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Certain invariant subspace structure of $L^2(\mathbb{T}^2)$ II

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Abstract. Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. Considering the largest z-invariant (resp. w-invariant) subspace $\mathfrak{F}_z$ (resp. $\mathfrak{F}_w$) in the wandering subspace $\mathfrak{M} \ominus zw\mathfrak{M}$ of $\mathfrak{M}$ with respect to the shift operator zw. If $\mathfrak{F}_w \neq \{0\}$ and $\mathfrak{F}_z \neq \{0\}$, then we consider the certain form of invariant subspaces $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$. Furthermore, we study certain classes of invariant subspaces of $L^2(\mathbb{T}^2)$.

Key words: invariant subspace, wandering subspace.

1. Introduction and preliminaries

Let $\mathbb{T}^2$ be the torus that is the cartesian product of 2 unit circles in $\mathbb{C}$. Let $L^2(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$ be the usual Lebesgue and Hardy space on the torus $\mathbb{T}^2$, respectively. A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is said to be invariant if $z\mathfrak{M} \subset \mathfrak{M}$ and $w\mathfrak{M} \subset \mathfrak{M}$. As is well known, the structure of invariant subspaces is much more complicated. In general, the invariant subspaces of $L^2(\mathbb{T}^2)$ are not necessarily of the form $\phi H^2(\mathbb{T}^2)$ with some unimodular function ϕ. The structure of Beurling-type invariant subspaces has been studied, and some necessary and sufficient conditions for invariant subspaces to be Beurling-type have been given (cf. [1, 2, 5], etc). Further, many authors had attempted to study the form of invariant subspaces of $L^2(\mathbb{T}^2)$ (cf. [4, 6, 7], etc).

In [4], we studied the structure of an invariant subspace $\mathfrak{M}$ as a zw-invariant subspace. We gave an alternative approach of Beuring-type invariant subspaces and a certain class of invariant subspace which contains the class of invariant subspaces of the form $\phi H_0^2(\mathbb{T}^2)$, where $H_0^2(\mathbb{T}^2) = \{f \in H^2(\mathbb{T}^2) : f(0, 0) = 0\}$ and ϕ is a unimodular function in $L^\infty(\mathbb{T}^2)$.

For $(m, n) \in \mathbb{Z}^2$ and $f \in L^2(\mathbb{T}^2)$, the Fourier coefficient of f is defined by

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$$\hat{f}(m,\,n) = \int_{\mathbb{T}^2} f(z,\,w)\overline{z}^m\overline{w}^n d\mu,$$

where μ is the Haar measure on $\mathbb{T}^2$. Let $\operatorname{supp}\hat{f} = \{(m,\,n) \in \mathbb{Z}^2 \colon \hat{f}(m,\,n) \neq 0\}$. For a subset A of $L^2(\mathbb{T}^2)$, we denote the closed subspace $[A]$ generated by A in $L^2(\mathbb{T}^2)$. We define several subspaces of $L^2(\mathbb{T}^2)$ which will be used later.

(i) $H^2(z)$ or $H^2(w)$ is the set of f (in $L^2(\mathbb{T}^2)$) with Fourier series:

$$\sum_{m=0}^{\infty} a_{m0}z^m \text{ or } \sum_{n=0}^{\infty} a_{0n}w^n,$$

respectively.

(ii) H^2_z or H^2_w is the set of f with Fourier series:

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} a_{mn}z^m w^n \text{ or } \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} a_{mn}z^m w^n,$$

respectively.

(iii) L^2_z or L^2_w is the set of f with Fourier series:

$$\sum_{m=-\infty}^{\infty} a_{m0}z^m \text{ or } \sum_{n=-\infty}^{\infty} a_{0n}w^n,$$

respectively.

Let $\mathfrak{M}$ be a zw-invariant subspace of $L^2(\mathbb{T}^2)$. Put $\mathfrak{F} = \mathfrak{M} \ominus zw\mathfrak{M}$, $\mathfrak{S}_z = \mathfrak{M} \ominus z\mathfrak{M}$ and $\mathfrak{S}_w = \mathfrak{M} \ominus w\mathfrak{M}$, respectively. Let $\mathfrak{F}_z$ (resp. $\mathfrak{F}_w$) be the largest z-invariant (resp. w-invariant) subspace of $\mathfrak{F}$. In § 2, we characterize invariant subspaces of $L^2(\mathbb{T}^2)$, where $\mathfrak{F}_z \neq 0$ and $\mathfrak{F}_w \neq 0$. Then there exist two unimodular functions ϕ_z and ϕ_w in $L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}_z = \phi_z H^2(z)$ and $\mathfrak{F}_w = \phi_w H^2(w)$. Putting $\varphi = \overline{\phi_w}\phi_z$, we consider the invariant subspace

$$\mathfrak{M}_\varphi = [H^2(\mathbb{T}^2) + \varphi H^2(\mathbb{T}^2)].$$

Then we remark that $\mathfrak{M}$ is of the form $\phi_w(\mathfrak{M}_\varphi \oplus N)$, where $N = \overline{\phi_w}\mathfrak{M} \ominus \mathfrak{M}_\varphi$ (see Theorem 2.8). In § 3, let φ be a unimodular function of $L^\infty(\mathbb{T}^2)$ such that $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$. Then we characterize the invariant subspace $\mathfrak{M}_\varphi$. Further, we consider the sufficient condition that $\mathfrak{F}_w = H^2(w)$ and $\mathfrak{F}_z = \varphi H^2(z)$ with respect to $\mathfrak{M} = \mathfrak{M}_\varphi$. In § 4, as a generalization of [4], we consider the invariant subspace

$$\mathfrak{M}_\alpha^{(m,n)} = [H^2(\mathbb{T}^2) + \psi_\alpha^{(m,n)} H^2(\mathbb{T}^2)]$$

(see the definition of $\psi_\alpha^{(m,n)}$ in § 4). Then we consider the necessary and suffcient condition that an invariant subspace $\mathfrak{M}$ is of the form $\mathfrak{M}_\alpha^{(m,n)}$ for some $\alpha \in \mathbb{D}$ where $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ (see Theorem 4.2).

