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Potential-theoretic study of functions on an infinite network

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Abstract. In the context of an infinite network N, the Dirichlet problem with respect to an arbitrary subset of vertices N is solved. Using this solution, some of the important potential-theoretic concepts like Balayage, Domination principle, and Poisson kernel are investigated in N.

Key words: infinite network, Dirichlet problem, Poisson kernel, Balayage, Green's formulas.

1. Introduction

In the context of an electric network, some of the important concepts that are dealt with include conductance, effective resistance, capacity measure, equilibrium measure etc. (Bendito et al. [2]). These notions have their obvious counterparts in the study of Newtonian potentials. There is also a parallel study of these notions with probabilistic interpretations. For example, the effective resistance has a close relation to the escape probability for a reversible Markov chain (Ponzio [8] and Tetali [9]) which is characterized by the transition probability from one state to another. The similarity between the conductance and the transition probability is obvious.

Keeping these facts in mind, a potential-theoretic study of functions on an infinite network has been undertaken by many researchers. (See for example, Yamasaki [10] and Premalatha et al. [7]). In this note, we solve the Dirichlet problem on an infinite network. Using this solution, we are able to arrive at some of the potential-theoretically important results in an infinite network: Balayage, Poisson kernel, Domination principle, Condenser principle, Green kernel on a set, Capacitary functions and Dirichlet-Poisson solution.

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2. Preliminaries

An infinite network (X, Y, r) consists of a countable set X of vertices, a countable set Y of paths joining some of the vertices and a strictly positive real function r on Y. The basic operator on the functions defined on X is the Laplacian which is defined so as to bring out a certain mean-value property of functions. In this framework, we can make enter the Cartier's study of harmonic functions on a tree (see Anandam and Bajunaid [1]) which he initiated in the context of studying the structure of the subgroups of $SL_2(K)$ for a given local field K (Cartier [4]).

Our interest is to study functions defined on a network (X, Y, r) from the classical potential-theoretic point of view which deal with Poisson kernel, Green kernel, Green's formulas, Balayage, Capacity, and the representation of positive superharmonic functions by means of measures using the Green kernel. (For these classical results we refer to the works of Kellogg [6] and Brelot [3].)

Let X be a countable set of vertices; a countable set Y of edges joining some pairs of nodes is given; the resulting graph is assumed to be connected, locally finite and without any self-loops. Two vertices x and z are said to be neighbours, denoted by $x \sim z$, if and only if there is an edge joining x and z. Assume that for each pair of distinct vertices a and b in X is associated a number $t(a, b) \geq 0$ such that $t(a, b) = t(b, a)$; $t(a, b) > 0$ if and only if $a \sim b$. Actually, the definition of $t(a, b)$ is based on the strictly positive real function r defined on the edge set Y in the given network $N = (X, Y, r)$. For a vertex x, let $B(x) = \{z \colon z \sim x\}$. We treat x as a vertex in $B(x)$. We also define $t(x_0) = \sum_{y \sim x_0} t(x_0, y)$, for $x_0 \in X$. Then $t(x) > 0$ for every $x \in X$. Let $u(x)$ be a real-valued function on $B(x_0)$ for some vertex $x_0 \in X$. Then the Laplacian of u at x_0 is defined by $\Delta u(x_0) = \sum_{z \sim x_0} t(x_0, z)[u(z) - u(x_0)]$.

Given a subset E of X, the interior $\mathring{E}$ of E is defined as $\mathring{E} = \{x \colon B(x) \subset E\}$; that is, $x \in \mathring{E}$ if and only if x and all its neighbours are in E. Write $\partial E = E \setminus \mathring{E}$. A real-valued function $s(x)$ on E is said to be superharmonic (respectively harmonic) on E if and only if $\Delta s(x) \leq 0$ (respectively $\Delta s(x) = 0$) for every $x \in \mathring{E}$; A real-valued function $w(x)$ on E is sid to be subharmonic if and only if $-w(x)$ is superharmonic on E. A superharmonic function $s \geq 0$ on E is called a potential if and only if for any subharmonic function $w(x)$ on E such that $w(x) \leq s(x)$, we have $w(x) \leq 0$. We do not always assume that there are positive potentials on X. We mention the existence of positive potentials on X as an extra condition, when it is required.

The following property of superharmonic functions defined on arbitrary set is useful in proving the uniqueness of certain solutions.

