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Doubly nonlinear Cahn-Hilliard-Gurtin equations

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Abstract. Our aim in this article is to study the long time behavior, in terms of finite-dimensional attractors, of doubly nonlinear Cahn-Hilliard type equations. In particular, we prove the existence of an exponential attractor. We also study the continuity of exponential attractors when the equations converge to the usual Cahn-Hilliard equation.

Key words: Doubly nonlinear Cahn-Hilliard-Gurtin equations, dissipativity, global attractor, exponential attractor.

1. Introduction

We are interested in this article in the study of the asymptotic behavior (in terms of finite-dimensional attractors) of doubly nonlinear equations of the form

$$\begin{cases} \partial_t u - \Delta v = 0, \\ \partial_t \alpha(u) - \Delta u + f(u) = v, \\ u\,|_{\partial\Omega} = v\,|_{\partial\Omega} = 0, \quad u\,|_{t=0} = u_0, \end{cases} \tag{1.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^3$. Such equations appear, e.g., in the study of phase separation in binary alloys. More precisely, M. Gurtin proposed in [23], based on a new balance law for internal microforces (i.e., for interactions at a microscopic level), generalizations of the Cahn-Hilliard equation of the form

$$\partial_t u = \kappa_1 \Delta v, \quad \kappa_1 > 0, \tag{1.2}$$

$$a(u, \nabla u, \partial_t u)\partial_t u - \kappa_2 \Delta u + f(u) = v, \quad \kappa_2 > 0, \tag{1.3}$$

where $a \geq 0$. Here, u is the order parameter (it corresponds to a density of atoms) and v is the chemical potential. Furthermore, (1.2) follows from the mass balance, while (1.3) is derived by considering the restrictions imposed

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by the second law of thermodynamics, together with the microforce balance. Thus, if the coefficient a only depends on u, we obtain a system of the form (1.1) (system (1.1) can also be seen as a (nonlinear) generalization of the viscous Cahn-Hilliard equation introduced in [31] and corresponding to $\alpha(s) = \alpha_0 s$, $\alpha_0 > 0$). Furthermore, if a only depends on $\partial_t u$, we obtain a system of the form

$$\begin{cases} \partial_t u - \Delta v = 0, \\ \alpha(\partial_t u) - \Delta u + f(u) = v, \end{cases} \tag{1.4}$$

which is also of interest. We will study such systems in a forthcoming article. We finally note that, usually, the Cahn-Hilliard equations are associated with Neumann boundary conditions, both for u and v. In that case, the treatment of the problem would be analogous, but technically more difficult, due to the presence of nonlocal terms; we chose to avoid such technicalities in this article, hence our choice of Dirichlet boundary conditions.

We can rewrite (1.1) in the following equivalent form:

$$\begin{cases} \partial_t(\alpha(u) + (-\Delta)^{-1} u) - \Delta u + f(u) = 0, \\ v = -(-\Delta)^{-1}\partial_t u, \quad u\,|_{\partial\Omega} = 0, \; u\,|_{t=0} = u_0, \end{cases} \tag{1.5}$$

where $(-\Delta)^{-1}$ is the inverse Laplacian (equipped with Dirichlet boundary conditions), so that we are led to the study of a compact perturbation of the following doubly nonlinear equation:

$$\partial_t \alpha(u) - \Delta u + f(u) = 0. \tag{1.6}$$

Such equations, which appear, e.g., in the Allen-Cahn theory in phase separation (one again considers generalizations of the Allen-Cahn equation based on the microforce balance proposed by M. Gurtin, see [23]), have been much studied and are now well understood (we refer the reader to, e.g., [1], [4], [6], [8], [9], [11], [14], [15], [19], [22], [28], [29], [33], [34] and [35]). In particular, one now has satisfactory results on the existence and uniqueness of solutions and on the asymptotic behavior of the system (existence of finite-dimensional attractors and convergence of solutions to steady states). Furthermore, such results concern both the cases where α does not degenerate (i.e., $\alpha'(s) \geq c_0 > 0$, $s \in \mathbb{R}$) and where α degenerates (with a finite number of degeneration points and also when the equation becomes elliptic in some region (typically, $\alpha'(s) = 0$ for $s \le 0$), see [29]).

Now, and especially when α degenerates, the mathematical study of (1.5) (or, equivalently, of (1.1)) does not simply follow from that of (1.6), in particular, as far as the study of the asymptotic behavior of the system is concerned. The main difficulty here is to obtain proper dissipative estimates and, contrary to (1.6), such estimates cannot be obtained directly by energy estimates (which only work under additional restrictive assumptions on α and f, see Sections 2 and 3 below; see also [29]). In particular, it is essential here to derive dissipative estimates in $L^\infty(\Omega)$. In order to deduce such estimates, we use a combination of Moser iterations techniques and of arguments based on the existence of a proper dissipation integral.

It is also worth noting that, in the degenerate case, the presence of the subordinated term $(-\Delta)^{-1}\partial_t u$ in (1.5) drastically changes the analytical and dynamical properties of the problem (when compared with (1.6)). In particular, equation (1.6) is well-posed and possesses the finite-dimensional global attractor *only* if the nonlinearity f is *monotone increasing* at all points where α *degenerates* (see [29]; see also [18] for examples of infinite-dimensional attractors when this condition is violated). Surprisingly, the presence of the nondegenerate lower order term $(-\Delta)^{-1}\partial_t u$ removes, in some sense, the degeneration and the dynamical behavior of the solutions to (1.5) is, a posteriori, closer to that of *nondegenerate* dissipative systems. In particular, we do not need the monotonicity assumption mentioned above (which is typical of degenerate systems) in order to establish the existence and the finite-dimensionality of the global attractor (this allows, in particular, to consider, contrary to the degenerate doubly nonlinear Allen-Cahn equation, the physically relevant (in the context of phase separation) cubic nonlinearity $f(s) = s^3 - s$ in the case of, say, one degeneration point at 0). At the same time, the analytical properties of equation (1.5) remain typical of degenerate equations (lack of further regularity on the solutions, lack of differentiability with respect to the initial data, ...).

This article is organized as follows. In Section 2, we study the well-posedness of the system. Then, in Section 3, we derive dissipative estimates which allow us to prove, in Section 4, the existence of finite-dimensional attractors. Section 5 is devoted to the study of the limit $\alpha \to 0$ (for $\alpha = 0$, the system reduces to the usual Cahn-Hilliard equation). However, we do not know how to obtain uniform (with respect to $\alpha \to 0$) estimates on the

L^∞-norm of the solutions and, consequently, we have to work with energy solutions only, which, in turn, requires essential growth restrictions on α and f. Finally, we prove, in the appendix, additional regularity results on the solutions when α does not degenerate.

Throughout this article, the same letter C (and, sometimes, C') denotes constants which may vary from line to line.

Setting of the problem

We recall that we consider the following equations, in a bounded smooth domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t u - \Delta v = 0, \tag{1.7}$$

$$\partial_t \alpha(u) - \Delta u + f(u) = v, \tag{1.8}$$

together with the boundary conditions

$$u\,|_{\partial\Omega} = v\,|_{\partial\Omega} = 0 \tag{1.9}$$

and the initial condition

$$u\,|_{t=0} = u_0. \tag{1.10}$$

We assume that the nonlinearity $\alpha \in C^1(\mathbb{R})$ is monotone increasing,

$$\alpha(0) = 0, \quad \alpha'(s) \geq 0, \quad s \in \mathbb{R}, \tag{1.11}$$

and satisfies the following nondegeneracy assumption at infinity:

$$\liminf_{|s|\to\infty} \alpha'(s) \geq C_0 > 0, \quad s \in \mathbb{R}, \tag{1.12}$$

and that the nonlinearity $f \in C^2(\mathbb{R})$ satisfies the standard dissipativity assumption

$$f(s)s \geq -C, \quad s \in \mathbb{R}, \quad C \geq 0. \tag{1.13}$$

Remark 1.1 For instance, the above assumptions are satisfied for $\alpha(s) = s^{2p+1}$, $p \in \mathbb{N}$, and $f(s) = \sum_{i=0}^{2q+1} a_i s^i$, $a_{2q+1} > 0$, $q \in \mathbb{N}$. For such an α, setting $w := \alpha(u)$, the Allen-Cahn type equation (1.6) reduces, say, for $f \equiv 0$, to the fast diffusion equation

$$\partial_t w - \Delta w^m = 0, \quad m = \frac{1}{2p+1}$$

(see, e.g., [2], [10], [12], [21], [32] and the references therein). We can note that $p = 2$, i.e., $m = \frac{1}{5}$, corresponds to the critical exponent in this fast diffusion equation (see also Remark 3.5 and (3.36) below). Furthermore, compared with the results of [29] for (1.6), the presence of the subordinated term $(-\Delta)^{-1}\partial_t u$ allows to prove the existence of finite-dimensional attractors for (1.5) for a much more general class of nonlinear terms f for the aforementioned “fast diffusion” term α.

2. Well-posedness for L^∞-solutions

In this section, we prove the unique solvability of system (1.7)–(1.8) in the following class of solutions:

$$u \in L^\infty([0,T] \times \Omega) \cap L^\infty\big([0,T], H^1_0(\Omega)\big), \quad v \in L^2\big([0,T], H^1_0(\Omega)\big). \tag{2.1}$$

The main result of the section is the following.

Theorem 2.1 *Let assumptions* (1.11), (1.12) *and* (1.13) *hold. Then, for every* $u_0 \in H^1_0(\Omega) \cap L^\infty(\Omega)$, *there exists a unique solution* (u, v) *to problem* (1.7)–(1.8) *belonging to the class* (2.1).

Proof. We rewrite the equations in the equivalent form

$$\begin{cases} \partial_t(\alpha(u) + (-\Delta)^{-1}u) - \Delta u + f(u) = 0, \\ v = -(-\Delta)^{-1}\partial_t u, \quad u\,|_{\partial\Omega} = 0,\ u\,|_{t=0} = u_0. \end{cases} \tag{2.2}$$

We first prove the uniqueness of solutions. To this end, we need the following lemma.

Lemma 2.2 *Let the assumptions of Theorem* 2.1 *hold. Then, for every solutions* (u_1, v_1) *and* (u_2, v_2) *to* (2.2) *belonging to the class* (2.1) (*with initial data* $u_{1,0}$ *and* $u_{2,0}$, *respectively, belonging to* $H^1_0(\Omega) \cap L^\infty(\Omega)$), *there holds*

$$\begin{aligned} &\big\|\alpha(u_1(t)) - \alpha(u_2(t)) + (-\Delta)^{-1}(u_1(t) - u_2(t))\big\|_{H^{-1}} \\ &\qquad \le Ce^{C't}\big\|\alpha(u_{1,0}) - \alpha(u_{2,0}) + (-\Delta)^{-1}(u_{1,0} - u_{2,0})\big\|_{H^{-1}}, \end{aligned} \tag{2.3}$$

where $t \in [0,T]$ and the constants C and C' depend on the norms of u_1 and u_2 in the class (2.1).

Proof. It follows from (2.2) that, setting $u := u_1 - u_2$,

$$\partial_t\big(\alpha(u_1) - \alpha(u_2) + (-\Delta)^{-1}u\big) - \Delta u + f(u_1) - f(u_2) = 0. \tag{2.4}$$

Multiplying (2.4) by $(-\Delta)^{-1}w$, $w = \alpha(u_1) - \alpha(u_2) + (-\Delta)^{-1}u$, and integrating over Ω and by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t\|w\|^2_{H^{-1}} + (\alpha(u_1) - \alpha(u_2), u) \\ &\qquad + \|u\|^2_{H^{-1}} + (f(u_1) - f(u_2), (-\Delta)^{-1}w) = 0, \end{aligned} \tag{2.5}$$

where $(\cdot,\cdot)$ denotes the usual L^2-scalar product. The difficulty here is to estimate the term

$$\big|(f(u_1) - f(u_2), (-\Delta)^{-1}w)\big|.$$

To do so, we write

$$\big|\big(f(u_1)-f(u_2), (-\Delta)^{-1}w\big)\big| \le \epsilon_1\|f(u_1)-f(u_2)\|^2_{H_\Delta^{-2+\delta}} + C_{\epsilon_1}\|w\|^2_{H^{-\delta}}, \tag{2.6}$$

where $\delta > 0$ is small enough (in particular, $\delta < 1/2$), $\epsilon_1 > 0$ is arbitrary and $H_\Delta^{-2+\delta} := [H^{2-\delta}(\Omega) \cap H_0^1(\Omega)]^*$.

