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Boundedness of multilinear singular integral operators on Hardy and Herz-type spaces

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Abstract. In this paper, the authors study a class of multilinear singular integral operators on product of Hardy spaces. The boundedness of another class of multilinear singular integral operators is also discussed on product of Herz-type spaces. Moreover, as their special cases, the corresponding results of multilinear fractional integral operator and multilinear Calderón-Zygmund operator can be obtained, respectively.

Key words: multilinear singular integral operator, Hardy space, Herz space, Herz-type Hardy space, Multilinear Calderón-Zygmund operator.

1. Introduction

The study of multilinear singular integrals was motivated not only as the generalization of the theory of linear ones but also their natural appearance in analysis. It has received increasing attention and well development in recent years, such as the study of bilinear Hilbert transform by Lacey and Thiele [7, 8] and the systemic treatment of multilinear Calderón-Zygmund operators by Grafakos-Torres [2, 3] and Grafakos-Kalton [1].

Let $m \in \mathbb{N}$ and $K(y_0, y_1, \ldots, y_m)$ be a function defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. We consider that T is a m-linear operator defined on product of test functions such that for K, the integral representation below is valid

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \ldots, y_m) \times \prod_{j=1}^{m} f_j(y_j) dy_1 \cdots dy_m, \quad (1.1)$$

whenever f_j, $j = 1, \ldots, m$, are smooth functions with compact support

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and $x \notin \cap_{j=1}^{m} \operatorname{supp} f_j$.

Especially, we call K a m-Calderón-Zygmund kernel if it satisfies the following size and smoothness estimates.

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{C}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn}}, \tag{1.2}$$

for some $C > 0$ and all $(y_0, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ away from the diagonal.

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \leq \frac{C|y_j - y_j'|^{\varepsilon}}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn+\varepsilon}}, \tag{1.3}$$

for some $\varepsilon > 0$, whenever $0 \leq j \leq m$ and $|y_j - y_j'| \leq (1/2) \max_{0 \leq k \leq m} |y_j - y_k|$.

If a multilinear operator T defined by (1.1) associated with a m-Calderón-Zygmund kernel K, and satisfies either of the following two conditions for given numbers $1 \leq t_1, t_2, \ldots, t_m, t < \infty$ with $1/t = 1/t_1 + 1/t_2 + \cdots + 1/t_m$.

(C1) T maps $L^{t_1,1} \times \cdots \times L^{t_m,1}$ into $L^{t,\infty}$ if $t > 1$,

(C2) T maps $L^{t_1,1} \times \cdots \times L^{t_m,1}$ into L^1 if $t = 1$,

where $L^{t_1,1}, \ldots, L^{t_m,1}$ and $L^{t,\infty}$ are Lorentz spaces. Then we say that T is a m-linear Calderón-Zygmund operator.

Grafakos and Torres [2] proved that the multilinear Calderón-Zygmund operator is bounded on product of Lebesgue spaces.

Theorem A ([2]) *Let T be a m-linear Calderón-Zygmund operator. Then for any numbers $1 < p_1, p_2, \ldots, p_m < \infty$ with $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_m$, T can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p.*

After the boundedness of multilinear Calderón-Zygmund operators on product of Lebesgue spaces has been established above, Grafakos and Kalton [1] discussed their boundedness on product of Hardy spaces successively. Inspired by them, in Section 2, we will discuss the boundedness of a class of multilinear operators, whose special case is multilinear fractional integral operator, i.e.

$$K(x, y_1, \ldots, y_m) = |(x - y_1, \ldots, x - y_m)|^{-mn+\alpha}, \quad 0 < \alpha < n, \tag{1.4}$$

on product of Hardy spaces.

In addition, the boundedness of multilinear Calderón-Zygmund operator on Herz-type spaces is also interesting.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $E_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{E_k}$ for $k \in \mathbb{Z}$, where by χ_E we denote the characteristic function of a set E.