Theorem 1 (Minimum Principle) *Let F be an arbitrary set that is not equal to X and let s be superharmonic on F. Suppose there is some $x_0 \in F$, such that $s(x_0) \leq s(x)$ for all $x \in F$. Then $\inf_{x \in F} s(x) = \inf_{z \in \partial F} s(z)$.*

Proof. Let $\alpha = \inf_{x \in F} s(x)$ and $\beta = \inf_{z \in \partial F} s(z)$. Clearly $\alpha \leq \beta$. Suppose $\beta > \alpha$. Then $x_0 \in \overset{\circ}{F}$. Choose $y \in X \setminus F$. Since X is connected, there is a path $\{x_0, x_1, \ldots, x_n = y\}$ connecting x_0 and y. Let i be the smallest index such that $x_i \notin \overset{\circ}{F}$. Since $x_{n-1} \sim x_n \notin F$, $x_{n-1} \notin \overset{\circ}{F}$, and hence $1 \leq i \leq n-1$.

Since $t(x_0)s(x_0) \geq \sum_{a \sim x_0} t(x_0, a)s(a)$ and $\alpha = s(x_0) \leq s(a)$, it is clear that $s(a) = s(x_0)$ for all $a \sim x_0$; in particular $s(x_1) = \alpha$. Repeating this argument, we prove that $s(x_i) = \alpha$. Remark that $x_i \in \partial F$; for $x_{i-1} \in \overset{\circ}{F}$ and $x_{i-1} \sim x_i$ so that $x_i \in F$, but $x_i \notin \overset{\circ}{F}$ by assumption. Hence $\beta = \inf_{z \in \partial F} s(z) \leq s(x_i) = \alpha$, contradicting the assumption that $\beta > \alpha$. We conclude that $\beta = \alpha$; that is $\inf_{x \in F} s(x) = \inf_{z \in \partial F} s(z)$. □

A similar argument proves also:

Proposition 2 *Let s be a superharmonic function defined on an arbitrary subset of X. If $\overset{\circ}{F}$ is connected and s attains its minimum at a vertex in $\overset{\circ}{F}$, then s is constant on $\overset{\circ}{F}$.*

3. Generalized Dirichlet problem

We shall consider the following problem: Suppose F is an arbitrary set in an infinite network X. Let $E \subset \overset{\circ}{F}$. Suppose $\phi(x)$ is a real-valued function on $F \setminus E$. Is it possible to find a unique function $\psi(x)$ on F such that $\psi = \phi$ on $F \setminus E$ and $\Delta\psi = 0$ on E? We call this the generalized Dirichlet problem in an infinite network. Clearly there is no solution in every possible case if the problem is posed in this generality. The following theorem establishes a solution under some restricted conditions.

Theorem 3 *Let F be an arbitrary set in X. Let $E \subset \overset{\circ}{F}$ and $f \geq 0$ be a function defined on $F \setminus E$. Suppose there exists a superharmonic function $s \geq 0$ on F such that $s \geq f$ on $F \setminus E$. Then there exists a function $h \geq 0$ on*

F such that $0 \le h \le s$ on F, $\Delta h = 0$ on E and $h = f$ on $F \setminus E$; moreover, if h_1 is another function on F such that $\Delta h_1 = 0$ on E and $h_1 = f$ on $F \setminus E$, then $h_1 \ge h$ on F.

Proof. Let $\mathcal{F}$ be the class of real-valued functions $u \ge 0$ on F such that $u = f$ on $F \setminus E$ and $\Delta u \le 0$ on E. Note $\mathcal{F}$ is nonempty. For if

$$v(x) = \begin{cases} s(x) & \text{if } x \in E \\ f(x) & \text{if } x \in F \setminus E, \end{cases}$$

then $v \in \mathcal{F}$; to see this, note that for any $x_0 \in E$, $\sum_{y \sim x_0} t(x_0, y)[s(y) - s(x_0)] \le 0$ since s is assumed to be superharmonic on F. This implies that if $x_0 \in E$ and $x_0 \sim z \notin E$, then $v(z) - v(x_0) = f(z) - s(x_0) \le s(z) - s(x_0)$, so that $\sum_{y \sim x_0} t(x_0, y)[v(y) - v(x_0)] \le 0$.