We then have, owing to classical interpolation inequalities,

$$\|w\|^2_{H^{-\delta}} \le \epsilon_2\|w\|^2_{L^2} + C_{\epsilon_2}\|w\|^2_{H^{-1}}, \tag{2.7}$$

where $\epsilon_2 > 0$ is arbitrary. Furthermore,

$$\begin{aligned} \|w\|^2_{L^2} &\le 2\big(\|\alpha(u_1) - \alpha(u_2)\|^2_{L^2} + \|(-\Delta)^{-1}w\|^2_{L^2}\big) \\ &\le C\big((\alpha(u_1) - \alpha(u_2), u) + \|w\|^2_{H^{-1}}\big), \end{aligned} \tag{2.8}$$

since $\alpha'(s) \ge 0$, $\forall s \in \mathbb{R}$, and $u_i \in L^\infty([0,T] \times \Omega)$, $i = 1,2$, which implies that $(\alpha(u_1) - \alpha(u_2), u) \ge 0$ and $\|\alpha(u_1) - \alpha(u_2)\|^2_{L^2} \le C(\alpha(u_1) - \alpha(u_2), u)$. We thus deduce from (2.7)–(2.8) that

$$\|w\|_{H^{-\delta}}^2 \le \epsilon_3(\alpha(u_1)-\alpha(u_2),u)+C_{\epsilon_3}\|w\|_{H^{-1}}^2,$$

where $\epsilon_3 > 0$ is arbitrary, so that (2.6) yields

$$\begin{aligned}&\big|(f(u_1)-f(u_2),(-\Delta)^{-1}w)\big|\\ &\quad\le \epsilon_1\|f(u_1)-f(u_2)\|^2_{H_\Delta^{-2+\delta}}+\epsilon_4(\alpha(u_1)-\alpha(u_2),u)+C_{\epsilon_1,\epsilon_4}\|w\|^2_{H^{-1}},\end{aligned} \tag{2.9}$$

where ϵ_1 and $\epsilon_4 > 0$ are arbitrary.

Now, we have, for $\varphi \in H_\Delta^{2-\delta}(\Omega) := H^{2-\delta}(\Omega)\cap H_0^1(\Omega)$ and using the fact that $u_i \in L^\infty([0,T]\times\Omega)\cap L^\infty([0,T],H_0^1(\Omega))$, $i=1,2$,

$$\begin{aligned}&\big|\langle f(u_1)-f(u_2),\varphi\rangle_{H_\Delta^{-2+\delta},H_\Delta^{2-\delta}}\big|\\ &\quad=|(f(u_1)-f(u_2),\varphi)|\\ &\quad=\left|\left(u,\int_0^1 f'(su_1+(1-s)u_2)ds\varphi\right)\right|\\ &\quad\le\|u\|_{H^{-1}}\left\|\nabla\left(\int_0^1 f'(su_1+(1-s)u_2)ds\varphi\right)\right\|_{L^2}\\ &\quad\le C\|u\|_{H^{-1}}\big((\|\nabla u_1\|_{L^2}+\|\nabla u_2\|_{L^2})\|\varphi\|_{L^\infty}+\|\nabla\varphi\|_{L^2}\big)\\ &\quad\le C\|u\|_{H^{-1}}\|\varphi\|_{H_\Delta^{2-\delta}},\end{aligned}$$

if $\delta < 1/2$ (so that the embedding $H^{2-\delta}\subset C$ holds), whence

$$\|f(u_1)-f(u_2)\|_{H_\Delta^{-2+\delta}}\le C\|u\|_{H^{-1}}. \tag{2.10}$$

We finally deduce from (2.9)–(2.10) that

$$\begin{aligned}&\big|(f(u_1)-f(u_2),(-\Delta)^{-1}w)\big|\\ &\quad\le\epsilon\big((\alpha(u_1)-\alpha(u_2),u)+\|u\|^2_{H^{-1}}\big)+C_\epsilon\|w\|^2_{H^{-1}},\end{aligned} \tag{2.11}$$

where $\epsilon > 0$ is arbitrary, and (2.3) follows from (2.5), (2.11), e.g., with $\epsilon = \frac{1}{2}$, and Gronwall's lemma. This finishes the proof of Lemma 2.2. □

The uniqueness of solutions to problem (2.2) is an immediate consequence of the above lemma. Indeed, if $u_{1,0} = u_{2,0}$, then

$$\alpha(u_1) - \alpha(u_2) + (-\Delta)^{-1}(u_1 - u_2) = 0,$$

which yields, multiplying this equation by $u_1 - u_2$ and noting that $\alpha' \geq 0$,

$$\|u_1 - u_2\|_{H^{-1}} = 0,$$

and, thus, $u_1 = u_2$.

We now establish the existence of a solution. To this end, we approximate the nonlinear term α by smooth functions α_ϵ, $\epsilon > 0$, which satisfy (1.11) and (1.12), with a (new) constant C_0 which is independent of ϵ, and

$$\alpha'_\epsilon(s) \geq C'_0(\epsilon) > 0, \quad s \in \mathbb{R}; \tag{2.12}$$

for instance, if α is of class $\mathcal{C}^2$, we can take $\alpha_\epsilon(s) = \alpha(s) + \epsilon s$, $s \in \mathbb{R}$.

We then consider the approximate problems

$$\partial_t u - \Delta v = 0, \tag{2.13}$$

$$\partial_t \alpha_\epsilon(u) - \Delta u + f(u) = v, \tag{2.14}$$

$$u\,|_{\partial\Omega} = v\,|_{\partial\Omega} = 0, \tag{2.15}$$

$$u\,|_{t=0} = u_{0,\epsilon}, \tag{2.16}$$

where $u_{0,\epsilon}$ smooth enough approximates u_0. Rewriting (2.13)–(2.16) in the equivalent form

$$\begin{cases} \partial_t(\alpha_\epsilon(u) + (-\Delta)^{-1}u) - \Delta u + f(u) = 0, \\ v = -(-\Delta)^{-1}\partial_t u, \quad u\,|_{\partial\Omega} = 0,\ u\,|_{t=0} = u_{0,\epsilon}, \end{cases} \tag{2.17}$$

we note that (2.17) is a compact perturbation of a second-order parabolic equation which possesses, owing to standard results (we can for instance use the Leray-Schauder fixed point theorem, see, e.g., [25]; see also the appendix) a unique solution u_ϵ, say, in $W^{(1,2),r}(\Omega \times (0,T))$, where $T > 0$ and $r < \infty$ is large enough. We thus deduce the existence and uniqueness of the regular solution (u_ϵ, v_ϵ) to (2.13)–(2.16).

Our task now is to deduce several uniform (with respect to ε) estimates which will allow us to establish the existence of a solution to the limit problem (2.2) by passing to the limit $\varepsilon \to 0$. For simplicity, we omit the index ϵ in what follows.

We start with a standard energy estimate. Multiplying (2.17) by $\partial_t u$ and integrating over $[0,T]\times\Omega$, we have

$$2\int_0^T \left[\left(\alpha'_\varepsilon(u(t))\partial_t u(t), \partial_t u(t)\right) + \|\partial_t u(t)\|^2_{H^{-1}}\right] dt \\ + \|\nabla u(T)\|^2_{L^2} + 2(F(u(T)),1) = \|\nabla u_0\|^2_{L^2} + 2(F(u_0),1), \tag{2.18}$$

where $F(s) := \int_0^s f(z)\,dz$. This, together with assumptions (1.11) and (1.13), gives

$$\|v\|_{L^2([0,T],H^1(\Omega))} + \|u\|_{L^\infty([0,T],H^1(\Omega))} \le Q(\|u_0\|_{L^\infty\cap H^1}), \tag{2.19}$$

where the monotonic function Q is independent of ε and T.

Our next task is to obtain an L^∞-estimate on the solution u. To this end, we need the following lemma.

Lemma 2.3 *Let $v \in L^2([0,T],H^1_0(\Omega))$ and let u be a sufficiently regular solution to equation* (2.14)*. Then, the following estimate holds*:

$$\|u(t)\|_{L^\infty} \le Q_T(\|u_0\|_{L^\infty}) + C_T\|v\|_{L^2([0,T],H^1(\Omega))}, \quad t\in[0,T], \tag{2.20}$$

where the monotonic function Q_T and the constant C_T depend on T, but are independent of ε.

Proof. Although the assertion of this lemma is rather standard, we give below, for the sake of convenience (and in order to check that the estimate is indeed uniform with respect to ε), the details of the proof. As usual, we obtain this estimate by a Moser-type iterations technique. Indeed, multiplying equation (2.14) by $u|u|^k$, $k>0$, and integrating over $[0,T]\times\Omega$, we have

$$(A_k(u(T)),1) + \frac{4(k+1)}{(k+2)^2}\int_0^T \left\|\nabla(u(t)|u(t)|^{\frac{k}{2}})\right\|^2_{L^2} dt \\ + \int_0^T \left(f(u(t)), u(t)|u(t)|^k\right) dt = (A_k(u(0)),1) + \int_0^T \left(v(t), u(t)|u(t)|^k\right) dt, \tag{2.21}$$

where $A_k(s) := \int_0^s \alpha_\varepsilon'(z)z|z|^k\,dz$. Using the embedding $H^1 \subset L^6$, we estimate the last integral in the right-hand side of (2.21) as follows:

$$\begin{aligned}\int_0^T (v(t), u(t)|u(t)|^k)\,dt &\\ \le \int_0^T \|v(t)\|_{L^6}\|u(t)\|^{k+1}_{L^{\frac{6}{5}(k+1)}}\,dt &\\ \le C\|v\|_{L^2([0,T],H^1(\Omega))}\|u\|^{k+1}_{L^{2(k+1)}([0,T],L^{\frac{6}{5}(k+2)}(\Omega))} &\\ \le C\Big(\|v\|^{k+2}_{L^2([0,T],H^1(\Omega))} + \|u\|^{k+2}_{L^{2(k+2)}([0,T],L^{\frac{6}{5}(k+2)}(\Omega))}\Big). &\qquad (2.22)\end{aligned}$$

Since the nonlinearity f satisfies the dissipativity assumption (1.13), we have

$$\begin{aligned}\int_0^T \big(f(u(t)), u(t)|u(t)|^{k+1}\big)\,dt &\ge -C\|u\|^{k+1}_{L^{k+1}([0,T]\times\Omega)}\\ &\ge -C\|u\|^{k+2}_{L^{2(k+2)}([0,T],L^{\frac{6}{5}(k+2)}(\Omega))} - C_T,\end{aligned}$$

for positive constants C and C_T which are independent of k (note however that C_T depends on T).

Finally, using assumption (1.12), we see that

$$A_k(s) \ge \frac{C}{k+2}|s|^{k+2} - A^{k+2}, \qquad (2.23)$$

where the constants $C > 0$ and $A \ge 0$ are independent of ε and k. Therefore, inserting the above estimates into (2.21), we end up with

$$\begin{aligned}&\max\big\{\|u\|_{L^\infty([0,T],L^{k+2}(\Omega))}, \|u\|_{L^{k+2}([0,T],L^{3(k+2)}(\Omega))}\big\}^{k+2}\\ &\quad\le C_T(k+2)\max\Big\{A^{k+2}, A_k(u_0), \|v\|^{k+2}_{L^2([0,T],H^1(\Omega))},\\ &\qquad\qquad \|u\|^{k+2}_{L^{2(k+2)}([0,T],L^{\frac{6}{5}(k+2)}(\Omega))}\Big\}, \qquad (2.24)\end{aligned}$$

where the constant C_T is also independent of k and ε. We now note that the interpolation embedding

$$L^\infty(L^{k+2})\cap L^{k+2}(L^{3(k+2)})\subset L^{\frac{7}{3}(k+2)}(L^{\frac{7}{5}(k+2)}) \tag{2.25}$$

gives the uniform (with respect to k) estimate

$$\|u\|_{L^{\frac{7}{3}(k+2)}([0,T],L^{\frac{7}{5}(k+2)}(\Omega))} \le C\max\big\{\|u\|_{L^\infty([0,T],L^{k+2}(\Omega))},\|u\|_{L^{k+2}([0,T],L^{3(k+2)}(\Omega))}\big\}.$$

Furthermore, we set, for $s\ge 0$,

$$Q(s):=\sup_k\Big(\max_{z\in[-s,s]}A_k(z)\Big)^{\frac{1}{k+2}}.$$

Then, obviously, Q is finite and

$$Q(s)\le s^2\Big\{\max_{z\in[-s,s]}\alpha'(z)\Big\}^{1/(k+2)}+C,$$

where C is independent of s. Using these estimates, we infer from (2.24) that

$$J_{\kappa(k+2)}\le (C_T(k+2))^{\frac{1}{k+2}}J_{k+2}, \tag{2.26}$$

where $\kappa=\frac{7}{6}$ and

$$J_{k+2}:=\max\big\{A,\|u\|_{L^{2(k+2)}([0,T],L^{\frac{6}{5}(k+2)}(\Omega))},\|v\|_{L^2([0,T],H^1(\Omega))},Q(\|u_0\|_{L^\infty})\big\}. \tag{2.27}$$

