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On generalized spin-boson models with singular perturbations

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Abstract. In this paper we consider generalized spin-boson models with singular perturbations. It is proven that under the infrared regularity condition Hamiltonians have the unique ground state for sufficiently small values of coupling constants. In addition it is shown that the asymptotic creation and annihilation operators of massless boson field exist.

Key words: quantum field theory, Fock spaces, spectral analysis.

1. Introduction and Main Theorem

1.1. Introduction

A generalized spin-boson model (GSB-models) is introduced by Arai and Hirokawa [5], which is a generalization of the so-called spin-boson model. It describes a general quantum system coupled to a boson field. A GSB-Hamiltonian is defined as a self-adjoint operator on the tensor product of a certain Hilbert space $\mathcal{K}$ and a Boson Fock space $\mathcal{F}_{\mathrm{b}}$, which consists of a decoupled Hamiltonian and an interaction term. The decoupled Hamiltonian is of the form:

$$H_0 = K \otimes I + I \otimes d\Gamma_{\mathrm{b}}(\omega), \tag{1}$$

where K is a self-adjoint operator on $\mathcal{K}$, and $d\Gamma_{\mathrm{b}}(\omega)$ the free Hamiltonian on $\mathcal{F}_{\mathrm{b}}$, which is given by the second quantization of a non-negative function ω. Then the GSB-Hamiltonian is given by

$$H_0 + \alpha \sum_{j=1}^{J} B_j \otimes \phi(f_j), \tag{2}$$

where $\alpha \in \mathbf{R}$ is a coupling constant, B_j a symmetric operator on $\mathcal{K}$, and $\phi(f_j)$ a field operator smeared by test function $f_j \in L^2(\mathbf{R}^d)$. The spectral

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properties of (2) are studied by [5], [7]. In particular the existence and uniqueness of the ground state is established under suitable conditions.

In this paper, instead of (2), we investigate Hamiltonians of the form:

$$H = H_0 + \beta \sum_{l=1}^{M} C_l \otimes \phi(g_l)^4, \tag{3}$$

where $\beta > 0$ and C_l is a self-adjoint but bounded operator on $\mathcal{K}$. The interaction term of (3) is singular compared with (2), and in particular it is not relatively bounded with respect to the decoupled Hamiltonian H_0. It is of interest to see the stability or instability of spectral properties of H_0 under singular perturbations.

Essential self-adjointness As is mentioned above, the interaction term in (3) is not relatively bounded with respect to H_0. Then it is not trivial to show the essential self-adjointness of H. In [3], [15], [23], an essential self-adjointness of a Hamiltonian in quantum field theory with a singular perturbation is considered. In this paper the essential self-adjointness of H is proven by applying [3].

Existence of a ground state We consider a ground state of H for the massless case:

$$\inf_{k \in \mathbf{R}^d} \omega(k) = 0. \tag{4}$$

Note that if the left hand side above is strictly positive we call it *massive*. Under (4) the bottom of the spectrum of H_0 is an eigenvalue *but* embedded in the continuous spectrum. Then it is not trivial to show the existence of a ground state of H even for sufficiently small but nonzero β, since the regular perturbation theory [22] for discrete spectra can not be applied. We prove that for sufficiently small β and under the infrared regularity condition:

$$g_j/\omega \in L^2(\mathbf{R}^d), \quad j = 1, \ldots, d, \tag{5}$$

H has a ground state such that the expectation of the number of bosons is finite.

Asymptotic fields For a massive case, the asymptotic field is constructed in e.g., [10], [11], [21]. For a massless case, however, it is not straightforward to construct it. Nevertheless it is also constructed in e.g., [12], [19]. Under (4) we prove the existence of the asymptotic field and construct a wave operator intertwining between H and $d\Gamma_{\rm b}(\omega) + E$ by the methods used in [19], where E denotes the ground state energy of H. From this we can also show the absence of the spectral gap of H. Namely it follows that the bottom of the spectrum, which is a point spectrum, also embedded in the continuous spectrum.

Uniqueness of the ground state By using the asymptotic field mentioned above, we can also prove the uniqueness of the ground state of H. Arai-Hirokawa [5] shows the uniqueness of the ground state of a *massive* GSB model. In this paper we show it for the massless case by applying the method used in [20].

Literatures of GSB-models and related works:
Miyao and Sasaki [24] consider a perturbation of a *massive* GSB-model:

$$\text{GSB Hamiltonian} + 1 \otimes \phi(f)^2. \tag{6}$$

They also show the existence of a ground state of (6). Arai, Hirokawa and Hiroshima [6] consider the absence of eigenvectors of a GSB-Hamiltonian (2) under the infrared singular condition:

$$g_j/\omega \notin L^2(\mathbf{R}^d) \quad \text{for some } j. \tag{7}$$

Arai and Kawano [8] prove the existence of a ground state even if the decoupled Hamiltonian has no ground state, but for a sufficiently *large* coupling constant. Hiroshima [20] proves the uniqueness of the ground state of Hamiltonians in some general class including GSB-Hamiltonians. Suzuki [29] investigates a scaling limit of GSB-Hamiltonians and derives effective Hamiltonians.

The existence of a ground state for related models is considered in e.g., [9], [13], [17], [19], [28]. In particular Bach, Fröhlich and Sigal [9] prove the existence of a ground state of the so-called non-relativistic quantum electrodynamics *without* infrared regular condition but sufficiently small coupling constants, and Griesemer, Lieb and Loss [17] extend it for arbitrary values of coupling constants.

Finally we give a short remark on a relationship between H and the ϕ^4 model. Our Hamiltonian H is close to the ϕ^4 model studied in Glimm and Jaffe [14], [15], [16], but we introduce cutoff functions to construct the Hamiltonian as an operator on a Hilbert space. The ϕ^4 model is defined on $\mathcal{F}_{\mathrm{b}}$ and massive, while H is massless, defined on $\mathcal{K} \otimes \mathcal{F}_{\mathrm{b}}$, and includes self-adjoint operators C_j's. So the analysis of H cannot be derived straightforwardly. Of course we need the infrared regular condition (5) in compensation for the massless assumption (4).

This paper is organized as follows:
In the remaining of Section 1, we define the total Hamiltonian H in (3) rigorously, and state the main results. In Section 2 we show the essential self-adjointness of H. In Section 3 we give a proof of the existence and uniqueness of the ground state of H. In Section 4 we give a proof of the existence of the asymptotic fields.

1.2. Boson Fock Space

Let $d \in \mathbf{N}$ denotes the spatial dimension. The boson Fock space over $L^2(\mathbf{R}^d)$ is defined by

$$\mathcal{F}_{\mathrm{b}} := \mathcal{F}_{\mathrm{b}}(L^2(\mathbf{R}^d)) := \oplus_{n=0}^{\infty} \big(\otimes_{\mathrm{s}}^n (L^2(\mathbf{R}^d))\big),$$

where $\otimes_{\mathrm{s}}^n L^2(\mathbf{R}^d)$ stands for the n-fold symmetric tensor product of $L^2(\mathbf{R}^d)$ and $\otimes_{\mathrm{s}}^0 (L^2(\mathbf{R}^d)) := \mathbf{C}$. The inner product of $\mathcal{F}_{\mathrm{b}}$ is given by

$$(\Phi, \Psi)_{\mathcal{F}_{\mathrm{b}}} = \sum_{n=0}^{\infty} (\Phi^{(n)}, \Psi^{(n)})_{\otimes^n L^2(\mathbf{R}^d)}. \tag{8}$$

In this paper the inner product $(y, x)_{\mathcal{X}}$ on the Hilbert space $\mathcal{X}$ is linear in x and antilinear in y. Unless confusions arise, we omit the subscript $\mathcal{X}$ of $(y, x)_{\mathcal{X}}$. Let $\Omega_{\mathrm{b}} = \{1, 0, 0, \dots\} \in \mathcal{F}_{\mathrm{b}}$ be the Fock vacuum. For $f \in L^2(\mathbf{R}^d)$, the creation operator is defined by

$$(a^*(f)\Psi)^{(n)} = \sqrt{n+1} S_{n+1}(f \otimes \Psi^{(n)}), \quad n \geq 1,$$

where S_n denotes a projection from $\otimes^n L^2(\mathbf{R}^d)$ onto $\otimes_{\mathrm{s}}^n L^2(\mathbf{R}^d)$ and $(a^*(f)\Psi)^{(0)} := 0$. The domain of $a^*(f)$ is given by

$$\mathcal{D}(a^*(f)) = \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \,\middle|\, \sum_{n=0}^{\infty} \|(a^*(f)\Psi)^n\|^2_{\otimes^n L^2(\mathbf{R}^d)} < \infty \right\}.$$

The annihilation operator $a(f)$ is defined by the adjoint operator of $a^*(f)$. Let $\mathcal{D} \subset L^2(\mathbf{R}^d)$ be a subspace. The finite particle subspace over $\mathcal{D}$ is given by

$$\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(\mathcal{D}) = \mathcal{L}\big\{ a^*(f_1)\cdots a^*(f_n)\Omega_{\mathrm{b}},\ \Omega_{\mathrm{b}} \mid f_j \in \mathcal{D},\ j = 1, \ldots, n,\ n \in \mathbf{N} \big\}.$$

In particular we call $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(L^2(\mathbf{R}^d))$ the finite particle subspace. It is seen that the domains of operators $a^*(f)$ and $a(g)$ include the finite particle subspace, leave it invariant, and satisfy the canonical commutation relations on it:

$$[a(f), a^*(g)] = (f, g), \tag{9}$$

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0. \tag{10}$$

The Segal operator is given by

$$\phi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f)). \tag{11}$$

It is well known that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(L^2(\mathbf{R}^d))$. By (9) and (10), it is seen that on $\mathcal{F}_{b}^{\mathrm{fin}}(L^2(\mathbf{R}^d))$

$$[\phi(f), \phi(g)] = i\,\mathrm{Im}(f, g). \tag{12}$$

In particular $[\phi(f), \phi(g)] = 0$, when f and g are real-valued functions.

Let T be an operator on $L^2(\mathbf{R}^d)$. We define the second quantization $d\Gamma_{\mathrm{b}}(T)$ of T by

$$d\Gamma_{\mathrm{b}}(T) = \oplus_{n=0}^{\infty} \left(\sum_{j=1}^{n} \Big(I \otimes \ldots I \otimes \underbrace{T}_{jth} \otimes I \ldots \otimes I \Big) \right).$$

1.3. Total Hamiltonian and Main Theorems

Let $\mathcal{K}$ be a Hilbert space over $\mathbf{C}$. Then the total Hilbert space is defined by

$$\mathcal{H} = \mathcal{K} \otimes \mathcal{F}_{\mathrm{b}}. \tag{13}$$

Let K be a operator on $\mathcal{K}$. The decoupled Hamiltonian of GSB models is defined by

$$H_0 = K \otimes I + I \otimes d\Gamma_{\mathrm{b}}(\omega), \tag{14}$$

where ω denotes the multiplication operator by a Lebesgue measurable function $\omega \neq 0$. We assume the following conditions:

(**S.1**) The operator K is self-adjoint and non-negative.
(**S.2**) The function ω is non-negative with $\inf_{\mathbf{k}\in\mathbf{R}^d} \omega(\mathbf{k}) = 0$.

