Hokkaido Mathematical Journal Vol. 38 (2009) p. 111–136

On the Stokes operator in general unbounded domains

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(Received June 27, 2007; Revised December 19, 2007)

Abstract. It is known that the Stokes operator is not well-defined in L^q-spaces for certain unbounded smooth domains unless $q = 2$. In this paper, we generalize a new approach to the Stokes resolvent problem and to maximal regularity in general unbounded smooth domains from the three-dimensional case, see [7], to the n-dimensional one, $n \geq 2$, replacing the space $L^q, 1 < q < \infty$, by $\tilde{L}^q$ where $\tilde{L}^q = L^q \cap L^2$ for $q \geq 2$ and $\tilde{L}^q = L^q + L^2$ for $1 < q < 2$. In particular, we show that the Stokes operator is well-defined in $\tilde{L}^q$ for every unbounded domain of uniform $C^{1,1}$-type in $\mathbb{R}^n$, $n \geq 2$, satisfies the classical resolvent estimate, generates an analytic semigroup and has maximal regularity.

Key words: General unbounded domains, domains of uniform $C^{1,1}$-type, Stokes operator, Stokes resolvent, Stokes semigroup, maximal regularity

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, denote a general unbounded domain with uniform $C^{1,1}$-boundary $\partial\Omega \neq \emptyset$, see Definition 1.1 below. As is well-known, the analysis of the instationary Navier-Stokes equations requires L^q-estimates, $q \neq 2$, to prove the strong energy estimate, the localized energy estimate involving also the pressure function and Leray's Structure Theorem for weak solutions. Unfortunately, the standard approach to the Stokes equations in L^q-spaces, $1 < q < \infty$, cannot be extended to general unbounded domains unless $q = 2$. On the one hand, the Helmholtz decomposition fails to exist for certain unbounded smooth domains on L^q, $q \neq 2$, see [4], [14]. On the other hand, in L^2 the Helmholtz projection and the Stokes operator are well-defined for every domain, the latter is self-adjoint, generates a bounded analytic semigroup and has maximal regularity.

In order to work locally in L^q-spaces, but globally, to be more precise, near space infinity, in L^2, the authors introduced in [7] in the three-dimensional case the function space

$$\tilde{L}^q(\Omega) = \begin{cases} L^q(\Omega) \cap L^2(\Omega), & 2 \leq q < \infty \\ L^q(\Omega) + L^2(\Omega), & 1 < q < 2 \end{cases}$$

2000 Mathematics Subject Classification : Primary 76D05; Secondary 35Q30.

to define the Helmholtz decomposition and the space

$$\tilde{L}^q_\sigma(\Omega) = \begin{cases} L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \le q < \infty \\ L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2 \end{cases}$$

of solenoidal vector fields in $\tilde{L}^q(\Omega)$ to define and to analyze the Stokes operator. It was proved that for every unbounded domain $\Omega \subseteq \mathbb{R}^3$ of uniform C^2-type the Stokes operator in $\tilde{L}^q_\sigma(\Omega)$ satisfies the usual resolvent estimate, generates an analytic semigroup and has maximal regularity. Moreover, for every dimension $n \ge 2$, the Helmholtz decomposition of $\tilde{L}^q(\Omega)$ exists for every unbounded domain $\Omega \subseteq \mathbb{R}^n$ of uniform C^1-type, see [8].

To describe this result, we introduce the space of gradients

$$\tilde{G}^q(\Omega) = \begin{cases} G^q(\Omega) \cap G^2(\Omega), & 2 \le q < \infty \\ G^q(\Omega) + G^2(\Omega), & 1 < q < 2 \end{cases},$$

where $G^q(\Omega) = \{\nabla p \in L^q(\Omega) : p \in L^q_{\mathrm{loc}}(\Omega)\}$, and recall the notion of domains of uniform C^k- and $C^{k,1}$-type.

Definition 1.1 *A domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, is called a uniform C^k-domain of type (α, β, K) where $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, $K > 0$, if for each $x_0 \in \partial\Omega$ there exists a Cartesian coordinate system with origin at x_0 and coordinates $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$, and a C^k-function $h(y')$, $|y'| \le \alpha$, with C^k-norm $\|h\|_{C^k} \le K$ such that the neighborhood*

$$U_{\alpha,\beta,h}(x_0) := \{y = (y', y_n) \in \mathbb{R}^n : |y_n - h(y')| < \beta,\ |y'| < \alpha\}$$

of x_0 implies $U_{\alpha,\beta,h}(x_0) \cap \partial\Omega = \{(y', h(y')) : |y'| < \alpha\}$ and

$$U^-_{\alpha,\beta,h}(x_0) := \{(y', y_n) : h(y') - \beta < y_n < h(y'),\ |y'| < \alpha\} = U_{\alpha,\beta,h}(x_0) \cap \Omega.$$

By analogy, a domain $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, is a uniform $C^{k,1}$-domain of type (α, β, K), $k \in \mathbb{N} \cup \{0\}$, if the functions h mentioned above may be chosen in $C^{k,1}$ such that the $C^{k,1}$-norm satisfies $\|h\|_{C^{k,1}} \le K$.

Theorem 1.2 ([8]) *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a uniform C^1-domain of type (α, β, K) and let $q \in (1, \infty)$. Then each $u \in \tilde{L}^q(\Omega)$ has a unique decomposition*

$$u = u_0 + \nabla p, \quad u_0 \in \tilde{L}^q_\sigma(\Omega),\ \nabla p \in \tilde{G}^q(\Omega),$$

satisfying the estimate

$$\|u_0\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \le c\|u\|_{\tilde{L}^q}, \tag{1.1}$$

where $c = c(\alpha, \beta, K, q) > 0$. *In particular, the Helmholtz projection* $\tilde{P}_q$ *defined by* $\tilde{P}_q u = u_0$ *is a bounded linear projection on* $\tilde{L}^q(\Omega)$ *with range* $\tilde{L}^q_\sigma(\Omega)$ *and kernel* $\tilde{G}^q(\Omega)$. *Moreover,* $\tilde{L}^q_\sigma(\Omega)$ *is the closure in* $\tilde{L}^q(\Omega)$ *of the space* $C^\infty_{0,\sigma}(\Omega) = \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\}$, *and the duality relations*

$$\big(\tilde{L}^q_\sigma(\Omega)\big)' = \tilde{L}^{q'}_\sigma(\Omega), \quad \big(\tilde{P}_q\big)' = \tilde{P}_{q'},$$

where $q' = \frac{q}{q-1}$, *hold.*

Using the Helmholtz projection $\tilde{P}_q$, $1 < q < \infty$, we define the Stokes operator $\tilde{A}_q$ as the linear operator with domain

$$\mathcal{D}(\tilde{A}_q) = \begin{cases} D^q(\Omega) \cap D^2(\Omega), & 2 \le q < \infty \\ D^q(\Omega) + D^2(\Omega), & 1 < q < 2 \end{cases},$$

where $D^q(\Omega) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$, by setting

$$\tilde{A}_q u = -\tilde{P}_q \Delta u, \quad u \in \mathcal{D}(\tilde{A}_q).$$

By analogy, we define the Sobolev space $\tilde{W}^{2,q}(\Omega)$ with norm

$$\|u\|_{\tilde{W}^{2,q}} = \|u\|_{\tilde{L}^q} + \|\nabla u\|_{\tilde{L}^q} + \|\nabla^2 u\|_{\tilde{L}^q},$$

see also (2.3) below. Let I be the identity and $\mathcal{S}_\varepsilon = \{0 \ne \lambda \in \mathbb{C}; |\arg \lambda| < \frac{\pi}{2} + \varepsilon\}$, $0 < \varepsilon < \frac{\pi}{2}$.

Then our first main result on the Stokes operator reads as follows:

Theorem 1.3 *Let* $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, *be a uniform* $C^{1,1}$*-domain of type* (α, β, K), *and let* $1 < q < \infty$, $\delta > 0$, $0 < \varepsilon < \frac{\pi}{2}$.

(i) *The operator*

$$\tilde{A}_q = -\tilde{P}_q\, \Delta : \mathcal{D}(\tilde{A}_q) \to \tilde{L}^q_\sigma(\Omega), \quad \mathcal{D}(\tilde{A}_q) \subset \tilde{L}^q_\sigma(\Omega),$$

is a densely defined closed operator.

(ii) *For all* $\lambda \in \mathcal{S}_\varepsilon$, *its resolvent* $(\lambda I + \tilde{A}_q)^{-1} : \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ *is well-defined. Moreover, for every* $f \in \tilde{L}^q_\sigma(\Omega)$ *the solution* $u \in \tilde{L}^q_\sigma(\Omega)$ *of the resolvent problem* $(\lambda I + \tilde{A}_q)u = f$ *satisfies the estimate*

$$\|\lambda u\|_{\tilde{L}^q_\sigma} + \|\nabla^2 u\|_{\tilde{L}^q} \le C\|f\|_{\tilde{L}^q_\sigma}, \quad |\lambda| \ge \delta, \tag{1.2}$$

where $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$.

(iii) *Given* $f \in \tilde{L}^q(\Omega)^n$, $\lambda \in \mathcal{S}_\varepsilon$, *the Stokes resolvent equation*

$$\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a unique solution $(u, \nabla p) \in \mathcal{D}(\tilde{A}_q) \times \tilde{G}^q(\Omega)$ *defined by* $u = (\lambda I + \tilde{A}_q)^{-1}\tilde{P}_q f$ *and* $\nabla p = (I - \tilde{P}_q)(f + \Delta u)$ *and satisfying*

$$\|\lambda u\|_{\tilde{L}^q} + \|\nabla^2 u\|_{\tilde{L}^q} + \|\nabla p\|_{\tilde{L}^q} \le C\|f\|_{\tilde{L}^q}, \quad |\lambda| \ge \delta, \tag{1.3}$$

with a constant $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$.

