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A generalization of antipodal point theorems for set-valued mappings

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Abstract. Let U be a bounded symmetric open neighborhood of the origin of $\mathbf{R}^{m+k}$ ($k \geqq 1$). We shall prove a generalization of the Borsuk's antipodal theorem for an admissible mapping $\varphi : \partial\overline{U} \to \mathbf{R}^m$ and the related topic. We shall generalize the theorem for the case of a bounded symmetric open neighborhood U of the origin of an infinite dimensional normed space $\mathbf{E}$. The Borsuk-Ulam theorem shall be studied for the case of a bounded symmetric open neighborhood U of the origin of an infinite dimensional normed space $\mathbf{E}$.

Key words: fixed point theorem, antipodal point theorem, Vietoris's theorem.

1. Introduction

When we assign each point x of X a non empty closed set $\varphi(x)$ in a topological space Y, we call the correspondence a set-valued mapping and write $\varphi : X \to Y$ by the Greek alphabet. For single-valued mappings, we write $f : X \to Y$ etc. by the Roman alphabet. In this paper, we assume that set-valued mappings are upper semi-continuous (cf. Section 14 in L. Górniewicz [8]).

Fixed point theorems for set-valued mappings have been developed by many mathematicians [3], [7]. L. Górniewicz defined admissible mappings (cf. Definition 2.5) in the class of set-valued mappings and proved a fixed point theorem and the Borsuk-Ulam theorem for admissible mappings (cf. Sections 40, 43 in [8]). M. Nakaoka proved many theorems concerning the equivariant theory [14]. In the previous paper [15], the author studied a fixed point theorem and estimated the dimension of the set of equivariant points for admissible mappings. They are generalizations of results of M. Nakaoka [11], [13] and K. Gęba and L. Górniewicz [5], [8]. In this paper, we shall prove a generalization of the Borsuk's antipodal theorem for an admissible mapping $\varphi : \partial\overline{U} \to \mathbf{R}^m$ where U is a bounded symmetric open

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neighborhood of the origin of $\mathbf{R}^{m+k}$ ($k \geqq 1$) and $\partial\overline{U}$ is the boundary of $\overline{U}$. Here symmetricity of U means $T(U) = U$ for the involution $T(x) = -x$. For the simplicity, we sometimes use the same symbol T for the involution. We also study the theorem for the case of a bounded symmetric open neighborhood U of the origin of an infinite dimensional normed space $\mathbf{E}$.

In the second section, we review various cohomology theories and some results. In this paper, we shall mainly use the Alexander-Spanier cohomology theory $\bar{H}^*(X; \mathbf{F})$ with coefficient in a field $\mathbf{F}$.

In the third section, we define equivariant mappings in the class of set-valued mappings (cf. Definition 3.4) and discuss generalizations of the Borsuk's antipodal theorem for admissible mappings. The first Stiefel-Whitney class $c = c(X, T)$ of a space X with a free involution T is defined by $c = f^*(\omega)$ where $f : X_\pi \to RP^\infty$ is the classifying mapping of the projection $p : X \to X_\pi$ and ω is the generator of $H^1(RP^\infty; \mathbf{F}_2)$ where $\mathbf{F}_2$ is the prime field of the order 2. Our main theorem is as follows: (cf. Theorem 3.5).

Main Theorem 1 *Let N be a paracompact Hausdorff space and N_0 its closed subspace with a free involution T_0 and $c^m \neq 0$ for $c = c(N_0, T_0)$ and M an m-dimensional closed manifold with a free involution T'. Assume that $\varphi : N \to M$ is an admissible mapping and is equivariant on N_0. Then, $k^* : \bar{H}^m(N; \mathbf{F}_2) \to \bar{H}^m(N_0; \mathbf{F}_2)$ is not trivial where $k : N_0 \to N$ is the inclusion.*

From the theorem, we obtain a generalization of the Borsuk's antipodal theorem (cf. Corollary 3.6). S. Y. Chang proved a generalization of the Borsuk's antipodal theorem (cf. Theorem 4 in [2]) for closed convex set-valued mappings by using method of general topology and analysis. By using the new definition of equivariant mappings for set-valued mappings which is a generalization of S. Y. Chang's definition, we shall prove the following theorem which is a generalization of his theorem (cf. Theorem 3.8).

Main Theorem 2 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 1$. Assume that $\varphi : \partial\overline{U} \to \mathbf{R}^m$ is an equivariant admissible mapping. Then there exists a point $x_0 \in \partial\overline{U}$ such that $\varphi(x_0) \ni 0$.*

We have the following theorem (cf. Theorem 3.10) which is a generalization of Theorem 6 in [2] and also a generalization of Theorem 9.1, 9.2 of Section 10 in [9] for set-valued mappings.

Main Theorem 3 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^m$. Assume that $\varphi : \overline{U} \to \mathbf{R}^m$ is an admissible mapping which is equivariant on the boundary $\partial\overline{U}$ of $\overline{U}$. Then there exist a point $x_0 \in \overline{U}$ such that $\varphi(x_0) \ni 0$ and a point $x_1 \in \overline{U}$ such that $\varphi(x_1) \ni x_1$.*

In the last section, we discuss a generalization of results of the section 3 to the case of an infinite dimensional normed space. We obtain the following theorem (cf. Theorem 4.4) which is a generalization of Theorem 7 in [2] for the case of an infinite dimensional normed space.

Main Theorem 4 *Let U be a bounded symmetric open neighborhood of the origin of an infinite dimensional normed space $\mathbf{E}$. Assume that $\varphi : \overline{U} \to \mathbf{E}$ is a compact admissible mapping which is equivariant on $\partial\overline{U}$. Then there exist a fixed point $z_0 \in \overline{U}$ such that $\varphi(z_0) \ni z_0$.*

In the above theorem, we can not deduce the existence of the zero value of φ contrary to the finite dimensional version.

We shall prove a generalization of the Borsuk-Ulam theorem for a compact field (cf. Theorem 4.5 and Section 2 in K. Geba and L. Górniewicz [6]).

Main Theorem 5 *Let $\mathbf{E}_k$ be a closed linear subspace of codimension $k \geqq 1$ of an infinite dimensional normed space $\mathbf{E}$ and U be a bounded symmetric open neighborhood of the origin of $\mathbf{E}$. If $\Phi : \partial\overline{U} \to \mathbf{E}_k$ is a compact admissible mapping, there is a point $x_0 \in \partial\overline{U}$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ where $\varphi(x) = x - \Phi(x)$.*

We shall also prove $\mathrm{Ind}A(\varphi) \geqq k-1$ where $A(\varphi) = \{x \in \partial\overline{U} \mid \varphi(x) \cap \varphi(T(x)) \neq \emptyset\}$ (cf. Corollary 4.6).

2. Various cohomology theories

To begin with, we give some remarks about several cohomology theories (cf. Y. Shitanda [15]). The Alexander-Spanier cohomology theory $\bar{H}^*(-;G)$ is isomorphic to the singular cohomology theory $H^*(-;G)$, that is,

$$\mu : \bar{H}^*(X;G) \cong H^*(X;G)$$

if the singular cohomology theory satisfies the continuity condition (cf. Theorem 6.9.1 in [16]).

For a paracompact Hausdorff space X, it holds also the isomorphism between the Čech cohomology theory $\check{H}^*(-;G)$ with coefficient in a constant sheaf and the Alexander-Spanier cohomology theory $\bar{H}^*(-;G)$ (cf. Theorem 6.8.8 in [16])

$$\check{H}^*(X;G) \cong \bar{H}^*(X;G).$$

An ANR space is an r-image of an open set of a normed space (cf. Proposition 1.8 in [8]). For an ANR space X, it holds also the isomorphisms:

$$\check{H}^*(X;G) \cong \bar{H}^*(X;G) \cong H^*(X;G)$$

by Theorem 6.1.10 of [16]. The remarkable feature of the Alexander-Spanier cohomology theory is that it satisfies the continuity property (cf. Theorem 6.6.2 in [16]). Hereafter we mainly use the Alexander-Spanier (co)homology theory with coefficient in a field $\mathbf{F}$.

