On Maslov's canonical operator

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Introduction

For a global treatment of quasi-classical approximation of the Schrödinger equation, V. P. Maslov [16] introduced and discussed quite systematically the notions of Lagrangean manifolds and associated canonical operators on them. The underlying ideas are known since long and their applications to the study of partial differential equations are not quite new. (e.g., Lax [11], Lewis [13], Ludwig [14]). Even very close considerations to Maslov's are not so rare, as are found in works of Keller [9], [10], Ludwig [15] and others, and Hörmander's recent notion of Fourier integral operators [7] is in a sense one of them, though apparently different. Maslov's original exposition [16], however interesting and stimulating its content be, seems to be not necessarily well arranged and even to contain certain unclearness, thus letting the reader sometimes difficult to grasp its validity. As to this J. Leray [12] gave a review but without any remark on the connections of Maslov's canonical operator and Hörmander's Fourier integral operator. However, we believe that these two notions are deeply concerned and in a sense variants of the same thing, and thus are not quite satisfied with this situation. So we describe below what Maslov's canonical operators should be. Our exposition will be thus quite close to Hörmander's Fourier integral operator. In fact, when I had completed my first draft I was then informed about Duistermaat [3]. The interpretation of Maslov's canonical operator by him and me are essentially the same. However, I choose a different symbol class (cf. Definition 2.1.1) to define canonical operators, and I personally believe that this choice of symbol class is an essential simplification from Maslov's original and with this I can smoothly apply Hörmander's method. On the other hand, Duistermaat [3] starts from a smaller symbol class and thus his discussion runs in a sense in the reversed order with respect to mine. Any way, I publish here only the definitions and elementary properties of the so-called canonical operators and omit their calculi, since their applications are done just in the same way as Fourier integral operators, that is, one needs only to construct canonical relations as an analogy to homogeneous canonical relations, and then to establish their calculi. Here, however, the degree of product symbols is not obtained as an simple addition of degrees of factors. In this way one can construct an approximate parametrix for the quasi-classical approximation of the Schrödinger equation by an integration of the Hamilton field with given $\frac{1}{2}$ density data on the diagonal set in the cotangent bundle of the product of the initial manifold. The quantization condition is thus interpreted as the condition which ensures the solution density to fall in the symbol class under consideration.

I note furthermore that detailed studies about the canonical operators near the caustics are executed in Guillemin-Schaeffer [6]. I also express my thanks to D. Fujiwara and K. Saito who showed me a copy of Duistermaat [3].

Chapter 1. Lagrangean manifolds and their properties.

1.1. Phase functions and related local properties

Let X be a paracompact manifold of class C^∞, $n=\dim X<\infty$, TX and T^*X denote the tangent and cotangent bundle over X, respectively. We introduce the Hamilton structure in T^*X in the following way. First consider the following diagram:

(1.1.1)
$$\begin{array}{ccccc} & & T(T^*X) & & \\ & {\scriptstyle d\pi_2}\swarrow & & \searrow{\scriptstyle \pi_2} & \\ TX & & & & T^*X \\ & {\scriptstyle \pi_1}\searrow & & \swarrow{\scriptstyle \pi_2} & \\ & & X & & \end{array}$$

Here π_j, $j=1,2,3$, are the canonical projections. We define a 1-form θ_X over T^*X by

(1.1.2) $$\langle v, \theta_X\rangle=\langle(d\pi_2)(v),\ \pi_3(v)\rangle,\quad v\in T(T^*X).$$

Here the left hand side of (1.1.2) is the coupling of $T(T^*X)$ and $T^*(T^*X)$ and the right hand side that of TX and T^*X. We shall simply write θ instead of θ_X if there is no fear of confusion.

Now the Hamilton structure in T^*X is given by the 2-form $d\theta$. The 2-form $d\theta$ is of rank $2n$ and induces a complex structure in each fiber of $T(T^*X)$. If $(x_1,\cdots,x_n)$ is a local coordinate system in X such that $x_1,\cdots,x_n$, $\xi_1,\cdots,\xi_n$ are the coresponding local coordinates in T^*X by the coupling $\langle\xi, dx\rangle$, then $\theta=\sum_{j=1}^n \xi_j dx_j$ and $d\theta=\sum_{j=1}^n d\xi_j\wedge dx_j$ in this coordinate system (cf. Sternberg [17], Gucnenheimer [5]).

An n-dimensional closed submanifold Λ of T^*X is called Lagrangean if $d\theta$ vanishes on Λ. On the other hand, $T^*X\backslash 0$ is a cone bundle by the action of the multiplicative group $\boldsymbol{R}_+$ in each fiber (cf. Hörmander [7], p. 86). A closed submanifold Λ of dimension n in $T^*X\backslash 0$ is called conic Lagrangean if $d\theta$ vanishes on Λ and if Λ is invariant under the action of $\boldsymbol{R}_+$ in the fibers of $T^*X\backslash 0$. These two classes of Lagrangean manifolds will be seen to be closely related. Lagrangean manifolds are defined locally by phase functions in the following way. Let U be a coordinate neighborhood in X with local coordinates $x_1,\cdots,x_n$ in U and V be an open subset of $\boldsymbol{R}^N$ with the coordinates $\sigma_1,\cdots,\sigma_N$. A C^∞ function ϕ: $U\times V\to\boldsymbol{R}$ is called a non-degenerate phase function if

(1. 1. 3) $$d\phi'_{\sigma_1}\wedge\cdots\wedge d\phi'_{\sigma_N}\neq 0$$

on the set

(1. 1. 4) $$C_\phi=\left\{(x,\sigma)\in U\times V;\ \phi'_{\sigma_j}(x,\sigma)=0,\ \ j=1,\cdots,N\right\}.$$

In particular, if V is a cone in $\boldsymbol{R}^N\backslash 0$ and $\phi(x,\sigma)$ is positively homogeneous of degree 1 in V and furthermore if ϕ has no critical points in $U\times V$, then ϕ is called a non-degenerate conic phase function. Note that except when we consider conic phase functions we may not assume that a phase function does not have critical points. Let

(1. 1. 5) $$\Lambda_\phi=\left\{(x,\phi'_x(x,\sigma));\ (x,\sigma)\in C_\phi\right\}.$$

If $\lambda_0\in\Lambda_\phi$, $\lambda_0=(x_0,\phi'_x(x_0,\sigma_0))$, $(x_0,\sigma_0)\in C_\phi$, we call the value $\phi(x_0,\sigma_0)$ the *level* of ϕ at λ_0 and sometimes denote it by $[\phi](\lambda_0)$.

PROPOSITION 1. 1. 1. *The differential of the map*

1. 1. 6) $$C_\phi\ni(x,\sigma)\to(x,\phi'_x)\in\Lambda_\phi$$

is bijective and $d\theta=0$ *on* Λ_ϕ. *That is,* ϕ *defines a Lagrangean germ in* T^*X. *In particular, if* ϕ *is conic, then* Λ_ϕ *is conic.*

PROOF. These facts are well-known (cf. Hörmander [7], Guckenheimer [5]). In fact, from $dx=0$, $d\phi'_x=0$ it follows $d\sigma=0$ by (1. 1. 3). The differential of the map (1. 1. 6) is thus bijective. Next, $\sum_j d\phi'_{x_j}\wedge dx_j=dd\phi=0$ since $d\phi=\sum\phi'_{x_j}dx_j$ on C_ϕ. It is clear that Λ_ϕ is conic if ϕ is conic. We note that $\sum_j\phi'_{x_j}dx_j=0$ on C_ϕ if ϕ is conic, since $\phi=\sum\phi'_{\sigma_j}\sigma_j=0$ on C_ϕ and thus $d\phi=\sum_j\phi'_{x_j}dx_j=0$.

We now consider how to relate general Lagrangean germs and conic

Lagrangean germs. For that purpose we introduce a map

$$\mu:\quad T^*X\ni(x,\xi)\to(x,\tau\xi)\in T^*X\,,\quad \tau\in\boldsymbol{R}\,.$$

Let ϕ be a non-degenerate phase function in $U\times V\subset X\times\boldsymbol{R}^N$ and Λ_ϕ the set defined by (1.1.5). Let $\tilde{X}=X\times\boldsymbol{R}$ and set

$$\begin{aligned}\tilde{\mu}^*\Lambda_\phi&=\left\{(\mu_\tau\lambda,\ -[\phi](\lambda),\ \tau);\ \tau>0,\ \lambda\in\Lambda_\phi\right\}\\&=\left\{(x,\ -\phi(x,\sigma),\ \tau\phi'_x(x,\sigma),\ \tau);\ (x,\sigma)\in C_\phi,\ \tau>0\right\}.\end{aligned}$$

We denote by p and q the projections from $T^*\tilde{X}$ onto T^*X and onto the fibers of $T^*\boldsymbol{R}$, respectively, by the decomposition $T^*\tilde{X}=T^*X\times T^*\boldsymbol{R}$.

PROPOSITION 1.1.2.

(i) *$\tilde{\mu}^*\Lambda_\phi$ is a conic Lagrangean germ in $T^*\tilde{X}\backslash 0$ and its phase function is given by*

$$\tilde{\phi}(x,t,\tilde{\sigma},\tau)=\tau\phi(x,\tilde{\sigma}/\tau)+\tau t$$

for $(x,t,\tilde{\sigma},\tau)\in U\times I\times\Gamma_V$. Here $I\supset\mathrm{Range}\,\phi$ and

$$\Gamma_V=\left\{(\tilde{\sigma},\tau);\ \tau>0,\ \tilde{\sigma}/\tau\in V\right\}\subset\boldsymbol{R}^{N+1}\backslash 0\,.$$

(ii) *For each $\tau_0>0$, $\tilde{\mu}^*\Lambda_\phi$ and $q^{-1}(\tau_0)$ intersect transversally and*

$$\Lambda_\phi=p\left\{\tilde{\mu}^*\Lambda_\phi\cap q^{-1}(1)\right\}.$$

PROOF. It is clear that $\tilde{\mu}^*\Lambda_\phi$ is conic and that $\dim\tilde{\mu}^*\Lambda_\phi=n+1$. Furthermore, on C_ϕ

$$\tau dt+\sum_j\xi_j dx_j=-\tau d\phi+\tau\sum\phi'_{x_j}dx_j=0\,,$$

so $\tilde{\mu}^*\Lambda_\phi$ is a conic Lagrangean germ. It is also clear that $\tilde{\phi}$ is a conic phase function for $\tilde{\mu}^*\Lambda_\phi$ since $\tilde{\phi}$ has no critical points, is homogeneous of degree 1 in $(\tilde{\sigma},\tau)$ and since

$$\begin{aligned}&\tilde{\phi}'_{\tilde{\sigma}_j}(x,t,\tilde{\sigma},\tau)=\phi'_{\sigma_j}(x,\tilde{\sigma}/\tau)\,,\\&\tilde{\phi}'_\tau(x,t,\tilde{\sigma},\tau)=t+\phi(x,\tilde{\sigma}/\tau)-\tau^{-1}\sum_j\phi'_{\sigma_j}(x,\tilde{\sigma}/\tau)\tilde{\sigma}_j\,,\\&\tilde{\phi}'_{x_j}(x,t,\tilde{\sigma},\tau)=\phi'_{x_j}(x,\tilde{\sigma}/\tau)\,,\ \tilde{\phi}'_t=\tau\,.\end{aligned}$$

To prove (ii) we need only to show that $\tilde{\mu}^*\Lambda_\phi\cap q^{-1}(\tau_0)$ is of dimension $(2n+1)+(n+1)-(2n+2)=n$, but this is clear.

Let $U\times I$ be a coordinate neighborhood of $\tilde{X}=X\times\boldsymbol{R}$ and Γ a cone in $\boldsymbol{R}^N\backslash 0$. A non-degenerate conic phase function $\phi(x,t,\sigma)$ is called locally non-stationary if $\phi'_t>0$ on $U\times I\times\Gamma$. The phase function $\tilde{\phi}(x,t,\tilde{\sigma},\tau)$ in

Proposition 1.1.2 is thus locally non-stationary. We have the following converse to Proposition 1.1.2.

PROPOSITION 1.1.3. *Let $\phi(x, t, \sigma)$ be a conic non-degenerate phase function in $U\times I\times\Gamma\subset X\times R\times \boldsymbol{R}^N\backslash 0$. Assume that ϕ be locally non-stationary. Then for each $\tau_0>0$, Λ_ϕ and $q^{-1}(\tau_0)$ intersect transversally and $p(\Lambda_\phi\cap q^{-1}(\tau_0))$ is a Lagrangean germ in T^*X.*

PROOF. It is immediately seen that $\phi(x, t, \sigma)-\tau_0 t$ is a non-degenerate phase function for $p(\Lambda_\phi\cap q^{-1}(\tau_0))$ if and only if Λ_ϕ and $q^{-1}(\tau_0)$ intersect transversally. Thus what we must show is that on $\phi'_\sigma=0$, $\phi'_t=\tau_0$

$$d\phi'_{\sigma_1}\wedge\cdots\wedge d\phi'_{\sigma_N}\wedge d\phi'_t\neq 0 \tag{1.1.7}$$

or equivalently the rank of the matrix

$$\begin{pmatrix}\phi''_{\sigma x} & \phi''_{\sigma t} & \phi''_{\sigma\sigma}\\ \phi''_{tx} & \phi''_{tt} & \phi''_{t\sigma}\end{pmatrix} \tag{1.1.8}$$

is equal to $N+1$ when $\phi'_\sigma=0$, $\phi'_t=\tau_0$. First assume $\phi''_{\sigma\sigma}=0$ when $\phi'_\sigma=0$. Since ϕ is non-degenerate, the matrix $(\phi''_{\sigma x}, \phi''_{\sigma t})$ is then of rank N when $\phi'_\sigma=0$. On the other hand, $\phi''_{t\sigma_j}\neq 0$ for some j by the Euler's identity and $\phi'_t\neq 0$. Hence, in such a case the rank of the matrix (1.1.8) is $N+1$. Next we increase σ-variables by Hörmander's device. If A is a non-degenerate quadratic form in $\boldsymbol{R}^M$, then

$$\tilde{\phi}(x, t, \tilde{\sigma})=\phi(x, t, \sigma)+A(\rho, \rho)/|\sigma|,$$
$$\tilde{\sigma}=(\sigma, \rho)\in\Gamma\times\boldsymbol{R}^M$$

is a non-degenerate conic phase function and $\phi'_\sigma=0$, $\phi'_t=\tau$, $\rho=0$ if and only if $\tilde{\phi}'_{\tilde{\sigma}}=0$, $\tilde{\phi}'_t=\tau_0$. Furthermore, on $\tilde{\phi}'_{\tilde{\sigma}}=0$, $\tilde{\phi}_t=\tau_0$,

$$|d\tilde{\phi}'_{\sigma_1}\wedge\cdots\wedge d\tilde{\phi}'_{\sigma_N}\wedge d\tilde{\phi}'_{\rho_1}\wedge\cdots\wedge d\tilde{\phi}'_{\rho_M}\wedge d\tilde{\phi}'_t|$$
$$=|\det A||d\phi'_{\sigma_1}\wedge\cdots\wedge d\phi'_{\sigma_N}\wedge d\phi'_t|\neq 0$$

by the first step. Finally, let $\phi_1(x, t, \sigma)$ and $\phi_2(x, t, \tilde{\sigma})$ be equivalent conic phase functions and assume that ϕ_2 satisfy (1.1.7). By the equivalence, there is a fiber preserving diffeomorphism $(x, t, \sigma)\rightarrow(x, t, \tilde{\sigma}(x, t, \sigma))$ such that $\phi_1(x, t, \sigma)=\phi_2(x, t, \tilde{\sigma}(x, t, \sigma))$. In particular, $\phi'_{1\sigma}=0, \phi'_{1t}=\tau_0$ if and only if $\phi'_{2\tilde{\sigma}}=0$, $\phi'_{2t}=\tau_0$. Furthermore, on $\phi'_{1\sigma}=0, \phi'_{1t}=\tau_0$,

$$|d\phi'_{1\sigma_1}\wedge\cdots\wedge d\phi'_{1\sigma_N}\wedge d\phi'_{1t}|$$
$$=\left|\det\left(\frac{\partial\tilde{\sigma}}{\partial\sigma}\right)\right||d\phi'_{2\tilde{\sigma}_1}\wedge\cdots\wedge d\phi'_{2\tilde{\sigma}_N}\wedge d\phi'_{2t}|\neq 0$$

by the assumption. Hence, for any locally non-stationary phase function we have (1.1.7).

