q_1\leq q_2 \leq\infty$, $0<p<\infty$, then $\|f\|^*_{(p,q_2)}\leq\|f\|^*_{(p,q_1)}$ holds and so $L(p, q_1)\subset L(p, q_2)$. Evidently $L(p,p)=L^p$ algebraically, in particular, if $1\leq p\leq\infty$, then $L(p,p)$ is isometrically isomorphic to L^p where $L(p,p)$ takes $\|\ \ \|^*_{(p,p)}$ as its norm.

Now we consider X to be a locally compact Hausdorff space and μ is a positive Borel measure, then it can be shown that: every function in $L(p, q)(X, \mu)$ is locally integrable if and only if any one of the cases $p=1=q$; $p=\infty=q$ or $1<p<\infty$, $1\leq q\leq\infty$ holds.

Throughout we will assume that X is locally compact Hausdorff space with positive Borel measure μ, and that function f in $L(p, q)$ is locally integrable.

It is known that the functional $\|\ \ \|^*_{(p,q)}$ endows a topology in $L(p, q)$ such that $L(p, q)$ is a topological vector space, and the limit $f_n\to f$ in this topology means that $\|f_n-f\|^*_{(p,q)}\to 0$ (see Hunt [2; p. 257]). We also introduce (see Hunt [2]) the following function

$$f^{**}(t) = \begin{cases} \sup_{\mu(E)\geq t} \frac{1}{\mu(E)}\int_E |f(x)|d\mu, & t\leq\mu(X) \text{ and } E\subset X \\ \frac{1}{t}\int_X |f(x)|d\mu, & t>\mu(X), \end{cases}$$

$\|f\|^\star_{(p,q)}=\|f^{\star\star}\|^*_{(p,q)}$ and $\|f\|_{(p,q)}=\|f^{**}\|^*_{(p,p)}$.

In [2; p. 258], Hunt proved that $L(p, q)$ is a Banach space under the norm $\|\ \ \|^\star_{(p,q)}$ for $1<p<\infty$, $1\leq q\leq\infty$. Moreover

$$\|f\|^*_{(p,q)}\leq\|f\|^\star_{(p,q)}\leq\|f^*\|^\star_{(p,q)}\leq\frac{p}{p-1}\|f\|^*_{(p,q)}).$$

It follows immediately that

PROPOSITION 2.1. *If $1<p<\infty$, $1\leq q\leq\infty$ then $\|\ \ \|^\star_{(p,q)}$ and $\|\ \ \|^*_{(p,q)}$ are equivalent norms on $L(p, q)$ and hence $L(p, q)$ is a Banach space under the norms $\|\ \ \|^\star_{(p,q)}$ or $\|\ \ \|_{(p,q)}$ which induces the same topology as $\|\ \ \|^*_{(p,q)}$ does.*

PROOF. It is immediately that $\| \ \|_{(p,q)}$ is a norm in $L(p, q)$. By previous inequalities, it is sufficient to show that

$$\|f\|^*_{(p,q)} \leq \|f\|_{(p,q)} \leq \frac{1}{p-1} \|f\|^*_{(p,q)} .$$

Indeed, if $q \neq \infty$ and since $f^* \leq f^{**} = (f^*)^{**}) = f^{**})^*$, it follows that

$$\begin{aligned}
\|f\|^*_{(p,q)} \leq \|f\|_{(p,q)} &= \left(\frac{q}{p}\int_0^\infty [t^{1/p} f^{**}(t)]^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\frac{q}{p}\int_0^\infty \left[\int_0^t f^*(x)\, dx\right]^q t^{(q/p)-1-q}\, dt\right)^{1/q} \\
&\leq \frac{p}{p-1}\left(\frac{q}{p}\int_0^\infty f^*(x)^q\, x^{(/pq)-1}\, dx\right)^{1/q} \quad \text{(Hardy inequality)} \\
&= \frac{p}{p-1} \|f\|^*_{(p,q)} .
\end{aligned}$$

If $q = \infty$, we have

$$\begin{aligned}
\|f\|^*_{(p,\infty)} &= \sup_{t>0}\ t^{1/p}\ f^*(t) \leq \sup_{t>0}\ t^{1/p}\ f^{**}(t) = \|f\|_{(p,\infty)} \\
&= \sup_{t>0}\ t^{(1/p)-1}\int_0^t f^*(s)\, ds \leq \sup_{t>0}\ t^{(1/p)-1}\int_0^t s^{-1/p} \|f\|^*_{(p,\infty)}\, ds \\
&= \frac{p}{p-1} \|f\|^*_{(p,\infty)} ,
\end{aligned}$$

this shows $\|f\|^*_{(p,\infty)} \leq \|f\|_{(p,\infty)} \leq \frac{p}{p-1} \|f\|^*_{(p,\infty)}$. Q.E.D.