(iv) *The Stokes operator* $\tilde{A}_q$ *satisfies the duality relation* $(\tilde{A}_q)' = \tilde{A}_{q'}$, *in particular,* $\langle \tilde{A}_q u, v\rangle = \langle u, \tilde{A}_{q'} v\rangle$ *for all* $u \in \mathcal{D}(\tilde{A}_q)$, $v \in \mathcal{D}(\tilde{A}_{q'})$ *and generates an analytic semigroup* $e^{-t\tilde{A}_q}$, $t \ge 0$, *in* $\tilde{L}^q_\sigma(\Omega)$ *with bound*

$$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q_\sigma} \le M e^{\delta t}\|f\|_{\tilde{L}^q_\sigma}, \quad f \in \tilde{L}^q_\sigma,\ t \ge 0, \tag{1.4}$$

where $M = M(q, \delta, \alpha, \beta, K) > 0$.

Note that the bound $\delta > 0$ in Theorem 1.3 may be chosen arbitrarily small, but that it is not clear whether $\delta = 0$ is allowed for a general unbounded domain and whether the semigroup $e^{-t\tilde{A}_q}$ is uniformly bounded in $\tilde{L}^q_\sigma(\Omega)$ for $0 \le t < \infty$.

Our second main result concerns the instationary Stokes system

$$\begin{aligned} u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \ \text{ in } \ \Omega \times (0, T) \\ u(0) = u_0, \quad u|_{\partial\Omega} = 0. \end{aligned} \tag{1.5}$$

Theorem 1.4 *Let* $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, *be a uniform* $C^{1,1}$*-domain of type* (α, β, K), *and let* $0 < T < \infty$, $1 < q$, $s < \infty$.

Then for each $f \in L^s(0, T; \tilde{L}^q_\sigma(\Omega))$ *and each* $u_0 \in \mathcal{D}(\tilde{A}_q)$ *there exists a* *unique solution* $u \in L^s(0,T;\mathcal{D}(\tilde{A}_q))$ *with* $u_t \in L^s(0,T;\tilde{L}^q_\sigma(\Omega))$ *of the system* (1.5) *satisfying the estimates*

$$\begin{aligned}\|u_t\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|u\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|\tilde{A}_q u\|_{L^s(0,T;\tilde{L}^q_\sigma)} \\ \le C\big(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{L^s(0,T;\tilde{L}^q_\sigma)}\big)\end{aligned} \tag{1.6}$$

and

$$\|u_t\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|u\|_{L^s(0,T;\tilde{W}^{2,q})} \le C\big(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{L^s(0,T;\tilde{L}^q_\sigma)}\big) \tag{1.7}$$

with $C = C(q,s,T,\alpha,\beta,K) > 0$.

Remark 1.5 (i) The assumption $u_0 \in \mathcal{D}(\tilde{A}_q)$ in Theorem 1.4 is used for simplicity and is not optimal. Actually, it may be replaced by the weaker properties $u_0 \in \tilde{L}^q_\sigma(\Omega)$ and $\int_0^T \|\tilde{A}_q e^{-t\tilde{A}_q} u_0\|^s_{\tilde{L}^q_\sigma}\, dt < \infty$. Then the term $\|u_0\|_{\mathcal{D}(\tilde{A}_q)}$ in (1.6), (1.7) can be substituted by the weaker norm

$$\Big(\int_0^T \|\tilde{A}_q e^{-t\tilde{A}_q} u_0\|^s_{\tilde{L}^q_\sigma}\, dt\Big)^{\frac{1}{s}},\ 1 < q < \infty. \tag{1.8}$$

(ii) Let $f \in L^s(0,T;\tilde{L}^q_\sigma(\Omega))$ in Theorem 1.4 be replaced by $f \in L^s(0,T;\tilde{L}^q(\Omega))$. Then $u \in L^s(0,T;\mathcal{D}(\tilde{A}_q))$, defined by $u_t + \tilde{A}_q u = \tilde{P}_q f$, $u(0) = u_0$, and ∇p, defined by $\nabla p(t) = (I - \tilde{P}_q)\big(f + \Delta u\big)(t)$, is a unique solution pair of the system

$$u_t - \Delta u + \nabla p = f,\ u(0) = u_0,$$

satisfying

$$\begin{aligned}\|u_t\|_{L^s(0,T;\tilde{L}^q_\sigma)} + \|u\|_{L^s(0,T;\tilde{W}^{2,q})} + \|\nabla p\|_{L^s(0,T;\tilde{L}^q)} \\ \le C\big(\|u_0\|_{D(\tilde{A}_q)} + \|f\|_{L^s(0,T;\tilde{L}^q)}\big)\end{aligned} \tag{1.9}$$

with $C = C(q,s,T,\alpha,\beta,K) > 0$.

Using (2.1) below we see that in the case $1 < q < 2$ the solution pair $u, \nabla p$ possesses a decomposition $u = u^{(1)} + u^{(2)}$, $\nabla p = \nabla p^{(1)} + \nabla p^{(2)}$ such that

$$\begin{aligned} &u^{(1)} \in L^s(0,T;W^{2,2}(\Omega)), \quad u_t^{(1)} \in L^s(0,T;L^2_\sigma(\Omega)),\\ &u^{(2)} \in L^s(0,T;W^{2,q}(\Omega)), \quad u_t^{(2)} \in L^s(0,T;L^q_\sigma(\Omega)), \\ &\nabla p^{(1)} \in L^s(0,T;L^2(\Omega)), \quad \nabla p^{(2)} \in L^s(0,T;L^q(\Omega)), \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &\|u_t\|_{L^s(0,T;\tilde L^q_\sigma)} + \|u\|_{L^s(0,T;\tilde L^q_\sigma)} + \|\nabla^2 u\|_{L^s(0,T;\tilde L^q)} + \|\nabla p\|_{L^s(0,T;\tilde L^q)} \\ &= \|u_t^{(1)}\|_{L^{s,2}} + \|u^{(1)}\|_{L^{s,2}} + \|\nabla^2 u^{(1)}\|_{L^{s,2}} + \|\nabla p^{(1)}\|_{L^{s,2}} \\ &\quad + \|u_t^{(2)}\|_{L^{s,q}} + \|u^{(2)}\|_{L^{s,q}} + \|\nabla^2 u^{(2)}\|_{L^{s,q}} + \|\nabla p^{(2)}\|_{L^{s,q}} \end{aligned}$$

where $L^{s,2} = L^s(0,T;L^2(\Omega))$, $L^{s,q} = L^s(0,T;L^q(\Omega))$.

(iii) Note that the constant C in (1.6), (1.7), (1.9) could depend on the given interval $(0,T]$. We do not know whether C can be chosen independently of T as in the usual L^q-theory in bounded and exterior domains, see [12].

2. Preliminaries

Let us recall some properties of sum and intersection spaces known from interpolation theory, cf. [3], [18].

Consider two (complex) Banach spaces X_1, X_2 with norms $\|\cdot\|_{X_1}$, $\|\cdot\|_{X_2}$, respectively, and assume that both X_1 and X_2 are subspaces of a topological vector space V with continuous embeddings. Further, we assume that $X_1 \cap X_2$ is a dense subspace of both X_1 and X_2. Then the intersection space $X_1 \cap X_2$ is a Banach space with norm

$$\|u\|_{X_1\cap X_2} = \max\big(\|u\|_{X_1},\ \|u\|_{X_2}\big).$$

The sum space

$$X_1 + X_2 := \{u_1 + u_2; u_1 \in X_1,\ u_2 \in X_2\} \subseteq V$$

is a well-defined Banach space with the norm

$$\|u\|_{X_1+X_2} := \inf\big\{\|u_1\|_{X_1} + \|u_2\|_{X_2}; u = u_1 + u_2,\ u_1 \in X_1,\ u_2 \in X_2\big\}.$$

If X_1 and X_2 are reflexive Banach spaces, an argument using weakly convergent subsequences yields the following property:

$$u \in X_1 + X_2 \quad \Rightarrow \exists\, u_1 \in X_1,\ u_2 \in X_2 : \quad \|u\|_{X_1+X_2} = \|u_1\|_{X_1} + \|u_2\|_{X_2}. \tag{2.1}$$

Concerning dual spaces we have

$$(X_1 \cap X_2)' = X_1' + X_2'$$

with the natural pairing $\langle u, f_1 + f_2\rangle = \langle u, f_1\rangle + \langle u, f_2\rangle$ for $u \in X_1 \cap X_2$ and $f = f_1 + f_2 \in X_1' + X_2'$, and

$$(X_1 + X_2)' = X_1' \cap X_2'$$

with the natural pairing $\langle u, f\rangle = \langle u_1, f\rangle + \langle u_2, f\rangle$ for all $u = u_1 + u_2 \in X_1 + X_2$, $f \in X_1' \cap X_2'$. Thus it holds

$$\|u\|_{X_1+X_2} = \sup\left\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|f\|_{X_1'\cap X_2'}}; 0 \neq f \in X_1' \cap X_2'\right\}$$

and

$$\|f\|_{X_1'\cap X_2'} = \sup\left\{\frac{|\langle u_1, f\rangle + \langle u_2, f\rangle|}{\|u\|_{X_1+X_2}}; 0 \neq u = u_1 + u_2 \in X_1 + X_2\right\};$$

see [3], [18].