Define $h(x) = \inf_{u \in \mathcal{F}} u(x)$, for every $x \in F$. Now we need to show that $h \in \mathcal{F}$. Clearly, $h(x) = f(x)$ on $F \setminus E$. For any $x_0 \in E$ and $\epsilon > 0$, there exists $u \in \mathcal{F}$ such that $h(x_0) + \epsilon > u(x_0)$. Since u is superharmonic, we have

$$u(x_0) \ge \frac{1}{t(x_0)} \sum_{x \sim x_0} t(x_0, x)u(x) \ge \frac{1}{t(x_0)} \sum_{x \sim x_0} t(x_0, x)h(x),$$

so that

$$h(x_0) + \epsilon > \frac{1}{t(x_0)} \sum_{x \sim x_0} t(x_0, x)h(x).$$

Since ϵ is arbitrary, we have $\Delta h(x_0) \le 0$ and thus $h \in \mathcal{F}$. To show that h is the required function, it is enough to prove that $\Delta h = 0$ on E. Now take $u \in \mathcal{F}$. If $x_0 \in E$, define

$$u_{x_0}(x) = \begin{cases} u(x) & \text{if } x \ne x_0 \\ \dfrac{1}{t(x_0)} \displaystyle\sum_{y \sim x_0} t(x_0, y)u(y) & \text{if } x = x_0. \end{cases}$$

Note that $u_{x_0}(x_0) - u(x_0) = (1/t(x_0)) \sum_{y \sim x_0} t(x_0, y)[u(y) - u(x_0)] \le 0$ and

$$\begin{aligned} \Delta u_{x_0}(x_0) &= \sum t(y, x_0)[u_{x_0}(y) - u_{x_0}(x_0)] \\ &= \Big[\sum t(y, x_0)u(y)\Big] - t(x_0)u_{x_0}(x_0) = 0. \end{aligned}$$

Thus, for every $u \in \mathcal{F}$ and $x_0 \in E$, there exists $u_{x_0} \in \mathcal{F}$ such that $u_{x_0} \le$

u and $\Delta u_{x_0}(x_0) = 0$. Since $h(x_0) \le h_{x_0}(x_0) \le h(x_0)$, we have $h(x_0) = h_{x_0}(x_0)$; that is, $\Delta h(x_0) = 0$.

By this construction, it is clear that h is the smallest of such functions; for, if h_1 is another such function, then $h_1 \in \mathcal{F}$ and hence $h_1 \ge h$. □

Corollary 4 *Let F be an arbitrary set in X. Let f be a real-valued function on F. Suppose there exists a potential p on X such that $|f| \le p$ on F. Then there exists a unique function h on X such that $h = f$ on F, $\Delta h = 0$ on F^c, and $|h|$ is majorized by a potential on X.*

Proof. In Theorem 3, replace F by X and E by F^c and consider f^+ and f^- separately. For the uniqueness, suppose that there is another function h_1 on X such that $h_1 = f$ on F, $\Delta h_1 = 0$ on F^c and $|h_1|$ is majorized by a potential on X. Then, take $u = h - h_1$ on F. Since $\Delta u = \Delta h - \Delta h_1 = 0$ on F^c, for $x \in F^c$, $t(x)u(x) = \sum t(x, y)u(y)$. Hence $t(x)|u(x)| \le \sum t(x, y)|u(y)|$, which implies that $\sum t(x, y)[|u(y)| - |u(x)|] \ge 0$; that is, $|u|$ is subharmonic on F^c. Since $|u| = 0$ on F, it follows that $|u|$ is subharmonic on X. Since h and h_1 are majorized by potentials on X, there exists a potential q on X such that $|u| \le q$. Hence $|u| \equiv 0$, so that $h = h_1$ on X. □

4. Consequences of the existence of the solution to the generalized Dirichlet problem

The above version of the Dirichlet problem and its solution in an infinite network (Theorem 3) has the following consequences:

Theorem 5 (Classical Dirichlet Problem) *Let F be an arbitrary set in X. Let $f \ge 0$ be a real-valued function on ∂F. Suppose there exists a superharmonic function $s \ge 0$ on F such that $s \ge f$ on ∂F. Then there exists a harmonic function $h \ge 0$ on F such that $0 \le h \le s$ on F and $h = f$ on ∂F; moreover, h is the smallest such solution.*

Proof. Take $E = \mathring{F}$ in Theorem 3. Recall that we say that a real-valued function h defined on F is harmonic on F if $\Delta h = 0$ on $\mathring{F}$. □

Corollary 6 *Let F be a finite subset in X. Let f be a real-valued function on ∂F. Then there exists a unique harmonic function h on F such that $h = f$ on ∂F.*

Proof. Assume first $f \geq 0$. Since F is finite, f is bounded above by a constant which is superharmonic. This proves the existence of h, when $f \geq 0$. For arbitrary f, write $f = f^+ - f^-$ to obtain a harmonic function h on F such that $h = f$ on ∂F. The uniqueness of h follows from the minimum principle on the finite set F (see Theorem 1). $\square$