REMARK. For a conic phase function $\phi(x, t, \sigma)$ such that ϕ'_t may vanish, $\Lambda_\phi \cap q^{-1}(0) \neq \emptyset$ and even $\Lambda_\phi \subset q^{-1}(0)$ may occur.

COROLLARY 1.1.4. *Let $\phi(x, t, \sigma)$ be a locally non-stationary phase function and set $\tilde{\phi}(x, t, \sigma) = \phi(x, t, \sigma) - \tau_0 t$. Then*

$$C_{\tilde{\phi}} = \left\{(x, t, \sigma);\ \tilde{\phi}'_\sigma(x, t, \sigma) = 0\,,\ \tilde{\phi}'_t(x, t, \sigma) = 0\right\}$$

is of dimension n and

$$\Lambda_{\tilde{\phi}} = \left\{(x, \tilde{\phi}'_x(x, t, \sigma));\ (x, t, \sigma) \in C_{\tilde{\phi}}\right\}$$

is a Lagrangean germ in T^*X.

COROLLARY 1.1.5. *Under the hypotheses of Proposition* 1.1.3, *the differential of the map*

$$p:\ \Lambda_\phi \cap q^{-1}(\tau_0) \to p\left(\Lambda_\phi \cap q^{-1}(\tau_0)\right)$$

is injective.

We call a conic Lagrangean germ $\tilde{\Lambda}_0$ in $T^*\tilde{X}\backslash 0$ locally non-stationary if it is defined by a locally non-stationary conic phase function. Let $\tilde{\pi}$ and π be respectively the canonical projections from $T^*\tilde{X}$ onto $\tilde{X}$ and from T^*X onto X. We denote by $\tilde{\pi}^{\tilde{\Lambda}_0}$ the restriction of $\tilde{\pi}$ on $\tilde{\Lambda}_0$. π^{Λ_0} is defined similarly if Λ_0 is a Larangean germ in T^*X.

PROPOSITION 1.1.6. *Let $\tilde{\Lambda}_0$ be a locally non-stationary Lagrangean germ in $T^*X\backslash 0$ at $\tilde{\lambda}_0 \in q^{-1}(1)$, $\tilde{\lambda}_0 \in \tilde{\Lambda}_0 \cap q^{-1}(1)$ so that $\Lambda_0 = p(\tilde{\Lambda}_0 \cap q^{-1}(1))$ is a Lagrangean germ in T^*X at $\lambda_0 = p(\tilde{\lambda}_0)$. Then we have*

$$\operatorname{rank}(d\tilde{\pi}^{\tilde{\Lambda}_0})_{\tilde{\lambda}_0} = \operatorname{rank}(d\,\pi^{\Lambda_0})_{\lambda_0}.$$

PROOF. Let $\phi(x, t, \sigma)$ be a non-stationary phase function defining $\tilde{\Lambda}_0$ and $\tilde{\lambda}_0 = (x_0, t_0, \phi'_x(x_0, t_0, ,\sigma_0), 1)$, $\phi'_\sigma(x_0, t_0, \sigma_0) = 0$, $\phi'_t(x_0, t_0, \sigma_0) = 1$. Assume that the matrix $(\phi''_{\sigma\sigma}(x_0, t_0, \sigma_0)) = 0$. Then by Hörmander [7, Th. 3.1.4],

$$N = n + 1 - \operatorname{rank}(d\tilde{\pi}^{\tilde{\Lambda}_0})_{\tilde{\lambda}_0}.$$

On the other hand, $\operatorname{rank}(d\pi^{\Lambda_0})_{\lambda_0} = n - \dim\ker(d\pi^{\Lambda_0})_{\lambda_0}$. Set $\psi(x, t, \sigma) = \phi(x, t, \sigma) - t$. Then by the (local) diffeomorphism

$$C_\psi \ni (x, t, \sigma) \to (x, \psi'_x) \in \Lambda_\psi = \Lambda_0\,,$$

we have

$$\begin{aligned}
&\dim \mathit{ker}(d\pi^{\Lambda_0})_{\lambda_0}\\
&\quad = \dim\left\{d\tilde{\sigma};\ \psi''_{\tilde{\sigma}\tilde{\sigma}}(x_0, t_0, \sigma_0)d\tilde{\sigma} = 0\,,\ \tilde{\sigma} = (\sigma, t)\right\}\\
&\quad = N + 1 - \operatorname{rank}(\psi''_{\tilde{\sigma}\tilde{\sigma}}(x_0, t_0, \sigma_0))\,.
\end{aligned}$$

We claim rank $(\phi''_{\tilde{\sigma}\tilde{\sigma}}(x_0, t_0, \sigma_0))=2$. In fact, this holds good since there is a j such that $\phi'_{t\sigma_j}(x_0, t_0, \sigma_0)\neq 0$ and since

$$\left(\phi''_{\tilde{\sigma}\tilde{\sigma}}(x_0, t_0, \sigma_0)\right)=\begin{pmatrix} \phi''_{tt}(x_0, t_0, \sigma_0) & \phi''_{t\sigma_1}(x_0, t_0, \sigma_0), \cdots \phi''_{t\sigma_N}(x_0, t_0, \sigma_0) \\ \phi''_{\sigma_1 t}(x_0, t_0, \sigma_0) & \\ \vdots & 0 \\ \phi''_{\sigma_N t}(x_0, t_0, \sigma_0) & \end{pmatrix}$$

Hence,

$$\operatorname{rank}(d\tilde{\pi}^{\tilde{\Lambda}_0})_{\tilde{\lambda}_0} = n-N+1 = \operatorname{rank}(d\pi^{\Lambda_0})_{\lambda_0}.$$

COROLLARY 1.1.7. *Let Λ_0 be a Lagrangean germ at λ_0 in T^*X. If $\phi(x, \sigma)$ is a phase function of Λ_0 at λ_0, $\lambda_0=(x_0, \phi'_x(x_0, \sigma_0))$, $\phi'_\sigma(x_0, \sigma_0)=0$, $(x_0, \sigma_0)\in U\times V\subset X\times \boldsymbol{R}^N$, then*

$$n-\operatorname{rank}(d\pi^{\Lambda_0})_{\lambda_0} = N-\operatorname{rank}\left(\phi''_{\tilde{\sigma}\tilde{\sigma}}(x_0, \sigma_0)\right).$$

PROOF. Let $\tilde{\phi}(x, t, \sigma, \tau)=\tau\phi(x, \sigma/\tau)+\tau t$, $(x, t, \sigma, \tau)\in U\times \Gamma_V$. Then $\tilde{\mu}^*\Lambda_\phi=\Lambda_{\tilde{\phi}}=\tilde{\Lambda}_0$ and there is a $\tilde{\lambda}_0\in\tilde{\Lambda}_0\cap q^{-1}(1)$ such that $p(\tilde{\lambda}_0)=\lambda_0$. More explicitly, $\tilde{\lambda}_0=(x_0, t_0, \phi'_x(x_0, \sigma_0), 1)$, $t_0=-\phi(x_0, \sigma_0)$. Then rank $(d\tilde{\pi}^{\tilde{\Lambda}_0})_{\tilde{\lambda}_0}=\operatorname{rank}(d\pi^{\Lambda_0})_{\lambda_0}$. On the other hand, by Hörmander [7,Th. 3.1.4]

$$n+1-\operatorname{rank}(d\tilde{\pi}^{\tilde{\Lambda}_0})_{\tilde{\lambda}_0} = N+1-\operatorname{rank}\left(\tilde{\phi}_{\rho\rho}(x_0, t_0, \sigma_0, 1)\right),\ \rho=(\sigma, \tau).$$

However, since

$$(\tilde{\phi}_{\rho\rho}) = \frac{1}{\tau}\begin{pmatrix} \phi''_{\sigma\sigma} & -\sigma''_{\sigma\sigma}\sigma \\ -{}^t\sigma\phi''_{\sigma\sigma} & {}^t\sigma\phi''_{\sigma\sigma}\sigma \end{pmatrix}, \tag{1.1.9}$$

$$\operatorname{rank}\left(\tilde{\phi}_{\rho\rho}(x_0, t_0, \sigma_0)\right) = \operatorname{rank}\left(\phi''_{\sigma\sigma}(x_0, \sigma_0)\right).$$

Hence,

$$n-\operatorname{rank}(d\pi^{\Lambda_0})_{\lambda_0} = N-\operatorname{rank}\left(\phi''_{\sigma\sigma}(x_\Lambda, \sigma_0)\right).$$

REMARK. Of course, a direct proof of Corollary 1.1.7 is possible and is done just in the same way as Hörmander [7, Th. 3.1.4].

We have used equivalence of conic phase functions in the proof of Proposition 1.1.3. We now discuss the equivalence of general phase functions. Let $\phi(x, \sigma)$ and $\tilde{\phi}(\tilde{x}, \tilde{\sigma})$ be non-degenerate phase functions in $U\times V\subset X\times \boldsymbol{R}^N$ and in $U\times\tilde{V}\subset X\times\boldsymbol{R}^{\tilde{N}}$, respectively. We say that $\phi(x, \sigma)$ and $\tilde{\phi}(x, \tilde{\sigma})$ are equivalent if there is a diffeomorphism $U\times V$ onto $U\times\tilde{V}$:

$$U\times V\ni(x, \sigma)\rightarrow\left(x, \tilde{\sigma}(x, \sigma)\right)\in U\times\tilde{V}$$

such that $\phi(x, \sigma)=\tilde{\phi}(x, \sigma))$ on $U\times V$.

PROPOSITION 1.1.8. *Let ϕ and $\tilde{\phi}$ be non-degenerate phase functions in $U\times V\subset X\times \boldsymbol{R}^N$ and in $U\times \tilde{V}\subset X\times \boldsymbol{R}^{\tilde{N}}$, respectively. Let $(x_0, \sigma_0)\in C_\phi$ and $(x_0, \tilde{\sigma}_0)\in C_{\tilde{\phi}}$ such that*

$$\lambda_0=(x_0, \xi_0)=\big(x_0, \phi'_x(x_0, \sigma_0)\big)=\big(x_0, \tilde{\phi}'_x(x_0, \tilde{\sigma}_0)\big)\in \Lambda_\phi\cap\Lambda_{\tilde{\phi}}\,.$$

Then ϕ and $\tilde{\phi}$ are equivalent near (x_0, σ_0) and $(x_0, \tilde{\sigma}_0)$ if and only if the following three conditions are satisfied.

(i) *The Lagrangean germs Λ_ϕ and $\Lambda_{\tilde{\phi}}$ at λ_0 are the same as well as the levels of ϕ and $\tilde{\phi}$ near λ_0 coincide.*

(ii) $N=\tilde{N}$.

(iii) $\operatorname{sgn}\big(\phi''_{\sigma\sigma}(x_0, \sigma_0)\big)=\operatorname{sgn}\big(\tilde{\phi}''_{\tilde{\sigma}\tilde{\sigma}}(x_0, \tilde{\sigma}_0)\big)$.

PROOF. The only if part is clear. We prove the sufficiency of the conditions (i)~(iii) by reducing the problem to conic phase functions. Take I, Γ_V, $\Gamma_{\tilde{V}}$ as in Proposition 1.1.2 and set

$$\begin{aligned}\phi_1(x, t, \sigma, \tau)&=\tau\phi(x, \sigma/\tau)+\tau t\,,\\ &\qquad (x, t, \sigma, \tau)\in U\times I\times \Gamma_V\\ \tilde{\phi}_1(x, t, \tilde{\sigma}, \tilde{\tau})&=\tilde{\tau}\tilde{\phi}(x, \tilde{\sigma}/\tilde{\tau})+\tilde{\tau}t\,,\\ &\qquad (x, t, \tilde{\sigma}, \tilde{\tau})\in U\times I\times \Gamma_{\tilde{V}}\,.\end{aligned}$$

By our assumptions, conic Lagrangean germs Λ_{ϕ_1} and $\Lambda_{\tilde{\phi}_1}$ at $(x_0, t_0, \tau\xi_0, \tau)$, $t_0=-\phi(x_0, \sigma_0)=-\tilde{\phi}(x_0, \tilde{\sigma}_0)$, coincide and $N+1=\tilde{N}+1$. We then verify that

$$\operatorname{sgn}\big(\phi''_{1\rho\rho}(x_0, t_0, \sigma_0, 1)\big)=\operatorname{sgn}\big(\tilde{\phi}''_{1\tilde{\rho}\tilde{\rho}}(x_0, t_0, \tilde{\sigma}_0, 1)\big), \tag{1.1.10}$$

$\rho=(\sigma, \tau)$, $\tilde{\rho}=(\tilde{\sigma}, \tilde{\tau})$. By (1.1.9), we have for the left hand side,

$$\begin{aligned}\operatorname{sgn}\big(\phi'_{1\rho\rho}(x_0, t_0, \sigma_0, 1)\big)&=\operatorname{sgn}\begin{pmatrix}\phi''_{\sigma\sigma}(x_0, \sigma_0) & 0\\ 0 & 0\end{pmatrix}\\ &=\operatorname{sgn}\big(\phi''_{\sigma\sigma}(x_0, \sigma_0)\big)\end{aligned}$$

and similarly for the right hand side of (1.1.10). Hence, (1.1.10) is true. Thus by Hörmander [7, Th. 3.1.6], there is a homogeneous diffeomorphims near $(x_0, t_0, \sigma_0, 1)$

$$\begin{aligned}U\times I\times\Gamma_V\ni(x, t, \sigma, \tau)&\to\\ &(x, t, \tilde{\sigma}, \tilde{\tau})\in U\times I\times \Gamma_{\tilde{V}}\,,\\ \tilde{\sigma}=\tilde{\sigma}(x, t, \sigma, \tau)\,, &\quad \tilde{\tau}=\tilde{\tau}(x, t, \sigma, \tau)\end{aligned} \tag{1.1.11}$$

mapping $(x_0, t_0, \sigma_0, 1)$ to $(x_0, t_0, \tilde{\sigma}_0, 1)$ and such that

$$\tilde{\tau}(x, t, \sigma, \tau)\tilde{\phi}\Big(x, t, \sigma, \tau)/\tilde{\tau}(x, t, \sigma, \tau)\Big) + \tilde{\tau}(x, t, \sigma, \tau)t$$
$$= \tau\phi(x, \sigma/\tau) + \tau t\,.$$

By this last equation, we have, in a neighborhood of (x_0, σ_0),

$$\tilde{\phi}\Big(x, \tilde{\sigma}(x, -\phi(x, \sigma), \sigma, 1)/\tilde{\tau}(x, -\phi(x, \sigma), \sigma, 1)\Big) = \phi(x, \sigma)$$

if we take $t = -\phi(x, \sigma), \tau = 1$ and note that $\tilde{\tau}(x, -\phi(x, \sigma), \sigma, 1) \neq 0$ near (x_0, σ_0) since $\tilde{\tau}(x_0, t_0, \sigma_0, 1) = 1$. Therefore, what we have to verify is that the mapping

(1. 1. 12) $$U \times V \ni (x, \sigma) \to$$
$$\Big(x, \tilde{\sigma}(x, -\phi(x, \sigma), \sigma, 1)/\tilde{\tau}(x, -\phi(x, \sigma), \sigma, 1)\Big) \in U \times \tilde{V}$$

is diffeomorphic near (x_0, σ_0). Since $\phi'_\sigma(x_0, \sigma_0) = 0$, we have

$$\left.\frac{D(\tilde{\sigma}(x, -\phi(x, \sigma), \sigma, 1)/\tilde{\tau}(x, -\phi(x, \sigma), \sigma, 1))}{D(\sigma)}\right|_{(x_0, \sigma_0)}$$
$$= \det\left\{\tilde{\tau}^{-2}(\tilde{\tau}\tilde{\sigma}'_\sigma - \tilde{\sigma}\tilde{\tau}'_\sigma)\right\}\Big|_{(x_0, -\phi(x_0, \sigma_0), 1))}.$$

The last determinant does not vanish and thus the mapping (1. 1. 12) is diffeomorphic near (x_0, σ_0). In fact, this follows from

$$\left.\frac{D(\tilde{\sigma}(x, t, \sigma, \tau),\ \tilde{\tau}(x, t, \sigma, \tau))}{D(\sigma, \tau)}\right|_{(x_0, t_0, \sigma_0, 1)} \neq 0$$

and

$$\begin{pmatrix} \tilde{\tau}E & -\tilde{\sigma} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{\sigma}'_\sigma & \tilde{\sigma}'_\tau \\ \tilde{\tau}'_\sigma & \tilde{\tau}'_\sigma \end{pmatrix}\begin{pmatrix} E & \sigma \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} \tilde{\tau}\tilde{\sigma}'_\sigma - \tilde{\sigma}\tilde{\tau}'_\sigma & 0 \\ \tilde{\tau}'_\sigma & \tilde{\tau} \end{pmatrix}.$$

Here E is the (n, n) unit matrix and we have used Euler's identities

$$\tilde{\sigma}'_\sigma\sigma + \tilde{\sigma}'_\tau\tau = \tilde{\sigma}\,, \qquad \tilde{\tau}'_\sigma\sigma + \tilde{\tau}'_\tau\tau = \tilde{\tau}\,,$$

which are consequences of the homogeneity assumptions of the mapping (1. 1. 11).