Consider closed subspaces $L_1 \subseteq X_1$, $L_2 \subseteq X_2$ with norms $\|\cdot\|_{L_1} = \|\cdot\|_{X_1}$, $\|\cdot\|_{L_2} = \|\cdot\|_{X_2}$ and assume that $L_1 \cap L_2$ is dense in both L_1 and L_2. Then $\|u\|_{L_1\cap L_2} = \|u\|_{X_1\cap X_2}$, $u \in L_1 \cap L_2$, and an elementary argument using the Hahn-Banach theorem shows that also

$$\|u\|_{L_1+L_2} = \|u\|_{X_1+X_2}, \quad u \in L_1 + L_2. \tag{2.2}$$

In particular, we need the following special case. Let $B_1 : \mathcal{D}(B_1) \to X_1$, $B_2 : \mathcal{D}(B_2) \to X_2$ be closed linear operators with dense domains $\mathcal{D}(B_1) \subseteq X_1$, $\mathcal{D}(B_2) \subseteq X_2$ equipped with graph norms

$$\|u\|_{\mathcal{D}(B_1)} = \|u\|_{X_1} + \|B_1 u\|_{X_1}, \quad \|u\|_{\mathcal{D}(B_2)} = \|u\|_{X_2} + \|B_2 u\|_{X_2},$$

respectively. Obviously each functional $F \in \mathcal{D}(B_i)'$, $i = 1, 2$, is given by some pair $f, g \in X_i'$ in the form $\langle u, F\rangle = \langle u, f\rangle + \langle B_i u, g\rangle$. We assume that the intersection $\mathcal{D}(B_1) \cap \mathcal{D}(B_2)$ is dense in both $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$ in the corresponding graph norms. Then (2.2) with $L_i = \{(u, B_i u); u \in \mathcal{D}(B_i)\} \subseteq X_i \times X_i$, $i = 1, 2$, and the equality of norms $\|\cdot\|_{(X_1\times X_1)+(X_2\times X_2)}$ and $\|\cdot\|_{(X_1+X_2)\times(X_1+X_2)}$ on $(X_1\times X_1)+(X_2\times X_2)$ yield the following result: For each $u \in \mathcal{D}(B_1) + \mathcal{D}(B_2)$ with decomposition $u = u_1 + u_2$, $u_1 \in \mathcal{D}(B_1)$, $u_2 \in \mathcal{D}(B_2)$, to be more precise, for an element $(u_1, B_1u_1) + (u_2, B_2u_2) \in L_1 \times L_2 \subset (X_1 \times X_1) + (X_2 \times X_2)$,

$$\|u\|_{\mathcal{D}(B_1)+\mathcal{D}(B_2)} = \|u_1 + u_2\|_{X_1+X_2} + \|B_1u_1 + B_2u_2\|_{X_1+X_2}. \qquad (2.3)$$

For instationary problems we need, given a Banach space X, the usual Banach space $L^s(0, T; X)$, $0 < T \le \infty$, of measurable X–valued (classes of) functions u with norm

$$\|u\|_{L^s(0,T;X)} = \left(\int_0^T \|u(t)\|_X^s \, dt\right)^{\frac{1}{s}}, \quad 1 \le s < \infty.$$

If X is reflexive and $1 < s < \infty$, then

$$L^s(0, T; X)' = L^{s'}(0, T; X'), \quad s' = \frac{s}{s-1},$$

with the natural pairing $\langle u, f\rangle_T = \int_0^T \langle u(t), f(t)\rangle \, dt$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and its dual X'.

Let $X = L^q(\Omega)$, $1 < q < \infty$. Then we use the notation

$$L^{s,q} := L^s(L^q(\Omega)) = L^s(0, T; L^q(\Omega)), \quad \|u\|_{L^{s,q}} = \left(\int_0^T \|u\|_q^s \, dt\right)^{1/s}.$$

The pairing of $L^s(0, T; L^q(\Omega))$ with its dual $L^{s'}(0, T; L^{q'}(\Omega))$ is given by $\langle u, f\rangle_T = \langle u, f\rangle_{\Omega,T} = \int_0^T \big(\int_\Omega u \cdot f \, dx\big) dt$. Moreover, we see that

$$L^{s,q} \cap L^{s,2} = L^s(0, T; L^q \cap L^2) \quad \text{and} \quad L^{s,q} + L^{s,2} = L^s(0, T; L^q + L^2)$$

since

$$(L^{s,q}+L^{s,2})' = (L^{s,q})' \cap (L^{s,2})' = L^{s'}(0,T;L^{q'} \cap L^2) = L^s(0,T;L^q+L^2)';$$

the pairing between $L^{s,q}+L^{s,2}$ and $(L^{s,q})' \cap (L^{s,2})'$ is given by $\langle u_1+u_2, f\rangle_T = \langle u_1, f\rangle_T + \langle u_2, f\rangle_T$ for $u_1 \in L^{s,q}$, $u_2 \in L^{s,2}$, $f \in (L^{s,q})' \cap (L^{s,2})'$. Furthermore, we can choose the decomposition $u = u_1 + u_2 \in L^s(0,T;L^q+L^2)$ in such a way that

$$\|u\|_{L^{s,q}+L^{s,2}} = \|u_1\|_{L^{s,q}} + \|u_2\|_{L^{s,2}}.$$

We conclude that

$$\|u_1+u_2\|_{L^{s,q}+L^{s,2}} = \sup\left\{ \frac{|\langle u_1+u_2, f\rangle_T\|}{\|f\|_{(L^{s,q})' \cap (L^{s,2})'}}; 0 \neq f \in L^{s'}(0,T;L^{q'} \cap L^2)\right\}.$$

Let us introduce the short notation

$$\tilde{L}^{s,q} = \begin{cases} L^{s,q} \cap L^{s,2}, & 2 \le q < \infty \\ L^{s,q} + L^{s,2}, & 1 < q < 2 \end{cases},$$

and note the duality relation $\big(\tilde{L}^{s,q}\big)' = \tilde{L}^{s',q'}$.

Concerning domains of uniform $C^{1,1}$-type (α,β,K), see Definition 1.1, we have to introduce further notations. Obviously, the axes e_i, $i = 1,\ldots,n$, of the new coordinate system (y', y_n) may be chosen in such a way that $e_1,\ldots,e_{n-1}$ are tangential to $\partial\Omega$ at x_0. Hence at $y' = 0$ the function $h \in C^{1,1}$ satisfies $h(y') = 0$ and $\nabla' h(y') = (\partial h/\partial y_1, \ldots, \partial h/\partial y_{n-1})(y') = 0$. By a continuity argument, for any given constant $M_0 > 0$, we may choose $\alpha > 0$ sufficiently small such that $\|h\|_{C^1} \le M_0$ is satisfied.

It is easily shown that there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ of fixed radius $r > 0$ with centers $x_j \in \overline{\Omega}$, such that with suitable functions $h_j \in C^{1,1}$ of type (α,β,K)

$$\overline{B}_j \subset U_{\alpha,\beta,h_j}(x_j) \;\text{ if } x_j \in \partial\Omega, \quad \overline{B}_j \subset \Omega \;\text{ if } x_j \in \Omega. \tag{2.4}$$

Here j runs from 1 to a finite number $N = N(\Omega) \in \mathbb{N}$ if Ω is bounded, and $j \in \mathbb{N}$ if Ω is unbounded. Moreover, as an important consequence, the covering $\{B_j\}$ of Ω may be constructed in such a way that not more than a fixed number $N_0 = N_0(\alpha,\beta,K) \in \mathbb{N}$ of these balls have a nonempty intersection:

$$\text{If } \ 1 \le j_1 < j_2 < \cdots < j_N \ \text{ and } \ N > N_0, \ \text{ then } \bigcap_{k=1}^{N} B_{j_k} = \emptyset. \tag{2.5}$$

Related to the covering $\{B_j\}$, there exists a partition of unity $\{\varphi_j\}$, $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, such that

$$0 \le \varphi_j \le 1, \ \operatorname{supp}\varphi_j \subset B_j, \quad \text{and} \quad \sum_{j=1}^{N} \varphi_j = 1 \text{ or } \sum_{j=1}^{\infty} \varphi_j = 1 \text{ on } \Omega. \tag{2.6}$$

The functions φ_j may be chosen so that $|\nabla\varphi_j(x)|+|\nabla^2\varphi_j(x)| \le C$ uniformly in j and $x \in \Omega$ with $C = C(\alpha, \beta, K)$.

If Ω is unbounded, then Ω can be represented as the union of an increasing sequence of bounded uniform $C^{1,1}$-domains $\Omega_k \subset \Omega$, $k \in \mathbb{N}$,

$$\Omega_1 \subset \cdots \subset \Omega_k \subset \Omega_{k+1} \subset \cdots, \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k, \tag{2.7}$$

where each Ω_k is of the same type (α', β', K'), see [13, p. 652]. Without loss of generality we assume that $\alpha = \alpha'$, $\beta = \beta'$, $K = K'$.

Using the partition of unity $\{\varphi_j\}$ we will perform the analysis of the Stokes operator by starting from well-known results for certain bounded and unbounded domains. For this reason, given $h \in C^{1,1}(\mathbb{R}^{n-1})$ satisfying $h(0) = 0$, $\nabla' h(0) = 0$ and with compact support contained in the $(n-1)$-dimensional ball of radius $r, 0 < r = r(\alpha, \beta, K) < \alpha$, and center 0, we introduce the bounded domain

$$\begin{aligned} H &= H_{\alpha,\beta,h;r} \\ &= \big\{ y = (y', y_n) \in \mathbb{R}^n : h(y') - \beta < y_n < h(y'), \ |y'| < \alpha \big\} \cap B_r(0); \end{aligned}$$

here we assume that $\overline{B_r(0)} \subset \{y \in \mathbb{R}^n : |y_n - h(y')| < \beta, |y'| < \alpha\}$.

On H we consider the classical Sobolev spaces $W^{k,q}(H)$ and $W_0^{k,q}(H)$, $k \in \mathbb{N}$, the dual space $W^{-1,q}(H) = \big(W_0^{1,q'}(H)\big)'$ and the space

$$L_0^q(H) = \left\{ u \in L^q(H) : \int_H u \, dx = 0 \right\}$$

of L^q-functions with vanishing mean on H.