We also note that, due to the analogue of estimate (2.19) for equation (2.14), we have a control of J_{k+2} for $k=0$,

$$J_2\le C_T(Q\big(\|u_0\|_{L^\infty})+\|v\|_{L^2([0,T],H^1(\Omega))}\big).$$

Thus, iterating estimate (2.26), we finally infer

$$\|u\|_{L^\infty([0,T]\times\Omega)}\le J_\infty:=\limsup_{k\to\infty}J_k\le C\big(\|v\|_{L^2([0,T],H^1(\Omega))}+Q(\|u_0\|_{L^\infty})\big)$$

and estimate (2.20) is proved, which finishes the proof of the lemma. □

We are now ready to finish the proof of existence. Indeed, according to Lemma 2.3 and estimate (2.19), we have

$$\|u_\epsilon\|_{L^\infty([0,T]\times\Omega)\cap L^\infty([0,T],H^1(\Omega))} + \|v_\epsilon\|_{L^2([0,T],H^1(\Omega))} \\ \leq Q_T(\|u_0\|_{L^\infty\cap H^1}), \tag{2.28}$$

where the monotonic function Q_T is independent of the approximation parameter ε. It then follows from (2.28) and classical compactness results (see, e.g., [26]) that, at least for a subsequence, $u_\epsilon \to u$ a.e. and in $L^\infty([0,T]\times\Omega)$ $\star$-weak and $L^\infty([0,T],H_0^1(\Omega))$ $\star$-weak and $v_\epsilon \to v$ in $L^\infty([0,T],H_0^1(\Omega))$ $\star$-weak, where (u,v) belongs to the class (2.1). This is sufficient to pass to the limit $\varepsilon \to 0$ in equations (2.17) and to have the existence of a solution to problem (2.2), which finishes the proof of Theorem 2.1. □

Remark 2.4 If we assume, in addition, that α does not degenerate,

$$\alpha'(s) \geq C > 0, \quad s \in \mathbb{R}, \tag{2.29}$$

then we have the additional term

$$\int_\Omega \alpha'_\epsilon(u_\epsilon)\|\partial_t u_\epsilon\|^2_{L^2}\,dx \geq C'\|\partial_t u_\epsilon\|^2_{L^2},$$

where $C' > 0$ is independent of ϵ, in (2.18). This yields that $\partial_t u \in L^2([0,T],L^2(\Omega))$ and $v \in L^2([0,T],H^2(\Omega)\cap H_0^1(\Omega))$.

Remark 2.5 Unfortunately, the function Q_T in estimate (2.28) depends on T (to be more precise, it grows polynomially as T grows). Therefore, we cannot extract any dissipativity from this estimate. However, we obtain this dissipativity in the next section under the additional assumption that α' grows at most polynomially as $|s| \to \infty$. We do not know whether or not the dissipativity holds without this asumption.

3. Dissipativity of L^∞-solutions

In this section, we establish the dissipativity of the L^∞-solutions constructed in the previous section. To this end, we need an additional assumption, namely,

$$\alpha'(s) \le C(1+|s|^p), \quad s \in \mathbb{R},\ C \ge 0, \tag{3.1}$$

for some nonnegative exponent p. Furthermore, the proof of the desired dissipativity is strongly based on the use of the dissipation integral

$$\int_0^\infty \|v(t)\|^2_{H^1}\,dt = \int_0^\infty \|\partial_t u(t)\|^2_{H^{-1}}\,dt \le Q(\|u_0\|_{L^\infty\cap H^1}), \tag{3.2}$$

for a monotonic function Q, which is an immediate consequence of estimate (2.19).

In order to avoid unnecessary technicalities, we derive all the estimates directly for equation (2.2), although, in order to justify them, one should use the approximating problem (2.17).

The next lemma gives the $L^{p+q} - L^\infty$-smoothing property for the solutions to equation (1.8) if $q = q(p)$ is large enough.

Lemma 3.1 *Let the assumptions of Theorem* 2.1 *hold and, in addition, inequality* (3.1) *be satisfied. Let also* $q > \frac{3p}{2}$. *Then, any sufficiently regular solution* u *to equation* (1.8) *satisfies the following smoothing property*:

$$\|u(1)\|_{L^\infty} \le C\big(1 + \|u_0\|_{L^{p+q}} + \|v\|_{L^2([0,1],H^1(\Omega))}\big)^P, \tag{3.3}$$

for positive constants P *and* C *which are independent of* u.

Proof. The proof of this estimate is, analogously to that of Lemma 2.3, based on a Moser iterations technique and essentially follows [19]. Let $\{k_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$ be two strictly increasing sequences of exponents and times, respectively, such that

$$k_0 = q-2, \quad k_n \to \infty, \quad t_0 = 0, \quad t_n \to 1,$$

and set

$$I_n := \max\big\{\|u\|_{L^\infty([t_n,2],\,L^{k_n+2}(\Omega))}, \|u\|_{L^{k_n+2}([t_n,2],L^{3(k_n+2)}(\Omega))}\big\}^{k_n+2}.$$

Then, multiplying equation (2.14) by $u|u|^{k_n}$ and integrating over $[t_*,t]\times\Omega$, where t_* is an arbitrary time in the interval $[t_{n-1},t_n]$ and $t \in [t_n,2]$, we have, as in (2.21),

$$\sup_{t\in[t_n,2]}(A_{k_n}(u(t)),1)+\frac{4(k_n+1)}{(k_n+2)^2}\int_{t_n}^2\left\|\nabla(u(t)|u(t)|^{\frac{k_n}{2}})\right\|_{L^2}^2\,dt$$
$$+\int_{t_{n-1}}^2\left|(f(u(t)),u(t)|u(t)|^{k_n})\right|\,dt$$
$$\le (A_{k_n}(u(t_*)),1)+\int_{t_{n-1}}^2\left(v(t),u(t)|u(t)|^{k_n}\right)dt. \tag{3.4}$$

Arguing as in the proof of Lemma 2.3 and using (3.1), we infer the following inequality:

$$I_n\le C(k_n+2)\Big(A^{k_n+2}+\|v\|^{k_n+2}_{L^2([0,2],H^1(\Omega))}$$
$$+\|u\|^{k_n+2}_{L^{2(k_n+2)}([t_{n-1},2],L^{\frac{6}{5}(k_n+2)}(\Omega))}+\|u(t_*)\|^{k_n+p+2}_{L^{k_n+p+2}}\Big). \tag{3.5}$$

We assume from now on that

$$3(k_{n-1}+2)>k_n+p+2 \tag{3.6}$$

(this assumption is not contradictory if and only if $q>\frac{3p}{2}$, see [19]; otherwise, the sequence k_n cannot be strictly increasing). Then, there exists $t^*\in[t_{n-1},t_n]$ such that

$$\|u(t_*)\|^{k_{n-1}+2}_{L^{k_n+p+2}}$$
$$\le\frac{C}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\|u(s)\|^{k_{n-1}+2}_{L^{3(k_{n-1}+2)}}\,ds\le\frac{C}{t_n-t_{n-1}}I_{n-1}. \tag{3.7}$$

Furthermore, using the interpolation embedding (2.25) with $(k+2)$ replaced by $\frac{6}{7}(k_n+2)$, we deduce that

$$\|u\|^{k_n+2}_{L^{2(k_n+2)}([t_{n-1},2],L^{\frac{6}{5}(k_n+2)})}$$
$$\le C\max\left\{\|u\|_{L^\infty([t_{n-1},2],L^{\frac{6}{7}(k_n+2)}(\Omega))},\|u\|_{L^{\frac{6}{7}(k_n+2)}([t_{n-1},2],L^{3\cdot\frac{6}{7}(k_n+2)}(\Omega))}\right\}^{k_n+2}$$
$$\le C_1I_{n-1}^{\frac{k_n+2}{k_{n-1}+2}}, \tag{3.8}$$

where we have assumed, in addition, that

$$\frac{7}{6}(k_{n-1}+2) \geq (k_n+2). \tag{3.9}$$

Inserting estimates (3.5), (3.7) and (3.8) into the right-hand side of (3.5), we obtain

$$I_n \leq C(k_n+2)\max\Big\{A^{k_n+2},\ \|v\|^{k_n+2}_{L^2([0,2],H^1(\Omega))},$$
$$I_{n-1}^{\frac{k_n+2}{k_{n-1}+2}},\ \Big(\frac{C}{t_n-t_{n-1}}I_{n-1}\Big)^{\frac{k_n+p+2}{k_{n-1}+2}}\Big\}. \tag{3.10}$$

Thus, setting $J_n := I_n^{\frac{1}{k_n+2}}$ and taking the root of power k_n+2 of both sides of this inequality, we obtain

$$J_n \leq [C(k_n+2)]^{\frac{1}{k_n+2}}\left[\frac{C}{t_n-t_{n-1}}\right]^{\frac{k_n+p+2}{(k_{n-1}+2)(k_n+2)}}$$
$$\cdot \max\big\{J_{n-1}, A, \|v\|_{L^2([0,2],H^1(\Omega))}\big\}^{\frac{k_n+p+2}{k_n+2}}. \tag{3.11}$$

We can now fix the sequence k_n in such a way that $k_n \sim \left(\frac{7}{6}\right)^n$ (satisfying both assumptions (3.6) and (3.9)) and finally fix the sequence t_n in such a way that

$$t_n - t_{n-1} = \frac{\beta}{k_n},$$

for some constant β which is independent of n. Then, inequality (3.11) reads

$$\max\big\{J_n, A, \|v\|_{L^2([0,2],H^1(\Omega))}\big\}$$
$$\leq [C_1(k_n+2)]^{\frac{k_n+p+2}{(k_{n-1}+2)(k_n+2)}}\max\big\{J_{n-1}, A, \|v\|_{L^2([0,2],H^1(\Omega))}\big\}^{\frac{k_n+p+2}{k_n+2}}$$

and, taking the logarithm of both sides of this inequality and setting

$$Z_n := \ln\max\big\{J_n, A, \|v\|_{L^2([0,2],H^1(\Omega))}\big\},$$

we finally end up with

$$Z_n \le a_n Z_{n-1} + b_n,$$
$$a_n := 1 + \frac{p}{k_n+2}, \quad b_n := \frac{k_n+p+2}{(k_{n-1}+2)(k_n+2)} \ln(C_1(k_n+2)) \tag{3.12}$$

(we can assume, without loss of generality, that $A \ge 1$ and, therefore, $Z_n \ge 0$). Iterating estimate (3.12), we find

$$Z_n \le P_n Z_0 + C_n, \quad P_n := \prod_{i=1}^n a_i, \quad C_n := a_1^{-1} \sum_{i=1}^n P_{n+1-i} b_i.$$

We finally set $P := \lim_{n\to\infty} P_n$ and $C := \lim_{n\to\infty} C_n$. These limits exist and are finite, since we have assumed that $k_n \sim \left(\frac{7}{6}\right)^n$. Then, the last inequality yields

$$\|u\|_{L^\infty([1,2],L^{k_n+2}(\Omega))} \le J_n \le \max\left\{J_n, A, \|v\|_{L^2([0,2],H^1(\Omega))}\right\}$$
$$\le e^C \max\left\{J_0, A, \|v\|_{L^2([0,2],H^1(\Omega))}\right\}^P$$

and, consequently,

$$\|u\|_{L^\infty([1,2]\times\Omega)} \le \limsup_{n\to\infty} \|u\|_{L^\infty([1,2],L^{k_n+2}(\Omega))}$$
$$\le e^C \big(1 + \|u\|_{L^\infty([0,2],L^q(\Omega))} + \|u\|_{L^q([0,2],L^{3q}(\Omega))}$$
$$+ \|v\|_{L^2([0,2],H^1(\Omega))}\big)^P, \tag{3.13}$$

for the positive constants C and P defined above. Furthermore, using estimate (2.24) with $k = q-2$ and the interpolation embedding (2.25), we can see that

$$\|u\|_{L^\infty([0,2],L^q(\Omega))} + \|u\|_{L^q([0,2],L^{3q}(\Omega))}$$
$$\le C'\big(1 + (A_{q-2}(u_0), 1)\big)^{\frac{1}{q}} \le C''\big(1 + \|u_0\|_{L^{p+q}(\Omega)}\big)^{1+\frac{p}{q}}$$

(here, we have also implicitly used (3.1)). Inserting this last estimate into the right-hand side of (3.13), we deduce (3.3) (up to a time scaling) and finish the proof of Lemma 3.1. □

Thus, in order to prove the dissipative estimate in $L^\infty(\Omega)$, it is sufficient to obtain it in $L^{p+q}(\Omega)$, for a fixed $q > \frac{3p}{2}$. Based on this observation and using the dissipation integral (3.2), we are now ready to obtain a uniform (as $t \to \infty$) estimate in $L^\infty(\Omega)$.

Lemma 3.2 *Let the assumptions of Lemma* 3.1 *hold. Then, the solution u to problem* (2.2) *satisfies*

$$\|u(t)\|_{L^\infty} \le Q(\|u_0\|_{L^\infty \cap H^1}), \quad t \ge 0, \tag{3.14}$$

where the monotonic function Q is independent of t and u.