2. Invariant subspaces as zw-invariant subspaces

Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. Since $z^n\mathfrak{M} \supset z^{n+1}\mathfrak{M}$ (resp. $w^n\mathfrak{M} \supset w^{n+1}\mathfrak{M}$) for $n \in \mathbb{Z}_+$, $\bigcap_{k=1}^\infty z^k\mathfrak{M}$ (resp. $\bigcap_{k=1}^\infty w^k\mathfrak{M}$) is also an invariant subspace. If $\bigcap_{k=1}^\infty z^k\mathfrak{M} = \{0\}$ (resp. $\bigcap_{k=1}^\infty w^k\mathfrak{M} = \{0\}$), we say that $\mathfrak{M}$ is z-pure (resp. w-pure). If $z\mathfrak{M} = \mathfrak{M}$ (resp. $w\mathfrak{M} = \mathfrak{M}$), we say that $\mathfrak{M}$ is z-reducing (resp. w-reducing). The structure of z-reducing (resp. w-reducing) invariant subspaces has been characterized in [7].

Since $\mathfrak{M}$ is an invariant subspace, $\mathfrak{M}$ is also a zw-invariant subspace and $(zw)^n\mathfrak{M} \supset (zw)^{n+1}\mathfrak{M}$ for $n \in \mathbb{Z}_+$. If $\bigcap_{k=1}^\infty (zw)^k\mathfrak{M} = \{0\}$, then we say that $\mathfrak{M}$ is zw-pure. If $zw\mathfrak{M} = \mathfrak{M}$, we say that $\mathfrak{M}$ is zw-reducing. First, we have the following proposition.

Proposition 2.1 *Let $\mathfrak{M}$ be an invariant subspace of $L^2(\mathbb{T}^2)$. Then*
(i) *If $\mathfrak{M}$ is either z-pure or w-pure, then $\mathfrak{M}$ is zw-pure.*
(ii) *$\mathfrak{M}$ is zw-reducing if and only if $\mathfrak{M}$ is z-reducing and w-reducing.*

If $\mathfrak{M}$ is zw-reducing, then by [6] and [7] the form of $\mathfrak{M}$ is well-known. Throughout this note, we assume without loss of generality that $\mathfrak{M}$ is zw-pure. Put $\mathfrak{F} = \mathfrak{M} \ominus zw\mathfrak{M}$, $\mathfrak{S}_z = \mathfrak{M} \ominus z\mathfrak{M}$ and $\mathfrak{S}_w = \mathfrak{M} \ominus w\mathfrak{M}$, respectively. Then we easily have

Proposition 2.2 *Keep the notations and assumptions as above. Then*
(i) $\mathfrak{M} = \sum_{k=0}^\infty \oplus z^k\mathfrak{S}_z \oplus \bigcap_{k=1}^\infty z^k\mathfrak{M} = \sum_{k=0}^\infty \oplus w^k\mathfrak{S}_w \oplus \bigcap_{k=1}^\infty w^k\mathfrak{M} = \sum_{k=0}^\infty \oplus (zw)^k\mathfrak{F}$.
(ii) $\mathfrak{F} = \mathfrak{S}_z \oplus z\mathfrak{S}_w = \mathfrak{S}_w \oplus w\mathfrak{S}_z$.

Let $\mathfrak{F}_z$ (resp. $\mathfrak{F}_w$) be the largest z-invariant (resp. w-invariant) subspace in $\mathfrak{F}$. It is clear that $\mathfrak{F}_z = \bigcap_{k=0}^\infty \overline{z}^k\mathfrak{F}$, $\mathfrak{F}_w = \bigcap_{k=0}^\infty \overline{w}^k\mathfrak{F}$, $\mathfrak{F}_z \subset \mathfrak{S}_w$ and $\mathfrak{F}_w \subset \mathfrak{S}_z$.

Proposition 2.3 *Keep the notations and the assumptions as above. Then $\mathfrak{F}_z$ (resp. $\mathfrak{F}_w$) is the largest z-invariant (resp. w-invariant) subspace in $\mathfrak{S}_w$ (resp. $\mathfrak{S}_z$).*

Proof. Since $\mathfrak{S}_z \subset \mathfrak{F}$, we have $\bigcap_{k=0}^{\infty} \overline{w}^k \mathfrak{S}_z \subset \bigcap_{k=0}^{\infty} \overline{w}^k \mathfrak{F} = \mathfrak{F}_w$. Conversely, for all $f \in \mathfrak{F}_w$ there exists $f_n \in \mathfrak{F}$ such that $f = \overline{w}^n f_n$. Then for all $g \in \mathfrak{M}$, we have

$$\langle w^n f, zg \rangle = \langle w^{n+1} f, zwg \rangle = \langle f_{n+1}, zwg \rangle = 0.$$