Lemma 2.1 *Let $1 < q < \infty$ and $H = H_{\alpha,\beta,h;r}$.*

(i) *There exists a bounded linear operator*

$$R : L^q_0(H) \to W^{1,q}_0(H)$$

such that $\operatorname{div} \circ R = I$ *on $L^q_0(H)$ and $R\big(L^q_0(H) \cap W^{1,q}_0(H)\big) \subset W^{2,q}_0(H)$. Moreover, there exists a constant $C = C(\alpha, \beta, K, q) > 0$ such that*

$$\begin{aligned} \|Rf\|_{W^{1,q}} &\le C\|f\|_{L^q} \quad \text{for all} \quad f \in L^q_0(H) \\ \|Rf\|_{W^{2,q}} &\le C\|f\|_{W^{1,q}} \quad \text{for all} \quad f \in L^q_0(H) \cap W^{1,q}_0(H)\,. \end{aligned} \tag{2.8}$$

(ii) *There exists $C = C(\alpha, \beta, K, q) > 0$ such that for every $p \in L^q_0(H)$*

$$\|p\|_q \le C\|\nabla p\|_{W^{-1,q}} = C \sup\left\{ \frac{|\langle p, \operatorname{div} v\rangle|}{\|\nabla v\|_{q'}} : 0 \ne v \in W^{1,q'}_0(H) \right\}. \tag{2.9}$$

(iii) *For given $f \in L^q(H)$ let $u \in L^q_\sigma(H) \cap W^{1,q}_0(H) \cap W^{2,q}(H)$, $p \in W^{1,q}(H)$ satisfy the Stokes resolvent equation $\lambda u - \Delta u + \nabla p = f$ with $\lambda \in \mathcal{S}_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$. Moreover, assume that* $\operatorname{supp} u \cup \operatorname{supp} p \subset B_r(0)$. *Then there are constants $\lambda_0 = \lambda_0(q, \alpha, \beta, K) > 0$, $C = C(q, \varepsilon, \alpha, \beta, K) > 0$ such that*

$$\|\lambda u\|_{L^q(H)} + \|u\|_{W^{2,q}(H)} + \|\nabla p\|_{L^q(H)} \le C\|f\|_{L^q(H)} \tag{2.10}$$

if $|\lambda| \ge \lambda_0$.

Proof. (i) It is well-known that there exists a bounded linear operator $R : L^q_0(H) \to W^{1,q}_0(H)$ such that $u = Rf$ solves the divergence problem $\operatorname{div} u = f$. Moreover, the estimate $(2.8)_1$ holds with $C = C(\alpha, \beta, K, q) > 0$, see [10, III, Theorem 3.1]. The second part follows from [10, III, Theorem 3.2].

(ii) A duality argument and (i) yield (ii), see [8], [16, II.2.1].

(iii) We extend u, p by zero so that $(u, \nabla p)$ may be considered as a solution of the Stokes resolvent system in a *bent half space*; then we refer to [6, Theorem 3.1, (i)]. □

The next lemma concerns the instationary Stokes systems

$$u_t - \Delta u + \nabla p = f, \quad u(0) = u_0 \quad \text{or} \quad -u_t - \Delta u + \nabla p = f, \quad u(T) = u_0, \tag{2.11}$$

in the domain H. To describe this crucial result we define the Stokes operator as usual by $A_q = -P_q\Delta$ with domain $\mathcal{D}(A_q) = L^q_\sigma(H) \cap W^{1,q}_0(H) \cap W^{2,q}(H)$.

Lemma 2.2 *Let $0 < T < \infty$, $u_0 \in \mathcal{D}(A_q)$ and $f \in L^q(0,T;L^q(H))$ be given. Assume that $u \in L^q(0,T,\mathcal{D}(A_q))$, $p \in L^q(0,T;W^{1,q}(H))$ solve one of the systems in* (2.11) *and satisfy* $\operatorname{supp} u_0 \cup \operatorname{supp} u(t) \cup \operatorname{supp} p(t) \subseteq B_r(0)$ *for a.a. $t \in [0,T]$.*

Then there is a constant $C = C(q,\alpha,\beta,K,T) > 0$ such that

$$\begin{aligned} &\|u_t\|_{L^q(0,T;L^q(H))} + \|u\|_{L^q(0,T;W^{2,q}(H))} + \|\nabla p\|_{L^q(0,T;L^q(H))} \\ &\quad \le C\big(\|u_0\|_{W^{2,q}(H)} + \|f\|_{L^q(0,T;L^q(H))}\big). \end{aligned} \tag{2.12}$$

Proof. In the case $u(0) = u_0$ this estimate follows from [17, Theorem 4.1, (4.2) and (4.21')], see also [15]. A careful inspection of the proofs shows that the constant C in (2.12) depends only on the type (α,β,K) and on q,T; actually, it suffices to assume the boundary regularity $C^{1,1}$ since only the boundedness of second order derivatives of functions locally describing the boundary is used.

The second case $-u_t - \Delta u + \nabla p = f$, $u(T) = u_0$, can be reduced to the first one by the transformation $\tilde{u}(t) = u(T-t)$, $\tilde{f}(t) = f(T-t)$, $\tilde{p}(t) = p(T-t)$. □

We note that the assumption $u_0 \in \mathcal{D}(A_q)$ is used for simplicity and can be weakened as in Remark 1.5 (i). Since $u_t \in L^q(0,T;L^q_\sigma)$, the conditions $u(0) = u_0$ or $u(T) = u_0$, resp., are well defined.

Next we collect several results on Sobolev embedding estimates and on the Stokes operator A_q, $1 < q < \infty$, on bounded $C^{1,1}$-domains.

Lemma 2.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain of type (α,β,K).*

(i) *Let $1 < q < \infty$. Then for every $M \in (0,1)$ there exists some constant $C = C(q,M,\alpha,\beta,K) > 0$ such that*

$$\|\nabla u\|_{L^q} \le M\|\nabla^2 u\|_{L^q} + C\|u\|_{L^q}, \quad u \in W^{2,q}(\Omega). \tag{2.13}$$

(ii) *Let $2 \le q < \infty$. Then for every $M \in (0,1)$ there exists a constant $C = C(q, M, \alpha, \beta, K) > 0$ such that*

$$\|u\|_{L^q} \le M\|\nabla^2 u\|_{L^q} + C\big(\|\nabla^2 u\|_{L^2} + \|u\|_{L^2}\big), \quad u \in W^{2,q}(\Omega). \tag{2.14}$$

Proof. The proofs of (i), (ii) are easily reduced to the case $u \in W^{2,q}(\mathbb{R}^n)$, using an extension operator on Sobolev spaces the norm of which is shown to depend only on q and (α, β, K). In (ii) we choose an $r \in [2, q)$ such that $\|u\|_{L^q} \le M\|\nabla^2 u\|_{L^r} + C\|u\|_{L^r}$ and use the interpolation inequality

$$\|v\|_{L^r} \le \gamma\left(\frac{1}{\varepsilon}\right)^{1/\gamma}\|v\|_{L^2} + (1-\gamma)\varepsilon^{1/(1-\gamma)}\|v\|_{L^q}, \tag{2.15}$$

with $\gamma \in (0,1)$, $\frac{1}{r} = \frac{\gamma}{2} + \frac{1-\gamma}{q}$, for $v = u$ and $v = \nabla^2 u$ for suitable $\varepsilon > 0$ to get (2.14). For basic details see [1, IV, Theorem 4.28], [9] and [16, II.1.3]. □

Lemma 2.4 *Let $1 < q < \infty$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain.*

(i) *The Stokes operator $A_q = -P_q\Delta : \mathcal{D}(A_q) \to L^q_\sigma(\Omega)$, where $\mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega)$, satisfies the resolvent estimate*

$$\|\lambda u\|_{L^q} + \|A_q u\|_{L^q} \le C\|f\|_{L^q}, \quad C = C(\varepsilon, q, \Omega) > 0, \tag{2.16}$$

where $u \in \mathcal{D}(A_q)$, $\lambda u + A_q u = f \in L^q_\sigma(\Omega)$, $\lambda \in \mathcal{S}_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$, and it holds the estimate

$$\|u\|_{W^{2,q}} \le C\|A_q u\|_{L^q}, \quad C = C(q, \Omega).$$

Moreover, $\langle A_q u, v\rangle = \langle u, A_{q'} v\rangle$ for all $u \in \mathcal{D}(A_q)$, $v \in \mathcal{D}(A_{q'})$ and $A_q' = A_{q'}$.

(ii) *If $q = 2$, then the resolvent problem $\lambda u + A_2 u = f \in L^2_\sigma(\Omega)$, $\lambda \in \mathcal{S}_\varepsilon$, has a unique solution $u \in \mathcal{D}(A_2)$ satisfying the estimate*

$$\|\lambda u\|_{L^2} + \|A_2 u\|_{L^2} \le C\|f\|_{L^2} \tag{2.17}$$

with the constant $C = 1 + 2/\cos\varepsilon$ independent of Ω. Moreover, A_2 is self-adjoint and $\langle A_2 u, u\rangle = \|A_2^{\frac{1}{2}} u\|^2_{L^2} = \|\nabla u\|^2_{L^2}$ for all $u \in \mathcal{D}(A_2)$.

Proof. For (i) see [6], [11], [17]. For (ii) – including even general unbounded domains – we refer to [16]. □

Finally we return to the instationary Stokes system for a bounded $C^{1,1}$-domain $\Omega \subseteq \mathbb{R}^n$, written in the form of the abstract evolution problem

$$u_t + A_q u = f, \quad u(0) = u_0, \tag{2.18}$$

with initial value $u_0 \in \mathcal{D}(A_q)$ and $f \in L^s(0,T; L^q_\sigma(\Omega))$, $1 < q,\ s < \infty$. In view of the variation of constants formula we define the operators $\mathcal{J}_{s,q}$, $\mathcal{J}'_{s,q}$ by

$$\mathcal{J}_{s,q} f(t) = \int_0^t e^{-(t-\tau)A_q} f(\tau)d\tau, \quad \mathcal{J}'_{s,q} f(t) = \int_t^T e^{-(\tau-t)A_q} f(\tau)d\tau. \tag{2.19}$$

Lemma 2.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain.*

(i) *Let $1 < q,\ s < \infty$ and $0 < T < \infty$. Then for every initial value $u_0 \in \mathcal{D}(A_q)$ and external force $f \in L^s(0,T; L^q_\sigma(\Omega))$ the nonstationary Stokes system* (2.18) *has a unique solution $u \in L^s(0,T; \mathcal{D}(A_q))$ given by*

$$u(t) = e^{-tA_q} u_0 + \mathcal{J}_{s,q} f(t)$$

satisfying the estimate

$$\|u_t\|_{L^{s,q}} + \|u\|_{L^{s,q}} + \|A_q u\|_{L^{s,q}} \le C\big(\|u_0\|_{\mathcal{D}(A_q)} + \|f\|_{L^{s,q}}\big) \tag{2.20}$$

with a constant $C = C(q,s,T,\Omega)$. Analogously, the nonstationary Stokes system $-u_t + A_q u = f$, $u(T) = u_0$, has a unique solution $u \in L^s(0,T; \mathcal{D}(A_q))$, namely, $u(t) = e^{-(T-t)A_q} u_0 + (\mathcal{J}'_{s,q} f)(t)$; this solution satisfies (2.20) *with the same constant C. Moreover, there holds the duality relation $(\mathcal{J}_{s,q})' = \mathcal{J}'_{s',q'}$.*

(ii) *In the case $q = 2$ the constant $C = C(2,s,T,\Omega) = C(s,T)$ in* (2.20) *does not depend on the domain Ω.*

Proof. For (i) see [12], [17]. The assertions on $\mathcal{J}'_{s,q}$ follow from the transformation $\tilde{u}(t) = u(T-t)$, $\tilde{f}(t) = f(T-t)$ and by duality arguments. For (ii) – including even general unbounded domains – we refer to [16, IV.1.6]. □

Note that in (2.16) and (2.20) it is not clear up to now how the constant C will depend on the underlying bounded domain Ω except for $q = 2$.