From now on we shall consider that the space $L(p, q)$ endows the norm $\| \ \|_{(p,q)}$ for $1<p<\infty$, $1\leq q\leq\infty$. In the cases $p=1=q$ and $p=\infty=q$, $\| \ \|^*_{(1,1)}$ and $\| \ \|^*_{(\infty,\infty)}$ are already the complete norms of $L(1, 1)$ and $L(\infty, \infty)$, thus for the discussion of $L(1, 1)$ and $L(\infty, \infty)$, we mean that they have complete norms $\| \ \|^*_{(1,1)}$ and $\| \ \|^*_{(\infty,\infty)}$ respectively. By Hunt [2 ; pp. 259–262], we have

PROPOSITION 2.2. (i) *For any one of the cases $p=1=q$; $p=\infty=q$ or $1<p<\infty$ and $1\leq q\leq\infty$, the space $L(p, q)$ is a Banach space with respect to the norm $\| \ \|_{(p,q)}$.* (ii) *The conjugate space $L(p, 1)$ is $L(p', \infty)$ where $1/p+1/p'=1$. The conjugate space $L(p, q)$, $1<p<\infty$, $1<q<\infty$, is $L(p', q')$* where $1/q+1/q'=1$ *and hence they are reflexive. The dual pair is of the form*

$$\langle f, g\rangle = \int fg\, d\mu, \quad f\in L(p, q),\ g\in L(p', q') .$$

3. Multiplierson $L(p, q)(G)$ spaces

Let A be a commutative normed algebra and B a (two sided) A-module normed linear space. We denote by $\mathfrak{M}(A, B)$ the set of all bounded linear mapping

$$T: A \to B \text{ such that } T(ab) = a(Tb) \text{ for any } a, b \in A.$$

Each element $T \in \mathfrak{M}(A, B)$ is called the multiplier of A to B.

Under the usual convolution, the space $L^p(G)$ for $1 \leq p \leq \infty$ and $A^p(G)$ for $1 \leq p \leq \infty$ are L^1-module and it was known that

$$\mathfrak{M}(L^1(G), L^1(G)) \cong M(G)$$
$$\mathfrak{M}(L^1(G), L^p(G)) \cong L^p(G) \qquad 1 < p < \infty$$
$$\mathfrak{M}(L^1(G), A^p(G)) \cong A^p(G) \qquad 1 \leq p \leq 2$$

where G is a locally compact abelian group, $M(G)$ denotes the bounded regular measures and $A^p(G)$ is, with the norm $\|f\|^p = \max(\|f\|_1, \|\hat{f}\|_p)$, a Banach algebra of all functions f in $L^1(G)$ whose Fourier transforms $\hat{f}$ being to $L^p(\Gamma)$.

Blozinsky]1] proved that the usual convolution $*$ on simple function space $S_0(G)$ can be uniquely extended to $L^1 * L(p, q)$ where every function in $L(p, q)$ is locally integrable and hence $L(p, q)(G)$ is an L^1-module Banach space with respect to $\| \; \|_{(p,q)}$.

Let $A(p, q)(G)$ be the subspace of $L^1(G)$ with Fourier transforms in $L(p, q)(\Gamma)$ provided each function in $L(p, q)(\Gamma)$ is locally integrable. For every $f \in A(p, q)(G)$, we supply a norm in $A(p, q)$ by

$$\|f\|_{A(p,q)} = \max\left(\|f\|_1, \; \|\hat{f}\|_{(p,q)}\right).$$

This norm is equivalent to the sum norm $\|f\|_1 + \|\hat{f}\|_{(p,q)}$. In particular, if $p = q$, then $A(p, q) = A^p$.

In [8], Yap showed that $A(p, q)$ $(1 < p < \infty, 1 \leq q \leq \infty)$ is a Segal algebra with respect to the sum norm, and so it is also a Segal algebra with respect to our given norm. In this section we investigate that whether the following identities hold

$$\mathfrak{M}(L^1(G), L(p, q)(G)) \cong L(p, q)(G)$$
$$\mathfrak{M}(L^1(G), A(p, q)(G)) \cong A(p, q)(G).$$

For convenient, we state some lemmas which are probably not all new.

LEMMA 3. 1. *Let λ be Haar measure of G. Then* (i) $\lambda_{f_s} = \lambda_f$, (ii) $f_s^* =$

f^*, $f_s^{**}=f^{**}$, $\|f_s\|_{(p,q)}^*=\|f\|_{(p,q)}^*$ *and* $\|f_s\|_{(p,q)}=\|f\|_{(p,q)}$ *where* $f_s(x)=f(x-s)$.

PROOF. (i) Since $\{x\in G\,;\,|f_s(x)|>t\}=\{x\in G\,;\,|f(x-s)|>t\}=\{y\in G\,;\,|f(y)|>t\}+s$, we have $\lambda_{f_s}(t)=\lambda_f(t)$.

(ii) This is a consequence of (i). Q.E.D.

LEMMA 3.2. *For every* $f\in L(p,q)(G)$, $1<p<\infty$, $1\le q<\infty$, *the mapping* $s\to f_s$ *of* G *into* $L(p,q)(G)$ *is continuous.*