Let $f : X \to Y$ be a continuous mapping. When $f^{-1}(K)$ is a compact set for any compact subset $K \subset Y$, f is called a proper mapping. f is called a perfect mapping, if f is a closed mapping and any preimage $f^{-1}(y)$ is a compact set for each $y \in Y$. A perfect mapping is a proper mapping by Theorem 3.7.2 in R. Engelking [4]. For the case that Y is a metric space, a proper mapping f is a closed mapping (cf. Proposition 1.8.1 in [8]). We can not find any proof on the proposition for the general case. Since perfect mappings behave better than proper mappings, we adopt perfect mappings for the definition of Vietoris mappings (cf. Definition 1.4.6 in [1]).

The following proposition is essentially proved in Theorem 3.3.22 in [4].

Proposition 2.1 *Let X be a Hausdorff space and Y a compactly generated Hausdorff space. If $f : X \to Y$ is a onto continuous and proper mapping, it is a closed mapping.*

Compactly generated Hausdorff spaces are k-space (cf. Corollary 3.3.19 in [4]). Metric spaces and CW-complexes are compactly generated Hausdorff spaces. Clearly there exists a Hausdorff space X which is not a compactly generated. We give a compactly generated Hausdorff topology for X. The space is denoted by $\hat{X}$. The identity mapping $f : \hat{X} \to X$ is a proper continuous mapping. Take F which is not a closed set in X and is a closed set in $\hat{X}$. Since $f(F)$ is not a closed set, f is not a closed mapping.

Let X and Y are Hausdorff spaces with free involutions. If an equivariant mapping $f : X \to Y$ is a perfect mapping, $f_\pi : X_\pi \to Y_\pi$ is also a perfect mapping. A mapping $f : X \to Y$ is called a compact mapping, if $f(X)$ is contained in a compact set of Y, or equivalently its closure $\overline{f(X)}$ is compact.

Definition 2.2 Let X and Y be paracompact Hausdorff spaces. A mapping $f : X \to Y$ is called a Vietoris mapping, if it satisfies the following conditions:

1. f is a perfect and onto continuous mapping.
2. $f^{-1}(y)$ is an acyclic space for any $y \in Y$, that is, it is a connected space and $\bar{H}^*(f^{-1}(y); \mathbf{F}) = 0$ for positive dimensions.

When f is closed and onto continuous mapping and satisfies the condition (2), we call it weak Vietoris mapping.

We prepare a Lemma.

Lemma 2.3 *Let $p : X \to Y$ a perfect mapping and $f : Z \to Y$ a continuous mapping. Then $q : W \to Z$ is a perfect mapping where*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"] & X \arrow[d, "p"] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$

is a pull-back square. Especially if p is a Vietoris mapping, q is also a Vietoris mapping.

Proof. Consider the following diagram:

$$\begin{tikzcd} W \arrow[r, "{(q,g)}"] \arrow[d, "q"] & Z \times X \arrow[r, "Pr_X"] \arrow[d, "Id_Z \times p"] & X \arrow[d, "p"] \\ Z \arrow[r, "{(Id_Z,f)}"] & Z \times Y \arrow[r, "Pr_Y"] & Y \end{tikzcd}$$

where Pr_X and Pr_Y are the projections to X and Y respectively, $(Id_Z, f)(z) = (z, f(z))$. Since p is a perfect mapping, $Id_Z \times p$ is also a perfect mapping by Theorem 3.7.9 in [4]. Z is a closed subspace of $Z \times Y$. Therefore q is a perfect mapping by Proposition 3.7.6 in [4]. $\square$

Theorems in the previous paper [15] are valid for the case of perfect mapping instead of proper mappings, for example Theorem 3.9 and 3.10 and Lemma 5.4 and therefore Theorem 5.5 and 6.3 etc.

The following theorem is called the Vietoris's theorem and is important for our purpose (cf. Theorem 6.9.15 in [16]).

Theorem 2.4 *Let $f : X \to Y$ be a weak Vietoris mapping between paracompact Hausdorff spaces X and Y. Then,*

$$f^* : \bar{H}^m(Y; \mathbf{F}) \to \bar{H}^m(X; \mathbf{F})$$

is an isomorphism for all $m \geqq 0$.

The graph of a set-valued mapping $\varphi : X \to Y$ is defined by $\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$. If φ is upper semi-continuous, Γ_φ is closed, but the converse is not true. If the image $\varphi(X)$ is contained in a compact set, the converse is true (cf. Section 14 in [8]).

Definition 2.5 An upper semi-continuous mapping $\varphi : X \to Y$ is admissible, if there exists a paracompact Hausdorff space Γ satisfying the following conditions:

1. there exist a Vietoris mapping $p : \Gamma \to X$ and a continuous mapping $q : \Gamma \to Y$,
2. $\varphi(x) \supset q(p^{-1}(x))$ for each $x \in X$.

A pair (p, q) of mappings p and q is called a selected pair of φ.

Define $\varphi^* : \bar{H}^*(Y; G) \to \bar{H}^*(X; G)$ by the set $\{(p^*)^{-1} q^*\}$ where (p, q) is a selected pair of admissible mapping $\varphi : X \to Y$. And φ_* is similarly defined by the set $\{q_*(p_*)^{-1}\}$.

3. Borsuk's antipodal theorem

Let X and Y be spaces with involutions T and T' respectively. $g : X \to Y$ is called an equivariant mapping, if it satisfies $g(Tx) = T'g(x)$ for $x \in X$. The classical Borsuk's antipodal theorem is stated as follows. A continuous mapping $f : S^m \to \mathbf{R}^m$ has at least one pair of antipodal points to the same point, that is, there exists a point $x_0 \in S^m$ such that $f(x_0) = f(-x_0)$ (cf. Theorem 5.2 of Section 5 in [9]). Equivalently an equivariant mapping $f : S^{m-1} \to S^{m-1}$ is not null-homotopic. In other words an equivariant mapping $f : S^m \to R^m$ has the zero value. The classical Borsuk's fixed point theorem is stated as follows. A continuous mapping $f : D^m \to \mathbf{R}^m$ which is equivariant on the boundary $\partial D^m = S^{m-1}$ has a fixed point (cf. Theorem 6.2 of Section 5 in [9]).

The Borsuk's antipodal theorem is generalized to the following theorem (cf. Theorem 9.2 of Section 10 in [9]).

Theorem 3.1 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^m$. Assume that the closure $\overline{U}$ of U is a finite polyhedron and $f : \overline{U} \to \mathbf{R}^m$ be a continuous mapping which is equivariant on the boundary $\partial\overline{U}$ of $\overline{U}$. Then f has the zero value, that is, there exists a point $x_0 \in \overline{U}$ such that $f(x_0) = 0$.*

S. Y. Chang proved the following Borsuk antipodal theorem for upper semi-continuous mappings which are closed convex set-valued (cf. Theorem 4 in [2]). Let X and Y be two normed spaces and N a closed subset of X. According to his paper, a convex set-valued mapping $\psi : N \to Y$ is called an antipodal mapping on a symmetric subset N_0 of N, if ψ satisfies $\psi(x) \cap (-\psi(-x)) \neq \emptyset$ for all $x \in N_0$.

Theorem 3.2 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^{m+1}$, and $\psi : \partial\overline{U} \to \mathbf{R}^m$ be upper semi-continuous, closed convex set-valued, and antipodal preserving. Then ψ has the zero value, that is, there exists a point $x_0 \in \overline{U}$ such that $\psi(x_0) \ni 0$.*

In this paper, we shall give a generalization of the above theorems. We prepare a theorem for our purpose (cf. Theorem 4.3 in [13]).