Let

$$H' = \sum_{l=1}^{M} C_l \otimes \phi(g_l)^4, \tag{15}$$

where C_l, $l = 1, \ldots, M$, is an operator on $\mathcal{K}$. We introduce the following assumption:

(**S.3**) C_l, $l = 1, \ldots, M$ is a bounded, non-negative self-adjoint operator.

Proposition 1.1 (Essential self-adjointness) *Assume* (**S.1**)–(**S.3**). *Then* $H_0 + \beta H'$, $\beta \geq 0$, *is essentially self-adjoint on*

$$\mathcal{D}_0 = \mathcal{D}(K)\hat{\otimes}\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(L^2_0(\mathbf{R}^d)), \tag{16}$$

where $\hat{\otimes}$ *denotes the algebraic tensor product and*

$$L^2_0(\mathbf{R}^d) = \left\{\psi \in L^2(\mathbf{R}^d) \mid \textit{supp } \psi \textit{ is compact}\right\}.$$

It is noted that

$$\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(L^2_0(\mathbf{R}^d)) \subset \cap_{n=1}^{\infty}\mathcal{D}(d\Gamma_{\mathrm{b}}(\omega)^n) \bigcap \cap_{n=1}^{\infty}\mathcal{D}(\phi(g)^n).$$

Let us define the total Hamiltonian by

$$H = \overline{(H_0 + \beta H')_{\restriction \mathcal{D}_0}}, \tag{17}$$

where $\overline{X}$ denotes the closure of X and $\beta > 0$. By Proposition 1.1, it is seen that H is self-adjoint, and hence $\sigma(H) \subset [E_0(H), \infty)$ follows where $E_0(H) = \inf \sigma(H)$.

Let us assume the following conditions.

(**S.4**) The function g_l is a real-valued continuous function and $g_l \in \mathcal{D}(\omega)$ for $l = 1, \ldots, M$.

(**S.5**) (I) C_l and $C_{l'}$ commute for all $l, l' = 1, \ldots, M$.

(II) $C_l^{1/2}$, $l = 1, \ldots, M$, leave $\mathcal{D}(K)$ invariant, i.e. $C_l^{1/2}\mathcal{D}(K) \subset \mathcal{D}(K)$. There exists a constant $\nu_l \in \mathbf{R}$, such that for all $\Psi \in \mathcal{D}(K)$,

$$\left(\Psi, \left[C_l^{1/2}, \left[C_l^{1/2}, K\right]\right]\Psi\right) \geqslant \nu_l \|\Psi\|^2.$$

It is noted that $C_l\mathcal{D}(K) \subset \mathcal{D}(K)$, $l = 1, \ldots, M$, follows from the condition (II) in (**S.5**).

By applying the methods used in [4], we can obtain the following proposition.

Proposition 1.2 (Absence of spectral gap) *Assume* (**S.1**)–(**S.5**). *Then for sufficiently small* β, $\sigma(H) = [E_0(H), \infty)$ *follows.*

To prove the existence of a ground state of H, we introduce the following assumptions:

(**S.6**) The function $\omega(\mathbf{k})$ is continuous and $\lim_{|\mathbf{k}| \to \infty} \omega(\mathbf{k}) = \infty$, and there exist constants $\tilde{c} > 0$ and $\tilde{r} > 0$ such that

$$|\omega(\mathbf{k}) - \omega(\mathbf{k}')| \leq \tilde{c}|\mathbf{k} - \mathbf{k}'|^{\tilde{r}}(1 + \omega(\mathbf{k}) + \omega(\mathbf{k}')), \quad \mathbf{k}, \mathbf{k}' \in \mathbf{R}^d.$$

(**S.7**) The operator K has a compact resolvent.

(**S.8**) (**Infrared regularity condition**) It holds that

$$\int_{\mathbf{R}^d} \left|\frac{g_l(\mathbf{k})}{\omega(\mathbf{k})}\right|^2 d\mathbf{k} < \infty, \quad l = 1, \ldots, M.$$

Theorem 1.3 (Existence of Ground states) *Assume* (**S.1**)–(**S.8**). *Then* H *has a ground state for sufficiently small* β.

We introduce the additional assumptions.

(**S.9**) There exists a closed set $O \subset \mathbf{R}^d$ such that $\omega \in C^2(\mathbf{R}^d \backslash O)$, $g_l \in C_0^2(\mathbf{R}^d \backslash O)$, $l = 1, \ldots, M$, $\frac{\partial \omega}{\partial k_{j_1}}(\mathbf{k}) \neq 0$ and $\frac{\partial \omega}{\partial k_{j_2}}(\mathbf{k}) \neq 0$ on $\mathbf{R}^d \backslash O$ for some $j_1, j_2 \in \{1, \ldots, d\}$.

Proposition 1.4 (Uniqueness of ground states) *Assume* (**S.1**)–(**S.9**). *Then* $\dim\ker(H - E_0(H)) \leq \dim\ker(K - E_0(K))$ *for sufficiently small* β, *where* $E_0(X) = \inf \sigma(X)$.

Theorem 1.5 (Existence of asymptotic fields) *Suppose* (**S.1**)–(**S.3**) *and* (**S.9**). *Let* $h \in C^2(\mathbf{R}^d \backslash O)$ *and* $\operatorname{supp} h$ *be compact. Then for* $\Psi \in \mathcal{D}(H)$, *the asymptotic field*

$$a^{\sharp}_{\pm\infty}(h)\Psi := s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} (I \otimes a^{\sharp}(h)) e^{itH_0} e^{-itH} \Psi,$$

exists.

Let us define the asymptotic in/out-going Fock space by $\mathcal{F}_{\pm\infty} = \bigoplus_n \mathcal{F}^n_{\pm\infty}$ with

$$\mathcal{F}^n_{\pm\infty} = \overline{\mathcal{L}\{a^*_{\pm\infty}(h_1) \cdots a^*_{\pm\infty}(h_n)\Psi_g, \mid h_i \in \mathcal{D}(\omega^{-1/2}),\ i = 1, \ldots, n\}}, \quad (18)$$

where Ψ_g is a ground state of H and $\overline{\mathcal{D}}$ denotes the closure of $\mathcal{D}$. Here in particular we set $\mathcal{F}^0_{\pm\infty} = \{z\Psi_g \mid z \in \mathbf{C}\}$. Let

$$\mathcal{F}^n = \overline{\mathcal{L}\{a^*(h_1) \cdots a^*(h_n)\Omega_{\mathrm{b}}, \Omega_{\mathrm{b}} \mid h_i \in \mathcal{D}(\omega^{-1/2}),\ i = 1, \ldots, n\}}. \quad (19)$$

We define the wave operator $W_{\pm\infty} = \oplus_n \overline{W^n_{\pm\infty}}$, $W^n_{\pm\infty} : \mathcal{F}^n \to \mathcal{F}^n_{\pm\infty}$ by

$$W^n_{\pm\infty} a^*(h_1) \cdots a^*(h_n)\Omega_{\mathrm{b}} := a^*_{\pm\infty}(h_1) \cdots a^*_{\pm\infty}(h_n)\Psi_g. \quad (20)$$

It is noted that by the commutation relations given by Lemma 4.2, $W^n_{\pm\infty}$ is isometry and then $\overline{W^n_{\pm\infty}}$ is the unitary operator from $\mathcal{F}^n$ onto $\mathcal{F}^n_{\pm\infty}$. From Theorem 1.5 and, we obtain the following corollary.

Corollary 1.6 *Suppose* (**S.1**)–(**S.9**). *Then* $d\Gamma_{\mathrm{b}}(\omega) + E_0(H) = W^*_{\pm\infty} \cdot H_{\restriction \mathcal{F}_{\pm\infty}} W_{\pm\infty}$.

2. Essential Self-adjointness of $H_0 + \beta H'$

2.1. Proof of Proposition 1.1

For $\Psi \in \mathcal{D}(N_{\rm b}+1)$ it is known that $\|a^\sharp(\xi)\Psi\| \le \|\xi\|\|(N_{\rm b}+1)^{1/2}\Psi\|$, where $a^\sharp(\xi) = a(\xi)$ or $a^*(\xi)$. Hence we have

$$\|\phi(\xi)\Psi\| \le \sqrt{2}\|\xi\|\big\|(N_b+1)^{1/2}\Psi\big\|. \tag{21}$$

Let $\tilde{\phi}(\xi) = \frac{1}{\sqrt{2}}(-a(\xi)+a^*(\xi))$. Then it follows that on $\mathcal{F}_{\rm b}^{\rm fin}(L^2(\mathbf{R}^d))$,

$$[N_{\rm b}, \phi(\xi)] = \tilde{\phi}(\xi) \tag{22}$$

and for $\Psi \in \mathcal{D}(N_{\rm b}^{1/2})$,

$$\|\tilde{\phi}(\xi)\Psi\| \le \sqrt{2}\|\xi\|\big\|(N_b+1)^{1/2}\Psi\big\|. \tag{23}$$

It is seen in ([2, Lemma 2.4]) that $a^\sharp(\xi)$ maps $\mathcal{D}(N_{\rm b}^{3/2})$ into $\mathcal{D}(N_{\rm b})$ and for $\Psi \in \mathcal{D}(N_{\rm b}^{3/2})$,

$$\big\|[(N_{\rm b}+1)^{1/2}, a^\sharp(\xi)]\Psi\big\| \le \tilde{c}\|\xi\|\big\|(N_{\rm b}+1)^{1/2}\Psi\big\|, \tag{24}$$

where $\tilde{c} = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{(\lambda+1)^2}d\lambda$. From (21), (23) and (24), we obtain the following lemma.