Definition 1.1 Let $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$.
(1) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f : f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \quad \text{and} \quad \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \bigg(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_q^p\bigg)^{1/p}$$

with usual modifications made when $p = \infty$.
(2) The inhomogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f : f \in L^q_{\text{loc}}(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \bigg(\|f\chi_{B_0}\|_q^p + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_q^p\bigg)^{1/p}$$

with usual modifications made when $p = \infty$.

It is easy to see that $\dot{K}_q^{0,q}(\mathbb{R}^n) = K_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ and $\dot{K}_q^{\alpha/q,q}(\mathbb{R}^n) = L^q_{|x|^\alpha}(\mathbb{R}^n)$ for $0 < q \le \infty$ and $\alpha \in \mathbb{R}$. Actually, for $1 < q < \infty$, $L^q_{|x|^\alpha}(\mathbb{R}^n)$ is a Lebesgue spaces with power weights if and only if $-n < \alpha < n(q-1)$. Thus, Herz spaces are generalizations of Lebesgue spaces. Moreover, the homogeneous Herz spaces include the Lebesgue spaces with power weights as special cases.

For the standard singular integral operator defined by

$$Tf(x) = \text{p.v.} \int K(x-y) f(y) dy, \tag{1.5}$$

Stein [12] shows that if T is bounded on $L^q(\mathbb{R}^n)$, $1 < q < \infty$, and

$$|K(x)| \le \frac{C}{|x|^n}, \quad \forall x \ne 0, \tag{1.6}$$

then T is also bounded on the weighted spaces $L^q_{|x|^\beta}(\mathbb{R}^n)$, $-n < \beta < n(q-1)$, where the range of β is the best.

In 1994, the above Stein's result was developed by Soria and Weiss [11] in the following way. The singular integral operator satisfying (1.6) will be replaced by any sublinear operator T satisfying the following size condition: For any $f \in L^1(\mathbb{R}^n)$ with compact support and for $x \notin \operatorname{supp} f$,

$$|Tf(x)| \le C \int \frac{|f(y)|}{|x-y|^n} dy. \tag{1.7}$$

It should be pointed out that (1.7) is satisfied by many operators in harmonic analysis, such as Calderón-Zygmund operator, C. Fefferman's singular multiplier, R. Fefferman's singular integral operator, Ricci-Stein's oscillatory singular integral, the Bochner-Riesz means at the critical index, and so on.

Inspired by them, recently the author in [14] established the corresponding result of multi-sublinear operator on product of Herz spaces, which are more extensive than Lebesgue spaces with power weights.

Theorem B ([14]) *Let T be a multi-sublinear operator satisfying*

$$|T(f_1, \ldots, f_m)(x)| \le C \int_{(\mathbb{R}^n)^m} \frac{|f_1(y_1)| \cdots |f_m(y_m)|}{|(x-y_1, \ldots, x-y_m)|^{mn}} dy_1 \cdots dy_m, \tag{1.8}$$

for any integrable functions $f_1, \ldots, f_m$ with compact support and $x \notin \cap_{j=1}^m \operatorname{supp} f_j$.

Suppose $0 < p_j \le \infty$, $1 < q_j < \infty$, $-n/q_j < \alpha_j < n(1-1/q_j)$, $j = 1, \ldots, m$, $\alpha = \sum_{j=1}^m \alpha_j$, $1/p = \sum_{j=1}^m 1/p_j$, $1/q = \sum_{j=1}^m 1/q_j$. If T is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, then T is bounded from $\dot{K}^{\alpha_1, p_1}_{q_1}(\mathbb{R}^n) \times \cdots \times \dot{K}^{\alpha_m, p_m}_{q_m}(\mathbb{R}^n)$ into $\dot{K}^{\alpha, p}_q(\mathbb{R}^n)$ and from $K^{\alpha_1, p_1}_{q_1}(\mathbb{R}^n) \times \cdots \times K^{\alpha_m, p_m}_{q_m}(\mathbb{R}^n)$ into $K^{\alpha, p}_q(\mathbb{R}^n)$.