PROOF. Since simple functions are dense in $L(p,q)(G)$, it is sufficient to show that for any simple function f, $s\to f_s$ is continuous. Let $f=\sum_{i=1}^n k_i\,\chi_{E_i}$, then $f_s=\sum_{i=1}^n k_i\,\chi_{E_i+s}$. Now

$$(f_s-f)^{**}\le\sum_{i=1}^n |k_i|\,(\chi_{E_i+s}-\chi_{E_i})^{**}$$

and

$$|\chi_{E_i+s}-\chi_{E_i}|(x)=\begin{cases}1 & x\in(E_i+s)\,\Delta E_i\\ 0 & \text{otherwise},\end{cases}$$

where Δ denotes the symmetric difference of sets and χ_E is the characteristic function on E. We have

$$\lambda(\chi_{E_i+s}-\chi_{E_i})(t)=\begin{cases}\lambda[E_i+s)\,\Delta E_i], & t<1\\ 0 & t\ge 1\end{cases}$$

$$(\chi_{E_i+s}-\chi_{E_i})^*(t)=\begin{cases}1 & t<\lambda((E_i+s)\,\Delta E_i)\\ 0 & t\ge\lambda((E_i+s)\,\Delta E_i).\end{cases}$$

This shows that

$$\|\chi_{E_i+s}-\chi_{E_i}\|_{(p,q)}^*=[\lambda((E_i+s)\,\Delta E_i)]^{1/p}\to 0\,(s\to 0).$$

Hence

$$\|f_s-f\|_{(p,q)}^*\to 0\,(s\to 0) \text{ and } \|f_s-f\|_{(p,q)}\to 0\,(s\to 0) \qquad \text{Q.E.D.}$$

LEMMA 3.3. *There is an approximate identity* $\{a_\alpha\}$ *of* $L^1(G)$ *such that* $\|a_\alpha\|_1=1$ *and* $f*a_\alpha\to f$ *for every* $f\in L(p,q)$, $1<p<\infty$, $1\le q<\infty$. *It holds also for* $f\in L(p,q)$, $1<p<\infty$, $1<q<\infty$.

PROOF. For a proof of simple function follows immediately from Lemma 3.2. Indeed, let $\{U_\alpha\}$ be a decreasing neighborhood system at the origin in G, for each α, we assume that a_α is non-negative continuous function with support in U_α such that $\int_G a_\alpha(x)\,d\lambda(x)=1$. If g is a simple function, then

$$\begin{aligned}\|g*a_\alpha-g\|_{(p,q)}&\le\int_G\|g_y-g\|_{(p,q)}\,a_\alpha(y)\,d\lambda(y)\\ &\le\sup_{y\in U_\alpha}\|g_y-g\|_{(p,q)}\to 0 \text{ (Lemma 3.2)}.\end{aligned}$$

the limit being taken over the net of α. As the simple fnctuions are dense in $L(p, q)$, thus for any $\varepsilon>0$ and $f\in L(p, q)$, there exists a simple function g such that $\|f-g\|_{(p,q)}<\varepsilon$ and

$$\begin{aligned}\|f * a_\alpha - f\|_{(p,q)} &\leq \|f * a_\alpha - g * a_\alpha\|_{(p,q)} + \|g * a_\alpha - g\|_{(p,q)} \\ &\quad + \|g-f\|_{(p,q)} \\ &< 2\varepsilon + \|g * a_\alpha - g\|_{(p,q)} \\ \limsup_\alpha \|f * a_\alpha - f\|_{(p,q)} &\leq 2\varepsilon .\end{aligned}$$

Since ε is arbitrary, we have $f * a_\alpha \to f$ for every $f\in L(p, q)$. Q.E.D.

One of the main results is the following

THEOREM 3.4. *The multiplier space $\mathfrak{M}(L^1(G), L(p, q)(G))$ is isometric isomorphic to $L(p, q)(G)$ for $1<p<\infty$, $1<q<\infty$. Moreover for any $T\in \mathfrak{M}(L^1(G), L(p, q)(G))$, there is a unique f in $L(p, q)(G)$ such that $Ta=a * f$ for every $a\in L^1(G)$.*

PROOF. By Proposition 2.2, we see that the dual pair is of the form

$$\langle f, g\rangle = \int_G f(x)\, g(x)\, d\lambda(x), \quad f\in L(p', q'),\ g\in L(p, q).$$

Moreover for every $\mathfrak{G}\in L'(p, q')$ there is a unique $g\in L(p, q)$ such that

$$\mathfrak{G}(f) = \int_G fg\, d\lambda \quad \text{for every } f\in L(p', q').$$

Now if $a\in L^1(G)$, $f\in L(p', q')$, we define $a \circledast f=\tilde{a} * f$, where a is the reflexive function of a. Then it follows from Blozinsky [1 ; Theorem 2.9] that $L(p', q')$ is an L^1-module under $\circledast$, naturaly if for every $a\in L^1(G)$, $\mathfrak{G}\in L'(p', q')$, we define

$$a \otimes \mathfrak{G}(f) = \mathfrak{G}(a \circledast f) \qquad \text{for } f\in L(p', q').$$

Then $L'(p', q')$ is also $L^1(G)$-module under $\otimes$. Since there is an approximate identity $\{a_\alpha\}$ in $L^1(G)$ such that $\|a_\alpha\|_1=1$ and

$$\|a_\alpha \circledast f - f\|_{(p',q')} = \|a_\alpha * \tilde{f} - \tilde{f}\|_{(p',q')} \to 0$$

it follows from Liu and Rooij [6 ; Lemma 2.8] that

$$\mathfrak{M}(L^1(G), L'(p'\, q')(G) \cong L'(p', q')(G),$$

and hence

$$\mathfrak{M}(L^1(G), L(p, q)(G)) \cong L(p, q)(G), \qquad 1<p, q<\infty .$$

We have to claim that the group algebra $L^1(G)$ acting on $L'(p', q')$ corresponds to acting on $L(p, q)$. That is for any $a\in L^1(G)$, $a \otimes \mathfrak{G}$ in