REMARK. Following the discussions of Hörmander [7] we can give a direct proof. Note that this proposition contains the Morse lemma (cf. our discussions below).

COROLLARY 1. 1. 9. *Let $\phi(x, \sigma)$ and $\tilde{\phi}(x, \tilde{\sigma})$ be non-degenerate phase functions such that $\phi''_{\sigma\sigma} = 0$ on C_ϕ and $\tilde{\phi}''_{\tilde{\sigma}\tilde{\sigma}} = 0$ on $C_{\tilde{\phi}}$. Then ϕ and $\tilde{\phi}$ are equivalent at (x_0, σ_0) and $(x_0, \tilde{\sigma}_0)$ if and only if the corresponding Lagrangean germs are the same and the levels there of ϕ and $\tilde{\phi}$ coincide.*

PROOF. By Corollary 1.1.7 we have $N=\tilde{N}$ and thus we can apply Proposition 1.1.8.

We can increase and decrease σ-variables of phase functions just in the same way as the case of conic phase functions. Let $\phi(x, \sigma)$ be a non-degenerate phase function in $U\times V$. Let W be an open set in $\boldsymbol{R}^M$ containing the origin. If A is a symmetric matrix in $\boldsymbol{GL}(M, \boldsymbol{R})$, then

$$\tilde{\phi}(x, \sigma, \rho)=\phi(x, \sigma)+\langle A\rho, \rho\rangle\,, \quad (\sigma, \rho)\in V\times W$$

is a non-degenerate phase function in $U\times V\times W$. Furthermore, $\Lambda_\phi=\Lambda_{\tilde{\phi}}$ and the levels of ϕ and $\tilde{\phi}$ are the same. To decrease the σ-variables of $\phi(x, \sigma)$, we assume that $V=V'\times V''$, $\sigma=(\sigma', \sigma'')$ and the matrix $(\phi''_{\sigma''\sigma''})$ is non-singular when $\phi'_\sigma=0$. Then we can solve $\sigma''=\psi(x, \sigma')$ from $\phi'_{\sigma''}=0$. If we set $\phi_1(x, \sigma')=\phi(x, \sigma', \psi(x, \sigma'))$, it is immediately seen that ϕ_1 is a non-degenerate phase function in $U\times V'$, $\Lambda_{\phi_1}=\Lambda_\phi$ and that the levels of ϕ and ϕ_1 are the same. As a related matter to this observation, we show the existence of the so-called *focal* coordinate system of a Lagrangean germ.

PROPOSITION 1.1.10. *Let Λ_0 be a Lagrangean germ at λ_0 in T^*X. Assume that* $\operatorname{rank}(d\pi^{\Lambda_0})_{\lambda_0}=m$. *Then we can choose local coordinates $x_1, \cdots, x_n$ in X at $\pi(\lambda_0)$ such that $(x_1, \cdots, x_m, \xi_{m+1}, \cdots, \xi_n)$ gives a local coordinate system in Λ_0 at λ_0 if $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$ are the local coordinates in T^*X induced by $x_1, \cdots, x_n$ by the coupling $\langle \xi, dx\rangle$.*

PROOF. We can choose a local coordinate system $(x_1, \cdots, x_n)$ in X at $\pi(\lambda_0)$ such that

$$\Lambda_0 \ni (x, \xi) \to \xi \in T_x(X)$$

is regular at λ_0 (cf. Hörmander [7, p. 136]). Thus a Lagrangean neighborhood of λ_0 in Λ_0 is given by a phase function

$$\phi(x, \xi)=\langle x, \xi\rangle-H(\xi)\,, \qquad \xi\in V\subset \boldsymbol{R}^n$$

(cf. Hörmander [7, Ramark 2 after Th. 3.1.3]). Then $\lambda_0=(0, \xi_0), H'_\xi(\xi_0)=0$. By the assumptions on $\operatorname{rank}(d\pi^{\Lambda_0})_{\lambda_0}$, we have $\operatorname{rank}(H''_{\xi\xi}(\xi_0))=m$. Thus we may assume that the matrix $(H''_{\xi'\xi'}(\xi_0))$ is non-singular if $\xi=(\xi', \xi'')\in V'\times V''$, $\xi'=(\xi_1, \cdots, \xi_m)$, $\xi''=(\xi_{m+1}, \cdots, \xi_n)$. Solving $\xi'=\psi(x', \xi'')$ from $x'-H'_{\xi'}(\xi', \xi'')=0$, $x'=(x_1, \cdots, x_m)\in U'$, we see that

$$\phi_1(x, \xi'')=\langle x', \psi(x', \xi'')\rangle+\langle x'', \xi''\rangle-H\big(\psi(x', \xi''), \xi''\big)\,,$$

$x''=(x_{m=1}, \cdots, x_n)$, is a non-degenerate phase function defining Λ_0 near λ_0. More explicitly, a neighborhood of λ_0 in Λ_0 is given by

$$\Big\{(x', H'_{\xi''}(\psi(x', \xi''), \xi''), \ \psi(x', \xi''), \xi'')\,;\ x'\in U', \xi''\in V''\Big\}.$$

That is, (x', ξ'') is a local coordinate system in Λ_0 at λ_0.

COROLLARY 1.1.11. *Let Λ_0 be a Lagrangean germ at λ_0 in T^*X. Then we can choose local coordinates $x_1, \cdots, x_n$ in X at $\pi(\lambda_0)$ such that Λ_0 is defined near λ_0 by a phase function of the form*

$$\phi(x, \sigma) = \langle x, \sigma\rangle - H(\sigma), \quad \sigma \in V \subset \boldsymbol{R}^n .$$

REMARK. We use Corollary 1.1.11 rather than focal coordinate systems in Lagrangean germs contrary to Maslov [16], or Leray [12]. It should be noted that out discussions are not valid generally when Λ_0 intersects with the zero section of T^*X. For, then, we cannot generally change local coordinates in X at π (λ_0). *Therefore, in the sequel, we assume all the Lagrangean germs never intersect with the zero section of T^*X.* Under these assumptions we see that given a non-degenerate phase function $\tilde{\phi}$ we can construct a phase function ϕ for $\Lambda_{\tilde{\phi}}$ with the same level as $\tilde{\phi}$ and of the form given in Corollary 1.1.11. In fact, if we consider a conic Lagrangean germ $\tilde{\mu}^*\Lambda_{\tilde{\phi}}$ and if $(x_0, t_0, \xi_0, \tau_0) \in \tilde{\mu}^*\Lambda_{\tilde{\phi}}$, then, since $\tilde{\mu}^*\Lambda_{\tilde{\phi}}$ is locally non-stationary so that $\tilde{\mu}^*\Lambda_{\tilde{\phi}}$ and $q^{-1}(\tau_1)$ intersect transversally for any $\tau_1 > 0$, we can choose local coordinates $x_1, \cdots, x_n$ in X at x_0 such that the mapping $\tilde{\mu}^*\Lambda_{\tilde{\phi}} \ni (x, t, \xi, \tau) \rightarrow (\xi, \tau)$ is regular at $(x_0, t_0, \xi_0, \tau_0)$. Here we have used the assumptons $\Lambda_{\tilde{\phi}} \subset T^*X \backslash 0$. Note that $(x_1, \cdots, x_n, t-t_0)$ is the local coordinate system in consideration. This means that $\tilde{\mu}^*\Lambda_{\tilde{\phi}}$ is given by a phase function of the form $\langle x, \xi\rangle + (t-t_0)\tau - H(\xi, \tau)$ near $(x_0, t_0, \xi_0, \tau_0)$. Then $\phi(x, \xi) = \langle x, \xi\rangle - H(\xi, 1) - t_0$ has the required property.

1.2. Some global considerations.

Let Λ be a Lagrangean submanifold of T^*X. Every point $\lambda \in \Lambda$ has a neighborhood in Λ defined by a non-degenerate phase function. We consider a class $\Phi(\Lambda)$ of non-degenerate phase functions defining germs of the Lagrangean manifold Λ. Let U be a coordinate neighborhood in X, V an open subset in $\boldsymbol{R}^N$ and $\phi: U \times V \rightarrow \boldsymbol{R}$ a non-degenerate phase function such that $\Lambda_\phi \subset \Lambda$. We write $U_\phi = U$, $V_\phi = V$, $N_\phi = N$ so that by $\phi \in \Phi(\Lambda)$ we can at the same time understand its defining quantities U_ϕ, V_ϕ, N_ϕ.

Now we require the class $\Phi(\Lambda)$ of phase functions to satisfy the following three conditions.

(i) *Let $\phi \in \Phi(\Lambda)$ and ψ be any non-degenerate phase function such that $\Lambda_\psi \subset \Lambda$ and $\Lambda_\phi \cap \Lambda_\psi \neq \emptyset$. If the levels of ϕ and ψ coincide on $\Lambda_\phi \cap \Lambda$, then $\psi \in \Phi(\Lambda)$.*

(ii) *Let ϕ, $\psi \in \Phi(\Lambda)$. Then there is a sequence of phase function $\phi_0, \phi_1, \cdots,$*

$\phi_l \in \Phi(\Lambda)$ *with* $\phi_0=\phi$, $\phi_l=\psi$ *such that for* $j=0, \cdots, l-1$, $\Lambda_{\phi_j} \cap \Lambda_{\phi_{j+1}} \neq \emptyset$ *and the levels of* ϕ_j *and* ϕ_{j+1} *coincide on* $\Lambda_{\phi_j} \cap \Lambda_{\phi_{j+1}}$.

(iii) $\Phi(\Lambda)$ *contains a subfamily consisting of phase functions* ϕ *for which* Λ_ϕ *form a locally finite covering of* Λ.

We note furthermore that we may assume Λ_ϕ connected if $\phi \in \Phi(\Lambda)$. We can form a subfamily in (iii) by phase functions of the form $\langle x, \xi\rangle - H(\xi)$ from Corollary 1.1.11. From (i) and (ii) it follows that if $\phi \in \Phi(\Lambda)$ and ψ a non-degenerate phase function defining a germ of Λ, then $\psi \in \Phi(\Lambda)$ if and only if there is a sequence $\phi_0=\phi, \phi_1, \cdots, \phi_l \in \Phi(\Lambda)$ such that $\Lambda_{\phi_j} \cap \Lambda_{\phi_{j+1}} \neq \emptyset$, the levels of ϕ_j and ϕ_{j+1} coincide there, $j=0, \cdots, l-1$, and that $\Lambda_{\phi_l} \cap \Lambda_\psi \neq \emptyset$ and the levels of ϕ_l and ψ coincide there. In particular, if Λ is conic, then conditions on levels are automatically satisfied if we take conic phase functions.

The difference of levels of phase functions has the following meaning.

PROPOSITION 1.2.1. *Let* ϕ *and* $\tilde{\phi} \in \Phi(\Lambda)$. *If* $(x_0, \xi_0) \in \Lambda_\phi \cap \Lambda_{\tilde{\phi}}$, $\xi_0=\phi'_x(x_0, \sigma_0) = \tilde{\phi}'_x(x_0, \tilde{\sigma}_0)$, $\phi'_\sigma(x_0, \sigma_0)=0$, $\tilde{\phi}'_\sigma(x_0, \tilde{\sigma}_0)=0$, *then there is a closed path in* Λ *passing* (x_0, ξ_0) *such that*

$$\phi(x_0, \sigma_0)-\tilde{\phi}(x_0, \tilde{\sigma}_0) = \int_\gamma \theta .$$

PROOF. This follows immediately from the following lemma.

LEMMA 1.2.2. *Let* $\phi_1, \phi_2 \in \Phi(\Lambda)$ *such that* $\Lambda_{\phi_1} \cap \Lambda_{\phi_2} \neq \emptyset$ *and the levels of* ϕ_1 *and* ϕ_2 *coincide on* $\Lambda_{\phi_1} \cap \Lambda_{\phi_2}$. *Let* $(x_1, \xi_1) \in \Lambda_{\phi_1}$, $(x_2, \xi_2) \in \Lambda_{\phi_2}$, $\xi_1=\phi'_{1x}(x_1, \sigma_1)$, $\phi'_{1\sigma}(x_1, \sigma_1)=0$, $\xi_2=\phi'_{2x}(x_2, \tilde{\sigma}_2)$, $\phi'_{2\tilde{\sigma}}(x_2, \tilde{\sigma}_2)=0$. *Then there is a path* γ_{12} *in* $\Lambda_{\phi_1} \cup \Lambda_{\phi_2}$ *connecting* (x_1, ξ_1) *and* (x_2, ξ_2) *such that*

$$\phi_2(x_2, \tilde{\sigma}_2)-\phi_1(x_1, \sigma_1) = \int_{\gamma_{12}} \theta .$$

PROOF. Let $(x_3, \xi_3) \in \Lambda_{\phi_1} \cap \Lambda_{\phi_2}$. Then $\xi_3=\phi'_{1x}(x_3, \sigma_3)=\phi'_{2x}(x_3, \tilde{\sigma}_3)$, $\phi'_{1\sigma}(x_3, \sigma_3) =0$, $\tilde{\phi}'_{2\tilde{\sigma}}(x_3, \tilde{\sigma}_3)=0$, $\phi_1(x_3, \tilde{\sigma}_3)=\phi_2(x_3, \tilde{\sigma}_3)$. Since we consider the differences of levels, we may choose ϕ_1 and ϕ_2 as given by Corollary 1.1.11. That is, we take

$$\phi_1(x, \sigma) = \langle x, \sigma\rangle - H_1(\sigma), \quad x \in U_1, \quad \sigma \in V_1 \subset \boldsymbol{R}^n,$$
$$\phi_2(y, \tilde{\sigma}) = \langle y, \tilde{\sigma}\rangle - H_2(\tilde{\sigma}), \quad y \in U_2, \quad \tilde{\sigma} \in V_2 \subset \boldsymbol{R}^n .$$

We may further assume that V_1 and V_2 be simply connected. Let $\sigma=\sigma(s)$ be a C^1-curve in V_1 connecting σ_1, and σ_3, $\sigma(0)=\sigma_1$, $\sigma(1)=\sigma_3$. Let $x(s)= H'_\sigma(\sigma(s))$. Then γ_1: $s \rightarrow (x(s), \sigma(s))$ is a C^1-curve in Λ_{ϕ_1} connecting (x_1, ξ_1) and (x_3, ξ_3). Thus

$$\phi_1(x_3, \sigma_3) - \phi(x_1, \sigma_1) = \int_0^1 \frac{d}{ds}\phi_1(x(s), \sigma(s))ds$$
$$= \int_0^1 \sum_j \phi'_{x_j}\dot{x}_j(s)ds = \int_{\gamma_1} \theta .$$

Similarly, there is a C^1-curve γ_2 in Λ_{ϕ_2} connecting (x_3, ξ_3) and (x_2, ξ_2) such that

$$\phi_2(x_2, \tilde{\sigma}_2) - \phi_2(x_3, \tilde{\sigma}_3) = \int_{\gamma_2} \theta .$$

Hence, $\gamma_{12} = \gamma_1 + \gamma^2$ has the required prperty.