Proof. We multiply equation (1.8) by $u|u|^k$, where $k > \frac{3p}{2} - 2$ is some *fixed* number, and integrate over Ω. Then, arguing as above and using the embedding $H^1 \subset L^6$, we infer

$$\partial_t(A_k(u(t)), 1) + 2\gamma\|u(t)\|^{k+2}_{L^{3(k+2)}} \le C_1 + C_2\|v(t)\|_{H^1}\|u(t)\|^{k+1}_{L^{\frac{6}{5}(k+2)}}, \tag{3.15}$$

for a positive constant γ. Using now the interpolation inequality

$$\|u\|_{L^{\frac{6}{5}(k+2)}} \le C\|u\|^{\frac{3}{4}}_{L^{k+2}}\|u\|^{\frac{1}{4}}_{L^{3(k+2)}},$$

together with Young's inequality with exponents $(k+2)$, 2 and $\frac{2(k+2)}{k}$, we see that, for any $\varepsilon, \gamma > 0$,

$$\begin{aligned}
&C_2\|v\|_{H^1}\|u\|^{k+1}_{L^{\frac{6}{5}(k+2)}} \\
&\quad\le C_3\left[\|v\|_{H^1}\|u\|^{\frac{k+2}{2}}_{L^{k+2}}\right]\cdot\left[\|u\|^{\frac{k-1}{4}}_{L^{k+2}}\right]\cdot\left[\|u\|^{\frac{k+1}{4}}_{L^{3(k+2)}}\right] \\
&\quad\le C'\left[\|v\|_{H^1}\|u\|^{\frac{k+2}{2}}_{L^{k+2}}\right]\cdot\left[\|u\|^{\frac{k}{2}}_{L^{3(k+2)}}\right] \\
&\quad\le \left[C'\varepsilon^{-1/2}\gamma^{-\frac{k}{2(k+2)}}\right]\cdot\left[\varepsilon^{1/2}\|v\|_{H^1}\|u\|^{\frac{k+2}{2}}_{L^{k+2}}\right]\cdot\left[\gamma^{\frac{k}{2(k+2)}}\|u\|^{\frac{k}{2}}_{L^{3(k+2)}}\right] \\
&\quad\le C_{\varepsilon,\gamma} + \varepsilon\|v\|^2_{H^1}\|u\|^{k+2}_{L^{k+2}} + \gamma\|u\|^{k+2}_{L^{3(k+2)}}
\end{aligned}$$

and, consequently,

$$\partial_t(A_k(u(t)),1)+\gamma\|u(t)\|^{k+2}_{L^{3(k+2)}}\le C_\epsilon+\epsilon\|v(t)\|^2_{H^1}\|u(t)\|^{k+2}_{L^{k+2}},\quad \forall\epsilon>0, \tag{3.16}$$

which, together with inequalities (2.23) and (3.1) and the inequality $3(k+2)>k+p+2$, implies that

$$\partial_t B_k(t)-\|v(t)\|^2_{H^1}B_k(t)+\gamma' B_k(t)^\theta\le C_3,\quad \gamma'>0, \tag{3.17}$$

where $\theta=\frac{k+2}{k+p+2}<1$, $B_k(t):=(A_k(u(t)),1)+C_4$ (for a constant C_4 which is independent of t and u) and C_3 is a constant which is independent of t and u. We claim that (3.17), together with the dissipation integral (3.2), implies the following estimate:

$$B_k(t)\le Q(\|u_0\|_{L^\infty\cap H^1}), \tag{3.18}$$

for a monotonic function Q which is independent of t and u. Indeed, setting $Z(t):=B_k(t)K(t)$ and $K(t):=e^{-\int_0^t\|v(s)\|^2_{H^1}\,ds}$, we have

$$\partial_t Z(t)+\gamma' K(t)^{1-\theta}Z(t)^\theta\le C_3K(t). \tag{3.19}$$

Furthermore, owing to the dissipation integral (3.2),

$$K_0\le K(t)\le 1, \tag{3.20}$$

for a constant K_0 which only depends on the $L^\infty\cap H^1$-norm of the initial datum. Thus, using the fact that Z is positive, we have

$$\partial_t Z(t)+\gamma' K_0^{1-\theta}Z(t)^\theta\le C_3, \tag{3.21}$$

which implies (by comparison arguments) estimate (3.18). More precisely, we have

$$Z(t)\le\tilde Z:=\max(Z(0),Z_\star),$$

where $Z_\star:=\left(\frac{C_3}{\gamma' K_0^{1-\theta}}\right)^{\frac{1}{\theta}}$ satisfies $\partial_t Z_\star+\gamma' K_0^{1-\theta}Z_\star^\theta=C_3$. Therefore, due to (2.23), we have

$$\|u(t)\|_{L^{k+2}}\le Q(\|u_0\|_{L^\infty\cap H^1}),\quad t\ge 0.$$

Thus, recalling that $k+2>\frac{3p}{2}$, Lemmas 2.3 and 3.1 give

$$\|u(t)\|_{L^\infty} \le Q(\|u_0\|_{L^\infty\cap H^1}), \quad t\ge 0, \tag{3.22}$$

for a monotonic function Q which is independent of t and u. This estimate, together with (2.19), gives the desired estimate (3.14) and finishes the proof of the lemma. □

The next simple lemma shows that the estimate on the L^∞-norm of u implies an analogous one on the H^1-norm.

Lemma 3.3 *Let the assumptions of Lemma* 3.1 *hold and let* u *be a solution to Equation* (2.2). *Then, the following estimate holds*:

$$\|u(t+1)\|_{H^1} \le Q(\|u\|_{L^\infty([t,t+1]\times\Omega)}), \quad t\ge 0, \tag{3.23}$$

for a monotonic function Q *which is independent of* t *and* u.

Proof. It is sufficient to prove estimate (3.23) for $t=0$ only. To this end, we first multiply equation (2.2) by u and integrate over $[0,1]\times\Omega$. Since the L^∞-norm of u is known, we have, after standard transformations,

$$\int_0^1 \|\nabla u(s)\|_{L^2}^2\,ds \le Q(\|u\|_{L^\infty([0,1]\times\Omega)}), \tag{3.24}$$

for a monotonic function Q. We then multiply the equation by $t\partial_t u$ and integrate over $[0,1]\times\Omega$. This gives

$$\begin{aligned}&\int_0^1 s\big[(\alpha'(u(s))\partial_t u(s), \partial_t u(s)) + \|\partial_t u(s)\|_{H^{-1}}^2\big]\,ds\\ &+\frac{1}{2}\|\nabla u(1)\|_{L^2}^2 + (F(u(1)),1)\\ &\quad= \int_0^1 \left(\frac{1}{2}\|\nabla u(s)\|_{L^2}^2 + (F(u(s)),1)\right)ds. \end{aligned}\tag{3.25}$$

Using (3.24) and exploiting again the fact that the L^∞-norm of the solution is known in order to estimate the second part of the integral in the right-hand side of (3.25), we deduce the required estimate (3.23) and finish the proof of the lemma. □

We are now ready to formulate and prove the main result of this section, namely, the dissipativity of u in $L^\infty(\Omega)\cap H^1(\Omega)$.

Theorem 3.4 *Let the assumptions of Lemma* 3.1 *hold. Then, the solution* u *to problem* (2.2) *satisfies the following estimate*:

$$\|u(t)\|_{L^\infty\cap H^1}\le Q(\|u_0\|_{L^\infty\cap H^1})e^{-\gamma t}+C_*,\quad t\ge 0, \tag{3.26}$$

where the positive constant γ *and the monotonic function* Q *are independent of* t *and* u.

Proof. As usual, in order to prove (3.26), it is sufficient to prove that there exist a constant C_* and a monotonic function Q_1 independent of the solution u such that

$$\|u(t)\|_{L^\infty\cap H^1}\le C_*,\quad \forall t\ge Q_1(\|u_0\|_{L^\infty\cap H^1}). \tag{3.27}$$

In order to prove (3.27), we again use the dissipation integral (3.2). Indeed, let u be a solution to (2.2). Then, due to (3.2), for every $N>0$, the interval $[0,T_0-N]$, with $T_0=T_0(\|u_0\|_{L^\infty\cap H^1}):=NQ(\|u_0\|_{L^\infty\cap H^1})$ (here, Q is the same as in (3.2)), contains at least one point $t_0=t_0(u)$ such that

$$\int_{t_0}^{t_0+N}\|v(t)\|^2_{H^1}\,dt\le 1. \tag{3.28}$$

We now consider equation (3.17) on the time interval $[t_0,t_0+N]$. Then, on the one hand, due to (2.23) and (3.18),

$$\begin{aligned}|B_k(t_0)|&\le Q(\|u_0\|_{L^\infty\cap H^1}),\\ B_k(t)&\ge\gamma\|u(t)\|^{k+2}_{L^{k+2}}-C,\quad t\in[t_0,t_0+N],\end{aligned} \tag{3.29}$$

where the monotonic function Q and the positive constants γ and C are independent of t and u. On the other hand, due to (3.28), estimate (3.20) is satisfied for $K(t):=e^{-\int_{t_0}^t\|v(s)\|^2_{H^1}\,ds}$, with $K_0=e^{-1}$ on this time interval, and

$$e^{-1}B_k(t)\le Z(t)\le B_k(t),\quad t\in[t_0,t_0+N]. \tag{3.30}$$

Therefore, due to the comparison principle for scalar ODEs,

$$B_k(t) \le eZ(t) \le ey(t), \quad t \in [t_0, t_0 + N], \tag{3.31}$$

where the function $y = y(t)$ solves the equation

$$y' + \gamma' K_0^{1-\theta} y^\theta = C_3, \quad C_3 > 0,\ y(t_0) \ge B_k(t_0), \tag{3.32}$$

We set, as above, $Z_* := \big(\frac{C_3}{\gamma' K_0^{1-\theta}}\big)^{\frac{1}{\theta}}$. In addition, we can assume, without loss of generality, that

$$y(t_0) = \max\big\{2Z_*, Q(\|u_0\|_{L^\infty \cap H^1})\big\},$$

where Q is the same as in (3.29). Then, the solution $y(t)$ is monotone decreasing and satisfies

$$y'(t) \le C_3 - \gamma' K_0^{1-\theta}(2Z_*)^\theta = -(2^\theta - 1)C_3$$

as long as $y(t) \ge 2Z_*$. Thus,

$$y(t) \le \max\big\{2Z_*, y(t_0) - (2^\theta - 1)C_3(t - t_0)\big\} \tag{3.33}$$

and, therefore,

$$B_k(t) \le e \max\big\{2Z_*, Q(\|u_0\|_{L^\infty \cap H^1}) - (2^\theta - 1)C_3(t - t_0)\big\}, \quad t \in [t_0, t_0 + N],$$

where the constants C_3 and Z_* are independent of u_0, t_0 and N. We now set

$$N = N(\|u_0\|_{L^\infty \cap H^1}) := \frac{Q(\|u_0\|_{L^\infty \cap H^1})}{(2^\theta - 1)C_3} + 2.$$

Then, the last estimate guarantees that

$$B_k(t) \le 2eZ_*, \quad t \in [t_0 + N - 2, t_0 + N]. \tag{3.34}$$

Inequalities (3.13), (3.28), (3.29) and (3.34) now give

$$\|u\|_{L^\infty([t_0+N-1, t_0+N]\times\Omega)} \le C', \tag{3.35}$$

where $t_0 \le T_0(\|u_0\|_{L^\infty \cap H^1})$ and the constant C' is independent of u_0. Therefore, due to Lemma 3.3, we obtain

$$\|u(t_0+N)\|_{L^\infty\cap H^1} \le C'',$$

where C'' is independent of u_0. Finally, using estimate (3.14), we infer

$$\|u(t)\|_{L^\infty\cap H^1} \le C_*, \quad t \ge t_0+N, \quad t_0 \le T_0(\|u_0\|_{L^\infty\cap H^1}).$$

Thus, (3.27) is verified and the theorem is proved. □

Remark 3.5 The proof of dissipativity given above is strongly based on the dissipation integral (3.2) and cannot be extended to a system with nonautonomous external forces for which this integral is infinite. This is related to the fact that "pathological" nonsmooth solutions to equation (1.8) may exist if α grows sufficiently fast as $|s| \to \infty$ (see [7] and [19] for situations in which one encounters such solutions). In that case, the energy solutions are not necessarily regular and we need, in addition, to establish the dissipativity in $L^q(\Omega)$, for $q > \frac{3p}{2}$, and, as far as the L^q-estimates are concerned, we can only treat the term $\partial_t(-\Delta)^{-1}u$ as a perturbation.