Theorem 7 (Potential-dominated Dirichlet Problem) *Let us suppose that positive potentials exist on X. Let F be an arbitrary subset in X. Let f be a real-valued function on ∂F. Suppose there is a positive potential on X such that $|f| \leq p$ on ∂F. Then there exists a unique harmonic function h on F such that $h = f$ on ∂F and $|h| \leq q$ on F where q is a potential on X.*

Proof. By Theorem 3, there exists a harmonic function $h_1 \geq 0$ on F such that $h_1 = f^+$ on ∂F and $h_1 \leq p$ on F; and similarly there is another harmonic function $h_2 \geq 0$ on F such that $h_2 = f^-$ on ∂F and $h_2 \leq p$ on F. Thus, $h = h_1 - h_2$ is harmonic on F, $h = f$ on ∂F and $|h| \leq 2p$ on F.

For the uniqueness, suppose (for $i = 1, 2$) H_i is harmonic on F, $H_i = f$ on ∂F and $|H_i| \leq q_i$ on F, where $q_i > 0$ is a potential on X. Let $u = H_1 - H_2$. Let

$$v = \begin{cases} |u| & \text{on } F \\ 0 & \text{on } X \setminus F. \end{cases}$$

Then, for $x_0 \in \overset{\circ}{F}$, $t(x_0)u(x_0) = \sum t(x_0, y)u(y)$ so that $t(x_0)|u(x_0)| \leq \sum t(x_0, y)|u(y)|$; and hence

$$\Delta|u|(x_0) = \sum t(x_0, y)[|u|(y) - |u|(x_0)] \geq 0.$$

That is, $\Delta v(x_0) \geq 0$ if $x_0 \in \overset{\circ}{F}$. Clearly, $\Delta v(x) \geq 0$ if $x \notin \overset{\circ}{F}$. Thus v is subharmonic on X and v is majorized by the potential $q_1 + q_2$ on X. Hence $v \leq 0$, which means $v \equiv 0$; that is, $H_1 = H_2$ on F. $\square$

Theorem 8 (Domination Principle) *Let p be a potential on X with harmonic support A (that is, $\Delta p(x) = 0$ for $x \in X \setminus A$). Suppose $s \geq 0$ is a superharmonic function on X such that $s \geq p$ on A. Then $s \geq p$ on X.*

Proof. Let $E = X \setminus A$ and $F = B(E) = \bigcup_{x \in E} B(x)$. Then $E \subset \overset{\circ}{F}$. By Theorem 3, there exists a function $h \geq 0$ on F such that $h = p$ on $F \setminus E$ and $\Delta h = 0$ on E; moreover, $h \leq s$ on F by the construction of h. For the same reason, $h \leq p$ on F also. Let

$$u = \begin{cases} p - h & \text{on } F \\ 0 & \text{on } X \setminus F. \end{cases}$$

Then, $u \geq 0$ on X, $u = 0$ on $X \setminus E$ and $\Delta u = 0$ on E. Therefore, u is subharmonic on X and $u \leq p$ on X. Hence $u \equiv 0$ on X and $h = p \leq s$ on $F \supset X \setminus A$. Since $p \leq s$ on A also, by hypothesis, we conclude $p \leq s$ on X. □

Theorem 9 (Green Potentials) *Suppose that positive potentials exist on X. Then, given $y \in X$, there exists a unique potential $G_y(x)$ on X such that $\Delta G_y(x) = -\delta_y(x)$ for all x in X; moreover $G_y(x) \leq G_y(y)$ for all $x \in X$.*

Proof. Let $p > 0$ be a potential on X. In Theorem 3, take $F = X$ and $E = X \setminus \{y\} \subset X = \overset{\circ}{X} = \overset{\circ}{F}$. Then, there exists $h \geq 0$ on F such that $h = p$ on $X \setminus E$ and $\Delta h(x) = 0$ for $x \in E$.
Hence on $X \setminus E = \{y\}$, $h(y) = p(y)$ so that

$$\begin{aligned} \Delta h(y) &= \sum t(y, x)[h(x) - h(y)] \\ &= \sum t(y, x)[h(x) - p(y)] \\ &\leq \sum t(y, x)[p(x) - p(y)] \\ &= \Delta p(y) \leq 0, \end{aligned}$$

since $h \leq p$ on E. Hence, h is superharmonic on X. Since h is majorized by a potential on X, h itself is a potential on X and $h \not\equiv 0$. Hence $h > 0$ on X and $\Delta h(x) = -c\delta_y(x)$ for some constant $c > 0$.