REMARK. The path γ in Proposition 1. 2. 1 is determined modulo the homotopy class in Λ since $d\theta = 0$ on Λ.

COROLLARY 1. 2. 3. *Let $\phi, \tilde{\phi} \in \Phi(\Lambda)$ and $\Lambda_\phi \cap \Lambda_{\tilde{\phi}} \neq \emptyset$. Then the difference of levels of ϕ and $\tilde{\phi}$ is locally constant in $\Lambda_\phi \cap \Lambda_{\tilde{\phi}}$.*

PROOF. Let $\lambda \in \Lambda_\phi \cap \Lambda_{\tilde{\phi}}$, and U_λ a simply connected neighborhood of λ in $\Lambda_\phi \cap \Lambda_{\tilde{\phi}}$. Then for any closed curve γ' in U_λ, $\int_{\gamma'} \theta = 0$ since $d\theta = 0$ on $\Lambda_\phi \cap \Lambda_{\tilde{\phi}}$.

We further note the following

PROPOSITION 1. 2. 4. *Let $\lambda \in \Lambda$ and γ a closed path in Λ passing λ. Then there are $\phi, \tilde{\phi} \in \Phi(\Lambda)$ defining Lagrangean germs in Λ at λ such that*

$$[\phi](\lambda) - [\tilde{\phi}](\lambda) = \int_\gamma \theta . \tag{1. 2. 1}$$

PROOF. We can cover the path γ by Lagrangean neighborhoods defined by $\phi_j \in \Phi(\Lambda)$, $j = 1, \cdots, l$ such that $\Lambda_{\phi_j} \cap \Lambda_{\phi_{j+1}} \neq \emptyset$ on which ϕ_j and ϕ_{j+l} have the same levels. Then $\gamma \subset \bigcup_{j=1}^{l} \Lambda_{\phi_j}$ and $\lambda \in \Lambda_{\phi_1} \cap \Lambda_{\phi_l}$. Let $\phi_l = \phi$ and $\phi_1 = \tilde{\phi}$. Then we see that (1. 2. 1) is true as in the proof of Proposition 1. 2. 1.

The following proposition will be very useful in the next chapter (cf. Remark after Corollary 1. 1. 11).

PROPOSITION 1. 2. 5. *Let $\lambda \in \Lambda$ and choose a local coordinate system $(x_1, \cdots, x_n)$ in X at $\pi(\lambda)$ so that Λ is defined near λ by a phase function $\phi(x, \xi) = \langle x, \xi \rangle - H(\xi)$ in $U \times V \subset X \times \boldsymbol{R}^n$. If we have another phase function $\phi_1(x, \xi) = \langle x, \xi \rangle - H_1(\xi)$ in $U \times V$ defining Λ near λ, then $H_1(\xi) - H(\xi) =$ constant.*

PROOF. By our assumptions we have

$$H_{1\xi_j}(\xi) = H_{\xi_j}(\xi), \qquad j = 1, \cdots, n,$$

for $\xi \in V$. Let $\xi_0 \in V$ be arbitrarily fixed. Then for ξ in a star like neighborhood of ξ_0 in V, we have

$$\begin{aligned}H_1(\xi)-H_1(\xi_0)&=\sum_j \xi_j\int_0^1 H_{1\xi_j}\Big((1-t)\xi+t\xi_0\Big)dt\\&=\sum_j \xi_j\int_0^1 H_{\xi_j}\Big((1-t)\xi+t\xi_0)\Big)dt\\&=H(\xi)-H(\xi_0),\end{aligned}$$

from which follows the proposition.

We end this section by a consideration of a particular class of Lagrangean submanifold in T^*X. Let $\tilde{X}=X\times \boldsymbol{R}$ and the projections p, q as before. That is, by $T^*\tilde{X}=T^*X\times T^*\boldsymbol{R}$, p is the projection from $T^*\tilde{X}$ onto T^*X and q from $T^*\tilde{X}$ onto the fibers of $T^*\boldsymbol{R}$. A conic Lagrangean submanifold $\tilde{\Lambda}$ in $T^*\tilde{X}\backslash 0$ is called non-stationary if $\tilde{\Lambda}\subset q^{-1}(\boldsymbol{R}\backslash 0)$. We assume $\tilde{\Lambda}\subset q^{-1}(\boldsymbol{R}_+)$ in the sequel.

PROPOSITION 1.2.6. *For every $\tau_0>0$, $\tilde{\Lambda}\cap q^{-1}(\tau_0)$ is a closed submanifold of dimension n in $T^*\tilde{X}\backslash 0$. $\Lambda=p(\tilde{\Lambda}\cap q^{-1}(\tau_0))$ is a Lagrangean submanifold in T^*X and the map p: $\tilde{\Lambda}\cap q^{-1}(\tau_0)\to\Lambda$ is locally homeomorphic.*

PROOF. By Proposition 1.1.3, $\tilde{\Lambda}$ and $q^{-1}(\tau_0)$ intersect transversally. Hence, $\tilde{\Lambda}\cap q^{-1}(\tau_0)$ is a closed submanifold of dimension $(n+1)+(2n+1)-(2n+2)=n$ in $T^*\tilde{X}\backslash 0$. By Corollary 1.1.5, p: $\tilde{\Lambda}\cap q^{-1}(\tau_0)\to\Lambda$ is locally homeomorphic and Λ is locally Lagrangean. Since $\tilde{\Lambda}\cap q^{-1}(\tau_0)$ is a transversal intersection, we can choose for every $\tilde{\lambda}_0=(x_0, t_0, \xi_0, \tau_0)\in\tilde{\Lambda}$ a local coordinate system $(x_1, \cdots, x_n)$ in $U\subset X$ at x_0 such that $\tilde{\Lambda}$ is defined near $\tilde{\lambda}_0$ by a phase function $\phi(x, t, \xi, \tau)=\langle x, \xi\rangle+(t-t_0)\tau-H(\xi, \tau)$, $x\in U$, $t\in I\subset\boldsymbol{R}$, $(\xi, \tau)\in\Gamma\subset\boldsymbol{R}^{n+1}\backslash 0$, Γ being an open cone and $H(\xi, \tau)$ positively homogeneous of degree 1 in (ξ, τ). In other words, a Lagrangean neighborhood of $\tilde{\lambda}_0$ in $\tilde{\Lambda}$ is given by $x-H'_\xi(\xi, \tau)=0$, $t-t_0-H_\tau(\xi, \tau)=0$, $(\xi, \tau)\in\Gamma$. Since $\tilde{\Lambda}$ is closed, $\tilde{\Lambda}$ is covered by such coordinate neighborhood $U_\alpha\times I_\alpha\times\Gamma_\alpha$ of $T^*\tilde{X}\backslash 0$. On the other hand, if $\tilde{\lambda}_0\in\tilde{\Lambda}\cap q^{-1}(\tau_0)$, then a Lagrangean neighborhood of $\lambda_0=p(\tilde{\lambda}_0)$ in Λ is given by $x-H'_\xi(\xi, \tau)=0$. Thus if we set $V_\alpha=\{\xi\in\boldsymbol{R}^n; (\xi, \tau_0)\in\Gamma_\alpha\}$, we see that Λ is covered by coordinates neighborhoods $U_\alpha\times V_\alpha$ in T^*X and thus Λ is a closed submanifold of T^*X.

We call a Lagrangean submanifold Λ of T^*X *stationary* if there is a non-stationary conic Lagrangean submanifold Λ in $T^*\tilde{X}\backslash 0$ such that $\Lambda=p(\tilde{\Lambda}\cap q^{-1}(\tau_0))$ for some $\tau_0>0$.

PROPOSITION 1.2.7. *Let Λ be a Lagrangean submanifold in T^*X. If $\pi_1(\Lambda)=\{0\}$ or $\boldsymbol{Z}$, then Λ is stationary.*

PROOF. We construct a non-stationary Lagrangean submanifold in

$T^*\tilde{X}\backslash 0$. First consider the case $\pi_1(\Lambda)=\{0\}$. Let $\lambda_0 \in \Lambda$ be arbitrarily fixed, and set $t(\lambda)=-\int_{\lambda_0}^{\lambda}\theta$, $\lambda\in\Lambda$. Since $d\theta=0$ on Λ, $t(\lambda)$ does not depend on the choice of paths connecting λ_0 and λ. Then $\tilde{\Lambda}=\{\mu_\tau\lambda, -t(\lambda), \tau), \lambda\in\Lambda, \tau>0\}$ is the corresponding non-stationary Lagrangean submanifold of $T^*\tilde{X}\backslash 0$, and $\Lambda=p(\tilde{\Lambda}\cap q^{-1}(1))$. The case when $\pi_1(\Lambda)=\boldsymbol{Z}$ is similar. Let $\lambda_0\in\Lambda$ be fixed, and for $\lambda\in\Lambda$ take a path γ in Λ connecting λ_0 and λ. Then set $t(\lambda;\gamma)=-\int_\gamma\theta$. If $\tilde{\gamma}$ is another path in Λ connecting λ_0 and λ, then $t(\lambda;\gamma)-t(\lambda;\tilde{\gamma})=\text{const.}$ $N(\gamma-\tilde{\gamma})$, where $N(\gamma-\tilde{\gamma})$ is the rotation number of the closed path $\gamma-\tilde{\gamma}$. If λ' is in a simply connected neighborhood of λ in Λ and if γ' is any path in this neighborhood connecting λ and λ', then $t(\lambda';\gamma+\gamma')$ does not depend on the choice of γ'. Hence, $\lambda\to t(\lambda;\gamma)$ is smooth. Therefore, if we set $\tilde{\Lambda}=\{(\mu_\tau\lambda, -t(\lambda;\gamma), \tau); \tau>0, \lambda\in\Lambda, \gamma$: a path in Λ from λ_0 to $\lambda\}$, then $\tilde{\Lambda}$ is the corresponding non-stationary Lagrangean submanifold in $T^*\tilde{X}\backslash 0$ and $\Lambda=p(\tilde{\Lambda}\cap q^{-1}(1))$. We note that this Lagrangean manifold $\tilde{\Lambda}$ is periodic in the t-direction.

Chapter 2. Canonical operators on Lagrangean manifolds

2.1. Symbols on Lagrangean germs.

Let Λ be a Lagrangean submanifold in T^*X. We want to define the canonical operator $\Gamma=\Gamma_\Lambda$ on Λ. For that purpose, we must consider an asymptotic class on Λ. When Λ is a conic Lagrangean submanifold in $T^*X\backslash 0$, then the action of $\boldsymbol{R}_+$ in the fibers of $T^*X\backslash 0$ permits us to define an asymptotic classes in Λ. Hewever, for general Lagrangean submanifolds, we must introduce a parameter set K which determines our asymptotic class. Let $K\subset\boldsymbol{R}_+$. We assume that K contains a sequence $k_j\to\infty$ and that the distance of K and 0 is positive, thus, $K\subset[\delta,\infty)$, $\delta>0$. We fix such a set K in the sequel.

In order to define the asymptotic class, we begin by a general consideration. Let W be an open set in $\boldsymbol{R}^M$ and consider an element $a(w,k)$ in $C_0^\infty(W)$ (resp. $C^\infty(W)$ depending on $k\in K$.

DEFINITION 2.1.1. *Let m be a real number. We write $a(w,k)\in S_0^m(W,K)$ (resp. $S^m(W,K)$) if the set*

$$k^{-m}a(w,k),\ k\in K,$$

forms a bounded set in $C_0^\infty(W)$ (resp. $C^\infty(W)$).

The following proposition is clear from Definition 2.1.1.

PROPOSITION 2.1.2. *Let m and m' be real. Then we have*

(a) $S_0^m(W, K)$ *is a linear space over* $\boldsymbol{C}$.

(b) *If* $m>m'$, *then* $S_0^{m'}(W, K)\subset S_0^m(W, K)$.

(c) $C_0^\infty(W)\subset S_0^0(W, K)$.

(d) *If* $a\in S_0^m(W, K)$, *then* $e^{ikc}a\in S_0^m(W, K)$ *for* $c\in\boldsymbol{R}$ *and* $k^{m'}a\in S_0^{m+m'}(W, K)$.

(e) *If* $a\in S_0^m(W, K)$ *and* $b\in S_0^{m'}(W, K)$, *then* $ab\in S_0^{m+m'}(W, K)$.

(f) *The assertions* (a) *to* (e) *hold good without the subscript* 0.

We shall in particular write $S_0^{-\infty}(W, K)=\bigcap_{m\in R} S_0^m(W, K)$ and $S_0^{-\infty}(W, K) =\bigcap_{m\in R} S^m(W, K)$. Note that if $a(w, k)=0$ for $k>k_1$ then $a(w, k)\in S_0^{-\infty}(W, K)$ or $S^{-\infty}(W, K)$.

We then have the following completeness property of the space $S^m(W, K)$ (cf. Hörmander [8], Th. 2.7).

PROPOSITION 2.1.3. *Let* $m_j, j=0, 1, 2, \cdots$, *be a strictly decreasing sequence tending to* $-\infty$. *If* $a_j(w, k)\in S^{m_j}(W, K)$, *then there is* $a(w, k)\in S^{m_0}(W, K)$ *such that*

$$a(w, k)-\sum_{j<l} a_j(w, k)\in S^{m_l}(W, K). \tag{2.1.1}$$

The function $a(w, k)$ *is uniquely determined modulo* $S^{-\infty}(W, K)$.

PROOF. Let B_i be an increasing sequence of compact subsets of W such that every compact subset of W is contained in one of them. Let $\psi(t)$ be a (continuous) function defined for $t\in\boldsymbol{R}$ such that $\psi(t)=1$ for $t\geq 1/2$ and $\psi(t)=0$ for $t\leq 0$, say. Then choose an increasing sequence $t_j\to+\infty$ such that

$$|\psi(k-t_j)D_w^\alpha a_j(w, k)|\leq k^{m_{j-1}}2^{-j}$$

for $w\in B_i$ and $|\alpha|+i\leqq j$. Here α is a multi-index $(\alpha_1, \cdots, \alpha_M)$ and $|\alpha|=\alpha_1+\cdots+\alpha_M$. Since $|k^{-m_j}D_w^\alpha a_j(w, k)|\leqq L_{\alpha,i}$ on B_i, and $m_{j-1}>m_j$, we only need to choose t_j such that $k^{m_j-m_{j-1}}L_{\alpha,i}\leqq 2^{-j}$ for $k\geqq t_j$. Then $a(w, k) =\sum_j \psi(k-t_j)a_j(w, k)$ satisfies the requirement.