Thus $w^n f \in \mathfrak{S}_z$ and so $f \in \overline{w}^n \mathfrak{S}_z$. This implies that $\bigcap_{k=0}^{\infty} \overline{w}^k \mathfrak{S}_z = \mathfrak{F}_w$. This completes the proof. □

Proposition 2.4 (cf. [4, Proposition 2]) *Let $\mathfrak{M}$ be a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$. Then:*

(i) *$z\mathfrak{F}_z \subsetneq \mathfrak{F}_z$ if and only if there exists a unimodular function $\phi_z \in L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}_z = \phi_z H^2(z)$.*

(ii) *$\mathfrak{F}_z = z\mathfrak{F}_z \neq \{0\}$ if and only if $\mathfrak{M} = \chi_E q H_z^2$, where q is a unimodular function of $L^\infty(\mathbb{T}^2)$, and χ_E is the characteristic function of a Borel subset E of $\mathbb{T}^2$ with $\chi_E\ (\neq 0) \in L_z^2$. In this case, $\mathfrak{F} = \mathfrak{F}_z$ and $\mathfrak{F}_w = \{0\}$.*

Similarly, we have the following result about $\mathfrak{F}_w$.

Proposition 2.5 (cf. [4, Proposition 3]) *Let $\mathfrak{M}$ be a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$. Then:*

(i) *$w\mathfrak{F}_w \subsetneq \mathfrak{F}_w$ if and only if there exists a unimodular function $\phi_w \in L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}_w = \phi_w H^2(w)$.*

(ii) *$\mathfrak{F}_w = w\mathfrak{F}_w \neq \{0\}$ if and only if $\mathfrak{M} = \chi_E q H_w^2$, where q is a unimodular function of $L^\infty(\mathbb{T}^2)$, and χ_E is the characteristic function of a Borel subset E of $\mathbb{T}^2$ with $\chi_E\ (\neq 0) \in L_w^2$. In this case, $\mathfrak{F} = \mathfrak{F}_w$ and $\mathfrak{F}_z = \{0\}$.*

Throughout this paper, we suppose that $\mathfrak{F}_z \neq \{0\}$ and $\mathfrak{F}_w \neq \{0\}$. Then we have $z\mathfrak{F}_z \subsetneq \mathfrak{F}_z$ and $w\mathfrak{F}_w \subsetneq \mathfrak{F}_w$. Otherwise, for example, assume that $\mathfrak{F}_z = z\mathfrak{F}_z \neq \{0\}$. Then, by Proposition 2.4(ii), we have $\mathfrak{M} = \chi_E q H_z^2$ and $\mathfrak{F}_w = \{0\}$. This is a contradiction. Thus there exist two unimodular functions ϕ_z and ϕ_w in $L^\infty(\mathbb{T}^2)$ such that $\mathfrak{F}_z = \phi_z H^2(z)$ and $\mathfrak{F}_w = \phi_w H^2(w)$. Put $\widetilde{\mathfrak{M}} = \overline{\phi}_w \mathfrak{M}$, then $\widetilde{\mathfrak{M}}$ is also an invariant subspace of $L^2(\mathbb{T}^2)$. Let $\widetilde{\mathfrak{F}} = \widetilde{\mathfrak{M}} \ominus zw\widetilde{\mathfrak{M}}$. Let $(\widetilde{\mathfrak{F}})_z$ (resp. $(\widetilde{\mathfrak{F}})_w$) be the largest z-invariant (resp. w-invariant) subspace of $\widetilde{\mathfrak{F}}$. Then we have

Proposition 2.6 *Keep the notations and assumptions as above. Then we have*

(i) $\widetilde{\mathfrak{F}} = \overline{\phi}_w \mathfrak{F}$.

(ii) $(\widetilde{\mathfrak{F}})_z = \overline{\phi}_w \phi_z H^2(z)$ *and* $(\widetilde{\mathfrak{F}})_w = H^2(w)$.
(iii) $H^2(\mathbb{T}^2) \subset \widetilde{\mathfrak{M}} \subset H^2_w$.

Proof. (i) and (ii) are clear.
(iii) Since $(\widetilde{\mathfrak{F}})_w = H^2(w)$, we have $H^2(\mathbb{T}^2) \subset \widetilde{\mathfrak{M}}$. Since $\widetilde{\mathfrak{M}} = \sum_{n=0}^{\infty} \oplus (zw)^n \widetilde{\mathfrak{F}}$ and $(\widetilde{\mathfrak{F}})_w = H^2(w) \subset \widetilde{\mathfrak{F}}$, we have

$$\widetilde{\mathfrak{M}} \perp \sum_{n=-\infty}^{-1} \oplus (zw)^n H^2(w).$$

If there exists an element f in $\widetilde{\mathfrak{M}}$ such that $\hat{f}(m, n) \neq 0$ for $m < n < 0$, then $\overline{w}^n f \in \widetilde{\mathfrak{M}}$. Since $(\overline{w}^n f)(m, 0) = \hat{f}(m, n) \neq 0$, $\overline{w}^n f$ is not orthogonal to

$$\sum_{n=-\infty}^{-1} \oplus (zw)^n H^2(w).$$

This is a contradiction. Therefore $\widetilde{\mathfrak{M}} \subset H^2_w$. This completes the proof. □