This problem can be overcome if the growth exponent p is not too large, namely,

$$\alpha'(s) \le C(1+|s|^4), \quad s\in\mathbb{R},\ C\ge 0. \tag{3.36}$$

Indeed, in that case, multiplying equation (2.2) by u, we have

$$\partial_t B(t) + \gamma B(t)^{\frac{1}{3}} + \gamma\|\nabla u(t)\|^2_{L^2} + (f(u(t)),u(t)) \le C, \tag{3.37}$$

where $B(t) := (A_0(u(t)),1) + \|u(t)\|^2_{H^{-1}}$ and γ and C are positive constants. This estimate gives the dissipativity in $L^2(\Omega)$,

$$\|u(t)\|^2_{L^2} + \int_t^{t+1} \left[\|\nabla u(s)\|^2_{L^2} + (f(u(s)),u(s))\right] ds \le Q(\|u_0\|_{L^6})e^{-\beta t} + C_*,$$
$$t\ge 0,\ \beta>0.$$

If, in addition, the nonlinearity f satisfies some regularity assumption, namely,

$$F(s) \le C(1+f(s)s), \quad s\in\mathbb{R},\ C\ge 0,$$

multiplying now equation (2.2) by $\partial_t u$ and $t\partial_t u$ and arguing in a standard way, we deduce the dissipativity in $H^1(\Omega)$,

$$\|u(t)\|_{H^1}^2 + \int_t^{t+1} \|v(s)\|_{H^1}^2 \, ds \le Q(\|u_0\|_{L^\infty \cap H^1}) e^{-\beta t} + C_*, \quad t \ge 0,\ \beta > 0. \tag{3.38}$$

Thus, we have a dissipative estimate on u in $L^6(\Omega)$. Now, recall that $p = 4$, so that we exactly have the limit case $q = 6 = \frac{3p}{2}$. If $p < 4$, the desired dissipativity in $L^\infty(\Omega)$ is an immediate consequence of Lemma 3.1. In order to handle the critical case $p = 4$, we need one more step before applying Lemma 3.1. To be more precise, owing to the dissipative estimate (3.38) and the interpolation embedding (2.25) (with $k = 0$), we have an analogue of inequality (3.15), with an additional function h in the right-hand side satisfying the estimate

$$\int_t^{t+1} |h(s)| \, ds \le Q(\|u_0\|_{L^\infty \cap H^1}) e^{-\beta t} + C_*, \quad t \ge 0,\ \beta > 0,$$

if $k \le \frac{4}{3}$. This, in turn, gives the following analogue of (3.16):

$$\partial_t B_k(t) + \gamma' [B_k(t)]^\theta + \gamma' \|u(t)\|_{L^{3(k+2)}}^{k+2} \le h(t), \quad \theta = \theta_k > 0,$$

which is sufficient to obtain a dissipative estimate of the form

$$\|u\|_{L^{k+2}([t,t+1], L^{3(k+2)}(\Omega))} \le Q(\|u_0\|_{L^\infty \cap H^1}) e^{-\beta t} + C_*, \quad t \ge 0,\ \beta > 0,$$

if $k \le \frac{4}{3}$. Thus, we have, for $k = \frac{4}{3}$, a control on the L^q-norm of $u(t_\star)$, for $t_\star \in [T, T+1]$, $\forall T \ge 0$, with $q = 3(k+2) = 10$, and we can now apply Lemma 3.1.

It is not difficult to see that this alternative scheme does not require the dissipation integral (3.2) to be finite and can be extended to the nonautonomous case (i.e., we can, in particular, consider nonautonomous external forces).

Another possibility to avoid the use of the dissipation integral (3.2) is to assume that f grows *faster* than α,

$$f(s)s \ge -C + \gamma |s|^{p+2}, \quad s \in \mathbb{R},\ \gamma > 0,\ C \ge 0.$$

In that case, an analogue of (3.37) still holds (but only the second term, now of the form $\gamma B(t)$, providing the dissipation appears; this dissipation is due to the nonlinearity, but not to the Laplacian). It is also worth mentioning that, in both cases, the above mentioned pathological energy solutions cannot exist, see [19].

4. Finite-dimensional attractors for L^∞-solutions

The aim of this section is to prove the existence of finite-dimensional attractors for the solution semigroup

$$S(t) : \Phi \to \Phi, \quad S(t)u_0 := u(t), \tag{4.1}$$

associated with equation (2.2) in the phase space

$$\Phi := L^\infty(\Omega) \cap H_0^1(\Omega). \tag{4.2}$$

We recall that, owing to Theorem 3.4, we already have the dissipativity of the semigroup (4.1) in Φ,

$$\|S(t)u_0\|_\Phi \le Q(\|u_0\|_\Phi)e^{-\gamma t} + C_*, \quad t \ge 0,\ \gamma > 0. \tag{4.3}$$

However, if α degenerates, it seems problematic (and even impossible) to obtain additional regularity on the solutions to problem (2.2) in order to establish the asymptotic compactness of this semigroup in a strong topology. Therefore, we will consider below the attractors in a *weak* topology only. For the reader's convenience, we recall the definition of such an attractor (see, e.g., [3], [5], [24], [30] and [36] for more details).

Definition 4.1 The set $\mathcal{A}$ is the weak global attractor for the semigroup (4.1) in the phase space Φ if:

1) $\mathcal{A}$ is closed in Φ and is compact in Φ endowed with the $*$-weak topology;

2) $\mathcal{A}$ is invariant, $S(t)\mathcal{A} = \mathcal{A}$, $t \ge 0$;

3) $\mathcal{A}$ attracts the images of all bounded sets of Φ in the $*$-weak topology, i.e., for every bounded set $B \subset \Phi$ and every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in the $*$-weak topology of Φ, there exists a time $T = T(B, \mathcal{O})$ such that

$$S(t)B \subset \mathcal{O}(\mathcal{A}), \quad \forall t \ge T. \tag{4.4}$$

In particular, owing to the compact embedding

$$\Phi \subset H^{1-\delta} \cap L^q,$$

for any $\delta > 0$ and $1 \leq q < \infty$, the weak attraction (4.4) implies the *strong* attraction in $H^{1-\delta}(\Omega) \cap L^p(\Omega)$,

$$\lim_{t\to\infty} \operatorname{dist}_{H^{1-\delta}\cap L^q}(S(t)B, \mathcal{A}) = 0, \tag{4.5}$$

where $\operatorname{dist}_V(\cdot,\cdot)$ denotes the Hausdorff semi-distance between sets in V, defined by

$$\operatorname{dist}_V(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_V.$$

The next theorem establishes the existence of such a weak attractor for the solution semigroup associated with equation (2.2).

Theorem 4.2 *Let the assumptions of Theorem* 3.4 *hold. Then, the semigroup* (4.1) *associated with problem* (2.2) *possesses the weak global attractor* $\mathcal{A}$ *in the sense of Definition* 4.1*. Furthermore, this attractor has the following standard structure*:

$$\mathcal{A} = \mathcal{K}\mid_{t=0}, \tag{4.6}$$

where $\mathcal{K} \subset L^\infty(\mathbb{R}\times\Omega) \cap L^\infty(\mathbb{R}, H_0^1(\Omega))$ *is the set of all bounded solutions to equation* (2.2) *defined for all* $t \in \mathbb{R}$.

Indeed, the asymptotic compactness in the $*$-weak topology is immediate, due to (4.3) and the fact that a bounded closed ball in Φ is compact in the $*$-weak topology, and the continuity of $S(t)$ (in the $*$-weak topology) on the absorbing ball can be verified in a standard way (we leave the details to the reader, see also [3] and [5]). Thus, usual global attractors' existence theorems give the attractor $\mathcal{A}$ for the solution semigroup and equality (4.6) (see again [3], [5], [24], [30] and [36]).

Our next aim is to prove the finite-dimensionality of the above attractor. Since it is compact in the $*$-weak topology only, the finite-dimensionality will also be understood in a weaker topology, say, in $H^{-1}(\Omega)$. Furthermore, we will also establish the existence of a so-called exponential attractor for the solution semigroup in the sense of the following definition.

Definition 4.3 A set $\mathcal{M}$ is an exponential attractor for the solution semigroup $S(t)$ in Φ if the following assumptions are satisfied:

1) $\mathcal{M}$ is bounded in Φ and compact in the $*$-weak topology of Φ;

2) $\mathcal{M}$ is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $t \geq 0$;

3) $\mathcal{M}$ has finite fractal dimension in the space $H^{-1}(\Omega)$,

$$\dim_f(\mathcal{M}, H^{-1}(\Omega)) \leq C < \infty; \tag{4.7}$$

4) $\mathcal{M}$ attracts exponentially the images of all bounded sets of Φ in the H^{-1}-topology, i.e., for every bounded subset $B \subset \Phi$,

$$\operatorname{dist}_{H^{-1}}(S(t)B, \mathcal{M}) \leq Q(\|B\|_\Phi)e^{-\gamma t}, \quad t \geq 0, \tag{4.8}$$

where the positive constant γ and the monotonic function Q are independent of t.

We note that, by interpolation between Φ and $H^{-1}(\Omega)$,

$$\|u\|_{H^{1-\delta}\cap L^q} \leq C\|u\|_{H^{-1}}^{\theta}\|u\|_{\Phi}^{1-\theta},$$

where the positive constants C and θ depend on $\delta > 0$ and $1 \leq q < \infty$, but are independent of u. This shows that an exponential attractor $\mathcal{M}$ is *automatically* finite-dimensional in the spaces $H^{1-\delta}(\Omega) \cap L^q(\Omega)$, for all $\delta > 0$ and $1 \leq q < \infty$, and the exponential attraction property (4.8) also holds in these spaces.

The next theorem, which establishes the existence of such an attractor, is the main result of this section.

Theorem 4.4 *Let the assumptions of Theorem* 3.4 *hold. Then, the solution semigroup $S(t)$ associated with equation* (2.2) *possesses an exponential attractor $\mathcal{M}$ in the sense of Definition* 4.3.

Proof. Keeping in mind the Lipschitz continuity property proved in Lemma 2.2, we introduce the following nonlinear transformation $\mathcal{B}$ of the phase space Φ:

$$\mathcal{B}(u) = \alpha(u) + (-\Delta)^{-1}u, \quad u \in \Phi. \tag{4.9}$$

The next result shows that this transformation is Hölder continuous in $H^{-1}(\Omega)$.

Lemma 4.5 *Let the assumptions of Theorem* 3.4 *hold. Then, the following inequalities hold, for every* u_1, $u_2 \in \Phi$:

$$\begin{cases} \|\mathcal{B}(u_1)-\mathcal{B}(u_2)\|_{H^{-1}} \le C\|u_1-u_2\|_{H^{-1}}^{\frac{1}{2}}, \\ \|u_1-u_2\|_{H^{-1}} \le C\|\mathcal{B}(u_1)-\mathcal{B}(u_2)\|_{H^{-1}}^{\frac{1}{2}}, \end{cases} \tag{4.10}$$

where the constant C *only depends on* $\|u_1\|_\Phi$ *and* $\|u_2\|_\Phi$.

Proof. We have, since $u_i \in \Phi,\ i=1,2$,

$$\begin{aligned} \|\mathcal{B}(u_1)-\mathcal{B}(u_2)\|_{H^{-1}} &\le C\big(\|\alpha(u_1)-\alpha(u_2)\|_{L^2}+\|u_1-u_2\|_{H^{-1}}\big) \\ &\le C_1\|u_1-u_2\|_{L^2} \le C_2\|u_1-u_2\|_{H^{-1}}^{\frac{1}{2}}\|u_1-u_2\|_{H^1}^{\frac{1}{2}} \\ &\le C_3\|u_1-u_2\|_{H^{-1}}^{\frac{1}{2}}. \end{aligned}$$

Conversely, using the monotonicity of α,

$$\begin{aligned} \|u_1-u_2\|_{H^{-1}}^2 &\le (\mathcal{B}(u_1)-\mathcal{B}(u_2),u_1-u_2) \\ &\le C\|\mathcal{B}(u_1)-\mathcal{B}(u_2)\|_{H^{-1}}\|u_1-u_2\|_{H^1} \\ &\le C'\|\mathcal{B}(u_1)-\mathcal{B}(u_2)\|_{H^{-1}} \end{aligned}$$

and the lemma is proved. □

The next lemma, which establishes some kind of smoothing property on the difference of solutions to (2.2), is the main technical tool to prove the existence of an exponential attractor (see also [27] and [37]).

Lemma 4.6 *Let the above assumptions hold and let* $l>0$, u_1 *and* u_2 *be two solutions to* (2.2) *and* $w_i := \mathcal{B}(u_i)$, $i=1,2$. *Then, the following estimate holds*:

$$\begin{aligned} &\|w_1-w_2\|_{L^2([l,2l]\times\Omega)}+\|\partial_t w_1-\partial_t w_2\|_{L^2([l,2l],H^{-3}(\Omega))} \\ &\quad \le C_l\|w_1-w_2\|_{L^2([0,l],H^{-1}(\Omega))}, \end{aligned} \tag{4.11}$$

where the constant C_l only depends on l and on the norms $\|u_i\|_\Phi$, $i = 1, 2$.