By a similar proof, a similar result holds for $S_0^m(W, K)$. We write (2.1.1) briefly $a(w, k)\sim\sum_j a_j(w, k)$.

The following is an analogue of Hörmander [7, Prop. 1.1.8] and the proof is essentially the same.

PROPOSITION 2.1.4. *Let* $a_1, \cdots, a_l$ *be real valued functions in* $S^0(W, K)$. *Let* f *be a* C^∞ *function in a neighborhood in* $\boldsymbol{R}^l$ *of all limit points of* $(a_1(w, k), \cdots, a_l(w, k))$ *when* $k\to\infty$ *while* w *may vary in* W. *Then* $(w, k)\to f(a_1(w, k), \cdots, a_l(w, k))$ *is in* $S^0(W, K)$ *for large values of* k.

Let $\phi\in\Phi(\Lambda)$. The space $S_{\mathrm{J}}^{m}(U_\phi\times V_\phi, K)$ will be the space of symbols we are to consider. We shall sometimes call the elements of $S^{m}(U_\phi\times V_\phi, K)$ K-symbols in $U_\phi\times V_\phi$. For later convenience, we study the effects on K-symbols by the applications of phase integral. Let $\tilde{V}$ be a neighborhood of the origin in $\boldsymbol{R}^{\tilde{N}}$ and A a non-singular symmetric $\tilde{N}\times\tilde{N}$ matrix. Then $\tilde{\phi}(x,\sigma,\rho)=\phi(x,\sigma)+\dfrac{1}{2}\langle A\rho,\rho\rangle$, $(x,\sigma,\rho)\in U_\phi\times V_\phi\times\tilde{V}$ is a non-degenerate phase function in $U_\phi\times V_\phi\times\tilde{V}$.

PROPOSITION 2.1.5. *Let* $a(x,\sigma,k)\in S_0^{m}(U_\phi\times V_\phi, K)$ *and* $\chi(\rho)\in C_0^\infty(\tilde{V})$. *Assume that* $\chi(\rho)=1$ *in a neighborhood of the origin. Then*

$$\tilde{a}(x,\sigma,\rho,k)=\chi(\rho)a(x,\sigma,k)\in S_0^{m}(U_\phi\times V_\phi\times\tilde{V}, K),$$

$$\left(\frac{k}{2\pi}\right)^{\tilde{N}/2}\int e^{ik<A\rho,\rho>/2}\,\tilde{a}(x,\sigma,\rho,k)d\rho\in S_{\mathrm{J}}^{m}(U_\phi\times V_\phi, K) \tag{2.1.2}$$

and

$$\left(\frac{k}{2\pi}\right)^{\tilde{N}/2}\int e^{ik<A\rho,\rho>}\,\tilde{a}(x,\sigma,\rho,k)d\rho-\frac{e^{\frac{\pi}{4}i\,\mathrm{sgn}\,A}}{|\det A|^{\frac{1}{2}}}a(x,\sigma,k) \tag{2.1.3}$$
$$\in S_{\mathrm{J}}^{m-1}(U_\phi\times V_\phi, K).$$

PROOF. That $\tilde{a}(x,\sigma,\rho,k)\in S_0^{m}(U_\phi\times V_\phi\times\tilde{V}, K)$ is trivial. (2.1.2) is true if we show (2.1.3). However, if $\zeta(x,\sigma)\in C_0^\infty(U_\phi\times V_\phi)$ such that $\zeta=1$ on supp $a(x,\sigma,k)$, then

$$\left(\frac{k}{2\pi}\right)^{\tilde{N}/2}\int e^{ik<A\rho,\rho>}\chi(\rho)d\rho\,\zeta(x,\sigma)\in S_0^{0}(U_\phi\times V_\phi, K)$$

and

$$\left(\frac{k}{2\pi}\right)^{\tilde{N}/2}\int e^{ik<A\rho,\rho>}\chi(\rho)d\rho\,\zeta(x,\sigma)=\frac{e^{\frac{\pi}{4}i\,\mathrm{sgn}\,A}}{|\det A|^{\frac{1}{2}}}\zeta(x,\sigma)$$

modulo $S_0^{-1}(U_\phi\times V_\phi, K)$ by the stationary phase method. Hence, (2.1.3) follows from Proposition 2.1.2.

Now we consider the case corresponding to decreasing the σ-variables in the phase function. Let $V_\phi=V'\times V''$, $V'\subset\boldsymbol{R}^{N'}$, $V''\subset\boldsymbol{R}^{N''}$ and $\sigma=(\sigma',\sigma'')\in V'\times V''$. Assume that the matrix $(\phi''_{\sigma''\sigma''})$ is non-singular on C_ϕ. Then from $\phi'_{\sigma''}=0$ we can solve $\sigma''=\psi(x,\sigma')$. If we set $\phi_1(x,\sigma')=\phi(x,\sigma',\psi(x,\sigma'))$, $\phi_1(x,\sigma')$ is a non-degenerate phase function in $U_\phi\times V'$. We note that when $\sigma''=\psi(x,\sigma')$, $\phi'_{\sigma'}=\phi'_{\sigma'}+\langle\phi'_{\sigma''},\psi'_{\sigma'}\rangle=\phi'_{\sigma'}$, and $\phi'_{1x}=\phi'_x+\langle\phi'_{\sigma''},\psi'_x\rangle=\phi'_x$.

PROPOSITION 2.1.6. *Let* $a(x,\sigma,k)\in S_0^{m}(U_\phi\times V_\phi, K)$. *Let*

(2. 1. 4) $$a_1(x,\sigma',k) = e^{-ik\phi_1(x,\sigma')}\left(\frac{k}{2\pi}\right)^{N''/2}\int e^{ik\phi(x,\sigma',\sigma'')}a(x,\sigma',\sigma'',k)d\sigma''.$$

Then $a_1(x,\sigma',k)\in S_0^m(U_\phi\times V',K)$ *and*

(2. 1. 5)

$$a_1(x,\sigma',k)-\frac{e^{\frac{\pi}{4}i\,\mathrm{sgn}\,(\phi''_{\sigma''\sigma''}(x,\sigma',\psi(x,\sigma')))}}{|\det(\phi''_{\sigma''\sigma''}(x,\sigma',\psi(x,\sigma')))|^{\frac{1}{2}}}a(x,\sigma',\psi(x,\sigma'),k)\in S_0^{m-1}(U_\phi\times V',K).$$

PROOF. Let A be a non-singular symmetric $N''\times N''$ matrix such that $\mathrm{sgn}\,A=\mathrm{sgn}\,(\phi''_{\sigma''\sigma''})$ and consider a non-degenerate phase function $\tilde\phi(x,\sigma',\rho)=\phi_1(x,\sigma')+\langle A\rho,\rho\rangle/2$, $\rho\in\tilde V''\subset \boldsymbol{R}^{N''}$, $\tilde V''\ni 0$. Then since $\tilde\phi'_x=\phi'_x$, $\tilde\phi'_{\sigma'}=\phi'_{\sigma'}$ and $\tilde\phi=\phi$ when $\tilde\phi'_{\sigma''}=0$, $\tilde\phi'_\rho=0$, we can apply Proposition 1. 1. 8. Thus there is a diffeomorphism $U_\phi\times V'\times\tilde V''\ni(x,\sigma',\rho)\to(x,\sigma',\sigma'')\in U_\phi\times V'\times V''$ such that $\tilde\phi(x,\sigma',\rho)=\phi(x,\sigma',\sigma''(x,\sigma',\rho))$. In particular,

$$\langle A\rho,\rho\rangle/2=\phi\left(x,\sigma',\sigma''(x,\sigma',\rho)\right)-\phi_1(x,\sigma').$$

Differentiation in ρ then gives on $\phi'_{\sigma''}=0$

(2. 1. 6) $$A={}^t\left(\frac{\partial\sigma''}{\partial\rho}\right)(\phi''_{\sigma''\sigma''})\left(\frac{\partial\sigma''}{\partial\rho}\right).$$

If we take $\sigma''=\sigma''(x,\sigma',\rho)$ in the integral on the right hand side of (2. 1. 4), then we have

(2. 1. 7) $$a_1(x,\sigma',k) =\left(\frac{k}{2\pi}\right)^{N''/2}\int e^{ik\langle A\rho,\rho\rangle/2}a\left(x,\sigma',\sigma''(x,\sigma',\rho),k\right)\left|\frac{D\sigma''}{D\rho}\right|d\rho.$$

Therefore, if we keep (2. 1. 6) in mind and apply the stationary phase method to (2. 1. 7), then we have $a_1(x,\sigma',k)\in S_0^m(U_\phi\times V',K)$ and (2. 1. 5).

2. 2. Canonical operators on Lagrangean manifolds.

Let Λ be a Lagrangean submanifold of T^*X and $\Phi(\Lambda)$ the class of phase functions of Λ defined in §1. 2. Let $\phi\in\Phi(\Lambda)$ and $a(x,\sigma,k)\in S_0^m(U_\phi\times V_\phi,K)$. For $u\in C_0^\infty(U_\phi)$, we consider

(2. 2. 1) $$\Gamma_\phi(a,u)=\left(\frac{k}{2\pi}\right)^{N/2}\iint e^{ik\phi(x,\sigma)}a(x,\sigma,k)\,u(x)dxd\sigma$$

where $N=N_\phi$. Note that the integral

$$(\Gamma_\phi\, a)(x, k)=\left(\frac{k}{2\pi}\right)^{N/2}\int e^{ik\phi(x,\sigma)}\, a(x, \sigma, k)\, d\sigma \tag{2.2.2}$$

is absolutely convergent for each $k\in K$ so that

$$\Gamma_\phi(a, u)=\langle \Gamma_\phi a, u\rangle\,.$$

However, we shall take u as a density of order 1/2 on X and we shall interpret (2. 2. 1) as a definitition of a distribution density of order 1/2 on X. Our purpose will be to determine the conditions for $a(x, \sigma, k)$ that $\Gamma_\phi(a, u)$ is "independent" of the choice of phase function ϕ.

We begin by supplementing Definition 2. 1. 1. Let $\alpha(k)$ be a map from K to C. For $m\in \boldsymbol{R}$, we say $\alpha(k)\in S^m(K)$ if $\sup_K |k^{-m}\alpha(k)|<\infty$. Then analogous properties to Propositions 2. 1. 2 and 2. 1. 3 are true, and we also write $S^{-\infty}(K)=\bigcap_{m\in\boldsymbol{R}} S^m(K)$.

Proposition 2. 2. 1. *If $\phi(x, \sigma)$ has no critical points on* supp $\{a(x, \sigma, k)\, u(x)\}$, *then $\Gamma_\phi(a, u)\in S^{-\infty}(K)$. In particular, if $\phi'_\sigma(x, \sigma)\neq 0$ on* supp $a(x, \sigma, k)$, *then $\Gamma_\phi a\in S_0^{-\infty}(U_\phi, K)$.*

Proof. We prove the latter part since the former part can be proved in a similar way. Let $\zeta(x, \sigma)\in C_0^\infty(U_\phi\times V_\phi)$ such that $\zeta=1$ on supp $a(x, \sigma, k)$ and $\phi'_\sigma(x, \sigma)\neq 0$ on supp ζ. Let $L=\sum_{j=1}^N c_j(x, \sigma)\partial/\partial\sigma_j$, $c_j(x, \sigma)=\zeta(x, \sigma)\phi'_{\sigma_j}(x, \sigma)/\sum_{l=1}^N |\phi'_{\sigma_l}(x, \sigma)|^2$, $N=N_\phi$. The differential operator L is well-defined by the choice of ζ. Furthermore, we have

$$Le^{ik\phi(x,\sigma)}=ik\, e^{ik\phi(x,\sigma)}$$

on supp $a(x, \sigma, k)$ on which $\zeta=1$. Therefore, integrating by parts, we have

$$(\Gamma_\phi a)(x, k)=\left(\frac{k}{2\pi}\right)^{N/2}(ik)^{-M}\int e^{ik\phi(x,\sigma)}({}^tL)^M a(x, \sigma, k)\, d\sigma$$

for $M=1, 2, \cdots$. Any derivative of $\Gamma_\phi a$ is treated similarly, thus we have $\Gamma_\phi a\in S_0^{-\infty}(U_\phi, K)$.

Remark. The first part of the above proposition seems to be annoying. However, we use 2. 2. 1 only to define $\Gamma_\phi a$ as a distribution density of order 1/2 on X for each fixed K.

The next proposition shows $\Gamma_\phi a$ is essentially determined by the values of a on C_ϕ (cf. Hörmander [7], Prop. 1. 2. 5).

Proposition 2. 2. 2. *Let $a(x ,\sigma, k)\in S_0^m(U_\phi\times V_\phi, K)$ vanishes on $C_\phi=\{(x, \sigma)\,;\, \phi'_\sigma(x, \sigma)=0\}$. Then there is a $b(x, \sigma, k)\in S_0^{m-1}(U_\phi\times V_\phi, K)$ such that*

$\Gamma_\phi a = \Gamma_\phi b$. *In particular, if* $a(x, \sigma, k)$ *vanishes of infinite order on* C_ϕ, *then* $\Gamma_\phi a \in S_0^{-\infty}(U_\phi, K)$.

PROOF. We may assume U_ϕ and V_ϕ sufficiently small so that if $\lambda_1, \cdots, \lambda_n$ are local coordinates on C_ϕ extended in a neighborhood of C_ϕ, then $\lambda_1, \cdots, \lambda_n$, $\phi'_{\sigma_1}, \cdots, \phi'_{\sigma_N}$, $N = N_\phi$, give local coorninates in $U_\phi \times V_\phi$. Since the class $S_0^m(U_\phi \times V_\phi, K)$ is defined independently of the choice of local coordinates, we apply the Taylor formula to $a(x, \sigma, k)$ regarded as a function in $\lambda_1, \cdots, \lambda_n$, $\phi'_{\sigma_1}, \cdots, \phi'_{\sigma_N}$ and so we have

$$a(x, \sigma, k) = \sum_{j=1}^{N} a_j(x, \sigma, k)\phi'_{\sigma_j}(x, \sigma)$$

since $a(x, \sigma, k) = 0$ when $\phi'_\sigma(x, \sigma) = 0$. Here $a_j(x, \sigma, k) \in S^m(U_\phi \times V_\phi, K)$. Thus if $\zeta \in C_0^\infty(U_\phi \times V_\phi)$ is such that $\zeta = 1$ on $\operatorname{supp} a(x, \sigma, k)$, then $\tilde{a}_j(x, \sigma, k) = \zeta(x, \sigma)a_j(x, \sigma, k) \in S_0^m(U_\phi \times V_\phi, K)$. Therefore, integration by parts gives

$$\begin{aligned}(\Gamma_\phi a)(x, k) &= \left(\frac{k}{2\pi}\right)^{N/2} \int e^{ik\phi(x,\sigma)} \sum_{j=1}^{N} \tilde{a}_j(x, \sigma, k)\phi'_{\sigma_j}(x, \sigma)d\sigma \\ &= \left(\frac{k}{2\pi}\right)^{N/2} \int e^{ik\phi(x,\sigma)} \left\{\frac{i}{k}\sum_{j=1}^{N} \frac{\partial}{\partial\sigma_j}\tilde{a}_j(x, \sigma, k)\right\} d\sigma\end{aligned}$$

but $b(x, \sigma, k) = ik^{-1}\sum_{j=1}^{N} \frac{\partial}{\partial\sigma_j}\tilde{a}_j(x, \sigma, k) \in S_0^{m-1}(U_\phi \times V_\phi, K)$ then. The second part follows immediately from the first part since $\Gamma_\phi a = \Gamma_\phi b$ with $b \in S_0^{-\infty}(U_\phi \times V_\phi, K)$.