3. Proofs

After a preliminary result on the norm $\|u\|_{W^{2,q}}$ and the graph norm $\|u\|_{\mathcal{D}(A_q)} = \|u\|_{L^q} + \|A_q u\|_{L^q}$, $u \in \mathcal{D}(A_q)$, for bounded domains $\Omega \subseteq \mathbb{R}^n$ we turn to the proofs of Theorem 1.3, see Subsection 3.1, and of Theorem 1.4, see Subsection 3.2. In both cases we consider first of all bounded domains for $q > 2$, then for $1 < q < 2$, and finally unbounded domains.

Lemma 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^{1,1}$-domain of type (α, β, K). Then there exists a constant $C = C(q, \alpha, \beta, K) > 0$ such that*

$$C\|u\|_{W^{2,q}} \le \|u\|_{\mathcal{D}(A_q)}, \quad u \in \mathcal{D}(A_q). \tag{3.1}$$

Proof. We use the system of functions $\{h_j\}$, $1 \le j \le N$, parametrizing $\partial\Omega$, the covering of Ω by balls $\{B_j\}$, and the partition of unity $\{\varphi_j\}$ as described in Section 2. Let

$$U_j = U^-_{\alpha,\beta,h_j}(x_j) \cap B_j \text{ if } x_j \in \partial\Omega \text{ and } U_j = B_j \text{ if } x_j \in \Omega,\ 1 \le j \le N. \tag{3.2}$$

Given $f \in L^q_\sigma(\Omega)$ and $u \in \mathcal{D}(A_q)$ satisfying $A_q u = f$, i.e. $-\Delta u + \nabla p = f$, $\operatorname{div} u = 0$ in Ω, let $w_j = R\big((\nabla\varphi_j)\cdot u\big) \in W^{2,q}_0(U_j)$ be the solution of the divergence equation $\operatorname{div} w_j = \operatorname{div}(\varphi_j u) = (\nabla\varphi_j)\cdot u$ in U_j, $1 \le j \le N$. Moreover, let $M_j = M_j(p)$ be the constant such that $p - M_j \in L^q_0(U_j)$. By Lemma 2.1 (i), (ii) and the equation $\nabla p = f + \Delta u$ we conclude that $\|w_j\|_{W^{1,q}(U_j)} \le C\|u\|_{L^q(U_j)}$, $\|w_j\|_{W^{2,q}(U_j)} \le C\|u\|_{W^{1,q}(U_j)}$ as well as

$$\|p - M_j\|_{L^q(U_j)} \le C\big(\|f\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)}\big)$$

with $C = C(q, \alpha, \beta, K) > 0$ independent of j. Finally, let $\lambda_0 > 0$ denote the constant in Lemma 2.1 (iii). Then $\varphi_j u - w_j$ satisfies the local resolvent equation

$$\begin{aligned} &\lambda_0(\varphi_j u - w_j) - \Delta(\varphi_j u - w_j) + \nabla\big(\varphi_j(p - M_j)\big) \\ &\quad = \varphi_j f + \Delta w_j - 2\nabla\varphi_j \cdot \nabla u - (\Delta\varphi_j)u + (\nabla\varphi_j)(p - M_j) + \lambda_0(\varphi_j u - w_j) \end{aligned}$$

in U_j. By (2.10) with $\lambda = \lambda_0$ and the previous *a priori* estimates we get the local inequalities

$$\|\varphi_j \nabla^2 u\|_{L^q(U_j)}^q + \|\varphi_j \nabla p\|_{L^q(U_j)}^q \le C\big(\|f\|_{L^q(U_j)}^q + \|u\|_{W^{1,q}(U_j)}^q\big), \tag{3.3}$$

$1 \le j \le N$. Taking the sum over $j = 1, \dots, N$ and exploiting the crucial property of the number N_0, see (2.5), we are led to the estimate

$$\begin{aligned}
\|\nabla^2 u\|_{L^q(\Omega)}^q + \|\nabla p\|_{L^q(\Omega)}^q &= \int_\Omega \bigg(\Big(\sum_j \varphi_j |\nabla^2 u|\Big)^q + \Big(\sum_j \varphi_j |\nabla p|\Big)^q\bigg)\, dx \\
&\le \int_\Omega N_0^{\frac{q}{q'}} \Big(\sum_j |\varphi_j \nabla^2 u|^q + \sum_j |\varphi_j \nabla p|^q\Big)\, dx \\
&\le C N_0^{\frac{q}{q'}} \Big(\sum_j \|f\|_{L^q(U_j)}^q + \sum_j \|u\|_{W^{1,q}(U_j)}^q\Big).
\end{aligned} \tag{3.4}$$

Next we use (2.13) for the term $\|u\|_{W^{1,q}(U_j)}$. Choosing $M > 0$ sufficiently small in (2.13), exploiting the absorption principle and again the property of the number N_0, (3.4) may be simplified to the estimate

$$\|\nabla^2 u\|_{L^q(\Omega)} \le C\big(\|f\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}\big) \tag{3.5}$$

where $C = C(q, \alpha, \beta, K) > 0$. Since $f = A_q u$, the proof is complete. □

3.1. Proof of Theorem 1.3

3.1.1 The Stokes resolvent in a bounded domain Ω when $q \ge 2$

We consider for $\lambda \in \mathcal{S}_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$, the resolvent equation

$$\lambda u + A_q u = \lambda u - \Delta u + \nabla p = f \quad \text{in} \quad \Omega$$

with $f \in L^q_\sigma(\Omega)$, where $2 \le q < \infty$. Our aim is to prove for its solution $u \in D(A_q)$ and $\nabla p = (I - P_q)\Delta u$ the estimate

$$\|\lambda u\|_{L^q \cap L^2} + \|\nabla^2 u\|_{L^q \cap L^2} + \|\nabla p\|_{L^q \cap L^2} \le C\|f\|_{L^q \cap L^2}, \quad |\lambda| \ge \delta > 0 \tag{3.6}$$

with a constant $C = C(q, \varepsilon, \delta, \alpha, \beta, K) > 0$. Note that this estimate is well-known for bounded domains with a constant $C = C(q, \varepsilon, \delta, \Omega) > 0$. As in Subsection 3.1 let $w_j = R\big((\nabla\varphi_j) \cdot u\big) \in W_0^{2,q}(U_j)$ and choose a constant $M_j = M_j(p)$ such that $p - M_j \in L_0^q(U_j)$. Then we obtain the local equation

$$\begin{aligned}&\lambda(\varphi_j u - w_j) - \Delta(\varphi_j u - w_j) + \nabla\big(\varphi_j(p - M_j)\big)\\&\qquad = \varphi_j f + \Delta w_j - 2\nabla\varphi_j \cdot \nabla u - (\Delta\varphi_j)u - \lambda w_j + (\nabla\varphi_j)(p - M_j)\end{aligned} \tag{3.7}$$

Concerning the term λw_j, we apply the embedding $W^{1,r}(U_j) \subset L^q(U_j)$ for some $r \in [2, q)$, then Lemma 2.1(i) and use the interpolation estimate (2.15) for $v = u$ to get for $M \in (0, 1)$ that

$$\|w_j\|_{L^q(U_j)} \le C_1\|w_j\|_{W^{1,r}(U_j)} \le M\|u\|_{L^q(U_j)} + C_2\|u\|_{L^2(U_j)};$$

here $C_i = C_i(M, q, r, \alpha, \beta, K) > 0$ $(i = 1, 2)$. Moreover, $\|\nabla^2 w_j\|_{L^q(U_j)} \le C\|\nabla u\|_{L^q(U_j)}$. For $p - M_j$ we use (2.9) and the equation $\nabla p = -\lambda u + \Delta u + f$ to see that

$$\begin{aligned}&\|p - M_j\|_{L^q(U_j)}\\&\qquad \le C\left(\|f\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)} + \sup\left\{\frac{|\langle \lambda u, v\rangle|}{\|\nabla v\|_{q'}} : 0 \ne v \in W_0^{1,q'}(U_j)\right\}\right),\end{aligned}$$

where $C = C(q, \alpha, \beta, K) > 0$. Again we choose $r \in [2, q)$, use the embedding $W^{1,q'}(U_j) \subset L^{r'}(U_j)$, then (2.15) for $v = \lambda u$ to get that

$$\|p - M_j\|_{L^q(U_j)} \le C\big(\|f\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)} + \|\lambda u\|_{L^2(U_j)}\big) + M\|\lambda u\|_{L^q(U_j)}.$$

Finally, we apply to the local resolvent equation (3.7) the estimate (2.10) with λ replaced by $\lambda + \lambda_0'$ where $\lambda_0' \ge 0$ is sufficiently large such that $|\lambda + \lambda_0'| \ge \lambda_0$ for $|\lambda| \ge \delta$, λ_0 as in (2.10).