Proof. Returning to (2.5) and (2.11), we have, for $\epsilon = \frac{1}{2}$ in (2.11),

$$\partial_t \|w\|^2_{H^{-1}} + (\alpha(u_1) - \alpha(u_2), u) + \|u\|^2_{H^{-1}} \le C\|w\|^2_{H^{-1}}, \tag{4.12}$$

where $w = w_1 - w_2$ and $u = u_1 - u_2$. Now, we have

$$\begin{aligned}(\alpha(u_1) - \alpha(u_2), u) + \|u\|^2_{H^{-1}} &\ge C\big(\|\alpha(u_1) - \alpha(u_2)\|^2_{L^2} + \|(-\Delta)^{-1}u\|^2_{L^2}\big) \\ &\ge C'\|w\|^2_{L^2},\end{aligned}$$

for $C' > 0$ small enough, so that (4.12) yields

$$\partial_t \|w\|^2_{H^{-1}} + C\|w\|^2_{L^2} + \frac{1}{2}\|u\|^2_{H^{-1}} \le C'\|w\|^2_{H^{-1}}. \tag{4.13}$$

Applying Gronwall's lemma to estimate (4.13), we deduce in a standard way that

$$\|w\|^2_{L^\infty([l,2l],H^{-1}(\Omega))} + \int_l^{2l} \big(\|w(t)\|^2_{L^2} + \|u(t)\|^2_{H^{-1}}\big)\, dt \le C_l \|w(l)\|^2_{H^{-1}}. \tag{4.14}$$

Furthermore, multiplying (4.13) by t and integrating over $[0, l]$, we find

$$\|w(l)\|^2_{H^{-1}} \le C_l \int_0^l \|w(t)\|^2_{H^{-1}}\, dt. \tag{4.15}$$

Combining the last two estimates, we obtain

$$\|w\|_{L^2([l,2l]\times\Omega)} + \|u\|_{L^2([l,2l],H^{-1}(\Omega))} \le C_l \|w\|_{L^2([0,l],H^{-1}(\Omega))}. \tag{4.16}$$

Thus, the first term in the left-hand side of (4.11) is estimated and we only need to estimate the H^{-3}-norm of the derivative $\partial_t w$. To this end, we note that the functions u and w solve

$$\partial_t w = \Delta u - [f(u_1) - f(u_2)], \tag{4.17}$$

so that, taking the H^{-3}-norm of both sides of this equation and using (2.10), we finally have

$$\|\partial_t w(t)\|_{H^{-3}} \le \|u(t)\|_{H^{-1}} + \|f(u_1(t)) - f(u_2(t))\|_{H^{-3}} \le C\|u(t)\|_{H^{-1}}. \quad (4.18)$$

Taking the $L^2([l, 2l])$-norm of both sides of (4.18) and using (4.16), we obtain the desired estimate for $\partial_t w$ and finish the proof of the lemma. □

We are now ready to finish the proof of the theorem. To this end, we note that, due to the dissipative estimate (4.3), it is sufficient to construct the exponential attractor for initial data belonging to the absorbing ball

$$\mathbb{B} := \{u_0 \in \Phi,\ \|u_0\|_\Phi \le 2C_*\}. \quad (4.19)$$

Furthermore, due to the same dissipative estimate, we can fix $l > 0$ in such a way that

$$S(l)\mathbb{B} \subset \mathbb{B}. \quad (4.20)$$

We now introduce the two spaces

$$\mathcal{H} := L^2([0, l], H^{-1}(\Omega)) \text{ and } \mathcal{H}_1 := L^2([0, l] \times \Omega) \cap H^1([0, l], H^{-3}(\Omega)) \quad (4.21)$$

and define a map $\mathbb{S} : \mathbb{B} \to \mathcal{H}$ by the following expression:

$$(\mathbb{S}u_0)(t) := \mathcal{B}(S(t)u_0), \quad t \in [0, l].$$

Then, due to Lemmas 2.2 and 4.5, this map is uniformly Hölder continuous as a map from $\mathbb{B}$, endowed with the H^{-1}-topology, into $\mathcal{H}$. Consequently, its image

$$\mathbb{B}_0 := \mathbb{S}(\mathbb{B}) \quad (4.22)$$

is a compact set of $\mathcal{H}$. We now consider the following map $\mathcal{L}$ which is conjugated with the solution operator $S(l)$ via the map $\mathbb{S}$:

$$\mathcal{L} := \mathbb{S} \circ S(l) \circ \mathbb{S}^{-1}.$$

Then, owing to (4.20), $\mathcal{L} : \mathbb{B}_0 \to \mathbb{B}_0$ and, owing to Lemma 4.6, we have, for every w_1, $w_2 \in \mathbb{B}_0$,

$$\|\mathcal{L}w_1 - \mathcal{L}w_2\|_{\mathcal{H}_1} \le C\|w_1 - w_2\|_{\mathcal{H}}, \quad (4.23)$$

where the constant C is independent of w_i, $i = 1, 2$.

Since the embedding $\mathcal{H}_1 \subset \mathcal{H}$ is compact, inequality (4.23) implies the existence of an exponential attractor $\mathbb{M} \subset \mathbb{B}_0$ for the discrete semigroup generated by the iterations of the map $\mathcal{L}$ which satisfies the following properties:

1) it is compact and positively invariant, $\mathcal{L}\mathbb{M} \subset \mathbb{M}$;

2) it attracts exponentially the set $\mathbb{B}_0$, i.e.,

$$\operatorname{dist}_{\mathcal{H}}(\mathcal{L}^n\mathbb{B}_0, \mathbb{M}) \le Ce^{-\gamma n}, \quad n \in \mathbb{N}, \tag{4.24}$$

where C and γ are positive constants;

3) its fractal dimension is finite,

$$\dim_f(\mathbb{M}, \mathcal{H}) \le C < \infty. \tag{4.25}$$

We refer the reader to [16] for more details.

We now note that, due to Lemma 4.5 and estimate (4.15), the map

$$\mathbb{S}^{-1} \circ \mathcal{L} : \mathbb{B}_0 \to \mathbb{B} \subset \Phi$$

is uniformly Hölder continuous with exponent $\frac{1}{2}$ (here, $\mathbb{B}$ is endowed with the H^{-1}-topology). Therefore, the set

$$\mathcal{M}_d := \mathbb{S}^{-1} \circ \mathcal{L}\mathbb{M} \tag{4.26}$$

is an exponential attractor for the discrete semigroup $\{S(ln),\ n \in \mathbb{N}\}$ acting on $\mathbb{B} \subset H^{-1}(\Omega)$.

As usual, the desired exponential attractor $\mathcal{M}$ for the semigroup $S(t)$ in the continuous framework of time can be constructed by the following expression:

$$\mathcal{M} := \cup_{t\in[0,l]} S(t)\mathcal{M}_d.$$

Indeed, due to Lemmas 2.2 and 4.5, the maps $S(t)$ are uniformly Hölder continuous (for the H^{-1}-topology) for every fixed t and, due to the control (2.19), every solution u to problem (2.2) satisfies

$$u \in C^{\frac{1}{2}-\delta}([0,T], H^{-1}(\Omega)), \quad \delta > 0.$$

Thus, the map $(t, u_0) \mapsto S(t)u_0$ is uniformly Hölder continuous on $[0, l] \times \mathcal{M}_d$. All the desired properties of the exponential attractor $\mathcal{M}$ are now immediate consequences of the analogous properties for the discrete attractor $\mathcal{M}_d$ and Theorem 4.4 is proved. □

Remark 4.7 Since, obviously, the global attractor is a subset of any exponential attractor, $\mathcal{A} \subset \mathcal{M}$, the above theorem guarantees that the global attractor $\mathcal{A}$ also has finite fractal dimension in $H^{-1}(\Omega)$.

5. Energy solutions and the Cahn-Hilliard limit $\alpha \to 0$

The aim of this section is to study the Cahn-Hilliard limit $\alpha \to 0$ in equations (2.2). However, the theory developed in the previous sections is not sufficient for this purpose, since our L^∞-estimates essentially depend on α and diverge as $\alpha \to 0$. Consequently, all further estimates, including also the dimension of the attractors, diverge as $\alpha \to 0$. In order to overcome this difficulty, we will consider weaker energy solutions to problem (2.2) and will avoid the use of the nonuniform L^∞-estimates. To this end, we need to impose the following restrictions on the nonlinearity $\alpha \in C^2(\mathbb{R})$:

$$0 \le \alpha'(s) \le \alpha_0, \quad |\alpha''(s)| \le \alpha_0,\ s \in \mathbb{R},\ \alpha_0 > 0. \tag{5.1}$$

Furthermore, in addition to the dissipativity assumption (1.13), we impose the following growth restriction and quasi-monotonicity condition on the nonlinearity f:

$$|f''(s)| \le C(1 + |s|), \quad f'(s) \ge -C,\ s \in \mathbb{R},\ C \ge 0. \tag{5.2}$$

We note that, in contrast to the previous sections, we now no longer need the nondegeneracy (1.12) of the nonlinearity α at infinity.

We say that a pair of functions (u, v),

$$u \in L^\infty\big([0,T], H_0^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big), \quad v \in L^2\big([0,T], H_0^1(\Omega)\big), \tag{5.3}$$

is an energy solution to problem (2.2) if u and v satisfy (2.2) in the sense of distributions.

The proof of existence of energy solutions is, in view of the assumptions made on α and f, straightforward.

The next lemma gives the uniform (as $\alpha \to 0$) dissipativity of such energy solutions.

Lemma 5.1 *Let assumptions* (1.13), (5.1) *and* (5.2) *be satisfied. Then, any energy solution* (u, v) *to problem* (2.2) *satisfies the following dissipative estimate*:

$$\|u(t)\|_{H^1}^2 + \int_t^{t+1} \big(\|v(s)\|_{H^1}^2 + \|u(s)\|_{H^2}^2\big)\, ds \le Q(\|u_0\|_{H^1})e^{-\gamma t} + C_*, \quad t \ge 0, \tag{5.4}$$

where the monotonic function Q *and the positive constants* γ *and* C_* *are independent of* $\alpha_0 \to 0$.

Proof. Multiplying equation (2.2) by u and integrating over Ω, we have

$$\partial_t\left[(A(u), 1) + \frac{1}{2}\|u\|_{H^{-1}}^2\right] + \|\nabla u\|_{L^2}^2 \le C, \tag{5.5}$$

where $A(s) := \int_0^s \alpha'(z)z\, dz$. Using the fact that

$$0 \le A(u) \le \frac{1}{2}\alpha_0|u|^2,$$

together with Friedrichs' inequality, we deduce from (5.5) that

$$\partial_t\left[(A(u), 1) + \frac{1}{2}\|u\|_{H^{-1}}^2\right] + \gamma\left[(A(u), 1) + \frac{1}{2}\|u\|_{H^{-1}}^2\right] + \gamma\|\nabla u\|_{L^2}^2 \le C,$$

where the positive constant γ is independent of $\alpha_0 \to 0$. Applying Gronwall's lemma to this relation, we obtain a dissipative estimate in $H^{-1}(\Omega)$,

$$\|u(t)\|_{H^{-1}}^2 + \int_t^{t+1} \|u(s)\|_{H^1}^2\, ds \le C\|u_0\|_{L^2}^2 e^{-\gamma t} + C_*, \quad t \ge 0,\ \gamma > 0, \tag{5.6}$$

where all the constants are uniform with respect to $\alpha_0 \to 0$.

In a next step, we multiply equation (2.2) by $\partial_t u$. Then, arguing as in (2.18) and (2.19), we find

$$\|u(t)\|_{H^1}^2 + \int_{t_0}^t \big[(\alpha'(u(s))\partial_t u(s), \partial_t u(s)) + \|v(s)\|_{H^1}^2\big]\, ds \le C\big(1 + \|u(t_0)\|_{H^1}^4\big), \tag{5.7}$$

where C is again independent of $\alpha_0 \to 0$ and $t \geq t_0 \geq 0$ (here, we have implicitly used the fact that $F(u) \leq C(1+|u|^4)$). Estimates (5.6) and (5.7) imply the following dissipative estimate in $H^1(\Omega)$:

$$\|u(t)\|_{H^1}^2 + \int_t^{t+1} \left[(\alpha'(u(s))\partial_t u(s), \partial_t u(s)) + \|v(s)\|_{H^1}^2\right] ds$$
$$\leq C(1+\|u_0\|_{H^1})^4 e^{-\gamma t} + C_*, \tag{5.8}$$

$t \geq 0$, $\gamma > 0$, where all the constants are again uniform with respect to $\alpha_0 \to 0$ (indeed, taking, e.g., $t = 1$ in (5.6), we note that there exists $t_0 \in (1,2)$ such that $\|u(t_0)\|_{H^1}^2 \leq C\|u_0\|_{L^2}^2 e^{-\gamma t_0} + C_*$). Thus, it only remains to deduce an $L^2(H^2)$-estimate on u. In order to do so, we note that, owing to the global boundedness of α',

$$\int_t^{t+1} \|\partial_t \alpha(u(s))\|_{L^2}^2\, ds \leq \alpha_0 \int_t^{t+1} (\alpha'(u(s))\partial_t u(s), \partial_t u(s))\, ds. \tag{5.9}$$

Furthermore, due to the estimate $|f(u)| \leq C(1+|u|^3)$ and the embedding $H^1 \subset L^6$, estimate (5.8) allows to control the $L^2([t,t+1]\times\Omega)$-norm of the terms $f(u)$ and $v = -(-\Delta)^{-1}\partial_t u$ in equation (2.2). The classical H^2-regularity theorem for the Laplacian now gives the desired control on the $L^2(H^2)$-norm of u and finishes the proof of the lemma. □

The next lemma establishes the uniqueness of the energy solutions.