We now put $\varphi = \overline{\phi}_w \phi_z$ and $\mathfrak{M}_\varphi = [H^2(\mathbb{T}^2) + \varphi H^2(\mathbb{T}^2)]$. Then $\mathfrak{M}_\varphi$ is a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$ such that $\mathfrak{M}_\varphi \subset \widetilde{\mathfrak{M}}$. Put $\mathfrak{F}^\varphi = \mathfrak{M}_\varphi \ominus zw\mathfrak{M}_\varphi$, $\mathfrak{S}^\varphi_z = \mathfrak{M}_\varphi \ominus z\mathfrak{M}_\varphi$ and $\mathfrak{S}^\varphi_w = \mathfrak{M}_\varphi \ominus w\mathfrak{M}_\varphi$, respectively. Let $\mathfrak{F}^\varphi_z$ (resp. $\mathfrak{F}^\varphi_w$) be the largest z-invariant (resp. w-invariant) subspace of $\mathfrak{F}^\varphi$.

Proposition 2.7 *Keep the notations and assumptions as above. Then*
(i) $\mathfrak{F}^\varphi_z = \varphi H^2(z)$ *and* $\mathfrak{F}^\varphi_w = H^2(w)$.
(ii) φ *is a unimodular function of* $L^\infty(\mathbb{T}^2)$ *such that* $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$.

Proof. By [4, Proposition 4], we have (i).
(ii) Since $\varphi \in \widetilde{\mathfrak{M}} \subset H^2_w$, we have $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times \mathbb{Z}$. Since $\mathfrak{F}^\varphi_z \subset \mathfrak{S}^\varphi_w$, we have $\varphi \perp wH^2(\mathbb{T}^2)$. Therefore, $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$. This completes the proof. □

Then we have the following

Theorem 2.8 *Let* $\mathfrak{M}$ *be a* zw*-pure invariant subspace of* $L^2(\mathbb{T}^2)$ *such that* $\mathfrak{F}_w = \phi_w H^2(w)$ *and* $\mathfrak{F}_z = \phi_z H^2(z)$*, where* ϕ_w *and* ϕ_z *are unimodular functions of* $L^2(\mathbb{T}^2)$*. Put* $\varphi = \overline{\phi}_w \phi_z$ *and* $N = \widetilde{\mathfrak{M}} \ominus \mathfrak{M}_\varphi$*. Then* $\mathfrak{M}$ *is of the* *form*

$$\mathfrak{M} = \phi_w(\mathfrak{M}_\varphi \oplus N),$$

where φ *is a unimodular function of* $L^\infty(\mathbb{T}^2)$ *such that* $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$.

Example 2.9 For m, $n \in \mathbb{Z}_+$, we consider an invariant subspace

$$H^2_{m,n}(\mathbb{T}^2) = [z^m H^2(\mathbb{T}^2) + w^n H^2(\mathbb{T}^2)].$$

Let $\mathfrak{M}$ be an invariant subspace such that $\mathfrak{F}_z = z^m H^2(z)$ and $\mathfrak{F}_w = w^n H^2(w)$. Then it is clear that $\mathfrak{M} \supset H^2_{m,n}(\mathbb{T}^2)$. Put $N = \overline{w^n}(\mathfrak{M} \ominus H^2_{m,n}(\mathbb{T}^2))$. Then

$$\mathfrak{M} = H^2_{m,n}(\mathbb{T}^2) \oplus w^n N.$$

If $m = 1$ or $n = 1$, then $N = 0$. If $m = n = 2$, then we easily show that N is one of the following forms:

(i) $N = \{0\}$;
(ii) $N = [z\overline{w}]$; and
(iii) $N = [z\overline{w},\ \alpha z\overline{w}^2 + \beta\overline{w}]$, where α and β are non-zero complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

3. Invariant subspace $\mathfrak{M}_\varphi$

Let φ be a unimodular function of $L^\infty(\mathbb{T}^2)$ such that $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$. Put $\mathfrak{M}_\varphi = [H^2(\mathbb{T}^2) + \varphi H^2(\mathbb{T}^2)]$. Then $\mathfrak{M}_\varphi$ is a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$ such that

$$H^2(\mathbb{T}^2) \subset \mathfrak{M}_\varphi \subset H^2_w.$$

Put $\mathfrak{F}^\varphi = \mathfrak{M}_\varphi \ominus zw\mathfrak{M}_\varphi$, $\mathfrak{S}^\varphi_z = \mathfrak{M}_\varphi \ominus z\mathfrak{M}_\varphi$ and $\mathfrak{S}^\varphi_w = \mathfrak{M}_\varphi \ominus w\mathfrak{M}_\varphi$, respectively. Further, let $\mathfrak{F}^\varphi_z$ (resp. $\mathfrak{F}^\varphi_w$) be the largest z-invariant (resp. w-invariant) subspace of $\mathfrak{F}^\varphi$. If $\varphi \in H^2(z)$, then $\mathfrak{M}_\varphi = H^2(\mathbb{T}^2)$. Thus we may suppose that $\varphi \notin H^2(z)$.

In this section, we consider the conditions that $\mathfrak{F}^\varphi_z = \varphi H^2(z)$ and $\mathfrak{F}^\varphi_w = H^2(w)$.

Proposition 3.1 *Let* φ *be a unimodular function of* $L^\infty(\mathbb{T}^2)$ *such that* $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$ *and* $\varphi \notin H^2(z)$. *Then* $\varphi H^2(z) \subset \mathfrak{F}^\varphi_z \subset \varphi L^2_z$ *and* $H^2(w) \subset \mathfrak{F}^\varphi_w \subset L^2_w$.