Thus, if we write $G_y(x) = (1/c)h(x)$ on $F = X$, we find that $G_y(x) > 0$ is a potential on X such that $\Delta G_y(x) = -\delta_y(x)$. Moreover, by the above Domination Principle, $G_y(x) \leq G_y(y)$ for all $x \in X$.

For the uniqueness, note that if $G'_y(x)$ is another such potential on X, then $H(x) = G_y(x) - G'_y(x)$ is harmonic on X. Consequently, $|H|$ is subharmonic on X, majorized by the potential $G_y(x) + G'_y(x)$, so that $H \equiv 0$. □

The following corollary is given in GowriSankaran and Singman [5, Corollary 4.1] in the context of a Cartier tree (recall that as mentioned earlier a tree can be considered as a network) with an additional assumption that the transition probability $p(x, y)$ for $x \sim y$ satisfies the condition $\delta \leq p(x, y) \leq 1/2 - \delta$ where δ is a constant such that $0 < \delta < 1/2$. We prove it on an infinite network and without putting any extra condition.

Corollary 10 *Let f be a real-valued function defined on a finite subset F of X. Assume there exist positive potentials on X. Then there exist potentials p_1 and p_2 on X, with harmonic support in F, such that $f = p_1 - p_2$ on F.*

Proof. Since F is finite and since there exist potentials on X, we can find a positive potential p on X such that $|f| \leq p$ on F. Then, by Corollary 4, there exists h on X such that $h = f$ on F, $\Delta h = 0$ on F^c and $|h| \leq Q$ where Q is a potential on X. Let $u(x) = \Delta h(x)$ on X, so that $u \equiv 0$ on F^c. Write $q(x) = -\sum_{a\in F} u(a)G_a(x) = \sum_{a\in F} u^-(a)G_a(x) - \sum_{a\in F} u^+(a)G_a(x) = p_1(x) - p_2(x)$, so that p_1 and p_2 are potentials on X, with harmonic support in F. Note that $\Delta q(x) = u(x) = \Delta h(x)$ for all $x \in X$, so that $h(x) = p_1(x) - p_2(x) + H(x)$ on X where $H(x)$ is a harmonic function on X. This implies that $|H| \leq p_1 + p_2 + |h| \leq p_1 + p_2 + Q$. Since $|H|$ is subharmonic and $p_1 + p_2 + Q$ is a potential on X, $H \equiv 0$. Hence $h(x) = p_1(x) - p_2(x)$ on X and in particular $f(x) = p_1(x) - p_2(x)$ on F. □

Theorem 11 (Poisson Kernel) *Let F be an arbitrary subset of a network with positive potentials. Then for $y \in \partial F$ and $x \in \mathring{F}$, there exists a unique harmonic function $P_y(x) \geq 0$ on F such that $P_y(z) = \delta_y(z)$ for $z \in \partial F$ and $P_y(x) \leq q(x)$ on F where q is a potential on X. Moreover, if F is finite, any harmonic function h on F is of the form $h(x) = \sum_{y\in\partial F} h(y)P_y(x)$ for $x \in \mathring{F}$.*

Proof. For $y \in \partial F$, let $f(z) = \delta_y(z)$, $z \in \partial F$, be the nonnegative function defined on ∂F. Clearly, there are potentials on X majorizing f on ∂F. Then (using Theorem 7), there exits a unique harmonic function $P_y(x)$ on F such that $P_y(z) = f(z)$ on ∂F and $P_y(x) \leq q(x)$ on F where q is a potential on X.

To prove the second part, suppose h is a harmonic function defined on a finite set F. Consider $u(x) = \sum_{y\in\partial F} h(y)P_y(x)$ for $x \in F$. For $x \in \mathring{F}$, note that $\Delta u(x) = \sum_{y\in\partial F} h(y)\Delta P_y(x) = 0$ and for $z \in \partial F$, $u(z) = \sum_{y\in\partial F} h(y)P_y(z) = h(z)$. Thus, $(u - h)$ is a harmonic function on F, vanishing on the boundary ∂F. Since F is finite, by the minimum principle, $u - h \equiv 0$ on F. Hence $h(x) = \sum_{y\in\partial F} h(y)P_y(x)$, for $x \in F$. □

Remark Let F be a finite subset in X with or without positive potentials. Then, Theorem 3 permits us to establish the existence of the unique Poisson kernel $P_y(x)$, $y \in \partial F$, $x \in F$. Consequently, given any real-valued function f on ∂F, $h(x) = \sum_{y \in \partial F} f(y) P_y(x)$ is the unique harmonic function on F, with boundary values f on ∂F.