Theorem 3.3 *Let N be a paracompact Hausdorff space with a free involution T and M an m-dimensional closed manifold with a free involution T'. Assume that $c^m \neq 0$ for $c = c(N,T) \in \bar{H}^1(N_\pi; \mathbf{F}_2)$ and $f : N \to M$ is an equivariant mapping. Then $f^* : \bar{H}^m(M; \mathbf{F}_2) \to \bar{H}^m(N; \mathbf{F}_2)$ is not trivial.*

Proof. Let $h : M \to S^\infty$ be an equivariant mapping such that $h_\pi^*(\omega) = c(M,T')$. Here ω is the generator of $\bar{H}^1(RP^\infty; \mathbf{F}_2)$. $hf : N \to S^\infty$ is also an equivariant mapping such that $(hf)_\pi^*(\omega) = c(N,T)$. From $c(N,T)^m \neq 0$, it holds $c(M,T')^m \neq 0$. By the Gysin-Smith exact sequence, we see $\phi^*(c_M) = c(M,T')^m$ where c_M is the dual cocycle of the m-dimensional fundamental cycle $[M]$. By

$$\phi^* f^*(c_M) = f_\pi^* \phi^*(c_M) = f_\pi^*(c(M,T')^m) = c(N,T)^m \neq 0,$$

we obtain $f^*(c_M) \neq 0$. $\square$

In this paper we adopt a new definition of an equivariant mapping for set-valued mappings.

Definition 3.4 Let X and Y be paracompact Hausdorff spaces with involutions T and T' respectively. An admissible mapping $\varphi : X \to Y$ is said to be equivariant, if there exist a paracompact Hausdorff space Γ with a free involution and an equivariant Vietoris mapping $p : \Gamma \to X$ and an equivariant continuous mapping $q : \Gamma \to Y$ such that $qp^{-1}(x) \subset \varphi(x)$ for $x \in X$. An admissible mapping $\varphi : X \to Y$ is said to be equivariant on a closed subspace X_0 of X, if there exists an equivariant Vietoris mapping $p_0 : \Gamma_0 \to X_0$ and equivariant mapping $q_0 : \Gamma_0 \to Y$ and satisfies the following commutativity:

$$\begin{tikzcd} X_0 \arrow[d, "k"] & \Gamma_0 \arrow[l, "p_0"'] \arrow[r, "q_0"] \arrow[d, "i"] & Y \arrow[d, "id"] \\ X & \Gamma \arrow[l, "p"'] \arrow[r, "q"] & Y \end{tikzcd} \tag{1}$$

where (p,q) is a selected pair of φ and k and i are closed inclusions. It holds also $q_0 p_0^{-1}(x) \subset \varphi_0(x)$ for $x \in X_0$ where $\varphi_0(x) = \varphi(x) \cap T'\varphi(T(x))$.

Our definition is a generalization of S. Y. Chang's definition. In fact for a convex set-valued mapping $\psi : N \to Y$ which is equivariant on a symmetric subset N_0 in the sense of S. Y. Chang, consider $\psi_0 : N_0 \to Y$ defined by $\psi_0(x) = \psi(x) \cap (-\psi(-x))$ for $x \in N_0$. It is an acyclic mapping and satisfies $\psi_0(-x) = -\psi_0(x)$ for $x \in N_0$. The projection $p_0 : \Gamma_{\psi_0} \to N_0$ is an equivariant Vietoris mapping and the projection $q_0 : \Gamma_{\psi_0} \to Y$ is an equivariant continuous mapping. p_0 and q_0 satisfy $q_0 p_0^{-1}(x) = \varphi_0(x)$ for $x \in N_0$. The projections $p : \Gamma_\psi \to N$, $q : \Gamma_\psi \to Y$ and p_0, q_0 satisfy our definition of an equivariant mapping.

In some cases we can relax the condition of equivariant mappings in Definition 3.4. For example we replace the commutativity to the homotopy commutativity in the diagram (1). For an equivariant mapping $\varphi : X \to Y$, a set $A(\varphi)$ is defined by $\{x \in X \mid \varphi(Tx) \cap T'\varphi(x) \neq \emptyset\}$.

Now we state our main result.

Theorem 3.5 *Let N be a paracompact Hausdorff space and N_0 its closed subspace with a free involution T_0 and $c^m \neq 0$ for $c = c(N_0, T_0)$ and M an m-dimensional closed manifold with a free involution T'. Assume that $\varphi : N \to M$ is an admissible mapping and is equivariant on N_0. Then, $k^* : \bar{H}^m(N; \mathbf{F}_2) \to \bar{H}^m(N_0; \mathbf{F}_2)$ is not trivial where $k : N_0 \to N$ is the inclusion.*

Proof. Let $p : \Gamma \to N$ and $q : \Gamma \to M$ be a selected pair of φ. Let $p_0 : \Gamma_0 \to N_0$ and $q_0 : \Gamma_0 \to M$ be a selected pair of $\varphi_0 : N_0 \to M$ where $p_0 : \Gamma_0 \to N_0$ is an equivariant Vietoris mapping and $q_0 : \Gamma_0 \to M$ is an equivariant mapping. Since $\varphi : N \to M$ is an admissible mapping and equivariant on N_0, we have the following diagram:

$$\begin{tikzcd} N_0 \arrow[d, "k"] & \Gamma_0 \arrow[l, "p_0"'] \arrow[r, "q_0"] \arrow[d, "i"] & M \arrow[d, "id"] \\ N & \Gamma \arrow[l, "p"'] \arrow[r, "q"] & M. \end{tikzcd}$$

Since p_0 is an equivariant Vietoris mapping, $p_{0\pi}$ is a Vietoris mapping and $(p_{0\pi})^*$ is an isomorphism. By $c(\Gamma_0, T_1)^m = (p_{0\pi})^*(c(N_0, T_0)^m) \neq 0$ where T_1 is a free involution of Γ_0, q_0^* is not trivial for the m-dimension by Theorem 3.3.

Let $i : \Gamma_0 \to \Gamma$ be the natural inclusion. If $k^* = 0$ for the m-dimension, we see $i^* = 0$ for the m-dimension by the isomorphisms $\bar{H}^*(N_0; \mathbf{F}_2) \cong \bar{H}^*(\Gamma_0; \mathbf{F}_2)$, $\bar{H}^*(N; \mathbf{F}_2) \cong \bar{H}^*(\Gamma; \mathbf{F}_2)$ and the commutativity $p_0^* k^* = i^* p^*$. Since it holds $q_0 = qi$ and $i^* = 0$ for the m-dimension, we obtain $q_0^* = 0$ for the m-dimension. This contradicts to the non triviality of q_0^*. Therefore we see $k^* \neq 0$ for the m-dimension. □

If we take $\mathbf{R}^{m+1} - \{0\}$ in the place of M in Theorem 3.3 and 3.5, we have the similar statements. Since the proofs are entirely similar to Theorem 3.3 and Theorem 3.5, we omit the proofs.

Corollary 3.6 *Let N be a paracompact Hausdorff space and N_0 its closed subspace with a free involution T_0 and $c^{m-1} \neq 0$ for $c = c(N_0, T_0)$. Assume that $k^* : \bar{H}^{m-1}(N; \mathbf{F}_2) \to \bar{H}^{m-1}(N_0; \mathbf{F}_2)$ is trivial for the inclusion $k : N_0 \to N$ and $\varphi : N \to \mathbf{R}^m$ is an admissible mapping and is equivariant on N_0. Then, there exists a point $x_0 \in N$ such that $\varphi(x_0) \ni 0$.*

Let N be a closed domain in $\mathbf{R}^m$ which is a symmetric polyhedron with respect to the involution $T(x) = -x$ and its boundary N_0 be a connected manifold. Since it holds the isomorphism $\delta^* : \bar{H}^{m-1}(N_0; \mathbf{F}_2) \to \bar{H}^m(N, N_0; \mathbf{F}_2)$, we see $i^* = 0 : \bar{H}^{m-1}(N; \mathbf{F}_2) \to \bar{H}^{m-1}(N_0; \mathbf{F}_2)$. Therefore we have a generalization of the Borsuk's antipodal theorem for an admissible mapping $\varphi : N \to \mathbf{R}^m$ from the above corollary.