Proof. Take any $f \in H^2(w)$. Then, for every $g \in H^2(\mathbb{T}^2)$,

$$\langle f,\, zwg\rangle = 0$$

and

$$\langle f,\, zw\varphi g\rangle = 0.$$

This implies that $H^2(w) \subset \mathfrak{F}^\varphi$. Since $H^2(w)$ is w-invariant, $H^2(w) \subset \mathfrak{F}^\varphi_w$. On the other hand, let $f \in H^2(z)$. Then for every $g \in H^2(\mathbb{T}^2)$,

$$\langle \varphi f,\, zwg\rangle = 0 \quad \text{and} \quad \langle \varphi f,\, zw\varphi g\rangle = \langle f,\, zwg\rangle = 0.$$

This implies that $H^2(z) \subset \mathfrak{F}^\varphi$, and so $\varphi H^2(z) \subset \mathfrak{F}^\varphi_z$.

Take any $f \in \mathfrak{F}^\varphi_w$. Since $\mathfrak{F}^\varphi_w = \bigcap_{n=0}^\infty \overline{w}^n \mathfrak{F}^\varphi$, we have $w^n f \in \mathfrak{F}^\varphi$ for any $n \geq 0$. This implies that $w^n f \perp zw\mathfrak{M}_\varphi$. In particular, $w^n f \perp zwH^2(\mathbb{T}^2)$. For any $n,\, k,\, l \geq 0$, we have

$$\langle f,\, z^{k+1}w^{l+1-n}\rangle = \langle w^n f,\, zwz^k w^l\rangle = 0.$$

Since $f \in \mathfrak{M}_\varphi \subset H^2_w$ by Proposition 2.6, $f \in L^2_w$. Thus we have $\mathfrak{F}^\varphi_w \subset L^2_w$.

Similarly, take any $f \in \mathfrak{F}^\varphi_z$. Since $z^n f \in \mathfrak{F}^\varphi$ for any $n \geq 0$, we have $z^n f \perp zw\varphi H^2(\mathbb{T}^2)$. For any $m,\, k,\, l \geq 0$, we have

$$\langle \overline{\varphi} f,\, z^{k+1-m}w^{l+1}\rangle = \langle z^m f,\, zw\varphi z^k w^l\rangle = 0.$$

Since $\overline{\varphi} f \in \overline{\varphi}\mathfrak{M}_\varphi = [\overline{\varphi} H^2(\mathbb{T}^2) + H^2(\mathbb{T}^2)] \subset H^2_z$, we have $\overline{\varphi} f \in L^2_z$. Thus $f \in \varphi L^2_z$ and so $\mathfrak{F}^\varphi_z \subset \varphi L^2_z$. This completes the proof. □

Theorem 3.2 *Keep the notations and assumptions as above. Then*
(i) $\mathfrak{F}^\varphi_w = H^2(w)$ *if and only if* $\mathfrak{M}_\varphi \cap \overline{wH^2(w)} = \{0\}$.
(ii) $\mathfrak{F}^\varphi_z = \varphi H^2(z)$ *if and only if* $\mathfrak{M}_\varphi \cap \overline{\varphi z H^2(z)} = \{0\}$.

Proof. (i) ($\Leftarrow$) By Proposition 3.1, we have

$$\mathfrak{F}^\varphi_w \ominus H^2(w) = \mathfrak{F}^\varphi_w \cap \overline{wH^2(w)} \subset \mathfrak{M}_\varphi \cap \overline{wH^2(w)} = \{0\}.$$

Thus $\mathfrak{F}^\varphi_w = H^2(w)$.
($\Rightarrow$) Suppose that $\mathfrak{M}_\varphi \cap \overline{wH^2(w)} \neq \{0\}$. Then there exists a nonzero element f in $\mathfrak{M}_\varphi \cap \overline{wH^2(w)}$. For all $n,\, k,\, l \geq 0$,

$$\langle w^n f,\, zwz^k w^l\rangle = \langle f,\, z^{k+1}w^{l-n+1}\rangle = 0$$

and

$$\langle w^n f,\, zw\varphi z^k w^l\rangle = \langle f,\, \varphi z^{k+1} w^{l-n+1}\rangle = 0.$$

Thus we have $w^n f \in \mathfrak{F}^{\varphi}$ for every $n \geq 0$, that is, $f \in \mathfrak{F}^{\varphi}_w$. Therefore $H^2(w) \subsetneq \mathfrak{F}^{\varphi}_w$. This is a contradiction. Similarly we have (ii). This completes the proof. □

Corollary 3.3 *Keep the notations and assumptions as above. Then*
(i) *If* $\mathfrak{M}_\varphi \perp \overline{wH^2(w)}$, *then* $\mathfrak{F}^{\varphi}_w = H^2(w)$.
(ii) *If* $\mathfrak{M}_\varphi \perp \overline{\varphi z H^2(z)}$, *then* $\mathfrak{F}^{\varphi}_z = \varphi H^2(z)$.