Corollary 12 *Let F be an arbitrary subset of a network with positive potentials. Let $f \geq 0$ be a function defined on ∂F, such that $f \leq p$ on ∂F where p is a potential on X. Then, $H(x) = \sum_{y \in \partial F} f(y) P_y(x)$ is the unique harmonic function on F with values f on ∂F.*

Proof. We know that there exists a unique harmonic function $H \geq 0$ on F, such that $H = f$ on ∂F and $H \leq p$ on F (see Theorem 7). Enumerate the vertices on ∂F as a_1, a_2, ...

Let $u_k(x) = \sum_{n=1}^{k} f(a_n) P_{a_n}(x)$. Then u_k is harmonic on F such that $u_k(a_n) = f(a_n)$ for $1 \leq n \leq k$ and $u_k(a_n) = 0$ if $n > k$. Consider $u_k - H$ which is harmonic on F with boundary values ≤ 0 on ∂F. Let $w = \sup(u_k - H, 0)$. Then w is nonnegative subharmonic on F such that $w = 0$ on ∂F.

Now for each n, there exists a potential q_n on X such that $P_{a_n}(x) \leq q_n(x)$ on F. Let

$$w_1 = \begin{cases} w & \text{on } F \\ 0 & \text{on } X \setminus F \end{cases} .$$

Then w_1 is nonnegative subharmonic on X and w_1 is majorized by a potential q also. Hence $w_1 \leq 0$, which implies that $w \equiv 0$. Consequently $u_k \leq H$ on F.

Hence $u(x) = \sum_{y \in \partial F} f(y) P_y(x) = \sup_k u_k(x) \leq H(x)$ on F, so that $u(x) \geq 0$ is harmonic on F and for any $y \in \partial F$, $u(y) = f(y)$. That is, u is a solution on F to the Dirichlet problem with boundary values f on ∂F. But H also is such a solution. Thus from the uniqueness of the solution (see Theorem 7), we conclude $H(x) = \sum_{y \in \partial F} f(y) P_y(x)$ on F. □

Corollary 13 *Let F be an arbitrary subset of an infinite network with positive potentials. Let f be a real-valued function defined on ∂F such that $|f| \leq p$ on ∂F, where p is a potential on X. Then $h(x) = \sum_{y \in \partial F} f(y) P_y(x)$ is the unique harmonic function on F with boundary values f on ∂F.*

Theorem 14 (Balayage) *Let $s \geq 0$ be a superharmonic function on X and let A be an arbitrary set in X. Then there exists a superharmonic*

function $R_s^A \geq 0$ *on* X *with the following properties*:
(1) $R_s^A \leq s$ *on* X.
(2) $R_s^A = s$ *on* A.
(3) $\Delta R_s^A(x) = 0$ *for each* $x \in X \setminus A$.
R_s^A *is the smallest nonnegative superharmonic function on* X *with the above three properties.*

Proof. Let $E = X \setminus A$. Let $F = \bigcup_{x \in E} B(x)$, so that $E \subset \overset{\circ}{F}$. Then by Theorem 3, there exists a function $h \geq 0$ on F such that $h \leq s$ on F, $h = s$ on $F \setminus E$ and $\Delta h(x) = 0$ for every $x \in E$. Define

$$R_s^A = \begin{cases} s & \text{on } X \setminus F \\ h & \text{on } F \end{cases}.$$

Then R_s^A has all the properties mentioned in the statement. □

Theorem 15 (Condenser Principle) *Let* A *and* B *be two disjoint sets in* X. *Let* $E = X \setminus (A \cup B)$. *Then there exists a bounded function* u *on* X *such that*
(1) $\Delta u(x) = 0$ *if* $x \in E$.
(2) $\Delta u(x) \leq 0$ *and* $u(x) = 1$ *if* $x \in A$.
(3) $\Delta u(x) \geq 0$ *and* $u(x) = 0$ *if* $x \in B$.
(4) $0 \leq u(x) \leq 1$ *for* $x \in X$.
Moreover,
(a) *if there are no positive potentials on* X, *then* u *is uniquely determined*;
(b) *if there are positive potentials on* X *and if* A *is finite, then there exists such a uniquely determined function* u *that is majorized by a potential on* X.

Proof. Let $F = \bigcup_{x \in E} B(x)$. Then $E \subset \overset{\circ}{F}$. Define a function $f(x)$ on $F \setminus E$ as follows

$$f(x) = \begin{cases} 1 & \text{if } x \in (F \setminus E) \cap A \\ 0 & \text{if } x \in (F \setminus E) \cap B \end{cases}.$$

Since $0 \leq f \leq 1$ on $F \setminus E$, by Theorem 3, there exists a function h on F such that $0 \leq h \leq 1$ on F, $h = f$ on $F \setminus E$ and $\Delta h(x) = 0$ if $x \in E$. Define

$$u(x) = \begin{cases} h(x) & \text{if } x \in F \\ 1 & \text{if } x \in (X \setminus E) \cap A \\ 0 & \text{if } x \in (X \setminus E) \cap B \end{cases}.$$

Then $u(x)$ has the properties (1) to (4).