Lemma 5.2 *Let the assumptions of Lemma* 5.1 *hold. Then, an energy solution is unique. Furthermore, any two energy solutions u_1 and u_2 to problem* (2.2) *satisfy estimate* (2.3)*, where the constants C and C' are uniform with respect to $\alpha_0 \to 0$.*

Proof. The proof of this lemma basically follows that of Lemma 2.2, but we now no longer have a control on the L^∞-norms of u_1 and u_2 and, consequently, we need to estimate the term

$$\left|(f(u_1) - f(u_2), (-\Delta)^{-1} w)\right|$$

in a different way, using the growth restriction on f instead of the L^∞-control.

We set $l(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds$, so that $f(u_1) - f(u_2) = l(t)u$. Then, using the embedding $H^1 \subset L^6$, we have, for every $\phi \in H_0^1(\Omega)$,

$$
\begin{aligned}
|(f(u_1) - f(u_2), \phi)| &= |(u, l(t)\phi)| \leq \|u\|_{H^{-1}} \|l(t)\phi\|_{H^1} \\
&\leq C\|u\|_{H^{-1}} \big(\|l(t)\|_{L^\infty} + \|\nabla l(t)\|_{L^3}\big) \|\phi\|_{H^1}
\end{aligned}
$$

and, consequently,

$$
\|f(u_1) - f(u_2)\|_{H^{-1}} \leq C\big(\|l(t)\|_{L^\infty} + \|\nabla l(t)\|_{L^3}\big)\|u\|_{H^{-1}}. \tag{5.10}
$$

Using now the growth restriction on f and the interpolation inequality

$$
\|u_i\|_{L^\infty}^2 \leq C\|u_i\|_{H^1}\|u_i\|_{H^2}, \quad i = 1, 2,
$$

we have

$$
\|l(t)\|_{L^\infty} \leq C(1 + \|u_1\|_{H^1} + \|u_2\|_{H^1})(1 + \|u_1\|_{H^2} + \|u_2\|_{H^2}).
$$

Analogously,

$$
\begin{aligned}
\|\nabla l(t)\|_{L^3} &\leq C \sup_{i=1,2} (1 + \|u_i\|_{L^6}\|\nabla u_i\|_{L^6}) \\
&\leq C'(1 + \|u_1\|_{H^1} + \|u_2\|_{H^1})(1 + \|u_1\|_{H^2} + \|u_2\|_{H^2}).
\end{aligned}
$$

Thus, using the uniform control on the H^1-norm of the solutions given in Lemma 5.1, we finally have

$$
\begin{aligned}
&\big|(f(u_1) - f(u_2), (-\Delta)^{-1}w)\big| \\
&\quad \leq \|f(u_1) - f(u_2)\|_{H^{-1}}\|w\|_{H^{-1}} \\
&\quad \leq \varepsilon\|u\|_{H^{-1}}^2 + C_\varepsilon\big(1 + \|u_1\|_{H^2}^2 + \|u_2\|_{H^2}^2\big)\|w\|_{H^{-1}}^2,
\end{aligned} \tag{5.11}
$$

where $\varepsilon > 0$ is arbitrary and C_ε depends on ε and on the H^1-norms of u_1 and u_2, but is independent of $\alpha_0 \to 0$.

Furthermore, since $\alpha'(u) \leq \alpha_0$,

$$\begin{aligned}\|w\|_{L^2}^2 &\le 2\big[\|\alpha(u_1)-\alpha(u_2)\|_{L^2}^2+\|u\|_{H^{-2}}^2\big]\\ &\le C\big[\alpha_0(u,\alpha(u_1)-\alpha(u_2))+\|u\|_{H^{-1}}^2\big]\\ &\le C'\big[(u,\alpha(u_1)-\alpha(u_2))+\|u\|_{H^{-1}}^2\big],\end{aligned} \tag{5.12}$$

where the constant C' is independent of $\alpha_0 \to 0$. Inserting estimates (5.11) and (5.12) into inequality (2.5), we deduce that

$$\begin{aligned}\partial_t\|w(t)\|_{H^{-1}}^2+\gamma\big(\|w(t)\|_{L^2}^2+\|u(t)\|_{H^{-1}}^2\big)\\ \le C\big(1+\|u_1(t)\|_{H^2}^2+\|u_2(t)\|_{H^2}^2\big)\|w(t)\|_{H^{-1}}^2,\end{aligned} \tag{5.13}$$

where the positive constants γ and C are uniform with respect to $\alpha_0 \to 0$. Applying Gronwall's lemma to this relation and using the control on the $L^2(H^2)$-norms of u_1 and u_2 obtained in Lemma 5.1, we deduce estimate (2.3) and finish the proof of the lemma. □

Corollary 5.3 *Let the assumptions of Lemma* 5.1 *hold. Then, for any two energy solutions u_1 and u_2 and $w_i := \mathcal{B}(u_i)$, $i = 1, 2$, we have estimate* (4.11)*, where the constant C_l depends on l and on the $L^\infty(H^1)$-norms of u_1 and u_2, but is independent of $\alpha_0 \to 0$. Furthermore, the constant C in the Hölder continuity inequality* (4.10) *is also uniform as $\alpha_0 \to 0$.*

Indeed, the uniform estimate (4.11) follows from (5.13) exactly as in the proof of Lemma 4.6 and the uniform estimate (4.10) can be obtained repeating word by word the proof of Lemma 4.5 (but now using the fact that α' is globally bounded).

Thus, under the assumptions of this section, equation (2.2) is well-posed in the energy phase space $\Phi_{en} := H_0^1(\Omega)$ and generates a dissipative solution semigroup $S_\alpha(t)$ in this space,

$$S_\alpha(t):\Phi_{en}\to\Phi_{en},\quad S_\alpha(t)u_0 := u(t), \tag{5.14}$$

where u solves equation (2.2) (since we are now interested in the limit $\alpha \to 0$, we explicitly indicate the dependence on the nonlinearity α). According to Lemma 5.1, this semigroup is dissipative in the energy phase space Φ_{en},

$$\|u(t)\|_{\Phi_{en}} \le Q(\|u_0\|_{\Phi_{en}})e^{-\gamma t}+C_*,\quad t\ge 0, \tag{5.15}$$

where the monotonic function Q and the positive constants γ and C_* are independent of t, u_0 and $\alpha_0 \to 0$. Moreover, due to Lemma 5.2 and Corollary 5.3, this semigroup is (uniformly with respect to $\alpha_0 \to 0$) Hölder continuous in Φ_{en} endowed with the H^{-1}-norm.

Furthermore, analogously to Theorem 4.2, this semigroup possesses the weak global attractor $\mathcal{A}_\alpha$ in the phase space Φ_{en}. We also note that the case $\alpha_0 = 0$ (i.e., $\alpha = 0$) corresponds to the classical Cahn-Hilliard equation

$$\partial_t(-\Delta)^{-1}u = \Delta u - f(u), \quad u\,|_{\partial\Omega} = 0,\ u\,|_{t=0} = u_0, \tag{5.16}$$

which, as is well-known, possesses smooth global ($\mathcal{A}_0$) and exponential ($\mathcal{M}_0$) attractors, even in the strong topology of the phase space Φ_{en} (see, e.g., [17] and [36]).

The following result, which gives a robust Hölder continuous family of exponential attractors for equations (2.2) as $\alpha \to 0$, can be considered as the main result of this section.

Theorem 5.4 *Let the assumptions of Lemma* 5.1 *hold. Then, for every nonlinearity* α *satisfying* (5.1), *there exists a weak exponential attractor* $\mathcal{M}_\alpha \subset \Phi_{en}$ *of the solution semigroup* $S_\alpha(t)$ *in the sense of Definition* 4.3 (*in which the space* Φ *is replaced by* Φ_{en}). *Furthermore, these attractors are uniformly* (*with respect to* $\alpha_0 \to 0$) *bounded in* Φ_{en} *and estimates* (4.7) *and* (4.8) *are also independent of* $\alpha_0 \to 0$. *Finally, the attractors* $\mathcal{M}_\alpha$ *tend to the limit Cahn-Hilliard attractor* $\mathcal{M}_0$ *as* $\alpha \to 0$ *in the sense of the symmetric Hausdorff distance in* $H^{-1}(\Omega)$,

$$\operatorname{dist}^{sym}_{H^{-1}}(\mathcal{M}_\alpha, \mathcal{M}_0) \le C\alpha_0^\theta, \tag{5.17}$$

where the positive constants θ *and* C *are independent of* α *and can be computed explicitly.*

Proof. To prove this result, we need one more lemma to estimate the difference between the solutions to equations (2.2) and (5.16).

Lemma 5.5 *Let the assumptions of the theorem hold and let* u_α *and* u_0 *be two solutions to problems* (2.2) *and* (5.16), *respectively. Then, the following estimate is valid*:

$$\|u_\alpha(t)-u_0(t)\|^2_{H^{-1}}+\int_0^t\|u_\alpha(s)-u_0(s)\|^2_{H^1}\,ds$$
$$\le Ce^{Kt}\big(\|u_\alpha(0)-u_0(0)\|^2_{H^{-1}}+\alpha_0\big), \tag{5.18}$$

where the constants C and K depend on the H^1-norms of $u_\alpha(0)$ and $u_0(0)$, but are independent of t and $\alpha_0\to 0$.

Proof. We set $u:=u_\alpha-u_0$. Then, this function solves

$$\partial_t(-\Delta)^{-1}u-\Delta u+[f(u_\alpha)-f(u_0)]=-\alpha'(u_\alpha)\partial_t u_\alpha. \tag{5.19}$$

Multiplying this equation by u and using the quasi-monotonicity assumption on f and a proper interpolation inequality, we deduce in a standard way that

$$\partial_t\|u(t)\|^2_{H^{-1}}+\gamma\|u(t)\|^2_{H^1}\le K\|u(t)\|^2_{H^{-1}}+C\|\alpha'(u_\alpha)\partial_t u_\alpha\|_{H^{-1}}\|u\|_{H^1}, \tag{5.20}$$

where the positive constants γ, C and K are independent of $\alpha_0\to 0$. Furthermore, arguing as in the proof of Lemma 5.2, we can check that

$$\big\|\alpha'(u_\alpha(t))\partial_t u_\alpha(t)\big\|_{H^{-1}}\le C\alpha_0(1+\|u_\alpha(t)\|_{H^2})\|\partial_t u_\alpha(t)\|_{H^{-1}}, \tag{5.21}$$

where the constant C depends on the H^1-norms of the initial data, but is independent of α_0. Applying now Gronwall's lemma to (5.20) and using the uniform control on the $L^2(H^2)\cap L^\infty(H^1)$-norms of u_α and u_0 and the control on the $L^2(H^{-1})$-norm of $\partial_t u_\alpha$ given in Lemma 5.1, we deduce the desired estimate (5.18) and finish the proof of the lemma. □

We are now ready to finish the proof of the theorem, proceeding exactly as in the end of that of Theorem 4.4. Indeed, according to the dissipative estimate (5.15), the ball

$$\mathbb{B}:=\{u_0\in\Phi_{en},\ \|u_0\|_{\Phi_{en}}\le 2C_*\}$$

is an absorbing ball for all the semigroups $S_\alpha(t)$ if α_0 is not too large (say, if $\alpha_0\in(0,1]$). Furthermore, according to the same uniform estimate, we can fix l (independently of α_0) in such a way that

$$S_\alpha(l)\mathbb{B}\subset\mathbb{B}$$

if α_0 is not too large. We also define the transformations $\mathcal{B}_\alpha : \Phi_{en} \to \Phi_{en}$ by

$$\mathcal{B}_\alpha(u_0) := \alpha(u_0) + (-\Delta)^{-1}u_0, \quad \mathcal{B}_0(u_0) = (-\Delta)^{-1}u_0,$$

and the lifting maps $\mathbb{S}_\alpha : \mathbb{B} \to \mathcal{H}$ by

$$(\mathbb{S}_\alpha u_0)(t) := \mathcal{B}_\alpha(S_\alpha(t)), \quad t \in [0, l]$$