Now let us make a change of variables

$$x = x(\tilde{x}), \quad \sigma = \sigma(\tilde{x}, \tilde{\sigma}) \tag{2.2.3}$$

in (2.2.1). Then if we set $\tilde{\phi}(\tilde{x}, \tilde{\sigma}) = \phi(x(\tilde{x}), \sigma(\tilde{x}, \tilde{\sigma}))$, we have

$$\Gamma_\phi(a, u) = \left(\frac{k}{2\pi}\right)^{N/2} \iint e^{ik\tilde{\phi}(\tilde{x},\tilde{\sigma})} a\left(x(\tilde{x}), \sigma(\tilde{x}, \tilde{\sigma}), k\right)u\left(x(\tilde{x})\right)\left|\frac{Dx}{D\tilde{x}}\right|\left|\frac{D\sigma}{D\tilde{\sigma}}\right| d\tilde{x}d\tilde{\sigma}.$$

Hence, if we transform u as densities of order 1/2 and set $\tilde{u}(\tilde{x}) = u(x(\tilde{x})) \times |Dx(\tilde{x})/D\tilde{x}|^{1/2}$, then we have $\Gamma_\phi(a, u) = \Gamma_{\tilde{\phi}}(\tilde{a}, \tilde{u})$ if we define

$$\tilde{a}(\tilde{x}, \tilde{\sigma}, k) = a\left(x(\tilde{x}), \sigma(\tilde{x}, \tilde{\sigma}), k\right)\left|\frac{Dx(\tilde{x})}{D\tilde{x}}\right|^{1/2}\left|\frac{D\sigma(\tilde{x}, \tilde{\sigma})}{D\tilde{\sigma}}\right|. \tag{2.2.4}$$

Note that $\tilde{a} \in S_0^m(U_{\tilde{\phi}} \times V_{\tilde{\phi}}, K)$ if $a \in S_0^m(U_\phi \times V_\phi, K)$.

The interpretation of (2.2.4) is done just in the same way as Hörmander [7]. Namely, we introduce a density dC_ϕ on C_ϕ as a pull back of the Dirac measure in $\boldsymbol{R}^N$ under the map $C_\phi \ni (x, \sigma) \to \phi'_\sigma(x, \sigma)$. Thus if

$\lambda_1, \cdots, \lambda_n$ are local coordinates on C_ϕ extended in a neighborhood of C_ϕ, then we have

$$dC_\phi = \left| \frac{D(\lambda_1, \cdots, \lambda_n, \phi'_{\sigma_2}, \cdots, \phi'_{\sigma_N})}{D(x_1, \cdots, x_n, \sigma_1, \cdots, \sigma_N)} \right|^{-1} d\lambda_1, \cdots, d\lambda_n . \tag{2.2.5}$$

Let $\tilde{\lambda}_j(x, \sigma) = \lambda_j(x(\tilde{x}), \sigma(\tilde{x}, \tilde{\sigma}))$ under the diffeomorphism (2.2.3). Then since $\tilde{\phi}(\tilde{x}, \tilde{\sigma}) = \phi(x, \sigma)$ and $\tilde{\phi}'_{\tilde{\sigma}} = \phi'_\sigma \, \partial\sigma/\partial\tilde{\sigma}$, we have

$$\begin{aligned} \left| \frac{D(\tilde{\lambda}, \tilde{\phi}'_{\tilde{\sigma}})}{D(\tilde{x}, \tilde{\sigma})} \right| &= \left| \frac{D\sigma}{D\tilde{\sigma}} \right| \left| \frac{D(\lambda, \phi'_\sigma)}{D(x, \sigma)} \right| \left| \frac{D(x, \sigma)}{D(\tilde{x}, \tilde{\sigma})} \right| \\ &= \left| \frac{D\sigma}{D\tilde{\sigma}} \right|^2 \left| \frac{D(\lambda, \phi'_\sigma)}{D(x, \sigma)} \right| \left| \frac{Dx}{D\tilde{x}} \right| \end{aligned} \tag{2.2.6}$$

on $C_{\tilde{\phi}}$. From (2.2.4) and (2.2.6), the diffeomorphism (2.2.3) thus implies

$$\tilde{a}(\tilde{x}, \tilde{\sigma}, k) \left| \frac{D(\tilde{\lambda}, \tilde{\phi}'_{\tilde{\sigma}})}{D(\tilde{x}, \tilde{\sigma})} \right|^{-1/2} = a(x, \sigma, k) \left| \frac{D(\lambda, \phi'_\sigma)}{D(x, \sigma)} \right|^{-1/2} \tag{2.2.7}$$

on $C_{\tilde{\phi}}$. This shows that the image of $a\sqrt{dC_\phi}$ under the map $C_\phi \ni (x, \sigma) \to (x, \phi'_x) \in \Lambda_\phi$ and that of $\tilde{a}\sqrt{dC_{\tilde{\phi}}}$ under the map $C_{\tilde{\phi}} \ni (x, \tilde{\sigma}) \to (\tilde{x}, \tilde{\phi}'_{\tilde{x}}) \in \Lambda_{\tilde{\phi}}$ are the same if (x, σ) and $(\tilde{x}, \tilde{\sigma})$ are connected by (2.2.3).

Now we consider the effect on (2.2.1) under the change of phase functions preserving their *levels*. Thus assume that $V_\phi = V' \times V''$, $V' \subset \boldsymbol{R}^{N'}$, $V'' \subset \boldsymbol{R}^{N''}$ and the matrix $(\phi''_{\sigma''\sigma''}(x, \sigma', \sigma''))$ is non-singular when $\phi'_\sigma(x, \sigma', \sigma'') = 0$, $\sigma = (\sigma', \sigma'') \in V' \times V''$. Solving $\sigma'' = \psi(x, \sigma')$ from $\phi'_{\sigma''}(x, \sigma', \sigma'') = 0$ and set $\phi_1(x, \sigma') = \phi(x, \sigma', \psi(x, \sigma'))$. Then $\phi_1 \in \Phi(\Lambda)$ if $\phi \in \Phi(\Lambda)$ and $V_{\phi_1} = V'$, $U_{\phi_1} = U_\phi$. (2.2.1) now becomes

$$\Gamma_\phi(a, u) = \left(\frac{k}{2\pi} \right)^{N'/2} \iint e^{ik\phi_1(x, \sigma')} a_1(x, \sigma', k) u(x) d\sigma' dx = \Gamma_{\phi_1}(a_1, u)$$

where $a_1(x, \sigma', k)$ is given by (2.1.4). Let $\lambda_1, \cdots, \lambda_n$ be local coordinates on C_ϕ extended in a neighborhood of C_ϕ so that $\lambda_1, \cdots, \lambda_n, \phi'_{\sigma_1}, \cdots, \phi'_{\sigma_N}$ give local coordinates in $U_\phi \times V_\phi$. Then since $C_\phi = C_{\phi_1}$, $U_{\phi_1} \times V_{\phi_1}$ is a submanifold corresponding to $\phi'_{\sigma''} = 0$ for which $\lambda_1, \cdots, \lambda_n, \phi'_{1\sigma_1}, \cdots, \phi'_{1\sigma_{N'}}$, form a local coordinate system. Therefore, on C_{ϕ_1} we have

$$\left| \frac{D(\lambda, \phi'_{1\sigma'})}{D(x, \sigma')} \right|^{-1} = |\det(\phi''_{\sigma''\sigma''})|^{-1} \left| \frac{D(\lambda, \phi'_\sigma)}{D(x, \sigma)} \right|^{-1} . \tag{2.2.8}$$

Let $\Omega_{1/2}$ be the half-volume bundle over Λ (cf. Hörmander [7]) and if we denote by $S^m(\Lambda, \Omega_{1/2})$ the space of those sections of $\Omega_{1/2}$ over Λ_ϕ which are in $S^m_0(U_\phi \times V_\phi, K)$ in each trivialization of $\Omega_{1/2}$ over Λ_ϕ, $\phi \in \Phi(\Lambda)$. Then we have by Propositions 2.1.6 and 2.2.8

$$a_1\sqrt{dC_{\phi_1}}-e^{\frac{\pi i}{4}\operatorname{sgn}(\phi''_{\sigma''\sigma''})}a\sqrt{dC_\phi}\in S^{m-1}(\Lambda,\Omega_{1/2})\,.$$

Summarizing, we have shown

PROPOSITION 2.2.3. *Let ϕ and $\phi_1\in\Phi(\Lambda)$ define the same Lagrangean germs at $\lambda_0\in\Lambda$ and assume that the levels of ϕ and ϕ_1 near λ_0 coincide. Let $u\in C_0^\infty(U_\phi)$ and $u_1\in C_0^\infty(U_{\phi_1})$. If u and u_1 are connected as densities of order 1/2 on X, then for $a\in S_0^m(U_\phi\times V_\phi, K)$ we can find $a_1\in S_0^m(U_{\phi_1}\times V_{\phi_1}, K)$ such that*

$$\Gamma_\phi(a,u)=\Gamma_{\phi_1}(a_1,u_1)$$

and

$$a_1\sqrt{dC_{\phi_1}}-e^{\frac{\pi i}{4}\{\operatorname{sgn}(\phi''_{\sigma\sigma})-\operatorname{sgn}(\phi''_{1\sigma_1\sigma_1})\}}a\sqrt{dC_\phi}\in S^{m-1}(\Lambda,\Omega_{1/2}) \tag{2.2.9}$$

at $(x,\sigma_1)\in C_{\phi_1}$ and $(x,\sigma)\in C_\phi$ where (x,σ_1) and (x,σ) are mapped to the same point in the Lagrangean germs at λ_0.

COROLLARY 2.2.4. *Let $\phi\in\Phi(\Lambda)$ define Λ near $\lambda_0\in\Lambda$. Choose local coordinates $x_1,\cdots,x_n$ at $\pi(\lambda_0)$ in X so that Λ is given near λ_0 by a phase function $\tilde\phi(x,\xi)=\langle x,\xi\rangle-H(\xi)$ with the same level as ϕ, $(x,\xi)\in U_{\tilde\phi}\times V_{\tilde\phi}\subset X\times\boldsymbol{R}^n$. Then for any $a\in S_0^m(U_\phi\times V_\phi, K)$, there is $\tilde a\in S_0^m(U_{\tilde\phi}\times V_{\tilde\phi}, K)$ such that $\Gamma_\phi(a,u)=\Gamma_{\tilde\phi}(\tilde a,u)$, $u\in C_0^\infty(U_\phi)$ and at the cooresponding points of C_ϕ and $C_{\tilde\phi}$*

$$\tilde a\sqrt{d\xi_1\cdots d\xi_n}-e^{\frac{\pi i}{4}\{\operatorname{sgn}(\phi''_{\sigma\sigma})+\operatorname{sgn}H''_{\xi\xi}\}}a\sqrt{dC_\phi}\in S^{m-1}(\Lambda,\Omega_{1/2})\,. \tag{2.2.10}$$

PROOF. Since we can take $\xi_1,\cdots,\xi_n$ as local coordinates on $C_{\tilde\phi}$ and since $|D(\xi,\tilde\phi'_\xi)/D(x,\xi)|=1$, (2.2.10) follows from (2.2.9).

Finally we consider the effect under *general change* of phase functions. Let ϕ and $\phi_1\in\Phi(\Lambda)$ define the same Lagrangean germs at $\lambda_0\in\Lambda$. If we choose local coordinates in X at $\pi(\lambda_0)$ so that ϕ and ϕ_1 are of the form $\phi(x,\xi)=\langle x,\xi\rangle-H(\xi)$ and $\phi_1(x,\xi)=\langle x,\xi\rangle-H_1(\xi)$, then we have by Proposition 1.2.5 $H(\xi)-H_1(\xi)=L$, thus $\phi(x,\xi)-\phi_1(x,\xi)=-L$. It is then clear that $\Gamma_\phi(a,u)=\Gamma_{\phi_1}(a_1,u)$, $a\in S_0^m(U_\phi\times V_\phi\, K)$, $u\in C_0^\infty(U_\phi)$ if we take $a_1=e^{-ikL}a\in S_0^m(U_{\phi_1}\times V_{\phi_1}, K)$. Therefore, with Corollary 2.2.4, we have shown the following

PROPOSITION 2.2.5. *Let ϕ and $\tilde\phi\in\Phi(\Lambda)$ define the same Lagrangean germs at $\lambda_0\in\Lambda$. Let $u\in C_0^\infty(U_\phi)$ and $\tilde u\in C_0^\infty(U_{\tilde\phi})$. If u and $\tilde u$ are connected as densities of order 1/2 on X and if $a\in S_0^m(U_\phi\times V_\phi, K)$ we can find $\tilde a\in S_0^m(U_{\tilde\phi}\times V_{\tilde\phi}, K)$ such that $\Gamma_\phi(a,u)=\Gamma_{\tilde\phi}(\tilde a,\tilde u)$ and*

$$\tilde a\sqrt{dC_{\tilde\phi}}-e^{i\sigma(\phi,\tilde\phi,k)}a\sqrt{dC_\phi}\in S^{m-1}(\Lambda,\Omega_{1/2}) \tag{2.2.11}$$

at $(x, \sigma) \in C_\phi$ *and* $(x, \tilde{\sigma}) \in C_{\tilde{\phi}}$ *which are mapped to the same point on* Λ. *Here*

$$(2.2.12) \qquad \sigma(\phi, \tilde{\phi}, k) = k\left\{\phi(x, \sigma) - \tilde{\phi}(x, \tilde{\sigma})\right\} + \frac{\pi}{4}\left\{\operatorname{sgn}\phi''_{\sigma\sigma}(x, \sigma) - \operatorname{sgn}\tilde{\phi}''_{\tilde{\sigma}\tilde{\sigma}}(x, \tilde{\sigma})\right\}.$$

REMARK. By Proposition 1.2.1, there is a closed path γ in Λ passing $(x, \phi'_x(x, \sigma)) = (x, \tilde{\phi}'_x(x, \tilde{\sigma}))$, $(x, \sigma) \in C_\phi$, $(x, \tilde{\sigma}) \in C_{\tilde{\phi}}$ such that

$$\phi(x, \sigma) - \tilde{\phi}(x, \tilde{\sigma}) = \int_\gamma \theta .$$

By the map $\gamma \ni \lambda \to T_\lambda(\Lambda) \in \Lambda(n) = U(n)/0(n)$ we obtain a closed curve γ^* in the Lagrangean Grassmann (cf. Arnol'd [1], Hörmander [7]). Then as Arnol'd showed, $H^1(\Lambda(n), \boldsymbol{Z})$ is generated by the pull back α^* by the map $\det^2 : \Lambda(n) \to S^1$ of the generator α of $H^1(S^1, \boldsymbol{Z})$ and

$$(2.2.13) \qquad \langle \gamma^*, \alpha^* \rangle = \frac{1}{2}\left\{\operatorname{sgn}\phi''_{\sigma\sigma}(x, \sigma) - \operatorname{sgn}\tilde{\phi}''_{\tilde{\phi}\tilde{\phi}}(x, \tilde{\sigma})\right\} \in \boldsymbol{Z}.$$

Maslov denoted $\langle \gamma^*, \alpha^* \rangle$ by ind γ. ind γ is thus a homotopy invariant of Λ.