As for the uniqueness: In part (a) suppose there are no positive potentials on X. Let $v(x)$ be another function on X, having the properties (1) to (4). Then, as remarked earlier, $|u-v|$ is a subharmonic function on X. Since $|u-v|$ is bounded and since there are no positive potentials on X, $|u-v|$ is a constant; this constant should be 0, since $|u-v|$ vanishes on $A \cup B$.

For part (b), suppose there are positive potentials on X. Then (see Balayage), since A is finite, $R_1^A(x)$ is a potential on X and the construction given in Theorem 3 shows that $u(x) \leq R_1^A(x)$ on X.

Now suppose $w(x)$ is another function on X, majorized by a potential and having the properties (1) to (4). Then, it follows that $|u-w|$ is a subharmonic function on X, majorized by a potential on X. Hence, $u - w \equiv 0$ on X. □

Theorem 16 (Green Kernel on a Set) *Let F be an arbitrary set in an infinite network with positive potentials. Then for any $y \in \mathring{F}$, there exists a unique potential $G_y^F(x)$ on F such that $\Delta G_y^F(x) = -\delta_y(x)$ for $x \in \mathring{F}$, $G_y^F(z) = 0$ for $z \in \partial F$ and $G_y^F(x) \leq q(x)$ on F where q is a potential on X.*

Proof. Take (see Theorem 9) the Green potential $G_y(x) > 0$ on X, which satisfies the condition $\Delta G_y(x) = -\delta_y(x)$ for all $x \in X$. Then (see Theorem 7) there exists a harmonic function h on F such that $h(z) = G_y(z)$ for $z \in \partial F$. Define $G_y^F(x) = G_y(x) - h(x)$ on F. Then $G_y^F(x)$ has the above stated properties. □

Theorem 17 (Dirichlet-Poisson Solution) *Let E be an arbitrary set in an infinite network with positive potentials. Let $F = \bigcup_{x \in E} B(x)$. Let f and g be real-valued functions on X such that $|f| \leq q$ on $F \setminus E$, where q is a potential on X and g vanishes outside a finite set. Then there exists a unique function u on X such that $\Delta u = -g$ on E, $u = f$ on $X \setminus E$ and $|u| \leq p$ on E where p is a potential on X.*

Proof. Since $|f| \leq q$ on $F \setminus E$ (see Theorem 3), there exists a unique harmonic function h on F such that $h = f$ on $F \setminus E$ and $|h| \leq q$ on F. Define

$$w(x) = \begin{cases} h(x) & \text{if } x \in E \\ f(x) & \text{if } x \in X \setminus E \end{cases}.$$

Then, define for each $a \in E$, the function $\phi_a(x)$ in F such that $\Delta\phi_a(x) = -\delta_a(x)$ for $x \in E$, $\phi_a(x) = 0$ for $x \in F \setminus E$ and $\phi_a(x) \leq q_a(x)$ on F where q_a is a potential on X. We shall assume that $\phi_a(x)$ is defined on X by giving the value 0 for vertices outside F. Let G_a be the Green function as in Theorem 9 and let $s(x) = \sum_{a\in E} g(a)\phi_a(x)$. Since g has a finite support, $s(x)$ is well-defined on X, $\Delta s(x) = -g(x)$ for $x \in E$, $s(x) = 0$ on $X \setminus E$ and

$$|s(x)| \leq \sum_{a\in E} |g(a)|G_a(x) = q_2(x).$$

Note that q_2 is a potential on X. Let $u(x) = w(x) + s(x)$. Then $\Delta u(x) = -g(x)$ if $x \in E$ and $u(x) = f(x)$ if $x \in X \setminus E$. Moreover $|u(x)| \leq |w(x)| + |s(x)| \leq q(x) + q_2(x) = p(x)$ on E and p is a potential.