Lemma 2.1 *Assume* (**S.1**)–(**S.3**). *Then there exists a constant $\tilde{M} \ge 0$ such that*

$$\big\|\phi(g)^4\Psi\big\| \le \tilde{M}\|g\|^4\big\|(N_{\rm b}+1)^2\Psi\big\|, \qquad \Psi \in \mathcal{D}(N_{\rm b}^2). \tag{25}$$

Let us identify $\mathcal{H} = \mathcal{K} \otimes \mathcal{F}_{\rm b}$ with

$$\mathcal{H}_\mathcal{K} := \oplus_{n=0}^\infty L^2_{\rm sym}(\mathbf{R}^{dn}; \mathcal{K}), \tag{26}$$

where $L^2_{\rm sym}(\mathbf{R}^{dn}; \mathcal{K})$ is the set of $\mathcal{K}$-valued, square integrable symmetric functions on $\mathbf{R}^{dn}$ with $L^2_{\rm sym}(\mathbf{R}^0; \mathcal{K}) := \mathcal{K}$. Let

$$\mathcal{H}_\mathcal{K}^{\rm fin} = \big\{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_\mathcal{K} \mid \Psi^{(k)} = 0_\mathcal{K} \text{ for all } k > J \text{ with some } J\big\}.$$

It is clear that for $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{H}_{\mathcal{K}}^{\mathrm{fin}}$,

$$\big(\Psi^{(m)}, H'\Psi^{(n)}\big) = 0 \quad \text{for } |m-n| \geqslant 5. \tag{27}$$

Lemma 2.2 *Assume* (**S.1**)–(**S.3**). *Then $H_0 + \beta H'$ is essentially self-adjoint on $\mathcal{D}(H_0) \cap \mathcal{H}_{\mathcal{K}}^{\mathrm{fin}}$.*

Proof. Let $\mathcal{X}$, be the Hilbert spaces and S_0, S', $N_{\mathcal{X}}$ and L be the operators in Appendix A. Now we apply $\mathcal{H}_{\mathcal{K}}$ to $\mathcal{X}$, H_0 to S_0, $\beta H'$ to S', $I \otimes N_b$ to $N_{\mathcal{X}}$, and $I \otimes I$ to L under the identification (26). Then, by (40) and Lemma 2.1, it is seen that $H_0 + \beta H'$ satisfies the assumptions (A.1) and (A.2) in Appendix A. In addition, by the definition of $H_0 + \beta H'$, (A.3) is satisfied. Hence by Theorem A, the proof is completed. □

Proof of Proposition 1.1. By Lemma 2.2, it is enough to show that

$$\mathcal{D}(H_0) \cap \mathcal{H}_{\mathcal{K}}^{\mathrm{fin}} \subset \mathcal{D}\big(\overline{H_{\restriction \mathcal{D}_0}}\big). \tag{28}$$

Let

$$E_K^j := E_K([E_0(K), j)), \quad \chi_{n,j}(\mathbf{k}_1, \dots, \mathbf{k}_n) := \chi_{I_j}(\mathbf{k}_1) \times \cdots \times \chi_{I_j}(\mathbf{k}_n),$$

where E_K denotes the spectral projection of K and χ_{I_j} the characteristic function on $I_j = [-j, j) \times \cdots \times [-j, j) \subset \mathbf{R}^d$. Let $\Psi = \{\Psi^{(0)}, \Psi^{(1)}, \dots, \Psi^{(J)}, 0, 0, \dots\} \in \mathcal{D}(H_0) \cap \mathcal{H}_{\mathcal{K}}^{\mathrm{fin}}$. Then $\Psi^{(n)}, n \leqslant J$, can be represented as

$$\Psi^{(n)} = \sum_{k=1}^{\infty} u_{n,k} \otimes \psi_{n,k}^{(n)} \in \mathcal{D}(K \otimes I) \cap \mathcal{D}\big(I \otimes d\Gamma_b(\omega)_{\restriction \otimes_s^n L^2(\mathbf{R}^d)}\big),$$

where $u_{n,k} \in \mathcal{H}$ and $\Psi_{n,k}^{(n)} \in \otimes_s^n L^2(\mathbf{R}^d)$. Let

$$\Psi_j^{(n)} := \sum_{k=1}^{\infty} \big(E_K^j u_{n,k}\big) \otimes \big(\chi_{n,j} \psi_{n,k}^{(n)}\big), \quad j = 1, 2, \dots, \tag{29}$$

$$\Psi_{j,q}^{(n)} := \sum_{k=1}^{q} \big(E_K^j u_{n,k}\big) \otimes \big(\chi_{n,j} \psi_{n,k}^{(n)}\big), \quad j = 1, 2, \dots. \tag{30}$$

It is seen that $\Psi_j^{(n)} = (E_K^j \otimes M_{\chi_{n,j}})\Psi^{(n)}$, where $M_{\chi_{n,j}}$ denotes the multiplication operator defined by $(M_{\chi_{n,j}}\Psi^{(n)})(\mathbf{k}_1,\ldots,\mathbf{k}_n) = \chi_{n,j}(\mathbf{k}_1,\ldots,\mathbf{k}_n)\cdot \Psi^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n)$. Hence $\lim_{j\to\infty}\|\Psi_j^{(n)} - \Psi^{(n)}\| = 0$. Since $\Psi^{(n)} \in \mathcal{D}(K\otimes I)$, it is seen that $\|(K\otimes I)\Psi_j^{(n)} - (K\otimes I)\Psi^{(n)}\| = \|(E_j\otimes M_{\chi_{n,j}} - I)(K\otimes I)\Psi^{(n)}\| \to 0$ as $j\to\infty$. In addition, we have $\|I\otimes d\Gamma_{\mathrm{b}}(\omega)\Psi_j^{(n)} - I\otimes d\Gamma_{\mathrm{b}}(\omega)\Psi^{(n)}\| = \|(E_j\otimes M_{\chi_{n,j}} - I)(I\otimes d\Gamma_{\mathrm{b}}(\omega))\Psi^{(n)}\| \to 0$ as $j\to\infty$. By Lemma 2.1, we see that

$$\begin{aligned}
&\big\|(C_l\otimes\phi(g_l)^4)\Psi_j^{(n)} - (C_l\otimes\phi(g_l)^4)\Psi^{(n)}\big\| \\
&\quad\leqslant M_d\big\|(C_l\otimes(N+1)^2)\big(\Psi_j^{(n)} - \Psi^{(n)}\big)\big\| \\
&\quad\leqslant (n+1)^2 M_d\|C_l\|\big\|\Psi_j^{(n)} - \Psi^{(n)}\big\| \to 0,
\end{aligned}$$

as $j\to\infty$. Hence, we have $\|H\Psi_j^{(n)} - H\Psi^{(n)}\| \to 0$ as $j\to\infty$. By the definition of $\Psi_{j,q}^{(n)}$, it can be also seen that $\|\Psi_{j,q}^{(n)} - \Psi_j^{(n)}\| \to 0$ and $\|H\Psi_{j,q}^{(n)} - H\Psi_j^{(n)}\| \to 0$ as $q\to\infty$. Since $\{\Psi_{j,q}^{(n)}\}$ is a sequence of $\mathcal{D}_0$, we obtain $\Psi \in \mathcal{D}(\overline{H_{\restriction\mathcal{D}_0}})$. Thus (28) is obtained. $\square$

2.2. Proof of Proposition 1.2

Let $\xi\in\mathcal{D}(\omega^{-1/2})$. It is well known that for $\Psi\in\mathcal{D}(d\Gamma_{\mathrm{b}}(\omega)^{1/2})$,

$$\|a(\xi)\Psi\| \le \left\|\frac{\xi}{\sqrt{\omega}}\right\|\big\|d\Gamma_{\mathrm{b}}(\omega)^{1/2}\Psi\big\|, \tag{31}$$

$$\|a^*(\xi)\Psi\| \le \left\|\frac{\xi}{\sqrt{\omega}}\right\|\big\|d\Gamma_{\mathrm{b}}(\omega)^{1/2}\Psi\big\| + \|\xi\|\|\Psi\|, \tag{32}$$

and hence

$$\|\phi(\xi)\Psi\| \le \sqrt{2}\left\|\frac{\xi}{\sqrt{\omega}}\right\|\big\|d\Gamma_{\mathrm{b}}(\omega)^{1/2}\Psi\big\| + \frac{1}{\sqrt{2}}\|\xi\|\|\Psi\|. \tag{33}$$

For $\xi\in\mathcal{D}(\omega)$, it follows that on $\mathcal{F}_{\mathrm{b}}^{\mathrm{fin}}(L^2(\mathbf{R}^d))$,

$$[d\Gamma_{\mathrm{b}}(\omega), a(\xi)] = -a(\omega\xi), \quad [d\Gamma_{\mathrm{b}}(\omega), a^*(\xi)] = a^*(\omega\xi). \tag{34}$$

Moreover it is seen in ([2, Lemma 2.4]) that for $\xi\in\mathcal{D}(\omega^{1/2})\cap\mathcal{D}(\omega)$,

$$\begin{aligned}&\big\|[(d\Gamma_{\mathrm{b}}(\omega)+1)^{1/2},a^{\sharp}(\xi)]\Psi\big\|\\&\qquad\leq\tilde{c}\big(\|\omega^{1/2}\xi\|+\|\omega\xi\|\big)\big\|(d\Gamma_{\mathrm{b}}(\omega)+1)^{1/2}\Psi\big\|\end{aligned}\tag{35}$$

where $\tilde{c}=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\lambda}}{(\lambda+1)^2}d\lambda$. By (31), (32) and (35), we obtain the following lemma.

Lemma 2.3 *Let $\xi\in\mathcal{D}(\omega^{-1/2})$ and $\eta\in\mathcal{D}(\omega^{k/2})$, $k=-1,1,2$. Then for $\Psi\in\mathcal{D}(d\Gamma_{\mathrm{b}}(\omega))$,*

$$\begin{aligned}\big\|a^{\sharp}(\xi)a^{\sharp}(\eta)\Psi\big\|\leq&\left\|\frac{\xi}{\sqrt{\omega}}\right\|\left\|\frac{\eta}{\sqrt{\omega}}\right\|\|(d\Gamma_{\mathrm{b}}(\omega)+1)\Psi\|\\&+Z(\xi,\eta)\big\|(d\Gamma_{\mathrm{b}}(\omega)+1)^{1/2}\Psi\big\|+\|\xi\|\|\eta\|\|\Psi\|,\end{aligned}$$

where $Z(\xi,\eta)=\tilde{c}\big\|\frac{\xi}{\sqrt{\omega}}\big\|(\|\omega^{1/2}\eta\|+\|\omega\eta\|)+\big\|\frac{\xi}{\sqrt{\omega}}\big\|\|\eta\|+\|\xi\|\big\|\frac{\eta}{\sqrt{\omega}}\big\|$.

From Lemma 2.3 the following corollary immediately follows.

Corollary 2.4 *Assume* (**S.1**) *and $g\in\mathcal{D}(\omega^{k/2})$, $k=-1,1,2$. Then there exist constants $\gamma_1>0$, $\gamma_2>0$ depending on g such that for $\Psi\in\mathcal{D}(d\Gamma_{\mathrm{b}}(\omega))$,*

$$\|\phi(g)^2\Psi\|\leq\gamma_1\|d\Gamma_{\mathrm{b}}(\omega)\Psi\|+\gamma_2\|\Psi\|.$$

By the algebraic identity $[XY,Z]=X[Y,Z]+[X,Z]Y$ and (34), we see that for $\Psi\in\mathcal{F}_b^{fin}(L^2(\mathbf{R}^d))$ and for $\xi\in\mathcal{D}(\omega)$,

$$\big[\phi(\xi)^2,[\phi(\xi)^2,d\Gamma(\omega)]\big]\Psi=-4(\xi,\omega\xi)\phi(\xi)^2\Psi.\tag{36}$$

Lemma 2.5 *Assume* (**S.1**)–(**S.5**). *Then for sufficiently small $\beta>0$, there exist constants $c_0>0$ and $d_0>0$, such that*

$$\|H_0\Psi\|+\|H'\Psi\|\leqslant c_0\|H\Psi\|+d_0\|\Psi\|,\quad\Psi\in\mathcal{D}(H).\tag{37}$$

Proof. Let $\Psi\in\mathcal{D}_0$. Then we see that

$$\|H\Psi\|^2=\|H_0\Psi\|^2+\beta^2\|H'\Psi\|^2+\beta(\Psi,\,(H'H_0+H_0H')\Psi).\tag{38}$$