Now we interprete (2.2.11). Let $\tau_0(\phi, \tilde{\phi}) = \frac{1}{2}\left\{\operatorname{sgn}(\phi''_{\sigma\sigma}) - \operatorname{sgn}(\tilde{\phi}_{\tilde{\phi}\tilde{\phi}})\right\}$ and $\tau_1 = \phi - \tilde{\phi}$, where the evaluation is done at $(x, \sigma) \in C_\phi$ and $(x, \tilde{\sigma}) \in C_{\tilde{\phi}}$ giving the same point in Λ. Let $\boldsymbol{G}$ be a subgroup of $\boldsymbol{R}$ consisting of elements of the form $\int_\gamma \theta$ where $\gamma \in \pi_1(\Lambda)$. It is clear that $\tau_0(\phi, \tilde{\phi})$ and $\tau_1(\phi, \tilde{\phi})$ respectively determine cochain $\tau_0 \in H^1(\Lambda, \boldsymbol{Z})$ and $\tau_1 \in H^1(\Lambda, \boldsymbol{G})$. Then we may consider $\tau_0 \in H^1(\Lambda \times K, \boldsymbol{Z})$ and $\tau_1 \in H^1(\Lambda \times K, \boldsymbol{G})$. Let L_0 be the complex line bundle on $\Lambda \times K$ defined by τ_0 by letting $1 \in \boldsymbol{Z}$ act on $\boldsymbol{C}$ by multiplicaiton with the imaginary unit i. Thus L_0 is determined by the image of τ_0 in $H^1(\Lambda \times K, \boldsymbol{Z}_4)$. Let L_1 be the $\boldsymbol{S}^1$ bundle on $\Lambda \times K$ defined by τ_1 by letting $\boldsymbol{G}$ act on $\boldsymbol{R}$ by multiplication followed by raising to the power of e^i. However, we only consider a particular section ε_1 of L_1 given by $e^{ik\tau_1(\phi, \tilde{\phi})}$. If we regard the densities $\Omega_{1/2}$ of order 1/2 on Λ as a bundle over $\Lambda \times K$, then (2.2.11) gives a section of $\Omega_{1/2} \otimes L_0 \otimes L_1$ over $\Lambda \times K$. We denote by $L_Q = L_0 \otimes L_1$ and sometimes call it the bundle of quantization. Then $S_0^m(\Lambda \times K, \Omega_{1/2} \otimes L_Q)$ denotes the spaces of sections of $\Omega_{1/2} \otimes L_0 \otimes L_1$ over $\Lambda \times K$ defined by $a_\phi \in S^m(U_\phi \times V_\phi, K)$ and $a_{\tilde{\phi}} \in S^m(U_{\tilde{\phi}} \times V_{\tilde{\phi}}, K)$ with

$$(2.2.14) \qquad a_{\tilde{\phi}} \sqrt{dC_{\tilde{\phi}}} = e^{ik\tau_1(\phi, \tilde{\phi})}\, i^{\tau_0(\phi, \tilde{\phi})}\, a_\phi \sqrt{dC_\phi}$$

on $\Lambda_\phi \cap \Lambda_{\tilde{\phi}}$. Note that, for a fixed $k \in K$, $e^{ik\tau_1(\phi, \tilde{\phi})} e^{\tau_0(\phi, \tilde{\phi})}$ can be taken as a transition function in a certain subgroup $\boldsymbol{G}_k$ of $\boldsymbol{S}^1$. In this respect, we note the following

PROPOSITION 2.2.6. *Assume Λ be connected. Let $\lambda_0 \in \Lambda$ and $\Phi_{\lambda_0}(\Lambda)$ the totality of phase functions in $\Phi(\Lambda)$ defining Λ near λ_0. Then $e^{ik\tau_1(\phi,\tilde{\phi})+\frac{i\pi}{2}\tau_0(\phi,\tilde{\phi})}$ for any ϕ, $\tilde{\phi} \in \Phi_{\lambda_0}(\Lambda)$ is independent of $k \in K$ if and only if either*

(i) *all the levels of phase functions in $\Phi_{\lambda_0}(\Lambda)$ are the same; or*

(ii) $K \subset \left\{k_0 + \dfrac{2\pi m}{T_0}\ ;\ m=0, 1, 2, \cdots\right\}$ *for some $k_0>0$, $T_0>0$ and we have a homomorphism from $\pi_1(\Lambda)$ onto $\boldsymbol{Z}$ mapping γ to $\dfrac{1}{T_0}\displaystyle\int_\gamma \theta$.*

PROOF. Let $k_0 \in K$ and $k = l + k_0 \in K$. Then we must determine the conditions that $e^{il\tau_1(\phi,\tilde{\phi})}=1$, ϕ, $\tilde{\phi} \in \Phi_{\lambda_0}(\Lambda)$. These are equivalent to

$$l\left\{\phi(x, \sigma) - \tilde{\phi}(x, \tilde{\sigma})\right\} \equiv 0 \mod 2\pi \tag{2.2.15}$$

if $\phi'_\sigma(x, \sigma)=0$, $\tilde{\phi}'_{\tilde{\sigma}}(x, \tilde{\sigma})=0$, $\phi'_x(x, \sigma)=\tilde{\phi}'_{\tilde{x}}(x, \tilde{\sigma})=\xi$. In particular, if $l \neq 0$, $G_{\lambda_0} = \{[\phi](\lambda_0) - [\tilde{\phi}](\lambda_0)\ ;\ \phi, \tilde{\phi} \in \Phi_{\lambda_0}(\Lambda)\}$ is a discrete subgroup of $\boldsymbol{R}$. Therefore, either $G_{\lambda_0}=\{0\}$ or isomorphic to $\boldsymbol{Z}$. In the latter case, $G_{\lambda_0}=T_0\boldsymbol{Z}$ for some $T_0>0$. T_0 is independent of $\lambda_0 \in \Lambda$ since it is locally constant and Λ is connected. On the other hand, $G_{\lambda_0} = \left\{\int_\gamma \theta\ ;\ \gamma \in \pi_1(\Lambda)\right\}$ and so we have a homomorphsms

$$\pi_1(\Lambda) \ni \gamma \to \frac{1}{T_0}\int_\gamma \theta \in \boldsymbol{Z}.$$

On the other hand, if k_0 is the minimal element of K we then have $(k-k_0)T_0 \equiv 0 \mod 2\pi$ for any $k \in K$. Since $k>0$, we thus have $k \in \{k_0 + 2\pi m/T_0\ ;\ m=0, 1, 2, \cdots\}$. That either (i) or (ii) implies (2.2.15) is clear.

REMARK 1. The condition (i) is valid when Λ is simply connected or when Λ is conic. Then we can take $L_Q = L_0$.

REMARK 2. In case of (ii), $G_R = G_{R_0} = \left\{m_1\dfrac{k_0 T_0}{2\pi} + \dfrac{m_2}{2}\ ;\ m_1, m_2 \in \boldsymbol{Z}\right\}\Big/\boldsymbol{Z}$. Thus $k_0T_0/2\pi$ determines the group. In particular, if and only if $k_0T_0/\pi \equiv 0$ mod, $\boldsymbol{Z}$, $G_k = \boldsymbol{Z}_4$. In this case $K \subset \left\{\dfrac{2\pi}{T_0}\left(m + \dfrac{1}{2}\right);\ m=0, 1, 2, \cdots\right\}$ and $L_Q = L_0$.

Now we are going to define the canonical operator on Λ. We begin by

DEFINITION 2.2.7. *We denote by $I^m(X, \Lambda, K)$ the set of all distributions densities of order* 1/2 $A \in \mathscr{D}'(X, \Omega_{1/2})$ *such that $A = \sum_\phi A_\phi$ with supp A_ϕ locally finite and*

$$\langle A_\phi, u\rangle = \Gamma_\phi(a_\phi, u), \qquad u\in C_0^\infty(X, \Omega_{1/2},$$
$$\operatorname{supp} u \subset U_\phi$$

for some $a_\phi\in S^m(U_\phi\times V_\phi, K)$. *Here we let run* ϕ *in a subfamily of* $\Phi(\Lambda)$ *giving a locally finite covering of* Λ *by* Λ_ϕ.

PROPOSITION 2.2.8. *Let* $A\in I^m(X, \Lambda, K)$. *Then* $A\in C^\infty(X, \Omega_{1/2})$ *for every* $k\in K$. *Furthermore, let* $\lambda_0\in\Lambda$ *be such that* π^Λ *is regular at* λ_0. *Then in a neighborhood* U *of* λ_0 *in* X. $A=e^{iks(x)}a(x, k)$. *Here ds defines* Λ *near* λ_0 *and* $a\in S_0^m(U, K)$ *interpreted as a density of order* 1/2 *on* U.

PROOF. We only need to discuss locally. Since $\langle A_\phi, u\rangle=\langle\Gamma_\phi a_\phi, u\rangle$ and $\Gamma_\phi a_\phi\in C_0^\infty(U_\phi)$ for each $k\in K$, the first assertion follows. Let $\phi\in\Phi(\Lambda)$ defining Λ near λ_0. Then by Corollary 1.1.7, $(\phi''_{\sigma\sigma})$ is non-singular at (x_0, σ_0) $\in C_\phi$, $\lambda_0=(x_0, \phi'_\phi(x_0, \sigma_0))$. The second assertion now follows from Propositions 2.1.6 and 2.2.3.

If a_ϕ is a trivialization of $\chi_\phi a$, where $a\in S^m(\Lambda\times K, \Omega_{1/2}\otimes L_Q)$ and $\{\chi_\phi\}$ a partition of unity relative to the covering $\{\Lambda_\phi\}$ of Λ, then we denote the mapping $a\to A=\sum A_\phi\in I^m(X, \Lambda, K)$ by Γ_Λ. Γ_Λ may depend on the choice of coverings of Λ but is well-defined modulo $S^{m-1}(\Lambda\times K, \Omega_{1/2}\otimes L_Q)$ as will be seen in the next proposition. If Λ is understood, then we omit the subscript and write Γ.

PROPOSITION 2.2.9. *The map* Γ *induces the isomorphism*

$$\tilde{\Gamma}:\ S^m(\Lambda\times K, \Omega_{1/2}\otimes L_Q)/S^{m-1}(\Lambda\times K, \Omega_{1/2}\otimes L_Q)$$
$$\to I^m(X, \Lambda, K)/I^{m-1}(X, \Lambda, K).$$

We call $\tilde{\Gamma}$ *the canonical operator on* Λ *following Maslov.*

PROOF. That $\tilde{\Gamma}$ is a well-defined onto map follows from Proposition 2.2.5 and maps $S^{m-1}(\Lambda\times K, \Omega_{1/2}\otimes L_Q)$ to 0. To prove that $\tilde{\Gamma}$ is injective, we need the following analogue of Hörmander [7, Th. 3.2.4].

LEMMA 2.2.10. *Let* $\phi\in\Phi(\Lambda)$ *and* $a(x, \sigma, k)\in S_0^m(U_\phi\times V_\phi, K)$. *Let* $u\in C_0^\infty(U_\phi)$, $\rho\in C_0^\infty(U_\phi)$. *Assume that* ρ *is real valued. Then* (i) *if there is no point* $(x, \sigma)\in\operatorname{supp} a(x, \sigma, k)$ *with* $x\in\operatorname{supp} u$ *such that* $\phi'_\sigma=0$, $\phi'_x=\rho'_x$, *then* $\Gamma_\phi(a, e^{-ik\rho}u)\in S^{-\infty}(K)$.

(ii) *If there is precisely one point* $(x_0, \sigma_0)\in\operatorname{supp} a(x, \sigma, k)$ *with* $x_0\in\operatorname{supp} u$ *such that* $\phi'_\sigma(x_0, \sigma_0)=0$, $\phi'_x(x_0, \sigma_0)=\rho'_x(x_0)$ *and if* $\det S(\phi, \rho)\neq 0$ *at* (x_0, σ_0) *where*

$$S(\phi, \rho)=\begin{pmatrix}\phi''_{\sigma\sigma} & \phi''_{\sigma x}\\ \phi''_{x\sigma} & \phi''_{xx}-\rho''_{xx}\end{pmatrix}, \tag{2.2.16}$$

then $e^{-ik\phi(x_0,\sigma_0)+ik\rho(x_0)}\Gamma_\phi(a, e^{-ik\rho}u)\in S^{m+n/2}(K)$ *and*

(2. 2. 17)

$$e^{-ik\phi(x_0,\sigma_0)+ik\rho(x_0)}\Gamma_\phi(a, e^{-ik\rho}u)-\left(\frac{k}{2\pi}\right)^{n/2}\frac{e^{\frac{\pi}{4}i\,\mathrm{sgn}Q}}{|\det Q|^{\frac{1}{2}}}a(x_0, \sigma_0, k)\,u(x_0)\in S^{m+\frac{n}{2}-1}(K)$$

where $Q=S(\phi, \rho)$ *at* (x_0, σ_0).

PROOF. The proof is done in the same way as that of Hörmander [7, Th. 3.2.4]. In fact, (ii) follows from the stationary phase method and (i) is true since the phase function $\phi(x, \sigma)-\rho(x)$ has no critical points on supp $\{a(x, \sigma, k)u(x)\}$ and so the proof of Proposition 2.2.1 is applicable.

REMARK. If we choose the phase function $\phi(x, \sigma)$ in the form $\phi(x, \sigma)=\langle x, \sigma\rangle-H(\sigma)$, $(x, \sigma)\in U_\phi\times V_\phi\subset X\times \boldsymbol{R}^n$, and $\rho(x)=\langle x, \eta\rangle$, $\eta\in V_\phi$, then the stationary point for $\phi(x, \sigma)-\rho(x)$ is given by $x=H'_\sigma(\eta)$, $\sigma=\eta$. Thus if $u=1$ in a neighborhood of the projection to U_ϕ of supp $a\,(x, \sigma, k)$, then

$$e^{ikH(\eta)}\left(\frac{k}{2\pi}\right)^{-n/2}\Gamma_\phi(a, e^{-ik\rho}u)\in S_0^m(V_\phi, K)$$

modulo $S^{-\infty}(V_\phi, K)$ as a function of $\eta\in V_\phi$. Furthermore

$$a(H_\eta(\eta), \eta, k)-e^{ikH(\eta)}\left(\frac{k}{2\pi}\right)^{-n/2}\Gamma_\phi(a, e^{-ik\rho}u)\in S_0^{m-1}(V_\phi, K)$$

modulo $S^{-\infty}(V_\phi, K)$. These follow from the stationary phase method, namely the part (ii) of Lemma 2.2.10, combined with the part (i). Therefore, we see from Proposition 2.2.2 that $e^{ikH(\eta)}\left(\frac{k}{2\pi}\right)^{-n/2}\Gamma_\phi(a, e^{-ik\rho}u)$ determines $a(x, \sigma, k)$ modulo $S^{m-1}(U_\phi\times V_\phi, K)$ in the above case.

To complete the proof of Proposition 2.2.9, let $A=\sum A_{\phi_j}$ and for a $\lambda_0\in\Lambda_{\phi_0}$, $\lambda_0=(x_0, \xi_0)\neq(x, \phi'_{jx}(x, \sigma_j)), \phi'_{j\sigma_j}(x, \sigma_j))=0$, $(x, \sigma_j)\in$ supp a_j, $j\neq 0$, $a_j=a_{\phi_j}$. We claim that if $A\in I^{m-1}(X, \Lambda, K)$, then $a_0(x, \sigma, k)\in S^{m-1}(U_{\phi_0}\times V_{\phi_0}, K)$. Let ϕ_0 and ρ be as in the above Remark with U_{ϕ_0} a small neighborhood of x_0 in X and V_{ϕ_0} a small neighborhood of ξ_0 in $\boldsymbol{R}^n$ so that $\Lambda_{\phi_0}\cap\Lambda_{\phi_j}=\emptyset$, $j\neq 0$. Then we have as a function of $\eta\in V_{\phi_0}$, $e^{ikH(\eta)}(A_0, e^{-ik\rho}u)\in S^{m-1+n/2}(V_\phi, K)$ since $\langle A_j, \partial^{i\rho k}u\rangle\in S^{-\infty}(V_\phi, K)$ if $j\neq 0$ and $A_0=B-\sum A_j$, $B\in I^{m-1}(X, \Lambda, K)$. Therefore, as a function of $\eta\in V_{\phi_0}$, $a_0(H'_\eta(\eta), \eta, k)\in S^{m-1}(V_{\phi_0}, K)$, whence $a_0(x, \eta, k)\in S^{m-1}(U_{\phi_0}\times V_{\phi_0}, K)$.