We shall generalize Theorem 3.1 and 3.2 in what follows. Let $\partial\overline{U}$ be the boundary of $\overline{U}$, that is, $\partial\overline{U} = \overline{U} - Int\overline{U}$.

Proposition 3.7 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^n$. It holds $c^{n-1}(\partial\overline{U}, T) \neq 0$.*

Proof. To begin with, we shall prove the case that $\partial\overline{U}$ is a topological manifold. Set $M = \overline{U - D}$ where D is an open disk centered at 0 with a small radius $r > 0$. M is a topological manifold with boundary which has the free involution T. We have $i^*(c(M,T)) = c(\partial\overline{U}, T)$ for the inclusion $i : \partial\overline{U} \to M$ and $j^*(c(M,T)) = c(\partial\overline{D}, T)$ for the inclusion $j : \partial\overline{D} \to M$. We can prove the following formula:

$$c^{n-1}(\partial\overline{U}, T)\big[(\partial\overline{U})_\pi\big] = c^{n-1}(S^{n-1}, T)\big[S^{n-1}_\pi\big]$$

by the method of Theorem 4.9 in J. Milnor [10]. Since $c^{n-1}(S^{n-1}, T)$ is not zero, we obtain

$$c^{n-1}(\partial\overline{U}, T) \neq 0. \tag{2}$$

We shall prove the general case. We cover $\overline{U}$ by finitely many open disks $\{V_\alpha\}_{\alpha\in A}$ with a small radius below $r > 0$ such that $\overline{U} \subset \cup_{\alpha\in A} V_\alpha$. Here we can assume that $\{V_\alpha\}_{\alpha\in A}$ contains both of V_α and TV_α for $\alpha \in A$. We may assume that $W = \cup_{\alpha\in A}\overline{V_\alpha}$ is a manifold with boundary. Moreover we may assume that the boundary ∂W is a manifold. If ∂W is not a manifold, it happened at a point x where two closed disks $\overline{V}_1$ and $\overline{V}_2$ are tangent each other. Since the point x is clearly outside of $\overline{U}$ or on $\partial\overline{U}$, it is sufficient to add two small disks symmetrically at x and $T(x)$. Therefore we have

$$c^{n-1}(\partial W, T) \neq 0$$

as the above proof.

Set $\overline{U}_r = \{x \in \overline{U} \mid d(x, \partial\overline{U}) \geqq 2r\}$ where $d(x, \partial\overline{U})$ is the distance between x and $\partial\overline{U}$. We cover $\overline{U}_r$ symmetrically by finitely many open disks $\{V'_\beta\}_{\beta\in B}$ with a small radius below $r > 0$ such that $\overline{U}_r \subset \cup_{\beta\in B} V'_\beta \subset \overline{U}$. Set $W' = \cup_{\beta\in B}\overline{V'_\beta}$. We may assume that W' is a manifold with boundary and satisfies $W' \subset Int\overline{U}$. By $\partial(W - IntW') = \partial W \cup \partial W'$, we obtain

$$c^{n-1}(\partial W', T) \neq 0, \quad c^{n-1}(W - IntW', T) \neq 0.$$

Since families $\{IntW - W'\}$ and $\{W - IntW'\}$ are cofinal coverings of $\partial\overline{U}$, we have the isomorphism:

$$\bar{H}^*(\partial\overline{U}) \cong \lim_{\rightarrow} \bar{H}^*(IntW - W') \cong \lim_{\rightarrow} \bar{H}^*(W - IntW')$$

by the continuity of the Alexander-Spanier cohomology theory. By the naturality of Stiefel-Whitney class with respect to $\{W - IntW'\}$, we have

$$c^{n-1}(\partial\overline{U}, T) \neq 0 \tag{3}$$

for general case. $\square$

The following theorem is a generalization of Theorem 4 of S. Y. Chang [2].

Theorem 3.8 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 1$. Assume that $\varphi : \partial\overline{U} \to \mathbf{R}^m$ is an equivariant admissible mapping. Then there exists a point $x_0 \in \partial\overline{U}$ such that $\varphi(x_0) \ni 0$.*

Proof. By our assumption, there exists an equivariant Vietoris mapping $p_0 : \Gamma_0 \to \partial\overline{U}$ and an equivariant mapping $q_0 : \Gamma_0 \to \mathbf{R}^m$ such that $q_0 p_0^{-1}(x) \subset \varphi(x)$ for $x \in \partial\overline{U}$. By our hypothesis, we obtain

$$c(\Gamma_0, T') = p^*_{0\pi}\big(c(\partial\overline{U}, T)\big) \neq 0. \tag{4}$$

Assume that $\varphi(x)$ does not contain the origin of $\mathbf{R}^m$. q_0 is considered as $q_0 : \Gamma_0 \to \mathbf{R}^m - \{0\}$. Since q_0 is equivariant, we obtain

$$q^*_{0\pi}(c) = c(\Gamma_0, T') \tag{5}$$

by the proof of Theorem 3.3 where c is the first Stiefel-Whitney class of $\mathbf{R}^m - \{0\}$. From the results (4), (5), we have

$$(q_{0\pi})^*(c^{m+k-1}) = c(\Gamma_0, T')^{m+k-1} = (p_{0\pi})^*\big(c(\partial\overline{U}, T)^{m+k-1}\big). \qquad (6)$$

The left side of the equation is zero by $c^m = 0$ and the right side is not zero by Proposition 3.7 and the bijectivity of $(p_{0\pi})^*$. From the contradiction, we obtain the conclusion. □

Let ∂U be defined by $\partial U = \overline{U} - U$. Generally ∂U and $\partial\overline{U}$ are different and $\partial\overline{U} \subset \partial U$. It is easily seen that the above theorem holds for the case ∂U.

From Proposition 3.7, we obtain the following corollary which is a generalization of Theorem 5 of S. Y. Chang [2].

Corollary 3.9 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 1$. If $\varphi : \partial\overline{U} \to \mathbf{R}^m$ is an admissible mapping, then there exists point $x_1 \in \partial\overline{U}$ such that $\varphi(x_1) \cap \varphi(T(x_1)) \neq \emptyset$.*

Proof. We consider $\mathbf{R}^m$ as the subspace of S^m. From Proposition 3.7, we see $c(\partial\overline{U}, T)^{m+k-1} \neq 0$. Since φ^* contains the trivial element, there exists an element $x_1 \in \partial\overline{U}$ such that $\varphi(x_1) \cap \varphi(T(x_1)) \neq \emptyset$ by Theorem 6.3 in [15]. □

Let A, B and C be paracompact Hausdorff spaces. Let $i : A \to B$ be a closed embedding and $f : A \to C$ a closed continuous mapping. Then we have a space D obtained by the push-out of $B \xleftarrow{i} A \xrightarrow{j} C$. That is, $D = (B \cup C)/\equiv$ where $i(a) \equiv a \equiv j(a)$. Then $l : C \to D$ is also closed embedding and $k : B \to D$ a closed continuous mapping. D is a paracompact Hausdorff space by Theorem 5.1.33 and 5.1.34 in [4]. Therefore we obtain the excision isomorphism $\bar{H}^*(B, A; \mathbf{F}) \cong \bar{H}^*(D, C; \mathbf{F})$ by Theorem 6.6.5 in [16] and Mayer-Vietoris exact sequence:

$$\to \bar{H}^*(D; \mathbf{F}) \to \bar{H}^*(B; \mathbf{F}) \oplus \bar{H}^*(C; \mathbf{F}) \to \bar{H}^*(A; \mathbf{F}) \to \bar{H}^{*+1}(D; \mathbf{F}) \to . \quad (7)$$

The following theorem is a generalization of Theorem 6 of S. Y. Chang [2]. We use the same symbol T for the involution $T(x) = -x$ of the Euclidean spaces $\{\mathbf{R}^n\}$. Note that $\mathbf{R}^m$ is the subspace of $\mathbf{R}^n$ as the first m-coordinates for $m < n$.