By using the equality $X^2Y+YX^2=[X,[X,Y]]+2XYX$ with applying $C_l^{1/2}$ to X, and H_0 to Y, we have

$$\begin{aligned}
&(\Psi,\, (H'H_0 + H_0H')\Psi) \\
&\quad \geqslant \sum_{l=1}^{M} \bigl(\Psi, \bigl[C_l^{1/2} \otimes \phi(g_l)^2, \bigl[C_l^{1/2} \otimes \phi(g_l)^2, H_0\bigr]\bigr]\Psi\bigr) \\
&\quad = \sum_{l=1}^{M} \bigl\{\bigl(\Psi, \bigl[C_l^{1/2}, \bigl[C_l^{1/2}, K\bigr]\bigr] \otimes \phi(g_l)^4\Psi\bigr) \\
&\qquad\qquad + \bigl(\Psi, C_l \otimes [\phi(g_l)^2, [\phi(g_l)^2, d\Gamma_{\mathrm{b}}(\omega_V)]]\Psi\bigr)\bigr\}. \qquad (39)
\end{aligned}$$

By the assumption (**S.4**)–(**S.5**) and (36), we have

$$\begin{aligned}
&(\Psi, (H'H_0 + H_0H')\Psi) \\
&\quad \geqslant \sum_{l=1}^{M} \bigl\{\nu_l(\Psi, I \otimes \phi(g_l)^4\Psi) - 4(g_l, \omega g_l)(\Psi, C_l \otimes \phi(g_l)^2\Psi)\bigr\}. \qquad (40)
\end{aligned}$$

Then, by (**S.3**), we have

$$\begin{aligned}
&(\Psi, (H'H_0 + H_0H')\Psi) \\
&\quad \geqslant -\sum_{l=1}^{M} \bigl\{|\nu_l|\|I \otimes \phi(g_l)^2\Psi\|^2 + 4\|C_l\|(g_l, \omega g_l)\|I \otimes \phi(g_l)\Psi\|^2\bigr\}. \qquad (41)
\end{aligned}$$

By (33) and Corollary 2.4, there exists a constant $R_l \geq 0$ such that

$$\begin{aligned}
&|\nu_l|\|I \otimes \phi(g_l)^2\Psi\|^2 + 4(g_l, \omega g_l)\|C_l\|\|I \otimes \phi(g_l)\Psi\|^2 \\
&\quad \leqslant R_l(\|H_0\Psi\|^2 + \|\Psi\|^2). \qquad (42)
\end{aligned}$$

Hence, by (41), (42) and (38), we have

$$\|H\Psi\|^2 \geqslant \Bigl(1 - \beta\sum_{l=1}^{M} R_l\Bigr)\|H_0\Psi\|^2 + \beta^2\|H'\Psi\|^2 - \beta\sum_{l=1}^{M} R_l\|\Psi\|^2$$

for $\Psi \in \mathcal{D}_0$. Let us take β sufficiently small such as $1 - \beta\sum_{l=1}^{M} R_l > 0$. Then (49) follows for all $\Psi \in \mathcal{D}_0$. Since $\mathcal{D}_0$ is a core of H_L, we can extend (49) for all $\Psi \in \mathcal{D}(H)$. Hence (37) is obtained. □

By the spectral decomposition theorem, it is seen that

$$\left\|C_l \otimes \phi(g_l)^3\Psi\right\| \leq \left\|C_l \otimes \phi(g_l)^4\Psi\right\| + \|C_l\|\|\Psi\|.$$

We also see that

$$\left\|C_l \otimes \phi(g_l)^4\Psi\right\| \leq \|H'\Psi\| \tag{43}$$

follows by (**S.5**). Then by (37) and (43) we see that

$$\left\|(C_l \otimes \phi(g_l)^3)\Psi\right\| \leqslant c_0\|H\Psi\| + (d_0 + \|C_l\|)\|\Psi\|, \quad \Psi \in \mathcal{D}(H). \tag{44}$$

Proof of Proposition 1.2. To complete the proof, we show that H satisfies the assumptions (**E.1**)–(**E.4**) in Appendix B with applying $H = H_0 + \beta H'$ to $X = X_0 + qX'$. It is seen that H satisfies (**E.1**)–(**E.4**). Then we check (**E.4**). By the canonical commutation relations, we see that for $\Phi, \Psi \in \mathcal{D}_0$,

$$\begin{aligned}(I \otimes a^*(h)\Phi, H'\Psi) &- (H'\Psi, I \otimes a(h)\Phi) \\ &= 2\sqrt{2}\beta \sum_{l=1}^{M} (h, g_l)(\Phi, C_l \otimes \phi(g_l)^3\Psi).\end{aligned} \tag{45}$$

By (44) and Lemma 2.5, we can extend (45) for all $\Psi \in \mathcal{D}(H)$. Let $\{f_n\}_{n=1}^{\infty}$ be the sequence of $\mathcal{D}(\omega) \cap \mathcal{D}(\omega^{-1/2})$ such that $\|f_n\| = 1$, $n \geq 1$, and $w - \lim_{n\to\infty} f_n = 0$. Then we have

$$\begin{aligned}(I \otimes a^*(f_n)\Phi, H'\Psi) &- (H'\Psi, I \otimes a(f_n)\Phi) \\ &= 2\sqrt{2}\beta \sum_{l=1}^{M} (f_n, g_l)(\Phi, C_l \otimes \phi(g_l)^3\Psi) \longrightarrow 0,\end{aligned}$$

as $n \to \infty$. Hence H satisfies the (**E.4**), and the proof is completed. □

3. Ground States

3.1. Massive Case

In this subsection we investigate the ground state of H in the massive cases:

$$m := \inf_{\mathbf{k} \in \mathbf{R}^d} \omega(\mathbf{k}) > 0.$$

Let $V > 0$ and $L > 0$. We set

$$\Gamma_V = \frac{2\pi}{V}\mathbf{Z}^d = \left\{ \mathbf{q} = (q_1, \ldots, q_d) \mid q_j = \frac{2\pi}{V} n_j,\ n_j \in \mathbf{Z},\ j = 1, \ldots, d \right\},$$

$$\Gamma_{V,L} = \left\{ \mathbf{q} = (q_1, \ldots, q_d) \in \Gamma_V \mid |q_j| + \frac{\pi}{V} \leq L,\ j = 1, \ldots, d \right\},$$

Let

$$\mathcal{F}_{\mathrm{b},V} := \mathcal{F}_{\mathrm{b}}(\ell^2(\Gamma_V)).$$

We can regard $\mathcal{F}_{b,V}$ as a closed subspace of $\mathcal{F}_b(L^2(\mathbf{R}^d))$. For a lattice point $\mathbf{q} = (q_1, \ldots, q_d) \in \Gamma_V$, we set the subset of $\mathbf{R}^d$ by

$$C(\mathbf{q}, V) := \left[q_1 - \frac{\pi}{V}, q_1 + \frac{\pi}{V} \right) \times \cdots \times \left[q_d - \frac{\pi}{V}, q_d + \frac{\pi}{V} \right).$$

Let us define the following functions

$$\omega_V(\mathbf{k}) = \sum_{\mathbf{q} \in \Gamma_V} \omega(\mathbf{q}) \chi_{C(\mathbf{q},V)}(\mathbf{k}), \quad g_{l,L,V}(\mathbf{k}) = \sum_{\mathbf{q} \in \Gamma_{V,L}} g_l(\mathbf{q}) \chi_{C(\mathbf{q},\mathbf{V})}(\mathbf{k}),$$

$$g_{l,L} = \chi_L(\mathbf{k}) g_l(\mathbf{k}), \tag{46}$$

where $\chi_{C(\mathbf{q},\mathbf{V})}$ is the characteristic function on $C(\mathbf{q}, \mathbf{V})$, and $\chi_L(\mathbf{k}) = \chi_{[-L,L]}(k_1) \cdots \chi_{[-L,L]}(k_3)$. Let

$$\mathcal{H}_V = \mathcal{K} \otimes \mathcal{F}_{\mathrm{b},V},$$

and

$$H_{L,V} = \overline{\left(H_{0,V} + \beta H'_{L,V} \right)_{\restriction \mathcal{D}_0}}, \tag{47}$$

where $H_{0,V} = K \otimes I + I \otimes d\Gamma(\omega_V)$, $H'_{L,V} = \sum_{l=1}^{M} C_l \otimes \phi(g_{l,L,V})^4$, and

$$H_L = \overline{\left(H_0 + \beta H'_L \right)_{\restriction \mathcal{D}_0}}, \tag{48}$$

where $H'_L = \sum_{l=1}^{M} C_l \otimes \phi(g_{l,L})^4$. In a similar way as the proof of Proposition 1.1, it is proven that $H_{L,V}$ and H_L are essentially self-adjoint on $\mathcal{D}_0$.

Proposition 3.1 *Assume that* (**S.1**)–(**S.5**) *holds. Let V and L are sufficiently large, and $\beta > 0$ sufficiently small. Then*

(1) *there exist constants $c_1 > 0$ and $d_1 > 0$ independent of both V and L such that*

$$\|H_{0,V}\Psi\| + \left\|H'_{L,V}\Psi\right\| \leqslant c_1\|H_{L,V}\Psi\| + d_1\|\Psi\|, \quad \Psi \in \mathcal{D}(H_{L,V}), \tag{49}$$

(2) *there exist constants $c_2 > 0$ and $d_2 > 0$ independent of L such that*

$$\|H_0\Psi\| + \left\|H'_L\Psi\right\| \leqslant c_2\|H_L\Psi\| + d_2\|\Psi\|, \quad \Psi \in \mathcal{D}(H_L). \tag{50}$$

Proof. In a similar way of Lemma 2.5, we have for $\Psi \in \mathcal{D}_0$,

$$\begin{aligned}\|H_{L,V}\Psi\|^2 = \|H_{0,V}\Psi\|^2 &+ \beta^2\|H'_{L,V}\Psi\|^2 \\ &+ \beta\big(\Psi, (H'_{L,V}H_{0,V} + H_{0,V}H'_{L,V})\Psi\big).\end{aligned}$$

In a similar way as (41), we have

$$\begin{aligned}&\big(\Psi, (H'_{L,V}H_{0,V} + H_{0,V}H'_{L,V})\Psi\big) \\ &\quad \geqslant \sum_{l=1}^{M}\left\{\nu_l(\Psi, I \otimes \phi(g_{l,L,V})^4\Psi) - 4(g_{l,L,V}, \omega_V g_{l,L,V})(\Psi, C_l \otimes \phi(g_{l,L,V})^2\Psi)\right\}.\end{aligned} \tag{51}$$

Since $\lim_{V\to\infty}(g_{l,L,V}, \omega_V g_{l,L,V}) = (g_{l,L}, \omega g_{l,L})$ and $\lim_{L\to\infty}(g_{l,L}, \omega g_{l,L}) = (g_l, \omega g_l)$, we have for sufficiently large $V \geq 0$ and $L \geq 0$,

$$\begin{aligned}&\big(\Psi, (H'_{L,V}H_{0,V} + H_{0,V}H'_{L,V})\Psi\big) \\ &\quad \geqslant -\sum_{l=1}^{M}\left\{|\nu_l|\|I \otimes \phi(g_{l,L,V})^2\Psi\|^2 + 4\|C_l\|(g_l, \omega g_l)\|I \otimes \phi(g_{l,L,V})\Psi\|^2\right\}.\end{aligned} \tag{52}$$