$L'(p', q')$ corresponds to $a*g$ in $L(p, q)$ whenever $\mathfrak{G}\in L'(p', q')$ corresponds to $g\in L(p, q)$. Indeed, for every $f\in L(p', q')$, $a\otimes\mathfrak{G}(f)=\mathfrak{G}(a\circledast f)=\int_G(\tilde{a}*f)\,g\,d\lambda=\int_G f(a*g)\,d\lambda$. Hence for each $T'\in\mathfrak{M}(L^1(G),\ L'(p', q')(G))$, there is a unique $\mathfrak{G}\in L'(p', q')$ such that

$$T'a=a\otimes\mathfrak{G}\qquad\text{for every } a\in L^1(G).$$

This implies that for every $T\in\mathfrak{M}(L^1(G), L(p, q)(G))$ there is a unique $g\in L(p, q)$ such that

$$Ta=a*g\qquad\text{for every } a\in L^1(G).$$

Thus the theorem is proved. Q.E.D.

In order to characterize the multipliers $\mathfrak{M}(L^1(G), A(p, q)(G))$, we define a space $M(p, q)(G)$ as follows

$$M(p, q)(G)=\left\{\mu\in M(G)\,;\ \hat{\mu}\in L(p, q)(\Gamma),\ 1\leq p\leq\infty,\ 1\leq q\leq\infty\right\}$$

where $\hat{\mu}$ is the Fourier Stieltjes transform of a bounded regular measure μ in $M(G)$. For every $\mu\in M(p, q)(G)$ we supply a norm by

$$\|\mu\|_{M(p,q)}=\max\left\{\|\mu\|,\ \|\hat{\mu}\|_{(p,q)}\right\},\ 1\leq p,\ q\leq\infty\,.$$

Denote by $M(p, q)(G)$ the space of $M(p, q)(G)$ with the norm $\|\ \|_{M(p,q)}$. In particular if $p=q$, we denote by $M^p(G)=M(p, p)(G)$ with the norm $\|\mu\|_{M^p}=\max\{\|\mu\|, \|\hat{\mu}\|_p\}$, $\mu\in M^p(G)$. Note that $M(\infty, \infty)(G)=M(G)$.

THEOREM 3.5. *The space $M(p, q)(G)$ $(1\leq p\leq\infty,\ 1\leq q\leq\infty)$ is a commutative Banach algebra.*

PROOF. Evidently $M(p, q)(G)$ is a normed linear space. Suppose that $\{\mu_n\}$ is a Cauchy sequence in $M(p, q)(G)\subset M(G)$. Then $\{\hat{\mu}_n\}$ is also a Cauchy sequence in $L(p, q)(\Gamma)$. By the completeness of $M(G)$ and $L(p, q)(\Gamma)$, there are $\mu\in M(G)$ and $h\in L(p, q)(\Gamma)$ such that

$$\|\mu_n-\mu\|_{M(G)}\to 0 \text{ and } \|\hat{\mu}_n-h\|_{(p,q)}\to 0\,.$$

It follows from Yap [8; Lemma 2.2] that there is a subsequence $\{\hat{\mu}_{n_i}\}$ of $\{\mu_n\}$ which converges pointwise almost everywhere to h. Thus

$$\|\hat{\mu}_n-\hat{\mu}\|_\infty\leq\|\mu_n-\mu\|_{M(G)}\to 0 \text{ implies } \hat{\mu}=h\,,$$

this shows that $M(p, q)(G)$ is complete with respect to the norm $\|\ \|_{M(p,q)}$. Further, for any μ_1, μ_2 in $M(p, q)(G)$, we have

$$\begin{aligned}\|\mu_1 * \mu_2\|_{M(p,q)} &= \max\left\{\|\mu_1 * \mu_2\|_{M(G)},\ \|\hat{\mu}_1 \cdot \hat{\mu}_2\|_{(p,q)}\right\} \\ &\leq \max\left\{\|\mu_1\|\ \|\mu_2\|,\ \|\hat{\mu}_1\|\ \|\hat{\mu}_2\|_{(p,q)}\right\} \\ &\leq \|\mu_1\|_{M(p,q)}\ \|\mu_2\|_{M(p,q)}\,.\end{aligned}$$

Hence $M(p, q)(G)$ is a commutative Banach algebra. Q.E.D.

It is immediately that $A(p, q)$ is an ideal of $M(p, q)$. The question arises that whether the space $A(p, q)$ is a proper ideal of $M(p, q)$ or not. Precisely we have the following

THEOREM 3.6. (i) *If* $1\leq q\leq p\leq 2$, *then* $M(p, q)=A(p, q)$ (ii) *If* $p>2$, $1\leq q\leq\infty$, *then* $M(p, q)\supsetneqq A(p, q)$.

PROOF. (i) For $1\leq q\leq p\leq 2$, $\mu\in M(p, q)$, we have $\mu\in L(p, q)\subset L(p, p)$. This implies $\mu\in M(p, p)$. It follows from Liu and Rooij [6; Lemma 2. 3] that μ is absolutely continuous, so that $\mu\in A(p, q)$.