Now we combine these estimates and are led to the local inequality

$$\begin{aligned}&\|\lambda\varphi_j u\|_{L^q(U_j)} + \|\varphi_j\nabla^2 u\|_{L^q(U_j)} + \|\varphi_j\nabla p\|_{L^q(U_j)}\\&\qquad \le C\big(\|f\|_{L^q(U_j)} + \|u\|_{L^q(U_j)} + \|\nabla u\|_{L^q(U_j)} + \|\lambda u\|_{L^2(U_j)}\big) + M\|\lambda u\|_{L^q(U_j)}\end{aligned} \tag{3.8}$$

with $C = C(M, q, \delta, \varepsilon, \alpha, \beta, K) > 0$. Raising each term in (3.8) to the qth power, taking the sum over $j = 1, \dots, N$ in the same way as in (3.3)–(3.5) and using the crucial property (2.5) of the integer N_0 we get the inequality

$$\begin{aligned}&\|\lambda u\|_{L^q(\Omega)} + \|\nabla^2 u\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)}\\ &\quad \le C\big(\|f\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)} + \|\nabla u\|_{L^q(\Omega)} + \|\lambda u\|_{L^2(\Omega)}\big) + M\|\lambda u\|_{L^q(\Omega)}\end{aligned} \tag{3.9}$$

with $C = C(M, q, \delta, \varepsilon, \alpha, \beta, K) > 0$, $|\lambda| \ge \delta$. For the proof of (3.9) we also used the reverse Hölder inequality $\big(\sum_j a_j^q\big)^{1/q} \le \big(\sum_j a_j^2\big)^{1/2}$ for the real numbers $a_j = \|\lambda u\|_{L^2(U_j)}$ valid for $q \ge 2$. Applying (2.13) and choosing M sufficiently small we remove the terms $\|\nabla u\|_{L^q(\Omega)}$ and $\|\lambda u\|_{L^q(\Omega)}$ from the right-hand side in (3.9) by the absorption principle. The term $\|u\|_{L^q(\Omega)}$ is removed with the help of (2.14). Hence we get that

$$\|\lambda u\|_q + \|\nabla^2 u\|_q + \|\nabla p\|_q \le C\big(\|f\|_q + \|\lambda u\|_2 + \|u\|_2 + \|\nabla^2 u\|_2\big).$$

Now we combine this inequality with the estimate (2.17) for $|\lambda| \ge \delta$ and we apply (3.1) with $q = 2$. This proves the desired estimate (3.6) for $2 \le q < \infty$.

3.1.2 The case Ω bounded, $1 < q < 2$

We consider for $f \in L^2_\sigma + L^q_\sigma = L^q_\sigma$ and $\lambda \in \mathcal{S}_\varepsilon$, $|\lambda| \ge \delta$, the equation $\lambda u - \Delta u + \nabla p = f$ and its unique solution $u \in \mathcal{D}(A_q) + \mathcal{D}(A_2) = \mathcal{D}(A_q)$, $\nabla p = (I - \tilde{P}_q)\Delta u$. Note that $A_q = \tilde{A}_q$, $P_q = \tilde{P}_q$ and that $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^{q'}_\sigma(\Omega) \cap L^2_\sigma(\Omega) = L^{q'}_\sigma(\Omega)$. Using $f = \lambda u - \tilde{P}_q \Delta u$, the density of $\mathcal{D}(A_{q'}) \cap \mathcal{D}(A_2) = \mathcal{D}(A_{q'})$ in $L^{q'}_\sigma \cap L^2_\sigma$, (3.6) with q replaced by $q' > 2$, and setting $g = \lambda v + \tilde{A}_{q'} v$ for $v \in \mathcal{D}(A_{q'}) \cap \mathcal{D}(A_2)$ we obtain that

$$\begin{aligned}\|f\|_{L^2_\sigma + L^q_\sigma} &= \sup\left\{\frac{|\langle \lambda u + \tilde{A}_q u, v\rangle|}{\|v\|_{L^{q'}_\sigma \cap L^2_\sigma}}; 0 \ne v \in \mathcal{D}(A_{q'}) \cap \mathcal{D}(A_2)\right\}\\ &= \sup\left\{\frac{|\langle u, \lambda v + \tilde{A}_{q'} v\rangle|}{\|v\|_{L^{q'}_\sigma \cap L^2_\sigma}}; 0 \ne v \in \mathcal{D}(A_{q'}) \cap \mathcal{D}(A_2)\right\}\\ &= \sup\left\{\frac{|\langle u, g\rangle|}{\|(\lambda I - \tilde{P}_{q'}\Delta)^{-1} g\|_{L^{q'}_\sigma \cap L^2_\sigma}}; 0 \ne g \in L^{q'}_\sigma \cap L^2_\sigma\right\}\\ &\ge |\lambda| C^{-1} \sup\left\{\frac{|\langle u, g\rangle|}{\|g\|_{L^{q'}_\sigma \cap L^2_\sigma}}; 0 \ne g \in L^{q'}_\sigma \cap L^2_\sigma\right\}.\end{aligned} \tag{3.10}$$

By Section 2 the last term $\sup\{\ldots\}$ in (3.10) defines a norm on $L^q_\sigma + L^2_\sigma$ which is equivalent to the norm $\|\cdot\|_{L^q_\sigma+L^2_\sigma}$; the constants in this norm equivalence are related to the norm of $\tilde{P}_{q'}$ and depend only on q and (α,β,K), cf. Theorem 1.2. Hence we proved the estimate $\|\lambda u\|_{L^q_\sigma+L^2_\sigma} \le C\|f\|_{L^q_\sigma+L^2_\sigma}$ and even

$$\|\lambda u\|_{L^q_\sigma+L^2_\sigma} + \|u\|_{L^q_\sigma+L^2_\sigma} + \|A_q u\|_{L^q_\sigma+L^2_\sigma} \le C\|f\|_{L^q_\sigma+L^2_\sigma}, \quad \lambda \in \mathcal{S}_\varepsilon,\ |\lambda| \ge \delta. \tag{3.11}$$

By virtue of Lemma 3.1 and (2.3) with $B_1 = A_q, B_2 = A_2$, we conclude that $\|u\|_{W^{2,q}+W^{2,2}} \le c\big(\|u\|_{L^q_\sigma+L^2_\sigma} + \|A_q u\|_{L^q_\sigma+L^2_\sigma}\big)$ with a constant $c > 0$ depending only on q and (α,β,K). Then (3.11) and the identity $\nabla p = f - \lambda u + \Delta u$ lead to the estimate

$$\|\lambda u\|_{L^q_\sigma+L^2_\sigma} + \|u\|_{W^{2,q}+W^{2,2}} + \|\nabla p\|_{L^q+L^2} \le C\|f\|_{L^q_\sigma+L^2_\sigma}$$

with $C = C(q,\delta,\varepsilon,\alpha,\beta,K) > 0$. Hence we proved for every $q \in (1,\infty)$ the inequality

$$\|\lambda u\|_{\tilde{L}^q_\sigma} + \|u\|_{\tilde{W}^{2,q}} + \|\nabla p\|_{\tilde{L}^q} \le C\|f\|_{\tilde{L}^q_\sigma}, \quad u \in \mathcal{D}(\tilde{A}_q), \tag{3.12}$$

with $C = C(q,\delta,\varepsilon,\alpha,\beta,K) > 0$ when $|\lambda| \ge \delta > 0$. Now the proof of Theorem 1.3 (i) – (iii) is complete for bounded domains.

3.1.3 The case Ω unbounded

Consider the sequence of bounded subdomains $\Omega_j \subseteq \Omega$, $j \in \mathbb{N}$, of uniform $C^{1,1}$-type as in (2.7), let $f \in \tilde{L}^q_\sigma(\Omega)$ and $f_j := \tilde{P}_q f|_{\Omega_j}$. Then consider the solution $(u_j, \nabla p_j)$ of the Stokes resolvent equation

$$\lambda u_j - \tilde{P}_q \Delta u_j = \lambda u_j - \Delta u_j + \nabla p_j = f_j, \quad \nabla p_j = (I - \tilde{P}_q)\Delta u_j \quad \text{in } \Omega_j.$$

From (3.12) we obtain the uniform estimate

$$\|\lambda u_j\|_{\tilde{L}^q_\sigma(\Omega_j)} + \|u_j\|_{\tilde{W}^{2,q}(\Omega_j)} + \|\nabla p_j\|_{\tilde{L}^q(\Omega_j)} \le C\|f\|_{\tilde{L}^q_\sigma(\Omega)} \tag{3.13}$$

with $|\lambda| \ge \delta > 0$, $C = C(q,\delta,\varepsilon,\alpha,\beta,K) > 0$. Extending u_j and ∇p_j by 0 to vector fields on Ω we find, suppressing subsequences, weak limits

$$u = \text{w–}\lim_{j\to\infty} u_j \quad \text{in } \tilde{L}^q_\sigma(\Omega), \quad \nabla p = \text{w–}\lim_{j\to\infty} \nabla p_j \quad \text{in } \tilde{L}^q(\Omega)^n$$

satisfying $u \in \mathcal{D}(\tilde{A}_q)$, $\lambda u - \Delta u + \nabla p = \lambda u - \tilde{P}_q \Delta u = f$ in Ω and the *a priori* estimates (1.2), (1.3). Note that each ∇p_j when extended by 0 need not be a gradient field on Ω; however, by de Rham's argument, the weak limit of the sequence $\{\nabla p_j\}$ is a gradient field on Ω. Hence we solved the Stokes resolvent problem $\lambda u + \tilde{A}_q u = \lambda u - \Delta u + \nabla p = f \quad \text{in } \Omega$.

Finally, to prove uniqueness of u we assume that there is some $v \in \mathcal{D}(\tilde{A}_q)$ and $\lambda \in \mathcal{S}_\varepsilon$ satisfying $\lambda v - \tilde{P}_q \Delta v = 0$. Given $f' \in \tilde{L}^{q'}(\Omega)^n$ let $u \in \mathcal{D}(\tilde{A}_{q'})$ be a solution of $\lambda u - \tilde{P}_{q'} \Delta u = \tilde{P}_{q'} f'$. Then

$$0 = \langle \lambda v - \tilde{P}_q \Delta v, u \rangle = \langle v, (\lambda - \tilde{P}_{q'} \Delta) u \rangle = \langle v, \tilde{P}_{q'} f' \rangle = \langle v, f' \rangle$$

for all $f' \in \tilde{L}^{q'}(\Omega)^n$; hence, $v = 0$.