By (33) and Corollary 2.4, there exists a constant $\tilde{R}_l \geq 0$ such that

$$\begin{aligned}&|\nu_l|\|I \otimes \phi(g_{l,L,V})^2\Phi\|^2 + 4(g_l, \omega g_l)\|C_l\|\|I \otimes \phi(g_{l,L,V})\Psi\|^2 \\ &\quad \leqslant \tilde{R}_l(\|H_{0,V}\Psi\| + \|\Psi\|).\end{aligned}$$

Hence, by (52), we have

$$\|H_{L,V}\Psi\|^2 \geqslant \left(1-\beta\sum_{l=1}^{M}\tilde{R}_l\right)\|H_{0,V}\Psi\|^2+\beta^2\left\|H'_{L,V}\Psi\right\|^2-\beta\sum_{l=1}^{M}\tilde{R}_l\|\Psi\|^2$$

for $\Psi \in \mathcal{D}_0$. Hence sufficiently small β such as $1-\beta\sum_{l=1}^{M}\tilde{R}_l > 0$, (49) follows for all $\Psi \in \mathcal{D}_0$. Since $\mathcal{D}_0$ is a core of $H_{L,V}$, we can extend (49) for all $\Psi \in \mathcal{D}(H)$. Hence (49) is obtained. We can prove (2) in a similar way as (1). □

Lemma 3.2 *Assume* (**S.1**)–(**S.5**) *and* (**S.7**). *Then $H_{L,V}$ is reduced by $\mathcal{H}_V$, and $H_{L,V}$ has purely discrete spectrum in $[E_0(H_{L,V}), E_0(H_{L,V})+m)$.*

Proof. It is similar to ([5, Lemma 3.9, Lemma 3.10]). □

Lemma 3.3 *Assume* (**S.1**)–(**S.6**). *Then for all $z \in \mathbf{C}\backslash\mathbf{R}$, it follows that*

$$\lim_{V\to\infty}\|(H_{L,V}-z)^{-1}-(H_L-z)^{-1}\| = 0, \tag{53}$$

$$\lim_{L\to\infty}\|(H_L-z)^{-1}-(H-z)^{-1}\| = 0. \tag{54}$$

Proof. We see that

$$(H_{L,V}-z)^{-1}-(H_L-z)^{-1} = L_{1,V}+L_{2,V},$$

where

$$L_{1,V} = (H_{L,V}-z)^{-1}(1\otimes d\Gamma_{\mathrm{b}}(\omega)-1\otimes d\Gamma_{\mathrm{b}}(\omega_V))(H_L-z)^{-1},$$

$$L_{2,V} = \beta\sum_{l=1}^{M}(H_{L,V}-z)^{-1}\big(C_l\otimes(\phi(g_{l,L,V})^4-\phi(g_{l,L})^4)\big)(H_L-z)^{-1}.$$

Let $\tilde{c} > 0$ and $\tilde{r} > 0$ be the constants in (**S.6**) and we set $R(V) := \tilde{c}\tilde{r}^{d/2}\big(\frac{\pi}{V}\big)^d\big(\frac{1}{2m}+1\big)$.

Then it is seen ([5, Lemma 3.1]) that $\|(d\Gamma_{\mathrm{b}}(\omega)-d\Gamma_{\mathrm{b}}(\omega_V))\Psi\| \leqslant \frac{2R(V)}{1-R(V)}\|d\Gamma_{\mathrm{b}}(\omega)\Psi\|$, for $\Psi \in \mathcal{D}(d\Gamma_{\mathrm{b}}(\omega))$.

Then we have

$$\|L_{1,V}\| \leqslant \frac{2R(V)}{|Imz|(1-R(V))}\|(I\otimes d\Gamma_{\mathrm{b}}(\omega))(H_L-z)^{-1})\| \to 0, \tag{55}$$

as $V \to \infty$. By (12) and the assumption (**S.2**), $\phi(g_{L,V})$ commutes with $\phi(g_L)$. Then

$$\sum_{l=1}^{M} C_l \otimes \left(\phi(g_{l,L,V})^4 - \phi(g_{l,L})^4\right) = \sum_{j=0}^{3} S_j,$$

where $S_j = \sum_{l=1}^{M} C_l \otimes ((\phi(g_{l,L,V}) - \phi(g_{l,L}))\phi(g_{l,L,V})^{3-j}\phi(g_{l,L})^j)$, $j = 0,\ldots,3$.

Let $\Psi = (H_L - z)\Xi$ for $\Xi \in \mathcal{D}_0$. Then for $\Phi \in \mathcal{D}_0$

$$|(\Phi, L_{2,V}\Psi)| \leqslant \beta \sum_{j=0}^{3} \left|(\Phi, (H_{L,V}-z)^{-1}S_j)\Xi)\right|. \tag{56}$$

We evaluate the right side of (56). Let $j = 0$. We see that

$$\begin{aligned} &\left|(\Phi, (H_{L,V}-z)^{-1}S_0\Xi)\right| \qquad (57)\\ &\quad\leqslant \sum_{l=1}^{M} \left\|C_l \otimes \phi(g_{l,L,V})^2 (H_{L,V}-z^{\dagger})^{-1}\Phi\right\| \\ &\qquad\qquad \cdot \|I\otimes \phi(g_{l,L,V}-g_{l,L})\phi(g_{l,L,V})\Xi\|. \qquad (58)\end{aligned}$$

It is seen that $\|\phi(g_{l,L,V})^2\Theta\| \leq \|\phi(g_{l,L,V})^4\Theta\| + \|\Theta\|$ for $\Theta \in \mathcal{D}(\phi(g_{l,L,V})^4)$. Then by (49), the first term of (58) can be estimated as

$$\begin{aligned} &\left\|C_l \otimes \phi(g_{l,L,V})^2 (H_{L,V}-z^{\dagger})^{-1}\Phi\right\| \\ &\quad\leqslant \left\|C_l \otimes \phi(g_{l,L,V})^4 (H_{L,V}-z^{\dagger})^{-1}\Phi\right\| + \|\|C_l\|\left\|(H_{L,V}-z^{\dagger})^{-1}\Phi\right\| \\ &\quad\leqslant \left(\left\|H'_{L,V}(H_{L,V}-z^{\dagger})^{-1}\right\| + \frac{\|C_l\|}{|Imz|}\right)\|\Phi\| \\ &\quad\leqslant \left(c_1\left(1+\frac{|z|}{|Imz|}\right) + \frac{d_1+\|C_l\|}{|Imz|}\right)\|\Phi\|. \qquad (59)\end{aligned}$$

On the second term of (58), by Lemma 2.3 we have

$$
\begin{aligned}
&\big\|I \otimes \phi(g_{l,L,V} - g_{l,L})\phi(g_{l,L,V})\Xi\big\| \\
&\quad \le \big(\|\omega^{-1/2}(g_{l,L,V} - g_{l,L})\|\|\omega^{-1/2} g_{l,L,V}\| + Z(g_{l,L,V} - g_L, g_{l,L,V})\big) \\
&\qquad \cdot \|(d\Gamma_{\mathrm{b}}(\omega) + 1)(H_L - z)^{-1}\|\|\Psi\| \\
&\qquad + \|g_{l,L,V} - g_{l,L}\|\|g_{l,L,V}\|\|(H_L - z)^{-1}\|\|\Psi\|. \qquad (60)
\end{aligned}
$$

By (59) and (60), there exists a constant $M_0(g_{l,L,V}, g_{l,L}) > 0$ such that

$$
\big|(\Phi, (H_{L,V} - z)^{-1} S_0 (H_L - z)^{-1}\Psi)\big| \leqslant \sum_{l=1}^{M} M_0(g_{l,L,V}, g_{l,L})\|\Phi\|\|\Psi\|, \quad (61)
$$

and $\lim_{V\to\infty} M_0(g_{L,V}, g_L) = 0$. Since Ran $[(H_L - z)_{\lceil \mathcal{D}_0}]$ is dense in $\mathcal{H}$, we obtain

$$
\lim_{V\to\infty} \|(H_{L,V} - z)^{-1} S_0 (H_L - z)^{-1}\| = 0.
$$

In a similar way as S_0, we have for $j = 1, 2, 3$

$$
\lim_{V\to\infty} \|(H_{L,V} - z)^{-1} S_j (H_L - z)^{-1}\| = 0.
$$

Thus (53) is obtained. (54) is also proven in a similar way as (53). □

Proposition 3.4 *Assume* (**S.1**)–(**S.7**). *Then H has purely discrete spectrum in $[E_0(H), E_0(H) + m)$. In particular H has a ground state.*

Proof. By Lemma 3.2, $H_{L,V}$ has purely discrete spectrum in $[E_0(H_{L,V}), E_0(H_{L,V}) + m)$. In addition $H_{L,V}$ converges to H_L in the norm resolvent sense as $V \to \infty$ by Lemma 3.3. Hence by the general theorem ([27, Lemma 4.6]) H_L has purely discrete spectrum in $[E_0(H_L), E_0(H_L) + m)$. It is also seen that H_L converges to H in the norm resolvent sense as $L \to \infty$ by Lemma 3.3. Hence H also has purely discrete spectrum in $[E_0(H), E_0(H) + m)$. □

3.2. Ground states in Massless Cases

In this subsection, we assume that

$$
\inf_{\mathbf{k}\in\mathbf{R}^d} \omega(\mathbf{k}) = 0.
$$

Let

$$\omega_m(\mathbf{k}) = \omega(\mathbf{k}) + m, \quad (m > 0),$$

and

$$H_{0,m} = K \otimes I + I \otimes d\Gamma_{\mathrm{b}}(\omega_m), \tag{62}$$

$$H_m = \overline{(H_{0,m} + \beta\, H')_{|\mathcal{D}_0}}. \tag{63}$$

By Proposition 1.1 and Theorem 1.2, H_m is essentially self-adjoint on $\mathcal{D}_0$ and has a ground state. Let Ψ_m be a normalized ground state of H_m:

$$H_m \Psi_m = E_m \Psi_m, \quad \|\Psi_m\| = 1, \tag{64}$$

where $E_m := E_0(H_m)$.

In a similar way as Proposition 3.1, we obtain the following lemma.

Lemma 3.5 *Assume* (**S.1**)–(**S.5**)*. Then there exist constants* $c_3 > 0$ *and* $d_3 > 0$ *independent of* m *such that*

$$\|H_{0,m}\Psi\| + \|H'\Psi\| \leqslant c_3 \|H_m \Psi\| + d_3 \|\Psi\|. \tag{65}$$

Remark 3.1 It is noted that the condition $m > 0$ is not used in the proof of Proposition 3.1. And hence we can prove (65) for $m = 0$.

Proposition 3.6 *Assume* (**S.1**)–(**S.8**)*. Then for sufficiently small* m*, there exists a constant* $c_4 > 0$ *independent of* m *such that*

$$\left\| (I \otimes N_b)^{1/2} \Psi_m \right\|^2 \leqslant c_4 \beta^2 \sum_{l=1}^{M} \left\| \frac{g_l}{\omega_m} \right\|^2. \tag{66}$$

Proof. Let $h \in \mathcal{D}(\omega_m) \cap \mathcal{D}(\omega_m^{-1/2})$ and

$$T_m(h) := I \otimes a(\omega_m h) + 2\sqrt{2}\beta \sum_{l=1}^{M} (h, g_l) C_l \otimes \phi(g_l)^3.$$

Since $\Psi_m \in \mathcal{D}(I \otimes d\Gamma_{\mathrm{b}}(\omega_m))$ and $h \in \mathcal{D}(\omega_m)$, $\Psi_m \in I \otimes a(\omega_m h)$ follows.