Theorem 3.10 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^m$. Assume that $\varphi : \overline{U} \to \mathbf{R}^m$ is an admissible mapping which is equivariant on the boundary $\partial\overline{U}$ of $\overline{U}$. Then there exist a point $x_0 \in \overline{U}$*

such that $\varphi(x_0) \ni 0$ *and a point* $x_1 \in \overline{U}$ *such that* $\varphi(x_1) \ni x_1$.

Proof. We define a new open neighborhood V of the origin in $\mathbf{R}^{m+1}$:

$$V = \{(x,s) \in \mathbf{R}^{m+1} \mid x \in Int\overline{U},\ |s| < d(x, \partial\overline{U})\}.$$

Clearly V is an open neighborhood of the origin in $\mathbf{R}^{m+1}$ and bounded symmetric with respect to the antipodal involution in $\mathbf{R}^{m+1}$. We easily see:

$$\overline{V} = \{(x,s) \in \mathbf{R}^{m+1} \mid x \in \overline{U},\ |s| \leqq d(x, \partial\overline{U})\}.$$

The boundary $\partial\overline{V}$ of $\overline{V}$ is

$$\partial\overline{V} = \{(x,s) \in \mathbf{R}^{m+1} \mid x \in \overline{U},\ |s| = d(x, \partial\overline{U})\}.$$

Define the mapping $J : \overline{U} \to \mathbf{R}^{m+1}$ by

$$J(x) = x + d(x, \partial\overline{U})e_{m+1}$$

where $x \in \mathbf{R}^m$ and e_{m+1} is the $(m+1)$-th unit vector in $\mathbf{R}^{m+1}$. Note $d(-x, \partial\overline{U}) = d(x, \partial\overline{U})$. Clearly we see $\partial\overline{V} = J(\overline{U}) \cup \{TJ(\overline{U})\}$. By Proposition 3.7, we have $c(\partial\overline{V}, T)^m \neq 0$ and $c(\partial\overline{U}, T)^{m-1} \neq 0$.

Let $\hat{\varphi} : \overline{U} \to \mathbf{R}^m$ be defined as follows:

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in Int\overline{U} \\ \varphi(x) \cup T\varphi(Tx) & \text{if } x \in \partial\overline{U}. \end{cases} \tag{8}$$

Here $\varphi(x) \cup T\varphi(Tx)$ is a push out of the inclusions $\varphi(x) \cap T\varphi(Tx) \to \varphi(x)$ and $\varphi(x) \cap T\varphi(Tx) \to T\varphi(Tx)$. Since φ is upper semi-continuous, we can easily verify that $\hat{\varphi}$ is upper semi-continuous. Since φ is an equivariant admissible mapping on $\partial\overline{U}$, we can easily verify that $\hat{\varphi}$ is equivariant admissible on $\partial\overline{U}$. Note $\hat{\varphi}(Tx) = T\hat{\varphi}(x)$ for $x \in \partial\overline{U}$.

Define $\Psi : \partial\overline{V} \to \mathbf{R}^m$ by

$$\Psi(z) = \begin{cases} \hat{\varphi}(J^{-1}(z)) & \text{if } z \in J(\overline{U}) \\ T\hat{\varphi}(J^{-1}(Tz)) & \text{if } z \in TJ(\overline{U}). \end{cases} \tag{9}$$

In other words, $\Psi(z) = \hat{\varphi}(x)$ if $z = J(x)$ and $\Psi(z) = T\hat{\varphi}(x)$ if $z = -J(x)$.

Ψ is well-defined and an upper semi-continuous mapping defined on $\partial\overline{V}$. It holds $\Psi(Tz) = T\Psi(z)$ for $z \in \partial\overline{V}$.

Let $p : \Gamma \to \overline{U}$ and $q : \Gamma \to \mathbf{R}^m$ be a selected pair of φ and $p_0 : \Gamma_0 \to \partial\overline{U}$ and $q_0 : \Gamma_0 \to \mathbf{R}^m$ be a selected pair of $\varphi_0(x) = \varphi(x) \cap T\varphi(T(x))$. We shall show that Ψ is equivariant on $\partial\overline{V}$. Let $i_1 : \Gamma_0 \to \Gamma_1$ be defined by the inclusion $i : \Gamma_0 \to \Gamma$. T_1, T_2 and i_2 are defined by the following first push-out diagram and $\hat{\Gamma}$ is defined by the second push-out diagram:

$$\begin{tikzcd} \Gamma_0 \arrow[r, "T"] \arrow[d, "i_1"] & \Gamma_0 \arrow[r, "T"] \arrow[d, "i_2"] & \Gamma_0 \arrow[d, "i_1"] \\ \Gamma_1 \arrow[r, "T_1"] & \Gamma_2 \arrow[r, "T_2"] & \Gamma_1 \end{tikzcd} \qquad \begin{tikzcd} \Gamma_0 \arrow[r, "i_2"] \arrow[d, "i_1"] & \Gamma_2 \arrow[d, "k_2"] \\ \Gamma_1 \arrow[r, "k_1"] & \hat{\Gamma}. \end{tikzcd} \tag{10}$$

The relations $T_2T_1 = Id_{\Gamma_1}$, $T_1T_2 = Id_{\Gamma_2}$ etc. hold. $\hat{\Gamma}$ has the involution $\hat{T}$ induced by the following diagram and the definition of $\hat{\Gamma}$:

$$\begin{tikzcd} \Gamma_1 \arrow[d, "T_1"] & \Gamma_0 \arrow[l, "i_1"'] \arrow[r, "i_2"] \arrow[d, "T"] & \Gamma_2 \arrow[d, "T_2"] \\ \Gamma_2 & \Gamma_0 \arrow[l, "i_2"'] \arrow[r, "i_1"] & \Gamma_1. \end{tikzcd} \tag{11}$$

$\hat{p} : \hat{\Gamma} \to \partial\overline{V}$ is defined by

$$\hat{p}(y) = \begin{cases} J(p(y)) & \text{if } y \in \Gamma_1 \\ TJ(p(T_2(y))) & \text{if } y \in \Gamma_2. \end{cases} \tag{12}$$

In other words, $\hat{p}(y) = J(p(y))$ if $y \in \Gamma_1$ and $\hat{p}(y) = TJ(p(y'))$ if $y = T_1(y') \in \Gamma_2$. We see easily $\hat{p} : \hat{\Gamma} \to \partial\overline{V}$ is a Vietoris mapping by (7). It holds $\hat{p}(\hat{T}(y)) = T\hat{p}(y)$ for $y \in \hat{\Gamma}$. Note that $\partial\overline{V}$ is defined as same as $\hat{\Gamma}$ (cf. diagram (10), (11)). Therefore $\hat{p}$ is well-defined by $pi_1 = j_1p_0$ and $j_1 : \partial\overline{U} \to \overline{U}$.

$\hat{q} : \hat{\Gamma} \to \mathbf{R}^m$ is defined by

$$\hat{q}(y) = \begin{cases} q(y) & \text{if } y \in \Gamma_1 \\ Tq(T_2(y)) & \text{if } y \in \Gamma_2. \end{cases} \tag{13}$$

In other words, $\hat{q}(y) = q(y)$ if $y \in \Gamma_1$ and $\hat{q}(y) = Tq(y')$ if $y = T_1(y') \in \Gamma_2$. It holds $\hat{q}(\hat{T}(y)) = T\hat{q}(y)$ for $y \in \hat{\Gamma}$. Therefore $\hat{q}$ is well-defined by $q_0 = qi_1$. By $c(\partial\overline{V}, T)^m \neq 0$, we apply Theorem 3.8 to our case. Then we obtain a point $z_0 \in \partial\overline{V}$ such that $\Psi(z_0) \ni 0$. This means $\varphi(x_0) \ni 0$ for a point $x_0 \in \overline{U}$.