Corollary 3.4 *Keep the notations and assumptions as above. Then*
(i) $1 \in \mathfrak{S}^{\varphi}_w$ *if and only if* $\hat{\varphi}(0,\, -n) = 0$ *for all* $n \geq 1$. *In this case,* $\mathfrak{F}^{\varphi}_w = H^2(w)$.
(ii) $\varphi \in \mathfrak{S}^{\varphi}_z$ *if and only if* $\hat{\varphi}(m,\, 0) = 0$ *for all* $m \geq 1$. *In this case,* $\mathfrak{F}^{\varphi}_z = \varphi H^2(z)$.
(iii) *If* $\hat{\varphi}(m,\, 0) = \hat{\varphi}(0,\, -n) = 0$ *for all* $m,\, n \geq 1$, *then* $\mathfrak{F}^{\varphi}_z = \varphi H^2(z)$ *and* $\mathfrak{F}^{\varphi}_w = H^2(w)$.

4. Certain classes of invariant subspaces

Keep the notations as in § 2. Suppose that $\mathfrak{F}_z \neq \{0\}$ and $\mathfrak{F}_w \neq \{0\}$. In general, we have $\mathfrak{F}_z + \mathfrak{F}_w \subset [\mathfrak{S}_z + \mathfrak{S}_w] \subset \mathfrak{F}$. In [4], we studied invariant subspace structure with the property $\mathfrak{F}_z + \mathfrak{F}_w = [\mathfrak{S}_z + \mathfrak{S}_w]$.

Let $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$. For any $\alpha \in \mathbb{D}$ and $m,\, n \in \mathbb{N}$, we define a function $\psi^{(m,n)}_\alpha$ by

$$\psi^{(m,n)}_\alpha(z,\, w) = \frac{z^m \overline{w}^n - \alpha}{1 - \overline{\alpha} z^m \overline{w}^n}.$$

Then $\psi^{(m,n)}_\alpha$ is a unimodular function in $L^\infty(\mathbb{T}^2)$ with $\widehat{\psi^{(m,n)}_\alpha}(k,\, l) = 0$ for every $(k,\, l) \in \mathbb{Z}_+ \times (-\mathbb{Z}_+)$. Then we define an invariant subspace $\mathfrak{M}^{(m,n)}_\alpha$ of $L^2(\mathbb{T})$ by

$$\mathfrak{M}^{(m,n)}_\alpha = [H^2(\mathbb{T}^2) + \psi^{(m,n)}_\alpha H^2(\mathbb{T}^2)].$$

At first we have the following

Theorem 4.1 *If* $\mathfrak{M} = \mathfrak{M}^{(m,n)}_\alpha$, *then* $\mathfrak{F}_w = H^2(w)$, $\mathfrak{F}_z = \psi^{(m,n)}_\alpha H^2(z)$,

$$\mathfrak{S}_w = \psi^{(m,n)}_\alpha H^2(z) + [1,\, z,\, \ldots,\, z^{m-1}]$$

and

$$\mathfrak{S}_z = H^2(w) + [\psi_\alpha^{(m,n)},\, w\psi_\alpha^{(m,n)},\, \ldots,\, w^{n-1}\psi_\alpha^{(m,n)}].$$

Therefore we have

$$\begin{aligned}\mathfrak{F} &= \mathfrak{F}_z + \mathfrak{F}_w + [z,\, \ldots,\, z^m] + [w\psi_\alpha^{(m,n)},\, \ldots,\, w^{n-1}\psi_\alpha^{(m,n)}] \\ &= \mathfrak{F}_z + \mathfrak{F}_w + [z,\, \ldots,\, z^{m-1}] + [w\psi_\alpha^{(m,n)},\, \ldots,\, w^{n}\psi_\alpha^{(m,n)}].\end{aligned}$$

Proof. By Corollary 3.4, we have $\mathfrak{F}_w = H^2(w)$ and $\mathfrak{F}_z = \psi_\alpha^{(m,n)} H^2(z)$. We show that $\mathfrak{S}_z = H^2(w) + [\psi_\alpha^{(m,n)},\, \ldots,\, w^{n-1}\psi_\alpha^{(m,n)}]$. For $0 \le j \le n-1$ and for any $f,\, g \in H^2(\mathbb{T}^2)$, we have

$$\begin{aligned}&\langle w^j\psi_\alpha^{(m,n)},\, z(f + \psi_\alpha^{(m,n)} g)\rangle \\ &= \langle \psi_\alpha^{(m,n)},\, w^{-j} z f\rangle + \langle w^j,\, zg\rangle = 0.\end{aligned}$$

Since $H^2(w) = \mathfrak{F}_w \subset \mathfrak{S}_z$, we have $H^2(w) + [\psi_\alpha^{(m,n)},\, \ldots,\, w^{n-1}\psi_\alpha^{(m,n)}] \subset \mathfrak{S}_z$. We put $\mathfrak{N} = (H^2(w) + [\psi_\alpha^{(m,n)},\, \ldots,\, w^{n-1}\psi_\alpha^{(m,n)}]) \oplus z\mathfrak{M}$. Then it is enough to show that $\mathfrak{N} = \mathfrak{M}$. Since $H^2(\mathbb{T}^2) + z\psi_\alpha^{(m,n)} H^2(\mathbb{T}^2) \subset \mathfrak{N}$, we only need to show that $w^n\psi_\alpha^{(m,n)} H^2(w) \subset \mathfrak{N}$. In fact,

$$\begin{aligned}w^n\psi_\alpha^{(m,n)} &= w^n\Big(\frac{z^m\overline{w}^n - \alpha}{1 - \overline{\alpha} z^m \overline{w}^n}\Big) \\ &= w^n(z^m\overline{w}^n - \alpha)\Big(1 + \frac{\overline{\alpha} z^m\overline{w}^n}{1 - \overline{\alpha} z^m \overline{w}^n}\Big) \\ &= z^m - \alpha w^n + \overline{\alpha} z^m \psi_\alpha^{(m,n)}.\end{aligned}$$

Thus we have $w^n\psi_\alpha^{(m,n)} \in \mathfrak{N}$. For every $k \ge 1$, we have

$$w^{n+k}\psi_\alpha^{(m,n)} = z^m w^k - \alpha w^{n+k} + \overline{\alpha} z^m w^k \psi_\alpha^{(m,n)} \in \mathfrak{N}.$$

This implies that $\mathfrak{N} = \mathfrak{M}$.