By $\|C_l \otimes \phi(g_l)^3 \Psi_m\| \le \|C_l \otimes \phi(g_l)^4 \Psi_m\| + \|C_l\| \|\Psi_m\|$, (43) and (65), $\Psi_m \in \mathcal{D}(H_m)$ implies that $\Psi_m \in C_l \otimes \phi(g_l)^3$. Then by the commutation relation (34), we have

$$(H_m - E_m)(I \otimes a(h))\Psi_m = -T_m(h)\Psi_m. \tag{67}$$

By (67), we see that

$$\begin{aligned} 0 &\leqslant \big(I \otimes a(h)\Psi_m, (H_m - E_m) I \otimes a(h)\Psi_m\big) \\ &= -(I \otimes a(\omega_m h)\Psi_m, I \otimes a(h)\Psi_m) \\ &\quad - 2\sqrt{2}\beta \sum_l (h, g_l)\big(I \otimes a(h)\Psi_m, C_l \otimes \phi(g_l)^3 \Psi_m\big). \end{aligned} \tag{68}$$

Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal system of $L^2(\mathbf{R}^d)$ such that $e_i \in \mathcal{D}(\omega^{1/2}) \cap \mathcal{D}(\omega^{-1/2})$. By (68), we have

$$\begin{aligned} \nu_{l,i} := \left(I \otimes a\left(\frac{e_i}{\sqrt{\omega_m}}\right)\Psi_m, I \otimes a(\sqrt{\omega_m e_l})\Psi_m\right) \\ + 2\sqrt{2}\beta\big(I \otimes a(\eta_l)\Psi_m, C_l \otimes \phi(g_l)^3 \Psi_m\big) \leqslant 0, \end{aligned}$$

where $\eta_i = \frac{1}{\sqrt{\omega_m}}\big(e_i, \frac{g}{\sqrt{\omega}}\big)e_i$. It is seen ([5, Lemma 4.2]) that for all $\Psi \in \mathcal{D}(I \otimes d\Gamma(\omega_m))$,

$$\sum_{i=1}^{\infty} \left(I \otimes a\left(\frac{e_i}{\sqrt{\omega_m}}\right)\Psi, I \otimes a(\sqrt{\omega_m} e_i)\Psi\right) = \big\|I \otimes N_{\mathrm{b}}^{1/2}\Psi\big\|^2. \tag{69}$$

Since $\{e_i\}_{i=1}^{\infty}$ is a complete orthonormal system, we see that

$$\begin{aligned} &\sum_{i=1}^{\infty} \big(I \otimes a(\eta_i)\Psi_m, C_l \otimes \phi(g_l)^3 \Psi_m\big) \\ &\quad = \left(I \otimes a\left(\frac{g}{\omega_m}\right)\Psi_m, C_l \otimes \phi(g_l)^3 \Psi_m\right). \end{aligned} \tag{70}$$

Then by (69) and (70) it follows that

$$0 \geqslant \sum_{i=1}^{\infty} \nu_{l,i} = \left\| I \otimes N_{\mathrm{b}}^{1/2} \Psi_m \right\|^2 + 2\sqrt{2}\beta \left(I \otimes a\left(\frac{g}{\omega_m}\right) \Psi_m, C_l \otimes \phi(g_l)^3 \Psi_m \right).$$

Thus we have

$$\left\| I \otimes N_{\mathrm{b}}^{1/2} \Psi_m \right\|^2 \leqslant 2\sqrt{2}\beta \left\| \left(I \otimes a\left(\frac{g_l}{\omega_m}\right)\right) \Psi_m \right\| \left\| C_l \otimes \phi(g_l)^3 \Psi_m \right\|. \tag{71}$$

Note that $\left\|\left(I \otimes a\left(\frac{g_l}{\omega_m}\right)\right)\Psi_m\right\| \leqslant \left\|\frac{g_l}{\omega_m}\right\| \|(I \otimes N_{\mathrm{b}}^{1/2})\Psi_m\|$. From (65) and (43) it follows that

$$\begin{aligned} \left\| C_l \otimes \phi(g_l)^3 \Psi_m \right\| &\leqslant \left\| C_l \otimes \phi(g_l)^4 \Psi_m \right\| + \|C_l\| \|\Psi_m\| \\ &\leqslant c_3 E_m + d_3 + \|C_l\|. \end{aligned} \tag{72}$$

By the definition of ω_m, we see that $d\Gamma(\omega_m) = d\Gamma(\omega) + mN_b$. Then for $m < m'$, $E_m < E_{m'}$ follows. Hence the right side of (72) is suppressed by some constant independent of m. Then (66) is obtained. □

Let $\dim \mathcal{K} = \infty$. By (**S.7**) we can take the sequence $\{\mu_r\}_{r=0}^{\infty}$ which is eigenvector of K with $\mu_r \leqslant \mu_{r+1}$ and $\mu_r \to \infty$ as $r \to \infty$. We define orthogonal projections by

$$\begin{aligned} &P_r : \text{the projection from } \mathcal{K} \text{ to } \oplus_{s=0}^{r} \mathcal{K}_s, \\ &P_r^{\perp} := 1 - P_r, \\ &P_{\Omega_{\mathrm{b}}} : \text{the projection from } \mathcal{F}_{\mathrm{b}} \text{ to } \{z\Omega_{\mathrm{b}} | z \in \mathbf{C}\}. \end{aligned}$$

Lemma 3.7 *Assume* (**S.1**)–(**S.8**). *Then*

(1) $\lim_{m\to 0} E_m = E_0(H)$,
(2) *Let m be sufficiently small. Then for sufficiently large r, there exists a constant c_5 such that*

$$\left(\Psi_m, (P_r^{\perp} \otimes P_{\Omega_{\mathrm{b}}})\Psi_m\right) \leqslant \frac{c_5}{(\mu_{r+1} - E_m)^2}. \tag{73}$$

Proof. (1) is proven in a similar way as ([5, Lemma 4.11]). (2) is also proven in a similar way as ([5, Lemma 4.3]). □

Proof of Theorem 1.3.

(**Case of dim $\mathcal{K} = \infty$**) By (1) in Lemma 3.7 and the general theorem ([5, Lemma 4.9]), it is enough to show that there exists a nonzero weak limit of Ψ_m as $m \to 0$. Since $\|\Psi_m\| = 1$, there exists a subsequence $\{\Psi_{m_j}\}$ such that $\Psi_0 := w - \lim_{j\to\infty} \Psi_{m_j}$. By the inequality $P_r \otimes P_{\Omega_{\rm b}} \geqslant I - I \otimes N_b - P_r^{\perp} \otimes P_{\Omega_{\rm b}}$, (66) and (73), we have

$$\begin{aligned}&\big(\Psi_{m_j}, (P_r \otimes P_{\Omega_{\rm b}})\Psi_{m_j}\big)\\ &\qquad \geqslant 1 - c_4|\beta|^2 \sum_{l=1}^{M} \left\| \frac{g_l}{\omega_{m_j}} \right\|^2 - \frac{c_5}{(\mu_{r+1} - E_0(H_{m_j}))^2}.\end{aligned} \tag{74}$$

Since $\mu_r \to \infty$ as $r \to \infty$, we see that $(\Psi_{m_j}, (P_r \otimes P_{\Omega_{\rm b}})\Psi_{m_j}) \geq 1 - c_4|\beta|^2 \sum_{l=1}^M \big\|\frac{g_l}{\omega_{m_j}}\big\|^2$ for sufficiently large r. Since $P_r \otimes P_{\Omega_{\rm b}}$ is a finite rank operator, $(P_r \otimes P_{\Omega_{\rm b}})\Psi_{m_j}$ strongly converges to $(P_r \otimes P_{\Omega_{\rm b}})\Psi_0$ as $j \to \infty$. Then $(\Psi_0, (P_r \otimes P_\Omega)\Psi_0) \geqslant 1 - c_4\beta^2 \sum_{l=1}^M \big\|\frac{g_l}{\omega_0}\big\|^2$. For sufficiently small β, we have $(\Psi_0, (P_r \otimes P_\Omega)\Psi_0) > 0$. Then $\Psi_0 \neq 0$ follows, and Ψ_0 is a ground state of H.

(**Case of dim $\mathcal{K} < \infty$**) By $I \otimes P_\Omega \geqslant I - I \otimes N_b$, we get $(\Psi_m, (I \otimes P_{\Omega_{\rm b}})\Psi_m) \geqslant 1 - c_4|\beta|^2 \sum_{l=1}^M \big\|\frac{g_l}{\omega_m}\big\|^2 > 0$ for sufficiently small β. Then $\Psi_0 \neq 0$, and Ψ_0 is a ground state of H. □

3.3. Uniqueness of Ground States

Lemma 3.8 *Assume* (**S.9**). *Then for* $\eta \in C_0^2(\mathbf{R}^d \backslash O) \cap L^1(\mathbf{R}^d)$,

$$\left| \int_{\mathbf{R}^d} \eta(\mathbf{k}) e^{-it\omega(\mathbf{k})} d\mathbf{k} \right| \leq \frac{1}{t^2} \int_{\mathbf{R}^d} |\tilde{\eta}(\mathbf{k})| d\mathbf{k}, \tag{75}$$

where $\tilde{\eta}(\mathbf{k}) = \frac{\partial^2}{\partial_{k_{j_1}} \partial_{k_{j_2}}} \big\{ \big(\frac{\partial \omega(\mathbf{k})}{\partial_{k_{j_1}}}\big)^{-1} \big(\frac{\partial \omega(\mathbf{k})}{\partial_{k_{j_2}}}\big)^{-1} \eta(\mathbf{k}) \big\}$ *with* $j_1 \neq j_2$.