For the second part, define $\varphi_1 : \overline{U} \to \mathbf{R}^m$ by $\varphi_1(x) = x - \varphi(x)$ for $x \in \overline{U}$. $p : \Gamma \to \overline{U}$ and $p - q : \Gamma \to \mathbf{R}^m$ are a selected pair of φ_1. We easily verify that φ_1 is equivariant on $\partial\overline{U}$ by our hypothesis on φ. By applying the former part of this theorem to the case, there exists an element $x_1 \in \overline{U}$ such that $\varphi_1(x_1) \ni 0$, i.e. $\varphi(x_1) \ni x_1$. □

For an open set U of a normed space $\mathbf{E}$, it is said to be balanced if satisfies $sU \subset U$ for all s, $(0 \leqq s \leqq 1)$. Since a bounded symmetric open balanced space U satisfies the condition of the above theorem, we obtain easily Theorem 6 of S. Y. Chang [2]. From Theorem 3.10, we easily obtain the following corollary which is a generalization of Theorem 3.1 and Theorem 3.2.

Corollary 3.11 *Let U be a bounded symmetric open neighborhood of the origin in $\mathbf{R}^{m+k}$ for $k \geqq 0$. Assume that $\varphi : \overline{U} \to \mathbf{R}^m$ is upper semi-continuous mapping which is a closed convex set-valued mapping and satisfies $T'\varphi(x) \cap \varphi(T(x)) \neq \emptyset$ for $x \in \partial\overline{U}$. Then there exist a point $x_0 \in \overline{U}$ such that $\varphi(x_0) \ni 0$ and a point $x_1 \in \overline{U}$ such that $\varphi(x_1) \ni x_1$.*

4. Generalization to normed spaces

In this section we shall generalize some results of the section 3 to the case of infinite dimensional normed spaces. In this section $\mathbf{E}$ means an infinite dimensional normed space. We state mainly the case of an infinite dimensional normed space and add the case of the finite dimensional vector space if necessary. A set-valued mapping $\varphi : X \to Y$ is called a compact set-valued mapping, if $\varphi(x)$ is a compact set for each $x \in X$. $\varphi : X \to Y$ is called a closed convex set-valued mapping, if $\varphi(x)$ is a closed convex set for each $x \in X$. $\varphi : X \to Y$ is called a compact mapping, if the closure $\overline{\varphi(X)}$ of the image $\varphi(X)$ is a compact set. A compact mapping is a compact set-valued mapping. The converse is not true.

We prepare the Schauder approximation theorem for our application (cf. Theorem 12.9 in [8]).

Theorem 4.1 *Let X be a Hausdorff space and U an open set of a normed space $\mathbf{E}$ and $f : X \to U$ a continuous compact mapping. Then, for any $\epsilon > 0$, there exists a continuous compact mapping $f_\epsilon : X \to U$ satisfying the following condition*:

1. $f_\epsilon(X) \subset \mathbf{E}^{n(\epsilon)}$ *for a finite dimensional subspace* $\mathbf{E}^{n(\epsilon)}$ *of* $\mathbf{E}$
2. $\|f_\epsilon(x) - f(x)\| < \epsilon$ *for any* $x \in X$
3. $f_\epsilon(x),\ f(x) : X \to U$ *are homotopic, noted by* $f_\epsilon \simeq f$.

For a normed space $\mathbf{E}$, set $D = \{x \in \mathbf{E} \mid \|x\| \leqq 1\}$ and $S = \partial D$.

Lemma 4.2 *S is acyclic for an infinite dimensional normed space $\mathbf{E}$.*

Proof. It is sufficient to prove that $\mathbf{E} - \{0\}$ is acyclic. We shall prove this lemma by using the singular homology theory because of Theorem 5.5.3 and Theorem 6.9.1 in [16].

Let C be any p-dimensional cycle of $\mathbf{E} - \{0\}$, i.e. $\partial C = 0$. C has a form $C = \sum_{k=1}^{m} n_k \sigma_k$ where $n_k \in \mathbf{F},\ \sigma_k : \Delta^p \to \mathbf{E} - \{0\}$. Set $P = \cup_{k=1}^{m} \Delta_k^p$ where Δ_k^p's are p-dimensional simplices corresponding σ_k's. Let Δ_k^q and $\Delta_{k'}^q$ be faces of Δ_k^p and $\Delta_{k'}^p$ $(q < p)$ respectively. We define the space Q from P which the faces Δ_k^q and $\Delta_{k'}^q$ are patched together by the relation $\sigma_k|\Delta_i^q = \sigma_{k'}|\Delta_{i'}^q$. The space Q is a CW complex. Let $\tau : Q \to \mathbf{E} - \{0\}$ be defined by using σ_k's. By the Schauder approximation theorem, we have $\tau_\epsilon : Q \to \mathbf{E}^{n(\epsilon)} - \{0\}$ such that $\tau \simeq \tau_\epsilon$. A cycle $D = \sum_{k=1}^{n} n_k \rho_k$ is defined by using $\tau_\epsilon : Q \to \mathbf{E}^{n(\epsilon)} - \{0\}$. Since two singular cycles C and D are homologous, we may assume that $\sigma_k : \Delta^n \to \mathbf{E}^{n(\epsilon)} - \{0\}$. Since $\mathbf{E}$ is infinite dimensional, we can construct a $(p+1)$-dimensional chain B such that $\partial B = C$ by considering the cone of C and a point of $\mathbf{E} - \mathbf{E}^{n(\epsilon)}$. □

Let S_π be the orbit space of S by the antipodal involution. The cohomology ring of S_π is the polynomial ring for the case of the infinite dimensional normed spaces. This is easily proved by using the Gysin-Smith exact sequence of a double covering space. The cohomology ring of S_π is the truncated polynomial ring for the case of the n-dimensional normed space $(n > 1)$. S_π consists a point for $n = 1$.

If U is a balanced open neighborhood of the origin of a normed space, $\overline{U}$ and $Int\overline{U}$ are balanced neighborhood of the origin.

Proposition 4.3 *Let U be a bounded symmetric balanced open neighborhood of the origin of an infinite dimensional normed space. Then ∂U and*

$\partial\overline{U}$ are acyclic spaces. $H^*((\partial U)_\pi;\mathbf{F}_2)$ and $H^*((\partial\overline{U})_\pi;\mathbf{F}_2)$ are the polynomial ring.

Proof. We shall prove the case of ∂U. It is similarly proved for the case of $\partial\overline{U}$. Define $R:\overline{U}-\{0\}\to S$ by $R(x)=\frac{x}{\|x\|}$ and R_0 by the restriction of R to ∂U. Then we assert that the fiber $R_0^{-1}(x)$ of $x\in S$ is a point or a closed interval. If y and y' are in $R_0^{-1}(x)\subset\partial U=\overline{U}-U$, we may assume $y'=sy$ for $0<s<1$. It is proved that any point ty $(s<t<1)$ is in ∂U. If ty is a point of U, sy is in U by $s<t<1$. This is a contradiction. Therefore we see $ty\in\partial U$ for $s\leqq t\leqq 1$.

U contains a disk D_r centered at 0 with a radius $r>0$ and is contained in D_s centered at 0 with a radius $s>0$. The restriction $R_1: D_s-IntD_r\to S$ of R is proper by the homeomorphism between D_s-IntD_r and $S\times[r,s]$. Therefore the mapping R_0 is proper. Since the normed space $\mathbf{E}$ and its subspace are metric spaces i.e. paracompact Hausdorff spaces, $H^*(\partial U;\mathbf{F}_2)$ is isomorphic to $H^*(S;\mathbf{F}_2)$ by the Vietoris' theorem. ∂U is acyclic by Lemma 4.2. The second part is easily proved by the above result and the Gysin-Smith exact sequence of double covering space. □

For the case of the n-dimensional normed space $(n>1)$, we see easily that $H^*((\partial U)_\pi;\mathbf{F}_2)$ and $H^*((\partial\overline{U})_\pi;\mathbf{F}_2)$ are the truncated polynomial ring $\mathbf{F}_2(c)/(c^n)$ where $\dim c=1$. For the case $n=1$, ∂U and $\partial\overline{U}$ consist two points.