COROLLARY 2.2.11. *Let* $\lambda_0=(x_0, \xi_0)\in\Lambda$ *and* $\phi, \tilde{\phi}\in\Phi(\Lambda)$ *phase functions defining* Λ *near* λ_0, $\xi_0=\phi'_x(x_0, \sigma_0)=\tilde{\phi}'_x(x_0, \tilde{\sigma}_0)$, $\phi'_\sigma(x_0, \sigma_0)=0$, $\phi'_{\tilde{\sigma}}(x_0, \tilde{\sigma}_0)=0$. *Let* $\rho\in C^\infty(U_\phi\cap U_{\tilde{\phi}})$ *such that* ρ *is real and* $d\rho$ *intersects with* Λ *transversally at* λ_0. *Let* $a(x, \sigma, k)\in S_0^m(U_\phi\times V_\phi, K)$ *and* $\tilde{a}(x, \tilde{\sigma}, k)\in S_0^m(U_{\tilde{\phi}}\times V_{\tilde{\phi}}, K)$. *Then for any* $u\in C_0^\infty(U_\phi\cap U_{\tilde{\phi}})$

(2.2.18)
$$\Gamma_{\phi}(a, e^{-ik\rho}u) = \Gamma_{\tilde{\phi}}(\tilde{a}, e^{-ik\rho}u) \mod S^{m+\frac{n}{2}-1}(K)$$

if and only if (2.2.11) *holds.*

PROOF. Since Λ and $d\rho$ intersect transversally at λ_0, it follows that matrices $S(\phi, \rho)$ and $S(\tilde{\phi}, \rho)$ as given by (2.2.16) are non-singular at (x_0, σ_0) and $(x_0, \tilde{\sigma}_0)$ respectively. By (2.2.17), (2.2.18) is then equivalent to

(2.2.19)
$$e^{ik\phi(x_0,\sigma_0)+\frac{\pi i}{4}\operatorname{sgn}Q}|\det Q|^{-1/2}a(x_0, \sigma_0, k) = e^{ik\tilde{\phi}(x_0,\tilde{\sigma}_0)+\frac{\pi i}{4}\operatorname{sgn}\tilde{Q}}|\det \tilde{Q}|^{-1/2}\tilde{a}(x_0, \tilde{\sigma}_0, k)$$

mod $S^{m-1}(K)$. Here Q and $\tilde{Q}$ are the matrices $S(\phi, \rho)$ and $S(\tilde{\phi}, \rho)$ evaluated at (x_0, σ_0) and $(x_0, \tilde{\sigma}_0)$ respectively. Thus what we have to verify is

(2.2.20)
$$\operatorname{sgn} Q - \operatorname{sgn} \tilde{Q} = \operatorname{sgn} \phi''_{\sigma\sigma}(x_0, \sigma_0) - \operatorname{sgn} \tilde{\phi}''_{\sigma\sigma}(x_0, \tilde{\sigma}_0)$$

and

(2.2.21)
$$\sqrt{dC_{\tilde{\phi}}} = \frac{|\det S(\phi, \rho)|^{1/2}}{|\det S(\tilde{\phi}, \rho)|^{1/2}}\sqrt{dC_{\phi}}\,.$$

However, (2.2.20) and (2.2.21) are clearly true when ϕ and $\tilde{\phi}$ have the form $\phi(x, \xi) = \langle x, \xi\rangle - H(\xi)$ and $\tilde{\phi}(x, \xi) = \langle x, \xi\rangle - \tilde{H}(\xi)$ since then $H(\xi) - \tilde{H}(\xi) =$ const. near $\xi = \xi_0$. By increasing or decreasing σ-variables and by equivalence, we can reduce all phase functions ϕ and $\tilde{\phi}$ to the above case.

Let $A \in I^m(X, \Lambda, K)$ and $B \in I^{m'}(X, \Lambda, K)$, $A = \Gamma a$, $B = \Gamma b$, $a \in S^m(\Lambda \times K, \Omega_{1/2} \otimes L_Q)$ and $b \in S^{m'}(\Lambda \times K, \Omega_{1/2} \otimes L_Q)$. Since for each fixed $k \in K$, A and B are C^∞ densities of order 1/2 on X, we can then form

$$(A, B) = \int_X AB$$

if supp $A \cap$ supp B is compact. In particular, if we note that the complex conjugate of a section of L_Q gives a section of L_Q^{-1}, we can then form a coupling $[a, b]$ for $a \in S^m(\Lambda \times K, \Omega_{1/2} \otimes L_Q)$ and $b \in S^{m'}(\Lambda \times K, \Omega_{1/2} \otimes L_Q)$ thus giving a density on Λ. In fact, if in local trivializations of Λ, a is given by (2.2.14) and b by

$$b_{\tilde{\phi}}\sqrt{dC_{\tilde{\phi}}} = e^{ik\tau_1(\phi,\tilde{\phi})}i^{\tau_0(\phi,\tilde{\phi})}b_{\phi}\sqrt{dC_{\phi}}\,,$$

then $[a, b]$ is given by

$$a_{\tilde{\phi}}\bar{b}_{\tilde{\phi}}dC_{\tilde{\phi}} = a_{\phi}\bar{b}_{\phi}dC_{\phi}\,.$$

then we have

PROPOSITION 2.2.12. *Let* $a\in S^{m}(\Lambda\times K,\ \Omega_{1/2}\otimes L_{Q})$ *and* $b\in S^{m'}(\Lambda\times K,\ \Omega_{1/2}\otimes L_{Q})$ *such that* supp $a\cap$ supp b *compact. then*

$$(\Gamma a, \Gamma b)=\int_{\Lambda}[a, b] \qquad \textit{modulo } S^{m+m'-1}(K). \tag{2.2.22}$$

Since (2.2.22) *is an identity modulo* $S^{m+m'-1}(K)$ *this does not depend on the definition of* Γ.

PROOF. Let a be given by a_{ϕ} and b by b_{ϕ}. We only need to prove

$$\int(\Gamma_{\phi}a_{\phi})(x, k)\overline{(\Gamma_{\phi}b_{\phi})(x, k)}\,dx=\int a_{\phi}\bar{b}_{\phi}\,dC_{\phi}$$

modulo $S^{m+m'-1}(K)$. Let us choose ϕ in the form $\langle x, \xi\rangle-H(\xi)$. Then

(2.2.23)

$$\int(\Gamma_{\phi}a_{\phi})(x, k)\overline{(\Gamma_{\phi}b_{\phi})(x, k)}\,dx$$
$$=\left(\frac{k}{2\pi}\right)^{n}\iiint e^{ik\langle x,\xi-\eta\rangle-ikH(\xi)+ikH(\eta)}\,a_{\phi}(x, \xi, k)\,\overline{b_{\phi}(x, \eta, k)}\,dx\,d\xi\,d\eta.$$

If we apply the stationary phase method in x and η, then (2.2.23) becomes

$$\int a_{\phi}(H_{\xi}'(\xi), \xi, k)\,\overline{b_{\phi}(H_{\xi}'(\xi), \xi, k)}\,d\xi$$

modulo $S^{m+m'-1}(K)$ as should be proved.

2.3. Some supplementary comments on the canonical operator $\tilde{\Gamma}$.

When Λ is a conic Lagrangean manifold in $T^{*}X\backslash 0$, then Hörmander [7] proved the following isomorphism

$$\tilde{\Phi}:\ S_{\rho}^{m+\frac{n}{4}}(\Lambda, \Omega_{1/2}\otimes L)/S_{\rho}^{m+\frac{n}{4}+1-2\rho}(\Lambda, \Omega_{1/2}\otimes L)$$
$$\rightarrow I_{\rho}^{m}(X, \Lambda)/I_{\rho}^{m+1-2\rho}(X, \Lambda).$$

Here the elements of $I_{\rho}^{m}(X, \Lambda)$ is locally defined by the oscillatory integral

$$\langle A, u\rangle=(2\pi)^{-(n+2N)/4}\iint e^{i\phi(x,\theta)}\,a(x, \theta)\,u(x)\,dx\,d\theta$$

for $a\in S_{\rho}^{m+\frac{n}{4}-\frac{N}{2}}(U\times\Gamma)$, $U\times\Gamma\subset X\times\boldsymbol{R}^{N}\backslash 0$. Therefore if we compare to our discussions we see that the ideas of Maslov's canonical operator and Fourier integral operators are very close. This is of the formal nature. However we have a more precise correspondence when we consider a stationary Lagrangean manifold. Namely, we have the following

PROPOSITION 2.3.1. *Let* $\tilde{\Lambda}$ *be a non-stationary Lagrangean submanifold* *of* $T^*\tilde{X}\backslash 0$, $\tilde{X}=X\times \boldsymbol{R}$, *and* Λ *a stationary Lagrangean manifold in* T^*X *corresponding to* $\tilde{\Lambda}$. *Let* $a\in S_1^{m+(n+1)/4}(\tilde{\Lambda}, \Omega_{1/2}\otimes L)$ *such that the projection of* supp a *to the fibers of* $T^*\boldsymbol{R}$ *by* $T^*\tilde{X}=T^*X\times T^*\boldsymbol{R}$ *is compact. Then there is* $\tilde{a}\in S^{\tilde{m}}(\Lambda\times K, \Omega_{1/2}\otimes L_Q)$, $\tilde{m}=m+\frac{n}{4}-\frac{1}{4}$, *such that*

$$\int e^{-itk}\,\Phi\, a dt \equiv \Gamma\tilde{a}$$

mod $I^{\tilde{m}-1}(X, \Lambda, K)$. *Here* Φa *and* $\Gamma\tilde{a}$ *denote respectively representatives in* $I_1^m(X, \Lambda)$ *and in* $I^{\tilde{m}}(X, \Lambda, K)$ *determined by* a *and* $\tilde{a}$.

PROOF. Let $\lambda_0\in\Lambda$ and U a neighborhood of $\pi(\lambda_0)$ in X. By the choice of $\tilde{\Lambda}$ and Λ, we have $p:\tilde{\Lambda}\cap q^{-1}(1)\to\Lambda$. Then supp $a\cap p^{-1}(\lambda)$ is finite when λ is in a neighborhood of λ_0 in Λ. Let $\phi_j(x, t, \theta_j)$ be conic non-degenerate phase functions in $\Gamma_j\subset X\times\boldsymbol{R}^{N_j}\backslash 0$ defining $\tilde{\Lambda}$ near $\tilde{\lambda}_j\in$ supp $a\cap q^{-1}(\lambda_0)$. Then we have

$$\langle\Phi a, u\rangle=\sum_{j=1}^{l}(2\pi)^{-(n+1+2N_j)/4}\iiint_{\Gamma_j}e^{i\phi_j(x,t,\theta_j)}\,a_j(x, t, \theta_j)\,u(x, t)\,dxdtd\theta_j,$$

where $a_j\in S_1^{m+(n+1-2N_j)/4}(\boldsymbol{R}^{n+1}\times\boldsymbol{R}^{N_j})$ with cone supp $a_j\subset\Gamma_j$. We may take $\phi_j(x_1, t, \theta_j)$ in the form $\phi_j(x, t, \theta_j)=\langle x, \xi\rangle+(t-L_j)\xi_0-H(\xi, \xi_0)$, $H(\xi, \xi_0)$ positively homogeneous of degree 1 in $(\xi, \xi_0)\in\Gamma\subset\boldsymbol{R}^{n+1}\backslash 0$. Then $a_j(x, t, \xi, \xi_0)\in S_1^{m-(n+1)/4}(\boldsymbol{R}^{n+1}\times\boldsymbol{R}^{n+1})$, cone supp $a_j\subset\Gamma$. Furthermore we may assume that the projection of supp a_j to the t-axis lies in a small neighborhood of L_j. Thus if we take $u(x, t)=u(x)\,e^{-ikt}$, then by Hörmander [7, Th. 3. 2. 4]

$$\langle\Phi a, ue^{-kt}\rangle=(2\pi)^{-3(n+1)/4}\int\cdots\int e^{i\{\langle x,\xi\rangle+t\xi_0-H(\xi,\xi_0)\}}$$
$$\times\sum_{j=1}^{l}e^{-ikL_j}\,a_j(x, t+L_j,\ \xi, \xi_0)\,u(x)\,e^{-ikt}\,dxdtd\xi d\xi_0\,.$$

Let $\chi\in C_0^\infty(\boldsymbol{R})$ with $\chi=1$ near $t=0$ so that $\chi(t)\,a_j(x, t+L_j, \xi, \xi_0)=a_j(x, t+L_j, \xi, \xi_0)$. Choose, $U\times I\times V\times\boldsymbol{R}$ so that $(x, t, \xi, \xi_0)\in U\times I\times V\times\boldsymbol{R}_+$ if and only if $(x, t, \xi\xi_0, \xi)\in\Gamma$. We may assume that V is a bounded open set in $\boldsymbol{R}^n$. Then

$$\langle\Phi a, ue^{-ikt}\rangle=(2\pi)^{-3(n+1)/4}\int\cdots\int_{U\times I\times V\times R_+}e^{i\xi_0\{\langle x\xi,\rangle+t-H(\xi,1)\}}$$
$$\times\,\xi_0^n\sum_{j=1}^{l}e^{-ikL_j}\,a_j(x, t+L_j, \xi_0\xi, \xi_0)\,u(x)\,e^{-ikt}\,dxdtd\xi d\xi_0\,.$$

Let

$$\tilde{a}_j(x, \xi, k)=(2\pi)^{-3/4}\,e^{ik\{H(\xi,1)-\langle x,\xi\rangle\}}\iint_{I\times R^n}e^{i\xi_0\{\langle x,\xi\rangle+t-H(\xi,1)\}}$$
$$\times\,\xi_0^n\,a_j(x, t+L_j, \xi_0\xi, \xi_0)\,e^{-ikt}\,\chi(t)\,dtd\xi_0\,.$$

for $(x, \xi) \in U \times V$. This is an oscillatory integral and for fixed $x, \xi, a_j(x, t+L_j, \xi_0\xi, \xi_0) \in S_1^{m-(n+1)/4}(\boldsymbol{R} \times \boldsymbol{R})$. Furthermore we may assume that $a_j = 0$ if (x, ξ) lies outside a compact set in $U \times V$. Then

$$\langle \Phi a, ue^{-ikt} \rangle =$$
$$= (2\pi)^{-3n/4} \int \cdots \int_{U \times V} e^{ik\{\langle x,\xi \rangle - H(\xi,1)\}} \sum_{j=1}^{l} e^{-ikL_j} \tilde{a}_j(x, \xi, k) u(x) dx d\xi.$$

However, by Hörmander [7, Th. 3. 2. 4],

$$\tilde{a}_j(x, \xi, k) \in S_0^{m+3n/4-1/4}(U \times V, K)$$

and if we set $\tilde{a}(x, \xi, k) = \left(\frac{k}{2\pi}\right)^{-n/2} (2\pi)^{-3n/4} \sum_{j=1}^{l} e^{-ikL_j} \tilde{a}_j(x, \xi, k)$, then $\tilde{a}(x, \xi, k) \in S_0^{m+n/4-1/4}(U \times V, K)$ and

$$\langle \Phi a, ue^{-ikt} \rangle = \langle \Gamma \tilde{a}, u \rangle .$$

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