Proof. It is seen that $e^{-it\omega} = -\frac{1}{t^2} \big(\frac{\partial \omega(\mathbf{k})}{\partial_{k_{j_1}}}\big)^{-1} \big(\frac{\partial \omega(\mathbf{k})}{\partial_{k_{j_2}}}\big)^{-1} \frac{\partial^2}{\partial_{k_{j_1}} \partial_{k_{j_2}}} e^{-it\omega}$. Using integration by parts, we have

$$\int_{\mathbf{R}^d} \eta(\mathbf{k}) e^{-it\omega(\mathbf{k})} d\mathbf{k} = \frac{1}{t^2} \int_{\mathbf{R}^d} \tilde{\eta}(\mathbf{k}) e^{-it\omega(\mathbf{k})} d\mathbf{k}.$$

Hence we can complete the proof. □

Proof of Proposition 1.4. We see that for $\Phi, \Psi \in \mathcal{D}(H)$,

$$
\begin{aligned}
(I \otimes a^*(h)\Phi, H'\Psi) - (H'\Psi, I \otimes a(h)\Phi) &= 2\sqrt{2}\beta \sum_{l=1}^{M} (h, g_l)(\Phi, C_l \otimes \phi(g_l)^3 \Psi) \\
&= \int_{\mathbf{R}^d} \overline{h(\mathbf{k})}(\Phi, T(\mathbf{k})\Psi) d\mathbf{k},
\end{aligned}
$$

where

$$
T(\mathbf{k})\Psi := 2\sqrt{2}\beta \sum_{l=1}^{M} g_l(\mathbf{k}) C_l \otimes \phi(g_l)^3 \Psi. \tag{76}
$$

To complete the proof it is enough to show that H and $T(\mathbf{k})$ satisfy the assumptions (**H.1**)–(**H.6**) in Appendix C with applying $H = H_0 + \beta H'$ to $X = X_0 + qX'$ and $T(\mathbf{k})$ to $S(\mathbf{k})$. But it is trivial to see (**H.1**)–(**H.3**) and (**H.5**). Hence it remains to show (**H.4**) and (**H.6**). Let $h \in C_0^2(\mathbf{R}^d \backslash O)$. We see that

$$
\begin{aligned}
&\left| \int_{\mathbf{R}^d} \overline{h(\mathbf{k})} \big(\Phi, e^{-it(H - E_0(H) + \omega(\mathbf{k}))} T(\mathbf{k})\Psi\big) d\mathbf{k} \right| \\
&\quad \leq 2\sqrt{2}\beta \|\Phi\| \sum_{l=1}^{M} \|C_l \otimes \phi(g)^3 \Psi\| |(h, e^{-it\omega} g_l)|.
\end{aligned}
$$

Then Lemma 3.8 implies that $\int_{\mathbf{R}^d} \overline{h(\mathbf{k})}(\Phi, e^{-it(H - E_0(H) + \omega(\mathbf{k}))} T(\mathbf{k})\Psi) d\mathbf{k} \in L^1([0, \infty), dt)$. We can also see that

$$
\|T(\mathbf{k})\Psi\| \leq 2\sqrt{2}\beta \sum_{l=1}^{M} |g_l(\mathbf{k})| \|C_l \otimes \phi(g_l)^3 \Psi\|, \tag{77}
$$

and hence $\int_{\mathbf{R}^d} \|T(\mathbf{k})\Psi\|^2 d\mathbf{k} < \infty$ follows. Thus **(H.4)** is satisfied. Let Ψ_g be a ground state of H. Then by (44) and (77), we have

$$
\begin{aligned}
&\|(H - E_0(H) + \omega(\mathbf{k}))^{-1} T(\mathbf{k})\Psi_g\| \\
&\quad \leq \sum_{l=1}^{M} (c_0 E_0(H) + d_0 + \|C_l\|) \frac{|g_l(\mathbf{k})|}{\omega(\mathbf{k})} \|\Psi_g\|,
\end{aligned}
$$

and hence (**H.6**) follows. □

4. Asymptotic fields

4.1. Existence of Asymptotic Fields

Let

$$a_t^\sharp(h) := e^{itH}e^{-itH_0}(I \otimes a^\sharp(h))e^{itH_0}e^{-itH},$$

where $a^\sharp(h) = a(h)$ or $a^*(h)$. Let us prepare the some inequalities for proving the existence of the asymptotic fields. It is noted that by the spectral decomposition theorem, for all $\epsilon > 0$, there exists $\lambda_\epsilon > 0$ such that for all $\Psi \in \mathcal{D}(d\Gamma_{\rm b}(\omega))$,

$$\|d\Gamma_{\rm b}(\omega)^{1/2}\Psi\| \leq \epsilon\|d\Gamma_{\rm b}(\omega)\Psi\| + \lambda_\epsilon\|\psi\|. \tag{78}$$

Proposition 4.1 *Assume* (**S.1**)–(**S.3**), (**S.5**) *and* (**S.9**). *Let* $\Psi \in \mathcal{D}(H)$ *and* $s' < s$. *Then*

$$\begin{aligned} &a_s(h)\Psi - a_{s'}(h)\Psi \\ &\quad = -\frac{4i\beta}{\sqrt{2}}\sum_{l=1}^{M}\int_{s'}^{s}(g_l, e^{-it\omega}h)e^{itH}(C \otimes \phi(g_l)^3)e^{-itH}\Psi\, dt, \end{aligned} \tag{79}$$

where the above integral is the Bochner integral.

Proof. Let $\Phi, \Psi \in \mathcal{D}(H)$ and $\Phi(t) = e^{-itH}\Phi$, $\Psi(t) = e^{-itH}\Psi$. It is seen that $e^{-itH_0}(I \otimes a(h))e^{itH_0} = I \otimes a(e^{-it\omega}h)$. Then by the strong differentiability of $e^{itH}\Psi$ and $e^{itH_0}\Psi$ with respect to t, we have

$$\begin{aligned} &\frac{d}{dt}(\Phi, a_t(h)\Psi) \\ &\quad = i\big(H\Phi(t), I \otimes a(e^{-it\omega}h)\Psi(t)\big) - i\big(H_0\Phi(t), I \otimes a(e^{-it\omega}h)\Psi(t)\big) \\ &\qquad + i\big(I \otimes a^*(e^{it\omega}h)\Phi(t), H_0\Psi(t)\big) - i\big(I \otimes a^*(e^{it\omega}h)\Phi(t), H\Psi(t)\big). \end{aligned} \tag{80}$$

Since $\Phi(t)$, $\Psi(t) \in \mathcal{D}(H)$, there exist sequences $\{\Phi_n\}_{n=1}^\infty$ and $\{\Psi_n\}_{n=1}^\infty$ such that $\Phi_n \in \mathcal{D}_0$, $\Phi_n \to \Phi(t)$, $H\Phi_n \to H\Phi(t)$, and $\Psi_n \in \mathcal{D}_0$, $\Psi_n \to \Psi(t)H\Psi_n \to H\Psi(t)$ as $n \to \infty$. Since Φ_n, $\Psi_n \in \mathcal{D}_0$, we have

$$\begin{aligned}\nu_n :&= i\big(\Phi_n, [H', I \otimes a(e^{-it\omega}h)]\Psi_n\big)\\ &= i\sum_{l=1}^{M}\big(\Phi_n, C_l \otimes [\phi(g_l)^4, a(e^{-it\omega}h)]\Psi_n\big)\\ &= \frac{-4i}{\sqrt{2}}\sum_{l=1}^{M}(g_l, e^{-it\omega}h)(\Phi_n, C_l \otimes \phi(g_l)^3\Psi_n). \end{aligned} \tag{81}$$

On the other hand, $H\Psi_n \to H\Psi(t)$ yields that $H_0\Psi_n \to H_0\Psi(t)$ by (37). Then by (31) and (78), we obtain that $I\otimes a(e^{-it\omega}h)\Psi_n \to I\otimes a(e^{-it\omega}h)\Psi(t)$ as $n\to\infty$. In addition, (44) implies that $\lim_{n\to\infty} C_l \otimes \phi(g_l)^3\Psi_n = C_l \otimes \phi(g_l)^3\Psi(t)$ as $n \to \infty$. Hence we see that

$$(80) = \lim_{n\to\infty}\nu_n = \frac{-4i}{\sqrt{2}}\beta\sum_{l=1}^{M}(g_l, e^{-it\omega}h)\big(\Phi(t), C_l\otimes\phi(g_l)^3\Psi(t)\big).$$

Thus

$$\begin{aligned}&(\Phi, (a_s(h) - a_{s'}(h))\Psi)\\ &\quad = \frac{-4i\beta}{\sqrt{2}}\sum_{l=1}^{M}\int_{s'}^{s}(g_l, e^{-it\omega}h)\big(\Phi, e^{itH}(C_l\otimes\phi(g)^3)e^{-itH}\Psi\big)dt,\end{aligned}$$

follows. Since $\mathcal{D}(H)$ is dense in $\mathcal{H}$, the proof is completed. $\square$

Proof of Theorem 1.5. Let $h \in C_0^2(\mathbf{R}^d\backslash O)$, and $\Psi \in \mathcal{D}(H)$. By Proposition 4.1,

$$\begin{aligned}&\|(a_s(h) - a_{s'}(h))\Psi\|\\ &\quad \leqslant 4\beta\sum_{l=1}^{4}\int_{s'}^{s}|(g_l, e^{-it\omega}h)|\big\|e^{itH}(C_l\otimes\phi(g_l)^3)e^{-itH}\Psi\big\|dt.\end{aligned} \tag{82}$$

It is seen by (44) that

$$\big\|(C_l\otimes\phi(g_l)^3)e^{-itH}\Psi\big\| \le c_0\|H\Psi\| + (d_0 + \|C_l\|)\|\Psi\|. \tag{83}$$

Then by (82), (83) and Lemma 3.8, we obtain that $\|(a_s(h) - a_{s'}(h))\Psi\| \leqslant$

const. $\int_{s'}^{s} \frac{1}{t^2} dt \to 0$, as $s, s' \to \infty$ for $h \in C^2(\mathbf{R}^d \backslash O)$. Let $\xi \in \mathcal{D}(\omega^{-1/2})$. Since $C_0^2(\mathbf{R}^d \backslash O)$ is a core of $\omega^{-1/2}$, there exists a sequence $\{\xi_n\} \subset C_0^2(\mathbf{R}^d \backslash O)$ such that $\|\xi_n - \xi\| \to 0$, and $\|\omega^{-1/2}\xi_n - \omega^{-1/2}\xi\| \to 0$ as $n \to 0$. It is seen that for $\Psi \in \mathcal{D}(H)$ and $t' < t$,

$$\begin{aligned} \big\|a_s(\xi)\Psi - a_{s'}(\xi)\Psi\big\| &\leq \big\|a(e^{-it\omega}(\xi - \xi_n))e^{-isH}\Psi\big\| + \big\|a_s(\xi_n)\Psi - a_{s'}(\xi_n)\Psi\big\| \\ &\quad + \big\|a(e^{-is'\omega}(\xi - \xi_n))e^{-is'H}\Psi\big\|. \end{aligned} \tag{84}$$

By (31), (37) and (78),

$$\begin{aligned} &\big\|a(e^{-is\omega}(\xi - \xi_n))e^{-isH}\Psi\big\| \\ &\quad \leq \left\|\frac{\xi - \xi_n}{\sqrt{\omega}}\right\|(\epsilon c_0\|H\Psi\| + (\epsilon d_0 + \lambda_\epsilon)\|\Psi\|) + \|\xi - \xi_n\|\|\Psi\| \to 0, \end{aligned}$$

as $n \to \infty$. Hence by (84), $\|a_s(\xi)\Psi - a_{s'}(\xi)\Psi\| \to 0$, as $s, s' \to \infty$. $\square$

4.2. Algebraic Properties of the Asymptotic Fields

Lemma 4.2 *Assume* (**S.1**)–(**S.3**) *and* (**S.9**).