We next show that $\mathfrak{S}_w = \psi_\alpha^{(m,n)} H^2(z) + [1,\, z,\, \ldots,\, z^{m-1}]$. For $0 \le j \le m-1$ and for every $f,\, g \in H^2(\mathbb{T}^2)$, we have

$$\begin{aligned}&\langle z^j,\, w(f + \psi_\alpha^{(m,n)} g)\rangle \\ &= \langle z^j,\, wf\rangle + \langle z^j,\, w\psi_\alpha^{(m,n)} g\rangle \\ &= \langle z^j,\, wf\rangle + \overline{\langle \psi_\alpha^{(m,n)}},\, z^{-j} wg\rangle = 0.\end{aligned}$$

Since $\psi_\alpha^{(m,n)} H^2(z) = \mathfrak{F}_z \subset \mathfrak{S}_w$, we have

$$\psi_\alpha^{(m,n)} H^2(z) + [1,\, z,\, \ldots,\, z^{m-1}] \subset \mathfrak{S}_w.$$

We put $\mathfrak{N}_1 = (\psi_\alpha^{(m,n)} H^2(z) + [1, z, \ldots, z^{m-1}]) \oplus w\mathfrak{M}$. We want to prove that $\mathfrak{N}_1 = \mathfrak{M}$. Since $\psi_\alpha^{(m,n)} H^2(\mathbb{T}^2) + wH^2(\mathbb{T}^2) \subset \mathfrak{N}_1$, we only show that $z^m H^2(z) \subset \mathfrak{N}_1$. In fact, $z^m = w^n \psi_\alpha^{(m,n)} + \alpha w^n - \overline{\alpha} z^m \psi_\alpha^{(m,n)} \in \mathfrak{N}_1$. Further, for every $k \geq 1$,

$$z^{m+k} = w^n z^k \psi_\alpha^{(m,n)} + \alpha w^n z^k - \overline{\alpha} z^{m+k} \psi_\alpha^{(m,n)} \in \mathfrak{N}_1.$$

This implies that $\mathfrak{N}_1 = \mathfrak{M}$. The remainder of this theorem is proved from $\mathfrak{F} = \mathfrak{S}_z \oplus z\mathfrak{S}_w = \mathfrak{S}_w \oplus w\mathfrak{S}_z$. This proof is complete. □

We next show the converse of Theorem 4.1.

Theorem 4.2 *Let $\mathfrak{M}$ be a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$. Let $m, n \geq 1$. Then $\mathfrak{M} = \mathfrak{M}_\alpha^{(m,n)}$ for some $\alpha \in \mathbb{D}$ if and only if $\mathfrak{F}_w = H^2(w)$, $\mathfrak{F}_z = \varphi H^2(z)$, $\mathfrak{S}_w = \varphi H^2(z) + [1, z, \ldots, z^{m-1}]$ and $\mathfrak{S}_z = H^2(w) + [\varphi, w\varphi, \ldots, w^{n-1}\varphi]$ for some unimodular function φ in $L^\infty(\mathbb{T}^2)$ such that* $\operatorname{supp} \hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$.

Proof. If $\mathfrak{M} = \mathfrak{M}_\alpha^{(m,n)}$, by Theorem 4.1, we have the results. Thus we prove the converse. To do it, we only prove that $\varphi = c\psi_\alpha^{(m,n)}$ for some $c \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. By the assumption, $[1, \varphi] \subset \mathfrak{S}_z \cap \mathfrak{S}_w$. Thus

$$\begin{aligned} &\langle \varphi, z^i w^j \rangle = 0 \quad (i \geq 1,\ j \geq 0 \text{ or } i \geq 0,\ j \geq 1), \\ &\langle \varphi, z^i w^j \rangle = \langle \overline{w}^j \varphi, z^i \rangle = 0 \quad (1 \leq i \leq m-1,\ j \leq -1) \end{aligned}$$

and

$$\langle \varphi, z^i w^j \rangle = \langle \overline{w}^j \varphi, z^i \rangle = 0 \quad (i \geq 0,\ -(n-1) \leq j \leq -1).$$

Put $\hat{\varphi}(0, 0) = a_{00}$ and $\varphi_0 = \varphi - a_{00}$, respectively. Put $\mathfrak{N} = H^2(w) + \varphi H^2(z) + [z, \ldots, z^{m-1}] + [w\varphi, \cdots, w^{n-1}\varphi]$. Since $\mathfrak{F} = \mathfrak{S}_w \oplus w\mathfrak{S}_z = \mathfrak{S}_z \oplus z\mathfrak{S}_w$, we have

$$\mathfrak{F} = \mathfrak{N} + [w^n \varphi] = \mathfrak{N} + [z^m].$$

Thus $\dim(\mathfrak{F} \ominus \mathfrak{N}) = 1$ and $[w^n\varphi, z^m] \subset \mathfrak{F}$. It is clear that $w^n\varphi_0 \in \mathfrak{F}$ and $w^n\varphi_0 \perp \mathfrak{F}_w$. Moreover, for $j \geq 1$ $(j \neq n)$, we have

$$\begin{aligned} \langle w^n \varphi_0, z^j \varphi \rangle &= \langle w^n \varphi, z^j \varphi \rangle - a_{00} \langle w^n, z^j \varphi \rangle \\ &= \langle w^n, z^j \rangle - a_{00} \langle w^n, z^j \varphi \rangle = 0. \end{aligned}$$