For (ii), we consider $G=T$ and $\Gamma=Z$. Wiener and Wintner [7] proved that there is a nonnegative singular measure μ on T such that $\mu(n)=0(n^{-1/2+\varepsilon})$ for any $\varepsilon>0$. Thus for $p>2$, $1\leq q\leq\infty$, we choose a singular measure having the property of [7] showed. Then there are constants $C_1>C_2>0$ and n_0 such that

$$|\hat{\mu}(n)|\leq f(n)=\begin{cases}C_1 & ,\ |n|<n_0\\ C_2|n|^{-1/2+\varepsilon}, & |n|\geq n_0\end{cases}$$

and so

$$f^*(t)=\begin{cases}C_1 & ,\ t<2n_0\\ C_2\, m^{-1/2+\varepsilon}, & 2m\leq t<2m+2,\ m\geq n_0\end{cases}$$

$$\begin{aligned}\|f\|^*_{(p,q)} &= \left\{\frac{q}{p}\int_0^{2n_0} C_1^q x^{q/p-1}\,dx+\sum_{n=n_0}^{\infty}(C_2\,n^{-1/2+\varepsilon})^q\cdot\right.\\ &\qquad\left.\cdot\left((2n+2)^{q/p}-(2n)^{q/p}\right)\right\}^{1/q}\\ &= \left\{K_1+K_2\sum_{n=n_0}^{\infty} n^{-q/2+\varepsilon q}(n+1)^{q/p}-n^{q/p})\right\}^{1/q}\end{aligned}$$

where K_1, K_2 are constants. If $p>2$, $1\leq q<\infty$, we have $-\frac{q}{2}+\varepsilon q+\frac{q}{p}-1$ <-1 provided $\varepsilon\left(<\frac{p-2}{2pq}\right)$ is sufficient small, and since

$$n^{-q/2+\varepsilon q}\cdot\left((n+1)^{q/p}-n^{q/p}\right)=0\left(n^{-q/2+\varepsilon q+q/p-1}\right),$$

the series $\sum\limits_{n=p_0}$ converges. Hence $\|f\|^*_{(p,q)}<\infty$ and $\|\mu\|_{(p,q)}<\infty$ proves $\mu\in$

$M(p, q)$ but $\mu \notin A(p, q)$.

If $p>2$, $q=\infty$, we have $\|f\|^*_{(p,q)}=\sup_{t>0} t^{1/p} f^*(t)<\infty$. Hence $\mu \in M(p, \infty)$, $\mu \notin A(p, \infty)$. Q.E.D.

REMARK. We do not know what happen for the case $1<p<q\leq 2$.

The following lemma is useful and the proof follows immediately from the properties of Segal algebras.

LEMMA 3.7. *There is an approximate identity* $\{e_\alpha\}$ *of* $A(p, q)(G)$, $1<p<\infty$, $1\leq q<\infty$, *which is a bounded approximate identity of* $L^1(G)$ *such that* $\|e_\alpha\|_1\leq 1$ *and* $\hat{e}_\alpha$ *have compact support for all* α.

We have seen in Lemma 3.3 that there is an approximate identity of $L(p, q)(G)$ which is the bounded approximate identity of $L_1(G)$. One can choose the approximate identity like as Lemma 3.7 that the Fourier transforms have compact supports. For convenient, we state it as following

LEMMA 3.8. *There is a bounded approximate identity* $\{e_\alpha\}$ *of* $L^1(\Gamma)$ *such that* $\hat{e}_\alpha$ *has compact support in G and*

$$\|e_\alpha * f-f\|_{(p,q)}\to 0 \text{ for any } f\in L(p, q)(\Gamma),\ 1<p<\infty,\ 1\leq q<\infty .$$

LEMMA 3.9. (i) *The space* $A^1(G)$ *is contained in* $A(p, q)(G)$ *for* $1\leq p<\infty$, $1\leq q\leq\infty$.

(ii) *The Fourier transforms* $\widehat{A^1(G)}$ *and* $\widehat{A(p, q)(G)}$ *are dense in* $L(p, q)(\Gamma)$ *for* $1\leq p<\infty$, $1\leq q<\infty$.

PROOF. Suppose that $f=\hat{g}$, $g\in A^1 G$, then $f\in L^1(\Gamma)$ and $\int_0^\infty f^*\, dt=\int_\Gamma |f|\, d\eta<\infty$. If $p\geq q$, we have

$$\begin{aligned}\int_0^\infty x^{q/p-1} f^*(x)\, dx &\leq \|f\|\int_0^1 x^{q/p-1}\, dx+\int_1^\infty x^{q/p-1} f^*(x)^q\, dx\\ &\leq \frac{q}{p}\|f\|_\infty+\int_1^\infty f^*(x)^q\, dx\\ &<\infty,\end{aligned}$$

and so $f\in L(p, q)(\Gamma)$. If $p<q$, then $f\in L(p, p)\subset L(p, q)$. This shows that $g\ni A(p, q)$ and $\widehat{A^1(G)}\subset \widehat{A(p, q)(G)}\subset L(p, q)(\Gamma)$. It is sufficient to show that $\widehat{A^1(G)}$ is dense in $L(p, q)(\Gamma)$. Let $\{e_\alpha\}$ is $L^1(\Gamma)$ be an approximate identity of $L(p, q)(\Gamma)$ such that $\hat{e}_\alpha$ has compact support. Then for every $f\in L(p, q)(\Gamma)$ and any $\varepsilon>0$, there is α_0 such that