Now Theorem 1.3 (i) – (iii) is proved. The assertions (iv) of this Theorem are proved by standard duality arguments and semigroup theory. $\square$

3.2. Proof of Theorem 1.4

Let $0 < T < \infty$, $1 < s, q < \infty$, and consider a domain $\Omega \subseteq \mathbb{R}^n, n \geq 2$, of uniform $C^{1,1}$-type (α, β, K). Then we define the subspace $\tilde{L}^{s,q}_\sigma := L^s(0,T;\tilde{L}^q_\sigma(\Omega))$ of $\tilde{L}^{s,q} := L^s(0,T;\tilde{L}^q(\Omega))$ with norm $\|\cdot\|_{\tilde{L}^{s,q}_\sigma} = \|\cdot\|_{L^s(0,T;\tilde{L}^q(\Omega)_\sigma)}$. In addition to the operators $\mathcal{J}_{s,q}$, $\mathcal{J}'_{s,q}$ for bounded domains, see Lemma 2.4, we define $\tilde{\mathcal{J}}_{s,q}, \tilde{\mathcal{J}}'_{s,q}$ by

$$\tilde{\mathcal{J}}_{s,q} f(t) = \int_0^t e^{-(t-\tau)\tilde{A}_q} f(\tau)\, d\tau, \qquad \tilde{\mathcal{J}}'_{s,q} f(t) = \int_t^T e^{-(\tau-t)\tilde{A}_q} f(\tau)\, d\tau,$$

for $f \in \tilde{L}^{s,q}_\sigma$ and $0 \leq t \leq T$. Since $(\tilde{A}_q)' = \tilde{A}_{q'}$, we obtain for all $f \in \tilde{L}^{s,q}_\sigma$, $g \in \tilde{L}^{s',q'}_\sigma$ that

$$\langle \tilde{\mathcal{J}}_{s,q} f, g \rangle_T = \langle f, \tilde{\mathcal{J}}'_{s',q'} g \rangle_T.$$

3.2.1 Maximal regularity in a bounded domain Ω when $s = q \geq 2$

First we consider the case $u_0 = 0$ and $s = q$. Then $u = \tilde{\mathcal{J}}_{q,q} f$ solves the equation $u_t + \tilde{A}_q u = f$, $u(0) = 0$, and $u = \tilde{\mathcal{J}}'_{q,q} f$ is the solution of the system $-u_t + \tilde{A}_q u = f$, $u(T) = 0$. Our aim is to prove in both cases the estimate (1.7) with a constant $C = C(T, q, \alpha, \beta, K) > 0$. Obviously it suffices to consider the case $u = \tilde{\mathcal{J}}_{q,q} f$ since the other case follows using the transformation $\tilde{u}(t) = u(T-t)$, $\tilde{f}(t) = f(T-t)$. By Lemma 2.5 we know that $u = \tilde{\mathcal{J}}_{q,q}$ solves the equation

$$u_t + \tilde{A}_q u = u_t - \Delta u + \nabla p = f \in L^q(0,T;\tilde{L}^q_\sigma), \quad u(0) = 0,$$

with $\nabla p = (I - \tilde{P}_q)\Delta u$, and that u satisfies (2.20) with a constant $C = C(\Omega, q) > 0$; note that the norms $\|u\|_{W^{2,q}}$ and $\|u\|_{\mathcal{D}(A_q)}$ are equivalent. Thus it remains to prove that C in (2.20) can be chosen depending only on T, q and (α, β, K).

For this reason, we use the system of functions $\{h_j\}$, $1 \le j \le N$, the covering of Ω by balls $\{B_j\}$, and the partition of unity $\{\varphi_j\}$ as described in Section 2 as well as the bounded sets $U_j \subset B_j$, cf. (3.2). On U_j define $w = R((\nabla\varphi_j)\cdot u) \in L^q(0,T;W^{2,q}_0(U_j))$, and let $M_j = M_j(p)$ be the constant depending on $t \in (0,T)$ such that $p - M_j \in L^q(0,T;L^q_0(U_j))$, see Lemma 2.1. Since $\operatorname{div} w = (\nabla\varphi_j)\cdot u$ and $\operatorname{div} w_t = (\nabla\varphi_j)\cdot u_t$ for a.a. $t \in (0,T)$, the term $(\varphi_j u - w)$ solves in U_j the local equation

$$\begin{aligned}(\varphi_j u - w)_t - \Delta(\varphi_j u - w) + \nabla\big(\varphi_j(p - M_j)\big) \\ = \varphi_j f - w_t + \Delta w - 2\nabla\varphi_j \cdot \nabla u - (\Delta\varphi_j)u + (\nabla\varphi_j)(p - M_j). \qquad (3.14)\end{aligned}$$

From (2.8), (2.9) using $w_t = R((\nabla\varphi_j)\cdot u_t)$ and $\nabla p = f - u_t + \Delta u$ we will prove for all $\varepsilon \in (0,1)$ the estimates

$$\begin{aligned}\|w_t\|_{L^q(L^q(U_j))} &\le C\|u_t\|_{L^q(L^2(U_j))} + \varepsilon\|u_t\|_{L^q(L^q(U_j))}, \\ \|\nabla^2 w\|_{L^q(L^q(U_j))} &\le C\big(\|u\|_{L^q(L^q(U_j))} + \|\nabla u\|_{L^q(L^q(U_j))}\big), \qquad (3.15)\\ \|p - M_j\|_{L^q(L^q(U_j))} &\le C\big(\|f\|_{L^q(L^q(U_j))} + \|u_t\|_{L^q(L^2(U_j))} + \|\nabla u\|_{L^q(L^q(U_j))}\big) \\ &\quad + \varepsilon\|u_t\|_{L^q(L^q(U_j))}\end{aligned}$$

with $C = C(q,T,\varepsilon,\alpha,\beta,K) > 0$. In fact, for the proof of $(3.15)_1$, choose $r \in [2,q)$ such that the embedding $W^{1,r}(U_j) \subset L^q(U_j)$ holds with an embedding constant $c = c(q,r,\alpha,\beta,K) > 0$ independent of j. Moreover,

$$\|w_t\|_{L^q(U_j)} \le c\|w_t\|_{W^{1,r}(U_j)} \le c\|u_t\|_{L^r(U_j)}$$

for a.a. $t \in (0,t)$. Then the interpolation inequality (2.15) proves $(3.15)_1$, and $(2.8)_2$ implies $(3.15)_2$. For the proof of $(3.15)_3$ we use (2.9), the embedding $W^{1,q'}(U_j) \subset L^{r'}(U_j)$ with an embedding constant $c = c(q, r, \alpha, \beta, K) > 0$ independent of j and apply the previous interpolation argument to u_t.

Applying the local estimate (2.12) to (3.14) and using (3.15) we get that

$$\begin{aligned}
&\|\varphi_j u_t\|_{L^q(L^q(U_j))} + \|\varphi_j u\|_{L^q(L^q(U_j))} + \|\varphi_j \nabla^2 u\|_{L^q(L^q(U_j))} + \|\varphi_j \nabla p\|_{L^q(L^q(U_j))} \\
&\quad \le C\big(\|f\|_{L^q(L^q(U_j))} + \|u\|_{L^q(W^{1,q}(U_j))} + \|u_t\|_{L^q(L^2(U_j))}\big) + \varepsilon \|u_t\|_{L^q(L^q(U_j))}
\end{aligned}$$

with $C = C(T, q, \varepsilon, \alpha, \beta, K) > 0$. Raising this inequality to its qth power, taking the sum over $j = 1, \dots, N$ and exploiting the crucial property of the number N_0, see (2.5), we are led to the estimate

$$\begin{aligned}
&\|u_t\|^q_{L^{q,q}} + \|u\|^q_{L^{q,q}} + \|\nabla^2 u\|^q_{L^{q,q}} + \|\nabla p\|^q_{L^{q,q}} \\
&= \int_0^T \int_\Omega \Big(\Big|\sum_j \varphi_j u_t\Big|^q + \Big|\sum_j \varphi_j u\Big|^q + \Big|\sum_j \varphi_j \nabla^2 u\Big|^q + \Big|\sum_j \varphi_j \nabla p\Big|^q \Big)\, dx\, dt \\
&\le \int_0^T \int_\Omega N_0^{\frac{q}{q'}} \Big(\sum_j |\varphi_j u_t|^q + \sum_j |\varphi_j u|^q + \sum_j |\varphi_j \nabla^2 u|^q + \sum_j |\varphi_j \nabla p|^q \Big)\, dx\, dt \\
&\le C N_0^{\frac{q}{q'}} \Big(\sum_j \|f\|^q_{L^q(0,T;L^q(U_j))} + \sum_j \|u\|^q_{L^q(0,T;W^{1,q}(U_j))} \\
&\qquad + \sum_j \|u_t\|^q_{L^q(0,T;L^2(U_j))} \Big) + \varepsilon N_0^{\frac{q}{q'}} \sum_j \|u_t\|^q_{L^q(0,T;L^q(U_j))}\,. \qquad (3.16)
\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, exploiting the absorption principle and again the property of the number N_0, we may simplify (3.16) to the estimate

$$\begin{aligned}
&\|u_t\|_{L^{q,q}} + \|u\|_{L^{q,q}} + \|\nabla^2 u\|_{L^{q,q}} + \|\nabla p\|_{L^{q,q}} \\
&\quad \le C\big(\|f\|_{L^{q,q}} + \|u\|_{L^{q,q}} + \|u_t\|_{L^{q,2}}\big) \qquad (3.17)
\end{aligned}$$

where $C = C(q, \alpha, \beta, K) > 0$; note that in order to deal with the sum of the terms $\|u_t\|_{L^q(0,T;L^2(U_j))}$ we also used the reverse Hölder inequality. Now, concerning the term $\|u\|_{L^{q,q}}$, we use (2.14) with $\varepsilon > 0$ sufficiently small and exploit the absorption principle. Finally we apply Lemma 2.5 (ii), i.e., we add the estimate (2.20) with $q = 2$ to (3.17), to prove the estimate (1.7) for bounded domains when $s = q > 2$, $u(0) = 0$. Since the operator norm of $\tilde{P}_q$ is bounded by a constant $c = c(q, \alpha, \beta, K) > 0$ we get (1.6) for $s = q$, $u(0) = 0$.