For the case of a bounded symmetric balanced neighborhood of the origin in a locally convex topological space, S. Y. Chang proved the next Theorem 4.4 for closed convex set-valued mappings (cf. Theorem 7 in [2]). We shall generalize his theorem to the case of admissible mappings and spaces which are not necessarily contractible. The following theorem is also called the Borsuk's fixed point theorem (cf. Theorem 3.3 in Section 6 in [9]).

Theorem 4.4 *Let U be a bounded symmetric open neighborhood of the origin of an infinite dimensional normed space $\mathbf{E}$. Assume that $\varphi:\overline{U}\to\mathbf{E}$ is a compact admissible mapping which is equivariant on $\partial\overline{U}$. Then there exist a fixed point $z_0\in\overline{U}$ such that $\varphi(z_0)\ni z_0$.*

Proof. Let (p,q) be a selected pair of φ where $p:\Gamma\to\overline{U}$ is a Vietoris mapping and $q:\Gamma\to\mathbf{E}$ is a compact mapping. $p_0:\Gamma_0\to\partial\overline{U}$ is an equivariant Vietoris mapping and $q_0:\Gamma_0\to\mathbf{E}$ is a compact equivariant mapping. Let $q_n : \Gamma \to \mathbf{E}_n$ is a Schauder approximation of q such that

$$\|q_n(y) - q(y)\| < \frac{1}{n} \quad (y \in \Gamma)$$

where $\mathbf{E}_n$ is a finite dimensional subspace of $\mathbf{E}$ and $\dim \mathbf{E}_n = i_n$.

Set $\Gamma_n = p^{-1}(\overline{\mathbf{E}_n \cap U})$ and $\Gamma_{n,0} = p_0^{-1}(\partial(\overline{\mathbf{E}_n \cap U}))$. Define $p_n : \Gamma_n \to \overline{\mathbf{E}_n \cap U}$ and $q_n : \Gamma_n \to \mathbf{E}_n$ by the restrictions of p and q_n respectively. Similarly we define $p_{n,0} : \Gamma_{n,0} \to \partial(\overline{\mathbf{E}_n \cap U})$ and $q_{n,0} : \Gamma_{n,0} \to \mathbf{E}_n$ by the restrictions of p_0 and q_n respectively.

If φ has a fixed point in $\partial\overline{U}$, the theorem is true. If φ has not a fixed point in $\partial\overline{U}$, we may assume $p_{n,0} - q_{n,0} : \Gamma_{n,0} \to \mathbf{E}_n - \{0\}$ for large n. Define $\hat{q}_n : \Gamma_{n,0} \to \mathbf{E}_n$ by

$$\hat{q}_n(y) = \frac{1}{2}\{q_{n,0}(y) - q_{n,0}(Ty)\}$$

which is an equivariant mapping. It is easily seen that

$$\|\hat{q}_n(y) - q(y)\| < \frac{1}{n}, \quad \|\hat{q}_n(y) - q_{n,0}(y)\| < \frac{1}{n} \quad (y \in \Gamma_{n,0}).$$

We may assume $p_{n,0} - \hat{q}_n : \Gamma_{n,0} \to \mathbf{E}_n - \{0\}$ for large n. Since $p_{n,0} - \hat{q}_n$ is an equivariant mapping, we see $\deg(p_{n,0} - \hat{q}_n) \neq 0$ by Theorem 3.3. Therefore we see $\deg(p_{n,0} - q_{n,0}) = \deg(p_{n,0} - \hat{q}_n) \neq 0$ for large n.

Let u be the generator of $\bar{H}^1((\partial(\overline{\mathbf{E}_n \cap U}))_\pi; \mathbf{F}_2)$. we see $u^{i_n - 1} \neq 0$ by Proposition 3.7. We find an element $v \in \bar{H}^{i_n-1}(\partial(\overline{\mathbf{E}_n \cap U}); \mathbf{F}_2)$ such that $\phi^*(v) = u^{i_n-1}$ by the Gysin-Smith exact sequence. Here $\phi^* : \bar{H}^{i_n-1}(\partial(\overline{\mathbf{E}_n \cap U}); \mathbf{F}_2) \to \bar{H}^{i_n-1}(\partial(\overline{\mathbf{E}_n \cap U})_\pi; \mathbf{F}_2)$ is the transfer mapping. Consider a disk D^{i_n} and a sphere S^{i_n-1} contained $\mathbf{E}_n \cap U$ as Proposition 3.7. By comparing the ladder of exact sequences of $(\overline{\mathbf{E}_n \cap U}, \partial(\overline{\mathbf{E}_n \cap U}))$ and (D^{i_n}, S^{i_n-1}), we see that v is not any image of $\bar{H}^{i_n-1}(\overline{\mathbf{E}_n \cap U}; \mathbf{F}_2)$. Let w be the element of $\bar{H}^{i_n-1}(\Gamma_{n,0}; \mathbf{F}_2)$ such that $p_{n,0}^*(v) = w$. Therefore w is not any image of $k^* : \bar{H}^{i_n-1}(\Gamma_n; \mathbf{F}_2) \to \bar{H}^{i_n-1}(\Gamma_{n,0}; \mathbf{F}_2)$ where $k : \Gamma_{n,0} \to \Gamma_n$ is the inclusion.

Let e be the generator of $\bar{H}^{i_n-1}(\mathbf{E}_n - \{0\}; \mathbf{F}_2)$. By Theorem 3.3, it holds $(p_{n,0} - \hat{q}_n)^*(e) = w$. Therefore it holds $(p_{n,0} - q_{n,0})^*(e) = w$. If φ has not a fixed point in $\overline{U}$, we may assume $p - q : \Gamma \to \mathbf{E} - \{0\}$ and also $p_n - q_n : \Gamma_n \to \mathbf{E}_n - \{0\}$ for large n. Since it holds $(p_n - q_n)k = p_{n,0} - q_{n,0}$, this contradicts the condition on w in the precedent paragraph. Therefore we have $y_n \in \Gamma_n$ such that $p_n(y_n) - q_n(y_n) = 0$. Since q is a compact mapping, subsequences $\{q(y_{n_i})\}$ and $\{q_n(y_{n_i})\}$ converge to $x_0 \in \overline{U}$. Since p is a Vietoris mapping, we assume that $\{y_{n_i}\}$ converge to y_0 such that $p(y_0) = x_0$. Therefore we have $q(y_0) = p(y_0) = x_0$, that is, $\varphi(x_0) \ni x_0$. $\square$

According to Theorem 3.10, $\varphi : \overline{U} \to \mathbf{E}$ has the zero value for the case of the finite dimensional vector space $\mathbf{E}$. We can not assert the existence of the zero value of φ for the case of an infinite dimensional normed space $\mathbf{E}$, Now we shall give some examples.

Let D be the unit disk in a Hilbert space $\mathbf{H}$. Let $f : D \to D$ be defined by

$$f(\{z_n\}) = \big(\sqrt{1 - \|z\|^2}, \{z_n\}\big).$$

Clearly f is a continuous mapping on D and equivariant on the boundary S and not a compact mapping. If f has a zero value, it holds the equations $\sqrt{1 - \|z\|^2} = 0$ and $z_n = 0$ for all n. We easily obtain the contradiction from the equations. Therefore f has not a zero value. Similarly we easily see that f has not a fixed point.

Let $g : D \to D$ be defined by

$$g(\{z_n\}) = \left(\sqrt{1 - \|z\|^2}, \left\{\frac{z_n}{n}\right\}\right).$$

Clearly g is a continuous mapping on D and equivariant on the boundary S and a compact mapping. If g has a zero value, it holds the equations $\sqrt{1 - \|z\|^2} = 0$ and $\frac{z_n}{n} = 0$ for all n. We obtain easily the contradiction from the equations. Therefore g has not the zero value. Of course g has a fixed point by Theorem 3.10 of [15] (cf. Section 12 in [8]).