$$\|e_{\alpha_0} * f-f\|^{(p,q)}<\varepsilon .$$

Since the simple functions are dense in $L(p, q)(\Gamma)$, thus for the given $\varepsilon>0$, there exists a simple function g such that $\|f-g\|_{(p,q)}<\varepsilon$, and

$$\|e_{\alpha_0}*g-f\|_{(p,q)}\leq\|e_{\alpha_0}\|_1\,\|g-f\|_{(p,q)}+\|e_{\alpha_0}*f-f\|_{(p,q)}$$
$$<(C+1)\,\varepsilon$$

where $\|e_{\alpha_0}\|_1\leq C$. But $\widehat{e_{\alpha_0}*g}=\hat{e}_{\alpha_0}\,\hat{g}\in C_c(G)\subset L^1(G)$, we see that $e_{\alpha_0}*g\in A^1(G)$. This shows that $\widehat{A^1(G)}$ is dense in $L(p,q)(\Gamma)$. Q.E.D.

We use the symboles appear in Liu and Rooij [6]. Thus by Lemma 3.9, Theorem 2.4, and [6; Lemma 2.8], we obtain

LEMMA 3.10. *Let H be the closures of $\{(\tilde{f},-\hat{f})\,;\,f\in A^1(G)\}$ in $C_0(G)\times L(p',q')(\Gamma)$ and*

$$J=\left\{(\mu,h)\,;\ \mu\in M(G),\ h\in L(p,q)(\Gamma),\ \int_G\tilde{f}\ d\mu=\int_\Gamma\hat{f}\ h\ d\eta,\ \ f\in A^1(G)\right\}.$$

Then

$$\left\{C_0(G)\ V_H\,L(p',q')(\Gamma)\right\}'\cong M(G)\ \Lambda_J\,L(p,q)(\Gamma),\ 1<p<\infty,\ 1<q<\infty\,.$$

If $p=1$, then $q=1$, and the same result holds for $C_0(\Gamma)$ in place of $L(\infty,\infty)(\Gamma)=L^\infty(\Gamma)$.

LEMMA 3.11. *For $1<p<\infty$, $1<q<\infty$ or $p=1=q$, then*

$$M(G)\ \Lambda_J\,L(p,q)(\Gamma)\equiv M(p,q)(G)$$

where any element μ in $M(p,q)(G)$ is regarded as the pair $(\mu,\hat{\mu})$ in $M(G)\times L(p,q)(\Gamma)$ so that $M(p,q)$ is embedded in $M(G)\times L(p,q)(\Gamma)$.

PROOF. If $(\mu,h)\in J$, then it follows from $\widehat{A^1(G)}$ dense in $L(p',q')(\Gamma)$ and $\int_\Gamma\hat{f}\ \hat{\mu}\ d\eta=\int_\Gamma\hat{f}\ h\ d\eta$ for $f\in A^1(G)$ that $h=\hat{\mu}$. (In detail see Liu and Rooij [6].) Q.E.D.

It is clear that $M(p,q)(G)$ is an $L^1(G)$-module under convolution thus we have following theorem.

THEOREM 3.12. *For $1<p<\infty$, $1<q<\infty$ or $p=1=q$, the multiplier algebra $\mathfrak{M}(L^1(G),\ M(p,q)(G))$ is isometrically isomorphic to $M(p,q)(G)$. Moreover for every multiplier T of $L^1(G)$ into $M(p,q)(G)$ can be expressed as the form*

$$T(a)=a*\mu\quad \text{for some } \mu\in M(p,q)(G) \text{ and any } a\in L^1(G)\,.$$

PROOF. From Lemma 3.10 and Lemma 3.11, we see that

$$\left\{C_0(G)\ V_H\,L(p',q')(\Gamma)\right\}'\cong M(p,q)(G)\,.$$

Using this identity and [6; Lemma 2.8], the theorem follows immediately. To this end, we have only to show that $C_0(G)\ V_H\,L(p',q')(\Gamma)$ is $L^1(G)$-module under certain operation and there is an approximate identity of

$L^1(G)$ with the property as in [6; Lemma 2.8].

For any $\mu\in M(p,q)(G)$, there corresponds an element $U_\mu\in[C_0(G)\,V_H\,L(p,q)(\Gamma)]'$ defined by

$$U_\mu(f,g)=\int_G f\,d\mu+\int_\Gamma g\hat{\mu}\,d\eta,\quad (f,g)\in C_0(G)\,V^H\,L(p',q')(\Gamma).$$