Since $w^n\varphi_0 \perp w^k\varphi$ for $1 \le k \le n-1$, this implies that

$$w^n\varphi_0 \perp \mathfrak{N}_0.$$

Similarly, we have $z^m\varphi \perp \mathfrak{F}_w$ and $z^m\varphi \perp z^k\varphi$ for $0 \le k < \infty$ and $k \ne m$. It is clear that $z^m\varphi \perp [w\varphi, \ldots, w^{n-1}\varphi]$ and $w^n\varphi_0 \perp \mathfrak{F}_w$. Thus we have $z^m\varphi \perp \mathfrak{N}_0$. Therefore we have

$$\mathfrak{F} = \mathfrak{N}_0 \oplus [z^m\varphi, w^n\varphi_0].$$

Since $z^m \perp \mathfrak{F}_w$ and $z^m \perp [w\varphi, \ldots, w^{n-1}\varphi]$, we have $z^m \perp \mathfrak{N}_0$. Since $z^m \in \mathfrak{F}$, we have $z^m \in [z^m\varphi, w^n\varphi_0]$. Thus

$$\begin{aligned} z^m &= \gamma z^m\varphi + \delta w^n\varphi_0 \\ &= \gamma z^m\varphi + \delta w^n(\varphi - a_{00}) \\ &= (\gamma z^m\varphi + \delta w^n)\varphi - \delta a_{00}w^n \end{aligned}$$

for some constants γ and δ in $\mathbb{C}$. Thus

$$(\gamma z^m + \delta w^n)\varphi = z^m + \delta a_{00}w^n.$$

Since φ is unimodular,

$$\varphi = \frac{z^m + \delta a_{00}w^n}{\gamma z^m + \delta w^n} = \frac{z^m\overline{w}^n + \delta a_{00}}{\delta + \gamma z^m\overline{w}^n} \quad \text{a.e.}$$

Put

$$h(\lambda) = \frac{\lambda + \delta a_{00}}{\delta + \gamma\lambda}.$$

Then $\varphi(z, w) = h(z^m\overline{w}^n)$. Since $\hat{\varphi}(m, n) = 0$ for every $(m, n) \in \mathbb{Z}_+ \times (-\mathbb{Z}_+)$, h is an analytic function. Since φ is not constant and h is unimodular, we show that h is a Blaschke product, that is,

$$h(\lambda) = c\frac{\lambda - \alpha}{1 - \overline{\alpha}\lambda}$$

for some constants $c \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. Thus $\varphi(z, w) = h(z^m\bar{w}^n) = c\psi_\alpha^{(m,n)}(z, w)$, that is, $\varphi = c\psi_\alpha^{(m,n)}$, and so $\mathfrak{M} = \mathfrak{M}_\alpha^{(m,n)}$. This completes the proof. □

If $\hat{\varphi}(0, 0) = 0$, then, from the proof of Theorem 4.2, we have $\alpha = 0$. Therefore we have

Corollary 4.3 *Let $\mathfrak{M}$ be a zw-pure invariant subspace of $L^2(\mathbb{T}^2)$. Let $m, n \geq 1$. Then $\mathfrak{M} = \overline{w^n} H^2_{m,n}(\mathbb{T}^2)$ if and only if $\mathfrak{F}_w = H^2(w)$, $\mathfrak{F}_z = \varphi H^2(z)$, $\mathfrak{S}_w = \varphi H^2(z) + [1, z, \ldots, z^{m-1}]$ and $\mathfrak{S}_z = H^2(w) + [\varphi, w\varphi, \ldots, w^{n-1}\varphi]$ for some unimodular function φ in $L^\infty(\mathbb{T}^2)$ such that $\operatorname{supp}\hat{\varphi} \subset \mathbb{Z}_+ \times (-\mathbb{Z}_+)$ and $\hat{\varphi}(0, 0) = 0$.*

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References

[1] Gaspar D. and Suciu N., *On invariant subspaces in the bitorus.* J. Operator Theory **30** (1993), 227–241.

[2] Ghatage P. and Manderkar V., *On Beurling type invariant subspaces of $L^2(\mathbb{T}^2)$ and their equivalence.* J. Operator Theory **20** (1988), 31–38.

[3] Helson H., *Lectures on invariant subspaces.* Academic Press, New York, 1964.

[4] Ji G., Ohwada T. and Saito K.-S., *Certain invariant subspace structure of $L^2(\mathbb{T}^2)$.* Proc. Amer. Math. Soc. **126** (1998), 2361–2368.

[5] Mandrekar R., *The validty of Beurling theorems in polidisc.* Proc. Amer. Math. Soc. **103** (1988), 145–148.

[6] Nakazi T., *Certain invariant subspaces of H^2 and L^2 on a bidisc.* Canadian J. Math. **40** (1988), 1722–1280.

[7] Nakazi T., *Invariant subspaces in the bidisc and commutators.* J. Austral. Math. Soc. **56** (1994), 232–242.

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