To prove (1.6) with $u_0 \in \mathcal{D}(\tilde{A}_q)$ we solve the system $\tilde{u}_t + \tilde{A}_q \tilde{u} = \tilde{f}$, $\tilde{u}(0) = 0$, with $\tilde{f} = f - \tilde{A}_q u_0$. Then $u(t) = \tilde{u}(t) + u_0$ yields the desired solution with $u_0 \in D(\tilde{A}_q)$. This proves Theorem 1.4 for bounded Ω and $s = q \geq 2$.

3.2.2 The case Ω bounded, $1 < s = q < 2$

In this case we consider for $f \in L^{q,q}_\sigma + L^{q,2}_\sigma = L^{q,q}_\sigma$ and the initial value $u_0 = 0$ the Stokes system $u_t + \tilde{A}_q u = f$, $u(0) = 0$. By Lemma 2.5 there exists a unique solution $u(t) = \mathcal{J}_{q,q} f(t) = \tilde{\mathcal{J}}_{q,q} f(t)$; here we used that $\tilde{P}_q = P_q$ and $\tilde{A}_q = A_q$. For the following duality argument we need that the space

$$C_0^\infty(C_{0,\sigma}^\infty) = \big\{ v \in C_0^\infty(\Omega \times (0,T));\ \operatorname{div} v(x,t) = 0 \ \ \forall t \in (0,T) \big\}$$

is dense in $L^{q',q'}_\sigma \cap L^{q',2}_\sigma = \big(L^{q,q}_\sigma + L^{q,2}_\sigma \big)'$. Then the identity

$$\langle u_t + \tilde{A}_q u, \tilde{A}_{q'} v \rangle = \langle u, (-\partial_t + \tilde{A}_{q'}) \tilde{A}_{q'} v \rangle = \langle \tilde{A}_q u, (-\partial_t + \tilde{A}_{q'}) v \rangle$$

holds for $u = \mathcal{J}_{q,q} f$ and every $v \in \tilde{A}_{q'}^{-1}\big(C_0^\infty(C_{0,\sigma}^\infty) \big)$, since $(\tilde{\mathcal{J}}'_{q',q'})' = \tilde{\mathcal{J}}_{q,q}$. Let $g = -v_t + \tilde{A}_{q'} v$. Then we obtain by (1.6) with $s = q$ replaced by $s' = q' \geq 2$ and u replaced by v that

$$\begin{aligned}
\|f\|_{L^{q,q}_\sigma + L^{q,2}_\sigma} &= \sup \left\{ \frac{|\langle u_t + \tilde{A}_q u, \tilde{A}_{q'} v \rangle_T|}{\|\tilde{A}_{q'} v\|_{L^{q',q'}_\sigma \cap L^{q',2}_\sigma}} ; 0 \neq v \in \tilde{A}_{q'}^{-1}\big(C_0^\infty(C_{0,\sigma}^\infty) \big) \right\} \\
&= \sup \left\{ \frac{|\langle \tilde{A}_q u, g \rangle_T|}{\|\tilde{A}_{q'} v\|_{L^{q',q'}_\sigma \cap L^{q',2}_\sigma}} ; 0 \neq v \in \tilde{A}_{q'}^{-1}\big(C_0^\infty(C_{0,\sigma}^\infty) \big) \right\} \\
&\geq \frac{1}{C} \|\tilde{A}_q u\|_{L^{q,q}_\sigma + L^{q,2}_\sigma}, \qquad (3.18)
\end{aligned}$$

where $C = C(T, q', \alpha, \beta, K) > 0$. Here we used that the estimate (1.6) with q, s replaced by q', s' also holds with u, u_0, f replaced by v, $v(T) = 0$, g due to the transformation in time in the proof of Lemma 2.5, and exploited the norm equivalence

$$\|\cdot\|_{L^q_\sigma+L^2_\sigma} \sim \sup\left\{\frac{|\langle\cdot,h\rangle|}{\|h\|_{L^{q'}_\sigma\cap L^2_\sigma}}; 0\neq h\in L^{q'}_\sigma\cap L^2_\sigma\right\}$$

with constants depending only on q and (α,β,K), cf. Theorem 1.2. Hence we obtain the estimate $\|\tilde{A}_q u\|_{L^{q,q}_\sigma+L^{q,2}_\sigma} \le C\|f\|_{L^{q,q}_\sigma+L^{q,2}_\sigma}$, and it follows

$$\|u_t\|_{L^{q,q}_\sigma+L^{q,2}_\sigma} + \|\tilde{A}_q u\|_{L^{q,q}_\sigma+L^{q,2}_\sigma} \le C\|f\|_{L^{q,q}_\sigma+L^{q,2}_\sigma}. \tag{3.19}$$

Since $\|u\|_{W^{2,q}+W^{2,2}} \le c\big(\|u\|_{L^q_\sigma+L^2_\sigma} + \|\tilde{A}_q u\|_{L^q_\sigma+L^2_\sigma}\big)$ with a constant $c>0$ depending only on q and (α,β,K), (3.19) and the identity $\nabla p = f - u_t + \Delta u$ lead to the estimate

$$\|u_t\|_{L^{q,q}_\sigma+L^{q,2}_\sigma} + \|u\|_{L^q(0,T;W^{2,q}+W^{2,2})} + \|\nabla p\|_{L^{q,q}+L^{q,2}} \le C\|f\|_{L^{q,q}_\sigma+L^{q,2}_\sigma} \tag{3.20}$$

with $C = C(q,\varepsilon,\alpha,\beta,K) > 0$.

Now the proof of Theorem 1.4 is complete for bounded domains in the case $s=q$, $u(0)=0$. The case $u_0\in\mathcal{D}(\tilde{A}_q)$ is treated as in 3.3.1.

3.2.3 The case Ω unbounded

Consider the sequence of bounded subdomains $\Omega_j \subseteq \Omega$, $j\in\mathbb{N}$, of uniform $C^{1,1}$-type as in (2.7), let $f\in\tilde{L}^{q,q}_\sigma$ and $f_j := \tilde{P}^{(j)}_q f|_{\Omega_j}$ where $\tilde{P}^{(j)}_q$ denotes the Helmholtz projection in $\tilde{L}^q(\Omega_j)$. Then consider the solution $(u_j,\nabla p_j)$ of the instationary Stokes equation

$$\partial_t u_j - \tilde{P}_q\Delta u_j = \partial_t u_j - \Delta u_j + \nabla p_j = f_j, \quad \nabla p_j = (I-\tilde{P}_q)\Delta u_j \quad \text{in} \quad \Omega_j\times(0,T)$$

with initial condition $u_j(0)=0$. From (1.6) with $s=q$ we obtain the estimate

$$\|\partial_t u_j\|_{\tilde{L}^{q,q}} + \|u_j\|_{L^q(0,T;\tilde{W}^{2,q}(\Omega_j))} + \|\nabla p_j\|_{\tilde{L}^{q,q}} \le C\|f\|_{\tilde{L}^{q,q}_\sigma} \tag{3.21}$$

on Ω_j with $C = C(T,q,\alpha,\beta,K)>0$ independent of $j\in\mathbb{N}$. Extending u_j and ∇p_j for a.a. $t\in(0,T)$ from Ω_j by 0 to vector fields on Ω we find, suppressing subsequences, weak limits

$$u = \text{w-}\lim_{j\to\infty} u_j \quad \text{in } \tilde{L}^{q,q}_\sigma(\Omega), \quad \nabla p = \text{w-}\lim_{j\to\infty}\nabla p_j \quad \text{in } \tilde{L}^{q,q}(\Omega)$$

satisfying $u \in L^q(0,T;\tilde{L}^q_\sigma(\Omega))$, $\partial_t u - \Delta u + \nabla p = \partial_t u + \tilde{A}_q u = f$ in $\Omega \times (0,T)$ and the *a priori* estimate (1.6) with $u_0 = 0$; it follows (1.7) for this case. Hence we solved the instationary Stokes equation $\partial_t u + \tilde{A}_q u = \partial_t u - \Delta u + \nabla p = f$, $u(0) = 0$, in $\Omega \times (0,T)$ and proved (1.6), (1.7).

Up to now we considered only the case when $s = q$, $u(0) = 0$. However, an abstract extrapolation argument shows that the validity of (1.6) with $s = q$ immediately extends to all $s \in (1,\infty)$, see [2, p. 191] and [5, (1.12)], where A has to be replaced by $-\tilde{A}_q - \delta I$ with $\delta > 0$ as in (1.4). The case $u(0) = u_0 \neq 0$ can be reduced to the case $u_0 = 0$ in the same way as before.

Finally, to prove uniqueness let $v \in L^s(0,T;\tilde{W}^{2,q})$ satisfy $\partial_t v + \tilde{A}_q v = 0$ and $v(0) = 0$. Given $f' \in \tilde{L}^{s',q'}$ let $u \in L^{s'}(0,T;\tilde{W}^{2,q'})$ be a solution of $-u_t + \tilde{A}_{q'} u = \tilde{P}_{q'} f'$, $u(T) = 0$. Then

$$0 = \langle v_t + \tilde{A}_q v, u\rangle_T = \langle v, (-\partial_t + \tilde{A}_{q'})u\rangle_T = \langle v, \tilde{P}_{q'} f'\rangle_T = \langle v, f'\rangle_T$$

for all $f' \in \tilde{L}^{s',q'}$; hence, $v = 0$.

Now Theorem 1.4 is proved. □

Acknowledgement The authors are grateful to the referee for several improvements of the paper, in particular in Lemma 3.1.

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