(1) *Let $h \in \mathcal{D}(\omega^{-1/2})$. Then for $\Phi, \Psi \in \mathcal{D}(H)$,*

$$(\Phi, a_{\pm\infty}(h)\Psi) = \big(a_{\pm\infty}^*(h)\Phi, \Psi\big).$$

(2) *Let $h, h' \in \mathcal{D}(\omega^{k/2})$, $k = -1, 1, 2$. Then on $\mathcal{D}(H)$,*

$$\begin{aligned} \big[a_{\pm\infty}(h), a_{\pm\infty}^*(h')\big] &= (h, h'), \\ \big[a_{\pm\infty}(h), a_{\pm\infty}(h')\big] = \big[a_{\pm\infty}^*(h), a_{\pm\infty}^*(h')\big] &= 0. \end{aligned}$$

(3) *Let $h \in \mathcal{D}(\omega^{-1/2}) \cap \mathcal{D}(\omega)$. Then it follows that on $\mathcal{D}(|H|^{3/2})$,*

$$[H, a_{\pm\infty}(h)] = -a_{\pm\infty}(\omega h), \quad [H, a_{\pm\infty}^*(h)] = a_{\pm\infty}^*(\omega h).$$

(4) *Let Ψ_E be an eigenvector of H with eigenvalue E. Then for $h \in \mathcal{D}(\omega^{-1/2})$,*

$$a_{\pm\infty}(h)\Psi_E = 0. \tag{85}$$

Proof. (1) is proven in a similar way as ([19, Lemma 4.5]), (2) is ([19, Lemma 4.8]), (3) is ([19, Lemma 4.10]) and (4) is ([19, Lemma 4.11, Lemma 4.12]). □

Proof of Corollary 1.6. Let $h_i \in \mathcal{D}(\omega^{-1/2})$, $i = 1, \ldots, n$. By Lemma 4.2,

$$\begin{aligned} & e^{itH} a^*_{\pm\infty}(h_1) \ldots a^*_{\pm\infty}(h_n)\Omega_g \\ & \quad = e^{itE_0(H)} a^*_{\pm\infty}(e^{it\omega} h_1) \ldots a^*_{\pm\infty}(e^{it\omega} h_n)\Omega_g. \end{aligned} \tag{86}$$

Then e^{itH} leaves $\mathcal{F}_{\pm\infty}$ invariant, and hence H is reduced by $\mathcal{F}_{\pm\infty}$. In addition, we see that

$$\begin{aligned} & W_{\pm\infty} e^{itd\Gamma_{\mathrm{b}}(\omega)} a^*(h_1) \ldots a^*(h_n)\Omega_{\mathrm{b}} \\ & \quad = e^{it(H - E_0(H))} W_{\pm\infty} a^*(h_1) \ldots a^*(h_n)\Omega_{\mathrm{b}}. \end{aligned}$$

Thus we obtain $W_{\pm\infty} e^{it(d\Gamma_{\mathrm{b}}(\omega) + E_0(H))} = e^{itH} W_{\pm\infty}$, on $\mathcal{F}_{\pm\infty}$ and $d\Gamma_{\mathrm{b}}(\omega) + E_0(H) = W^*_{\pm\infty} H_{\restriction \mathcal{F}_{\pm\infty}} W_{\pm\infty}$. Hence the proof is completed. □

5. Concluding Remarks

In this paper we analyzed the GSB-Hamiltonian with a singular perturbation. But this model does not include the Hamiltonian of the system of non-relativistic particles coupled to bose fields

$$H = -\triangle + V + d\Gamma_{\mathrm{b}}(\omega) + \kappa \phi_\Lambda(\mathbf{x})^4,$$

where

$$\phi_\Lambda(\mathbf{x}) = \int_{\mathbf{R}^d} \frac{\chi_\Lambda(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} \big(a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\big) d\mathbf{k},$$

and χ_Λ is the ultraviolet cutoff. Indeed, the singular perturbation H' defined in (15) is finite tensor product. This problem is left for future study.

Appendix A. ([3, Self-adjointness])

Let $\mathcal{X}_n$, $n \geqslant 0$, be a sequence of a Hilbert space. We consider the infinite direct sum of $\mathcal{X}_n$:

$$\mathcal{X} = \bigoplus_{n=0}^{\infty} \mathcal{X}_n$$

with the inner product $(y, x)_{\mathcal{X}} = \sum_{n=0}^{\infty}(y^{(n)}, x^{(n)})_{\mathcal{X}_n}$ for $x = \{x^{(n)}\}_{n=0}^{\infty}$, $\{y^{(n)}\}_{n=0}^{\infty} \in \mathcal{X}$. Let S_0 be a self-adjoint operator on $\mathcal{X}$ with $S_0(\mathcal{D}(S_0) \cap \mathcal{X}_n) \subset \mathcal{X}_n$ and S' be a symmetric operator on $\mathcal{X}$ such that $\mathcal{D}(S') \supset \mathcal{D}_{\mathcal{X}}^{\mathrm{fin}}$, where

$$\mathcal{D}_{\mathcal{X}}^{\mathrm{fin}} := \left\{ x = \{x^{(n)}\}_{n=0}^{\infty} \in \mathcal{X} \mid x^{(k)} = 0 \text{ for all } k > J \text{ with some } J \right\}.$$

The number operator $N_{\mathcal{X}}$ on $\mathcal{X}$ is defined by $(N_{\mathcal{X}} x)^{(n)} = n x^{(n)}$. Let us define S the symmetric operator by

$$S = S_0 + S'.$$

We introduce the following assumptions:

(A.1) There exist a constant $c > 0$ and linar operator L on $\mathcal{X}$ such that $\mathcal{D}((S_{\restriction \mathcal{D}(S_0) \cap \mathcal{D}_{\mathcal{X}}^{\mathrm{fin}}})^*) \subset \mathcal{D}(L)$, $L(\mathcal{D}(L) \cap \mathcal{X}_n) \subset \mathcal{X}_n$, for all $n \geq 0$, and

$$|(y, x)| \leqslant c \|Ly\| \|(N_{\mathcal{X}} + 1)^2 x\|, \quad x, y \in \mathcal{D}(S_0) \cap \mathcal{D}_{\mathcal{X}}^{\mathrm{fin}}.$$

(A.2) There exists an integer $p \geqslant 0$ such that for all $x \in \mathcal{D}_{\mathcal{X}}^{fin}$,

$$\left(x^{(m)}, S' x^{(n)}\right)_{\mathcal{X}} = 0 \quad \text{for } |m - n| \geqslant p + 1.$$

(A.3) S is bounded from below.

Theorem A ([3, Theorem 2.1]) *Suppose that* (A.1)–(A.3). *Then S is essentially self-adjoint on $\mathcal{D}(S_0) \cap \mathcal{D}_{\mathcal{X}}^{fin}$.*

Appendix B. [4, Essential spectrum]

Let $X_0 = A \otimes I + I \otimes d\Gamma_{\mathrm{b}}(\omega)$, and

$$X = X_0 + qX', \quad q \in \mathbf{R}.$$

We assume the following conditions:

(E.1) The operator A is self-adjoint and bounded from below.

(E.2) X' is a symmetric operator on $\mathcal{H}$.
(E.3) X is self-adjoint and bounded from bellow.
(E.4) For the sequence $\{\xi_n\}_{n=1}^{\infty}$ of $\mathcal{D}(\omega) \cap \mathcal{D}(\omega^{-1/2})$ such that $\|\xi_n\| = 1$, $n \geq 1$, and $w - \lim_{n\to\infty} \xi_n = 0$, it follows that for $\Psi \in \mathcal{D}(X)$,

$$\lim_{n\to\infty} \left\{ ((X')^*\Psi, I \otimes a(\xi_n)^*\Psi) - (I \otimes a(\xi_n)\Psi, X'\Psi) \right\} = 0.$$

Theorem B ([4, Theorem 1.3]) *Assume* (E.1)–(E.4) *and* $\sigma(\omega) = [0.\infty)$. *Then* $\sigma(X) = \sigma_{ess}(X) = [E_0(X), \infty)$.

Appendix C. [20, Uniqueness of ground states]

Let $X_0 = A \otimes I + I \otimes d\Gamma_{\mathrm{b}}(\omega)$, and

$$X = X_0 + qX', \quad q \in \mathbf{R}.$$

We introduce the following assumptions:

(H.1) The operator A is self-adjoint and bounded from below.
(H.2) X' is a symmetric operator on $\mathcal{H}$, and there exist constants $a \geq 0$ and $b \geq 0$ such that

$$\|X'\Psi\| \leq a\|X_0\Psi\| + b\|\Psi\|, \quad \Psi \in \mathcal{D}(H_0).$$

(H.3) There exists an operator $S(\mathbf{k}) : \mathcal{H} \to \mathcal{H}$, $\mathbf{k} \in \mathbf{R}^3$, such that for $\Phi, \Psi \in \mathcal{D}(H_0)$,

$$(I \otimes a^*(f)\Phi, X'\Psi) - (X'\Phi, I \otimes a(f)\Psi) = \int_{\mathbf{R}^d} \overline{f(\mathbf{k})} (\Phi, S(\mathbf{k})\Psi) d\mathbf{k}.$$

(H.4) Let $\Phi \in \mathcal{D}(X_0)$, $f \in C^{\infty}(\mathbf{R}^d)$ and $S(\mathbf{k})$ in (H.3). Then for any ground state φ of X, it follows that

$$\int_{\mathbf{R}^d} \overline{f(\mathbf{k})} \big(\Phi, e^{it(X(q) - E_0(X(q)) + \omega(\mathbf{k}))} S(\mathbf{k})\varphi\big) d\mathbf{k} \in L^1([0, \infty), dt),$$

and $\int_{\mathbf{R}^d} \|S(\mathbf{k})\varphi\|^2 d\mathbf{k} < \infty$.

Theorem C.1 ([20, Theorem 2.9]) *Assume* (H.1)–(H.4). *Let* φ *be an ground state of* X. *Then* (a) *and* (b) *are equivalent.*

(a) $\varphi \in \mathcal{D}(I \otimes N_b^{1/2})$.
(b) $\int_{\mathbf{R}^d} \|(X - E_0(X) + \omega(\mathbf{k}))^{-1} S(\mathbf{k})\varphi\|^2 d\mathbf{k} < \infty$.
In particular, if (a) or (b) holds, then

$$\left\|(I \otimes N_{\mathrm{b}}^{1/2})\varphi\right\|^2 = q^2 \int_{\mathbf{R}^d} \|(X - E_0(X) + \omega(\mathbf{k}))^{-1} S(\mathbf{k})\varphi\|^2 d\mathbf{k}.$$

In addition, we introduce following assumptions:

(H.5) (*Existence of positive spectral gap of* A) *It holds that* $\inf \sigma_{ess}(A) - E_0(A) > 0$.
(H.6) *It follows that*

$$\lim_{q \to 0} \sup_{\varphi \in ker(X - E_0(X)) \setminus \{0\}} q^2 \int_{\mathbf{R}^d} \cdot \left\|(X - E_0(X) + \omega(\mathbf{k}))^{-1} S(\mathbf{k})\varphi\right\|^2 d\mathbf{k} / \|\varphi\|^2 = 0.$$

Theorem C.2 ([20, Theorem 4.2]) *Assume* (H.1)–(H.6)*. Then there exists a constant* $\tilde{q} > 0$ *such that for* $|q| < \tilde{q}$,

$$dim\ ker(X(q) - E_0(X(q))) \leq dim\ ker(A - E_0(A)).$$

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