(the spaces $\mathcal{H}$ and $\mathcal{H}_1$ are defined in (4.21)). Then, according to Corollary 5.3 and Lemma 5.2, the sets

$$\mathbb{B}_0(\alpha) := \mathbb{S}_\alpha(\mathbb{B}) \tag{5.22}$$

are uniformly bounded and compact in $\mathcal{H}$. Finally, we define the maps $\mathcal{L}_\alpha$ by

$$\mathcal{L}_\alpha := \mathbb{S}_\alpha \circ S_\alpha(l) \circ [\mathbb{S}_\alpha]^{-1}$$

and construct the desired family of exponential attractors for the maps $\mathcal{L}_\alpha$. Indeed, according to Corollary 5.3,

$$\|\mathcal{L}_\alpha w_1 - \mathcal{L}_\alpha w_2\|_{\mathcal{H}_1} \le C\|w_1 - w_2\|_{\mathcal{H}}, \quad w_i \in \mathbb{B}_0(\alpha),\ i = 1, 2, \tag{5.23}$$

where the constant C is independent of $\alpha_0 \to 0$. However, the phase spaces $\mathbb{B}_0(\alpha)$ for the maps $\mathcal{L}_\alpha$ now depend on α. In order to overcome this difficulty, we introduce the projectors $\Pi_{\alpha\to 0} : \mathbb{B}_0(\alpha) \to \mathbb{B}_0(0)$ and $\Pi_{0\to\alpha} := \Pi_{\alpha\to 0}^{-1}$ defined by

$$\Pi_{\alpha\to 0} w := \mathbb{S}_0 \circ [\mathbb{S}_\alpha]^{-1} w.$$

Then, due to Lemma 5.5, it is not difficult to see that

$$\begin{aligned} &\|\Pi_{\alpha\to 0} w - w\|_{\mathcal{H}} \le C\alpha_0^{\frac{1}{2}}, \quad w \in \mathbb{B}_0(\alpha); \\ &\|\Pi_{0\to\alpha} w - w\|_{\mathcal{H}} \le C\alpha_0^{\frac{1}{2}}, \quad w \in \mathbb{B}_0(0), \end{aligned} \tag{5.24}$$

where C is independent of α_0 and w. The last estimate shows, in particular, that

$$\mathrm{dist}^{sym}_{\mathcal{H}}(\mathbb{B}_0(\alpha), \mathbb{B}_0(0)) \leq C\alpha_0^{\frac{1}{2}}.$$

Furthermore, Lemma 5.5 also implies that

$$\|\mathcal{L}_\alpha(\Pi_{0\to\alpha} w) - \mathcal{L}_0 w\|_{\mathcal{H}} \leq C\alpha_0^{\frac{1}{2}}, \tag{5.25}$$

where C is again independent of $w \in \mathbb{B}_0(0)$ and α_0.

Thus, due to estimates (5.23), (5.24) and (5.25) and using an abstract theorem on the existence of robust families of exponential attractors, see, e.g., [20] and [30], the maps $\mathcal{L}_\alpha$ possess a family of exponential attractors $\mathbb{M}_\alpha \subset \mathcal{H}$ which satisfy estimates (4.24) and (4.25), uniformly with respect to α, and, in addition,

$$\mathrm{dist}^{sym}_{\mathcal{H}}(\mathbb{M}_\alpha, \mathbb{M}_0) \leq C\alpha_0^{\theta},$$

where $\theta > 0$ and C are independent of α.

Returning to the phase space Φ_{en} (exactly as in the end of the proof of Theorem 4.4), we obtain the desired family of exponential attractors $\mathcal{M}_\alpha \subset \Phi_{en}$ and finish the proof of Theorem 5.4. □

Remark 5.6 Since the attractors $\mathcal{M}_\alpha$ are uniformly bounded in $H^1(\Omega)$, estimate (5.17), together with a proper interpolation inequality, gives

$$\mathrm{dist}^{sym}_{H^{1-\delta}}(\mathcal{M}_\alpha, \mathcal{M}_0) \leq C_\delta \alpha_0^{\theta_\delta},$$

where the constants C_δ and θ_δ depend on $\delta > 0$, but are independent of α_0. In particular, we have the Hölder continuity of the family $\mathcal{M}_\alpha$ in $L^2(\Omega)$.

Remark 5.7 We assume that, in addition, the nonlinearity α does not degenerate at infinity (i.e., (1.12) holds). Then, arguing as in Section 3 (see also Remark 3.5), we can prove that

$$S_\alpha(t) : \Phi_{en} \to \Phi := L^\infty(\Omega) \cap H^1_0(\Omega), \tag{5.26}$$

for every $t > 0$, and, consequently, the global attractor constructed in Section 4 for the L^∞-solutions coincides with that based on the energy solutions (since any energy solution becomes an L^∞-solution as soon as $t > 0$). However, as already mentioned, the L^∞-estimates are *not uniform* with respect to $\alpha_0 \to 0$ and, consequently, are useless for the purposes of this section.

6. Appendix. The nondegenerate case

The aim of this appendix is to show how to obtain additional regularity results on the solutions to (2.2) in the nondegenerate case where $\alpha \in C^2(\mathbb{R})$, $u_0 \in C^2(\Omega)$ and

$$\alpha'(s) \geq \alpha_0 > 0, \quad s \in \mathbb{R}. \tag{6.1}$$

We first recall that, due to Theorem 3.4 and estimate (2.18), we have the following a priori estimate in $L^\infty(\Omega) \cap H_0^1(\Omega)$:

$$\|u(t)\|^2_{L^\infty \cap H^1} + \int_t^\infty \|\partial_t u(s)\|^2_{H^{-1}}\, ds \leq Q(\|u_0\|_{C^2})e^{-\gamma t} + C_*, \quad t \geq 0, \tag{6.2}$$

for positive constants γ and C_* and a monotonic function Q.

Our aim is to exploit the fact that α does not degenerate in order to establish additional regularity on the solutions. To be more precise, the following result holds.

Proposition 6.1 *Let the assumptions of Theorem* 3.4 *hold and let, in addition, $\alpha \in C^2(\mathbb{R})$, $u_0 \in C^2(\Omega)$ and* (6.1) *be satisfied. Then, the solution u to problem* (2.2) *belongs to the parabolic Sobolev space $W^{(1,2),q}([T,T+1]\times\Omega)$, for any $1 \leq q < \infty$ and $T \geq 0$, and the following estimate holds*:

$$\|u\|_{L^q([T,T+1]\times\Omega)} + \|u\|_{L^q([T,T+1],W^{2,q}(\Omega))} \leq Q(\|u_0\|_{C^2})e^{-\gamma T} + C_*, \tag{6.3}$$

for positive constants γ and C_* and a monotonic function Q which may a priori depend on q, but are independent of T and u.

Proof. We rewrite equation (2.2) in the form of a linear equation,

$$a(t,x)\partial_t u - \Delta u = h_u(t,x) := -(-\Delta)^{-1}\partial_t u(t,x) - f(u(t,x)), \tag{6.4}$$

where

$$a(t,x) := \alpha'(u(t,x)).$$

We note that, on the one hand, due to (6.1) and (6.2), we have the following nondegeneracy condition on a:

$$0<\alpha_0\le a(t,x)\le Q(\|u_0\|_{C^2})e^{-\gamma t}+C_*,\quad t\ge 0,\ x\in\Omega, \tag{6.5}$$

for positive constants γ and C_* and a monotonic function Q. On the other hand, estimate (6.2), together with the embedding $H^1\subset L^6$, guarantees that the right-hand side h_u is regular enough,

$$\|h_u\|_{L^2([T,T+1],L^6(\Omega))}\le Q(\|u_0\|_{C^2})e^{-\gamma T}+C_*. \tag{6.6}$$

These two estimates allow to apply the classical De Giorgi technique to the linear parabolic equation (6.4) and to establish the uniform Hölder continuity of the solution u,

$$\|u\|_{C^\delta([T,T+1]\times\Omega)}\le Q(\|u_0\|_{C^2})e^{-\gamma T}+C_*, \tag{6.7}$$

for positive constants δ, γ and C_* and a monotonic function Q (see, e.g., [25]).

The above Hölder continuity property implies that the coefficient $a(t,x)$ is also uniformly Hölder continuous with respect to t and x. This, in turn, implies that we can apply the standard localization technique in order to extend the maximal regularity estimates for the heat equation to equation (6.4) (see [25]). In particular, we have the following anisotropic maximal regularity result:

$$\begin{aligned}&\|\partial_t u\|_{L^2([T,T+1],L^6(\Omega))}+\|u\|_{L^2([T,T+1],W^{2,6}(\Omega))}\\&\qquad\le C\big(\|h_u\|_{L^2([T-1,T+1],L^6(\Omega))}+\|u\|_{L^\infty([T-1,T+1]\times\Omega)}\big),\end{aligned} \tag{6.8}$$

for $T\ge 1$, and

$$\begin{aligned}&\|\partial_t u\|_{L^2([0,1],L^6(\Omega))}+\|u\|_{L^2([0,1],W^{2,6}(\Omega))}\\&\qquad\le C\big(\|h_u\|_{L^2([0,1],L^6(\Omega))}+\|u_0\|_{C^2}\big).\end{aligned} \tag{6.9}$$

These two estimates, together with (6.6), give

$$\begin{aligned}&\|\partial_t u\|_{L^2([T,T+1],L^6(\Omega))}+\|u\|_{L^2([T,T+1],W^{2,6}(\Omega))}\\&\qquad\le Q(\|u_0\|_{C^2})e^{-\gamma T}+C_*,\qquad\qquad T\ge 0,\end{aligned} \tag{6.10}$$

for positive constants γ and C_* and a monotonic function Q.

Our next step is to obtain an estimate on the $L^\infty(L^2)$-norm of $\partial_t u$. To this end, we differentiate equation (2.2) with respect to t and set $v := \partial_t u$. Then, this function solves

$$\alpha'(u)\partial_t v + \alpha''(u)v^2 + (-\Delta)^{-1}\partial_t v = \Delta v - f'(u)v. \tag{6.11}$$

Multiplying this equation by v and integrating over Ω, we have

$$\partial_t\big[(\alpha'(u)v,v)+\|v\|^2_{H^{-1}}\big]+2\|\nabla v\|^2_{L^2}+2(f'(u)v,v)+(\alpha''(u),v^3)=0. \tag{6.12}$$

Integrating this equality with respect to $t\in[0,1]$ and using the non-degeneracy assumption, the L^∞-estimate on u and the obvious fact that $\|\partial_t u(0)\|_{L^2}\le Q(\|u_0\|_{C^2})$, we deduce that

$$\|v\|^2_{L^\infty([0,1],L^2(\Omega))}+\|v\|^2_{L^2([0,1],H^1(\Omega))}\le C\big(\|v\|^3_{L^3([0,1]\times\Omega)}+1\big)+Q(\|u_0\|_{C^2}), \tag{6.13}$$

where the constant C only depends on the L^∞-norm of u. Writing now

$$\begin{aligned}\|v\|^3_{L^3([0,1],L^3(\Omega))}&\le C\|v\|_{L^\infty([0,1],L^2(\Omega))}\|v\|^2_{L^2([0,1],L^4(\Omega))}\\&\le\varepsilon\|v\|^2_{L^\infty([0,1],L^2(\Omega))}+C_\varepsilon\|v\|^4_{L^2([0,1],L^6(\Omega))},\quad \epsilon>0,\end{aligned}$$

fixing $\varepsilon>0$ small enough and using (6.10), we deduce from (6.13) that

$$\|\partial_t u\|_{L^\infty([0,1],L^2(\Omega))}\le Q(\|u_0\|_{C^2}), \tag{6.14}$$

for a monotonic function Q. Furthermore, multiplying (6.12) by $t-T+1$, integrating over $[T-1,T+1]$ and arguing as above, we obtain the following smoothing property on $\partial_t u$:

$$\|\partial_t u\|_{L^\infty([T,T+1],L^2(\Omega))}\le C\big(\|\partial_t u\|^2_{L^2([T-1,T+1],L^6(\Omega))}+1\big), \tag{6.15}$$

where $T\ge 1$ and the constant C only depends on $\|u\|_{L^\infty([T-1,T+1]\times\Omega)}$.

Estimates (6.14) and (6.15), together with the dissipative estimates (6.2) and (6.10), give the desired dissipative estimate on the L^2-norm of $\partial_t u$,

$$\|\partial_t u(t)\|_{L^2}\le Q(\|u_0\|_{C^2})e^{-\gamma t}+C_*,\quad t\ge 0, \tag{6.16}$$

for a monotonic function Q and positive constants γ and C_*.

It is now not difficult to finish the proof of the proposition. Indeed, the last estimate, together with the embedding $H^2 \subset C$, gives a dissipative estimate on the L^∞-norm of the function h_u in the right-hand side of (6.4),

$$\|h_u(t)\|_{L^\infty} \le Q(\|u_0\|_{C^2})e^{-\gamma t} + C_*, \quad t \ge 0,\ \gamma > 0. \tag{6.17}$$

Applying then the usual L^q-regularity estimate to the parabolic equation (6.4), we deduce the desired estimate (6.3). The existence of a solution follows from this a priori estimate exactly as for second order semilinear parabolic PDEs (see [25]). □

Remark 6.2 The above scheme shows that the maximal regularity of a solution u is restricted by the regularity of α, f and Ω (and u_0 if we are interested in the regularity near $t = 0$) only. In particular, if α, f and Ω are of class C^∞, then the solution u is also of class C^∞ with respect to x and t.

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