We will define an operation $\otimes$ for which $C_0(G)\,V_H\,L(p',q')(\Gamma)$ is an $L^1(G)$-module. This operation $\otimes$ induces an operation over its dual space which is defined by

$$a\otimes U_\mu(f,g)=U_\mu(a\otimes(f,g)) \text{ and } a\otimes(f,g)=(\tilde{a}*f,\hat{a}g)$$

for any $(f,g)\in C_0(G)\,V_H\,L(p',q')(\Gamma)$ and $a\in L^1(G)$. At first we have to show $[C_0(G)\,V_H\,L(p',q')(\Gamma)]'$ and $M(p,q)(G)$ are the same space acted by $L^1(G)$ under the operations $\otimes$ and $*$ respectively. That is $a\otimes U_\mu=U_{a*\mu}$ holds for any $a\in L^1(G)$ and $\mu\in M(p,q)(G)$. In fact for $(f,g)\in C_0(G)\,V_H\,L(p',q')(\Gamma)$, we have

$$\begin{aligned}U_{a\,\mu}(f,g)&=\int_G f(x)\,d(a*\mu)(x)+\int_\Gamma \hat{a}\hat{\mu}g\,d\eta\\&=\int_G f(x)\,a*\mu(x)\,d\lambda(x)+\int_\Gamma \hat{a}\hat{\mu}g\,d\eta\ (\text{for } a*\mu\in L^1(G)),\end{aligned}$$

and

$$\begin{aligned}a\otimes U_\mu(f,g)&=U_\mu(a\otimes(f,g))=U_\mu(\tilde{a}*f,\hat{a}g)\\&=\int_G \tilde{a}*f(x)\,d\lambda(x)+\int_\Gamma \hat{a}\hat{\mu}g\,d\eta\\&=\int_G f(y)\,a*\mu(y)\,d\lambda(y)+\int_\Gamma \hat{a}\hat{\mu}g\,d\eta\\&=U_{a*\mu}(f,g).\end{aligned}$$

Now we show that $C_0(G)\,V_H\,L(p',q')(\Gamma)$ is $L^1(G)$-module under $\otimes$. In fact, if $(\tilde{f},-\hat{f})\in H'$, then it is clearly $a\otimes(\tilde{f},-\hat{f})=\tilde{a}*\tilde{f},-\hat{a}\hat{f})\in H$. Further for $(f,g)\in C_0(G)\times L(p',q')(\Gamma)$, we have

$$\begin{aligned}\|a\otimes(f,g)\|&=\|(\tilde{a}*f,\hat{a}g)\|\\&=\inf\left\{\|f'\|_\infty+\|g'\|_{(p',q')};\ (f',g')\cong(\tilde{a}*f,\hat{a}g)\bmod H\right\}\\&=\inf\left\{\|\tilde{a}*f+\tilde{h}\|_\infty+\|\hat{a}g-\hat{h}\|_{(p',q')};\ h\in A^1(G)\right\}\\&\le\inf\left\{\|\tilde{a}*f+\widetilde{a*k}\|+\|\hat{a}g-\widehat{a*k}\|_{(p',q')};\ k\in A^1(G)\right\}\end{aligned}$$

since $a*k\in L^1*A^1(G)\subset A^1(G)$,

$$\begin{aligned}&\le\inf\left\{\|a\|_1\,\|f+\tilde{k}\|_\infty+\|\hat{a}\|_\infty\,\|g-\hat{k}\|_{(p',q')};\ k\in A^1(G)\right\}\\&\le\|a\|_1\inf\left\{\|f'\|_\infty+\|g'\|_{(p',q')};\ (f',g')\cong(f,g)\bmod H\right\}\\&=\|a\|_1\,\|(f,g)\|.\end{aligned}$$

Hence

$$\|a \otimes (f, g)\| \leq \|a\|_1 \|(f, g)\| .$$

To complete the proof, it remains to show that there is a bounded approximate identity $\{e_\alpha\}$ of $L^1(G)$ with $\|e_\alpha\|_1 \leq 1$ such that

$$\|e_\alpha \otimes (f, g) - (f, g)\| \to 0 \text{ for every } (f, g) \in C_0(G)\, V_H\, L(p', q')(\Gamma) .$$

Let $\{e_\alpha\}$ be a bounded approximate identity of $L^1(G)$ with $\|e_\alpha\|_1 \leq 1$ and the Fourier transform $\hat{e}_\alpha$ has compact support such that

$$\|e_\alpha * f - f\|_{A(p', q')} \to 0 \text{ for every } f \in A(p', q')(G) \text{ (Lemma 3. 7.)} .$$

Since $\widehat{A(p', q')(G)}$ is dense in $L(p', q')(\Gamma)$, thus for any $g \in L(p', q')(\Gamma)$ and $\varepsilon > 0$ there is $h \in \widehat{A(p', q')(G)}$ such that $\|g - h\|_{(p', q')} < \varepsilon$, it follows that

$$\begin{aligned} \|\hat{e}_\alpha g - g\|_{(p', q')} &\leqq \|\hat{e}_\alpha g - \hat{e}_\alpha h\|_{(p', q')} + \|\hat{e}_\alpha h - h\|_{(p', q')} + \|h - g\|_{(p', q')} \\ &< 2\varepsilon + \|\hat{e}_\alpha h - h\|_{(p', q')} . \end{aligned}$$

Since for $\check{h} \in A(p', q')(G)$ with $\hat{\check{h}} = h$,

$$\|\hat{e}_\alpha h - h\|_{(p', q')} = \|\widehat{e_\alpha * \check{h}} - h\|_{(p', q')} \leq \|e_\alpha * \check{h} - \check{h}\|_{A(p', q')} \to 0 ,$$

and ε is arbitrary, we have

$$\|\hat{e}_\alpha g - g\|_{(p', q')} \to 0 .$$

Therefore for $(f, g) \in C_0(G)\, V_H\, L(p', q')(\Gamma)$, we have

$$\begin{aligned} \|e_\alpha \otimes (f, g) - (f, g)\| &= \|(\tilde{e}_\alpha * f - f, \hat{e}_\alpha g - g\| \\ &\leq \|\tilde{e}_\alpha * f - f\|_\infty + \|\hat{e}_\alpha g - g\|_{(p', q')} \\ &\to 0 , \end{aligned}$$

the limit being taken over all α. This proof is completed. Q.E.D.