Let X be a subset of a normed space $\mathbf{E}$ and $\Phi : X \to \mathbf{E}$ be a compact admissible mapping. A set-valued mapping $\varphi : X \to \mathbf{E}$ is called an admissible compact field, if φ is defined by $\varphi(x) = x - \Phi(x)$ (cf. Section 28 in [8]). Let $\mathbf{E}_k$ be a closed subspace of codimension k of a normed space $\mathbf{E}$. K. Geba and L. Górniewicz [6] proved the following theorem for the case of the unit sphere of a normed space (cf. Theorem 43.34 in [8]). Our method is different from their method.

Theorem 4.5 *Let $\mathbf{E}_k$ be a closed linear subspace of codimension $k \geqq 1$ of* *an infinite dimensional normed space* $\mathbf{E}$ *and* U *be a bounded symmetric open neighborhood of the origin of* $\mathbf{E}$*. If* $\Phi : \partial\overline{U} \to \mathbf{E}_k$ *is a compact admissible mapping, there is a point* $x_0 \in \partial\overline{U}$ *such that* $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ *where* $\varphi(x) = x - \Phi(x)$*.*

Proof. Let (p, q) be a selected pair of Φ where $p : \Gamma \to \partial\overline{U}$ is a Vietoris mapping and $q : \Gamma \to \mathbf{E}_k$ continuous mapping. There is a k-dimensional subspace $\mathbf{L}_k$ such that $\mathbf{E} = \mathbf{E}_k \oplus \mathbf{L}_k$. By the approximation theorem of Schauder, there are finite dimensional vector subspace $\mathbf{V}_n \subset \mathbf{E}_k$ and $q_n : \Gamma \to \mathbf{V}_n$ such that

$$\|q(y) - q_n(y)\| < \frac{1}{n}$$

for $y \in \Gamma$ and each $n \geqq 1$. We may assume that $i_n = \dim \mathbf{V}_n$ increases and $\mathbf{V}_n \subset \mathbf{V}_{n+1}$. Let $\Phi_n : \partial\overline{U} \to \mathbf{V}_n$ be a set-valued mapping defined by

$$\Phi_n(x) = B_n(\Phi(x)) \cap \mathbf{V}_n$$

where $B_n(\Phi(x)) = \{y \in \mathbf{E} \mid d(\Phi(x), y) \leqq \frac{1}{n}\}$. Since the graph of Φ_n is closed and $\Phi_n(\partial\overline{U})$ is compact, Φ_n is upper semi-continuous. Clearly Φ_n has a selected pair $p : \Gamma \to \partial\overline{U}$ and $q_n : \Gamma \to \mathbf{V}_n$. Therefore Φ_n is a compact and admissible mapping. Set $\varphi_n(x) = x - \Phi_n(x)$ for $x \in \partial\overline{U}$.

Set $Z_n = U \cap (\mathbf{V}_n \oplus \mathbf{L}_k)$ and $W_n = \partial\overline{Z}_n$. Z_n and W_n are subspaces of the $(i_n + k)$-dimensional Euclidean space $\mathbf{V}_n \oplus \mathbf{L}_k$. Consider $\hat{\Phi}_n : W_n \to \mathbf{V}_n$ defined by the restriction of Φ_n to W_n Note that $c(W_n, T)^{i_n+k-1} \neq 0$ by Proposition 3.7. Set $\hat{\varphi}_n(x) = x - \hat{\Phi}_n(x)$ for $x \in W_n$. By applying Theorem 6.3 of Y. Shitanda [15] to $\hat{\varphi}_n(x)$, we have a point $x_n \in W_n$ such that $\hat{\varphi}_n(x_n) \cap \hat{\varphi}_n(T(x_n)) \neq \emptyset$. This means $x_n - y_n = -x_n - z_n$ for some $y_n \in \hat{\Phi}_n(x_n)$ and $z_n \in \hat{\Phi}_n(T(x_n))$.

From the conditions, we can choose $y'_n \in \Phi(x_n)$ and $z'_n \in \Phi(T(x_n))$ such that $\|y_n - y'_n\| < \frac{1}{n}$ and $\|z_n - z'_n\| < \frac{1}{n}$. Since Φ is compact mapping, we can take proper subsequences $\{y'_{n_i}\}$ and $\{z'_{n_i}\}$ of $\{y'_n\}$ and $\{z'_n\}$ such that $\{y'_{n_i}\}$ and $\{z'_{n_i}\}$ converge to y_0 and z_0 respectively. Therefore $\{y_{n_i}\}$ and $\{z_{n_i}\}$ converge to y_0 and z_0 respectively. There is a convergent point x_0 of the subsequence $\{x_{n_i}\}$ from the equation $x_n - y_n = -x_n - z_n$. We have $x_0 = \dfrac{y_0 - z_0}{2}$. We easily see $y_0 \in \Phi(x_0)$ and $z_0 \in \Phi(T(x_0))$. By $x_0 - y_0 = -x_0 - z_0$, we obtain $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$, i.e. $A(\varphi) \neq \emptyset$. □

Let X be a space with a free involution T and S^k a k-dimensional sphere with the antipodal involution. Define $\gamma(X)$ and $\mathrm{Ind}(X)$ by

$$\gamma(X) = \inf\{k \mid f : X \to S^k \text{ equivariant mapping}\}$$
$$\mathrm{Ind}(X) = \sup\{k \mid c^k \neq 0\}$$

respectively, where $c \in \bar{H}^1(X_\pi; \mathbf{F}_2)$ is the class $c = f_\pi^*(\omega)$ for an equivariant mapping $f : X \to S^\infty$. If X is a compact space with a free involution, it holds the following formula (cf. Section 3 in [5]):

$$\mathrm{Ind}(X) \leqq \gamma(X) \leqq \dim X.$$

K. Gęba and L. Górniewicz proved $\mathrm{Ind}A(\varphi) \geqq k-1$ (cf. Theorem 2.5 in [6]). We shall generalize their result.

Corollary 4.6 *Under the hypothesis of Theorem* 4.5*, it holds*

$$\mathrm{Ind}A(\varphi) \geqq k-1.$$

Proof. We use the notation of Theorem 4.5. There exists a point $x_0 \in A(\varphi)$ by Theorem 4.5. Note that $A(\varphi)$ is a closed set because of the upper semicontinuity of φ. We can choose $\{\mathbf{V}_n\}$ such that $x_0 \in \mathbf{V}_n$ for any n.

Since $\hat{\varphi}_n : W_m \to \mathbf{V}_m$ is the restriction of φ_n to W_m, it holds $A(\hat{\varphi}_n) \neq \emptyset$ as Theorem 4.5. Set $\tilde{\Phi}_n(x) = x - B_n(\Phi(x))$ for $x \in \partial\overline{U}$. It holds $A(\hat{\varphi}_n) \subset A(\tilde{\Phi}_n)$. Clearly $A(\hat{\varphi}_n)$ and $A(\tilde{\Phi}_n)$ are paracompact Hausdorff spaces with free involutions. By Corollary 6.6 of [15], we have $\mathrm{Ind}(A(\hat{\varphi}_n)) \geqq k-1$. It holds also $\mathrm{Ind}(A(\tilde{\Phi}_n)) \geqq k-1$ by the naturality of the cup product. By the continuity of Alexander-Spanier cohomology theory (cf. Theorem 6.6.2 in [16]), we see $\mathrm{Ind}(A(\varphi)) \geqq k-1$ by $A(\varphi) = \bigcap_{n \geqq 1} A(\tilde{\Phi}_n)$. □

Though we state Theorem 4.5 and Corollary 4.6 for the case of the infinite dimensional normed spaces, the corresponding results (cf. [5]) for the finite dimensional normed spaces are proved by Theorem 6.3 and Corollary 6.6 of Y. Shitanda [15].

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