For the uniqueness of the solution, we follow the proof of Theorem 7. □

Corollary 18 *Let E be a finite subset of an infinite network, with or without positive potentials. Let f and g be real-valued functions on X. Then there exists a unique function u on X such that $\Delta u = -g$ on E and $u = f$ on $X \setminus E$.*

Corollary 19 *Let E be an arbitrary set in an infinite network X with potentials, such that ∂E is a finite set. Let $F = \bigcup_{x\in E} B(x)$. Suppose h harmonic on $X \setminus E$. Then there exist a harmonic function H on X and two potentials p_1 and p_2 on X with finite harmonic support in $F \setminus E$ such that $h = p_1 - p_2 + H$ outside E.*

Proof. Remark first that ∂E may be a finite set, even if E and $X\setminus E$ are not finite sets. Take for example, X as a star domain (See Cartier [4, p. 255]) with $N \geq 2$ branches and E as the infinite branch $C_1 = \{s_{0,1}, s_{1,1}, s_{2,1}, \ldots\}$. Then $\partial E = \{s_{0,1}\}$ and neither E nor $X \setminus E$ is finite.

Secondly, remark that $F \setminus E = \bigcup_{z\in\partial E} B(z)$. For if $x \in \mathring{E}$, then $B(x) \subset E$ and hence $B(x) \cap (F \setminus E) = \emptyset$. Consequently, since we are assuming that ∂E is finite, $F \setminus E$ should also be a finite set.

Now in the above theorem, take $g \equiv 0$ and f as the function h on $X \setminus E$ extended by 0 on E. Since $F \setminus E$ is finite, there exists a potential q such that $|f| \leq q$ on $F \setminus E$. Then there exists a function u on X such that $\Delta u = 0$ on E and $u = f$ on $X \setminus E$.

By the definition of f, $\Delta u = 0$ at each interior point of $X \setminus E$; and remark that F^c is contained in the interior of $X \setminus E$. (For, it is clear that

$F^c \subset X \setminus E$. Let $x \in F^c$, and $y \sim x$. Then $y \in X \setminus E$; for, otherwise, $y \in E$ and $x \in B(y) \subset F$, a contradiction. This means that if $x \in F^c$, then x is an interior point of $X \setminus E$.) Consequently, $\Delta u = 0$ on F^c. Thus, if $\Delta u \neq 0$, then $x \in F \setminus E$ which is a finite set. Write $s(x) = -\sum_{a \in F \setminus E} \Delta u(a) G_a(x)$ on X.

Then $s(x) = p_1(x) - p_2(x)$ on X, where p_1 and p_2 are potentials on X with harmonic support in $F \setminus E$; further, $\Delta s(x) = \Delta u(x)$ for all $x \in X$, so that $u(x) = s(x) + H(x)$ where $H(x)$ is harmonic on X. Consequently, $h(x) = p_1(x) - p_2(x) + H(x)$ on $X \setminus E$. □

Remark Since every potential with finite harmonic support in X is bounded (Domination principle, Theorem 8), from the above corollary we conclude that if h is a harmonic function defined outside a finite set E in a network X with potentials, then there exists a unique harmonic function H on X such that $|h - H| \leq p$ outside a finite set, where p is a bounded potential on X with finite harmonic support.

For the uniqueness, note that if $|h - H'| \leq p'$ outside a finite set for another pair (H', p'), then $|H - H'| \leq p + p'$ outside a finite set and hence $H - H' \equiv 0$.

Theorem 20 (Generalized Capacitary Functions) *Let A and B be two finite disjoint sets in an infinite network X with positive potentials. Let ϕ and ψ be nonnegative functions defined on A and B respectively. Then there exits a unique function u on X majorized by a potential such that $u = \phi$ on A, $u = \psi$ on B and $\Delta u = 0$ on $X \setminus (A \cup B)$.*

Proof. Let $E = X \setminus (A \cup B)$. Let $F = \bigcup_{x \in E} B(x)$, so that $E \subset \overset{\circ}{F}$. Define

$$f(x) = \begin{cases} \phi(x) & \text{if } x \in (F \setminus E) \cap A \\ \psi(x) & \text{if } x \in (F \setminus E) \cap B \end{cases}.$$

Since A and B are finite we can find a positive potential p on X such that $p(x) \geq \phi(x)$ on A and $p(x) \geq \psi(x)$ on B. Hence $p(x) \geq f(x)$ on $F \setminus E$. Then by Theorem 3, there exists a function h on F such that $\Delta h = 0$ on E, $h = f$ on $F \setminus E$ and $h \leq p$ on F. Define

$$u(x) = \begin{cases} h(x) & \text{if } x \in F \\ \phi(x) & \text{if } x \in A \\ \psi(x) & \text{if } x \in B \end{cases}.$$

Then $u(x)$ is a well defined function on X, having the stated properties. The uniqueness of u follows, as in the previous cases, from the fact that u is majorized by a potential on X. $\square$

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