THEOREM 3. 13. *For* $1 < p,\ q < \infty$ *or* $p = 1 = q$, *the identity*

$$\mathfrak{M}(L^1(G),\ A(p, q)(G)) \cong M(p, q)(G)$$

holds. Furthermore, if $T \in \mathfrak{M}(L^1(G), A(p, q)(G))$, *there is a unique* $\mu \in M(p, q)(G)$ *such that*

$$Ta = a * \mu \text{ for any } a \in L^1(G) .$$

PROOF. Evidently, for any $T \in \mathfrak{M}(L^1(G), A(p, q)(G))$, it follows that $T \in \mathfrak{M}(L^1(G), M(p, q)(G))$. Conversely for $T \in \mathfrak{M}(L^1(G), M(p, q)(G))$, it follows from Theorem 3. 12 that there is a unique $\mu \in M(p, q)(G)$ such that

$$Ta = a * \mu ,$$

holds for every $a\in L^1(G)$. Thus

$$\|\widehat{a*\mu}\|_{(p,q)}=\|\hat{a}\hat{\mu}\|_{(p,q)}\leq\|\hat{a}\|_{\infty}\ \|\hat{\mu}\|_{(p,q)}$$
$$\leq\|a\|_1\ \|\mu\|_{M(p,q)}<\infty\ ,$$

and since $a*\mu\in L^1(G)$, we see that $a*\mu\in A(p,q)(G)$. That is $T\in\mathfrak{M}(L^1(G), A(p,q)(G))$. Hence by Theorem 3.12 again, we obtain

$$\mathfrak{M}(L^1(G), A(p,q)(G))\cong\mathfrak{M}(L^1(G), M(p,q)(G))\cong M(p,q)(G).$$

Q.E.D.

Applying Theorem 3.6 and Theorem 3.13, we have

COROLLARY 3.14. (i) *If* $1<q\leq p<2$ or $p=1=q$, *then*

$$\mathfrak{M}(L^1(G), A(p,q)(G))\cong A(p,q)(G)\,.$$

(ii) *If* $2<p$, $1<q<\infty$, then

$$\mathfrak{M}(L^1(G), A(p,q)(G))\cong M(p,q)(G)\supsetneqq A(p,q)(G)\,.$$

REMARK 1. If $p=q$, $1\leq p<\infty$, then Theorem 3.6 and Theorem 3.12 (or Corollary 3.14) induce $\mathfrak{M}(L^1(G), A^p(G))\cong A^p(G)$ for $1\leq p\leq 2$ and $\mathfrak{M}(L^1(G), A^p(G))\cong M^p(G)\supsetneqq A^p(G)$ for $2<p<\infty$.

This answers the question risen in Lai [4]. The classical multiplier problem in $L^1(G)$ is a special case in our context. That is

$$\mathfrak{M}(L^1(G), L^1(G))\cong\mathfrak{M}(L^1(G), M(G))\cong M(G)\,.$$

REMARK 2. From Corollary 3.14 (ii), one sees that, in general, if $S(G)$ is a proper Segal algebra, the multiplier algebra $\mathfrak{M}(L^1(G), S(G))$ needs not be isometrically isomorphic to $S(G)$ itself.

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References

[1] A. P. BLOZINSKY: *On a convolution theorem for $L(p,q)$ spaces*, Trans. Amer. Math. Soc., 164 (1972), 2555-265.

A. P. BLOZINSKY: *Convolution of $L(p,q)$ functions*, Proc. Amer. Math. Soc. 32 (1972), 237-240.

[2] R. A. HUNT: *On $L(p,q)$ spaces*, Enseignement Math., 12 (1966), 249-276.

[3] H. C. LAI: *On some properties of $A^p(G)$-algebras*, Proc. Japan Acad., 23 (1969), 572-576.

[4] H. C. LAI: *On the multipliers of $A^p(G)$-algebras*, Tôhoku Math. J., 23 (1971), 641-662.

[5] R. LARSEN: "An introduction to the Theory of Multipliers", Springer-Verlag Berlin Heidelberg 1971.

[6] T. S. LIU and A. C. M. Van ROOIJ: *Sum and intersections of normed linear spaces*, Math. Nachr., 42 (1969), 29–42.

[7] N. WIENER and A. WINTNER: *Fourier-Stieltjes transforms and singular infinite convolution*, Amer. J. Math., 60 (1938), 513–522.

[8] L. Y. H. YAP: *On two classes of subalgebras in* $L_1(G)$, Proc. Japan Acad., 48 (1972), 315–319.

[9] L. Y. H. YAP: *Some remarks on convolution operators and* $L(p, q)$ *spaces*, Duke Math. J., 36 (1969), 647–658.

[10] L. Y. H. YAP: *On the impossibility of representing certain functions by convolutions*, Math. Scand., 26 (1970), 132–140.

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