As their special cases, the boundedness of the multilinear Calderón-Zygmund operator on product of Herz spaces can be deduced immediately. On the other hand, the author in [14] also discussed the situation when $\alpha_j \ge n(1-1/q_j)$, $j = 1, \ldots, m$, but only focused on the multilinear Calderón-Zygmund operator itself.

In Section 3, we will keep on studying the behaviors of the more general multilinear operators with weaker kernel condition and a class of multi-sublinear operators when $\alpha_j = n(1 - 1/q_j)$ and $\alpha_j > n(1 - 1/q_j)$, $j = 1, \ldots, m$.

2. Boundedness on product of Hardy spaces

In this section, we will discuss the boundedness of a class of multilinear operators on product of Hardy spaces. First introduce some necessary notations and requisite lemmas.

For any cube Q in $\mathbb{R}^n$ with side length $l(Q)$, let $\widetilde{Q}$ be the cube with the same center of Q and side length $l(\widetilde{Q}) = 2\sqrt{n}\, l(Q)$. For every set $G \subset \mathbb{R}^n$, denote G^c the complementary set of G in $\mathbb{R}^n$. Here and in what follows, for $t \in \mathbb{R}$, $[t]$ is the largest integer no more than t.

Lemma 2.1 ([1]) *Let $0 < p \le 1$. Then there is a constant $C(p)$ such that for all finite collections of cubes $\{Q_k\}_{k=1}^M$ in $\mathbb{R}^n$ and all nonnegative integrable functions g_k with $\operatorname{supp} g_k \subset Q_k$, we have*

$$\Big\|\sum_{k=1}^M g_k\Big\|_{L^p} \le C(p)\Big\|\sum_{k=1}^M \Big(\frac{1}{|Q_k|}\int_{Q_k} g_k(x)dx\Big)\chi_{\widetilde{Q}_k}\Big\|_{L^p}.$$

Lemma 2.2 ([6]) *Let $I_{\alpha,m}$ be a m-linear fractional integral operator with kernel K satisfying* (1.4). *Suppose $1 \le p_1, p_2, \ldots, p_m \le \infty$, $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n > 0$.*

(1) *If each $p_j > 1$, $j = 1, \ldots, m$, then*

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{L^q} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

(2) *If $p_j = 1$ for some j, then*

$$\|I_{\alpha,m}(f_1, \ldots, f_m)\|_{L^{q,\infty}} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

Let us state our main result of this section.

Theorem 2.1 *Suppose $0 < p_1, \ldots, p_m, q \le 1$ satisfying $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$, $0 < \alpha < n$. Let T be a m-linear operator defined by* (1.1) *with kernel K and satisfy the following two conditions.*

(2.1) *$K(y_0, y_1, \ldots, y_m)$ is differentiable up to $N+1$, where*

$$N = \max_{1\le j\le m}\Big\{\Big[mn\Big(\frac{1}{p_j} - 1\Big)\Big]\Big\},$$

and

$$|\partial_{y_0}^{\beta_0}\partial_{y_1}^{\beta_1}\cdots\partial_{y_m}^{\beta_m}K(y_0, y_1, \ldots, y_m)| \le \frac{C}{(\sum_{k,l=0}^{m}|y_k - y_l|)^{mn-\alpha+|\beta|}},$$

for all $|\beta| = N+1$, $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$.

(2.2) *There exists a* $1 < p^* < n/\alpha$ *such that* T *is bounded from* $L^\infty \times \cdots \times L^\infty \times L^{p^*} \times L^\infty \times \cdots \times L^\infty$ *into* L^{q^*}, *where* $1/q^* = 1/p^* - \alpha/n$ *and* L^{p^*} *is the* j^{th} *space of the* m *product of Lebesgue spaces,* $j = 1, \ldots, m$.

Then T *is a bounded operator from* $H^{p_1}(\mathbb{R}^n)\times\cdots\times H^{p_m}(\mathbb{R}^n)$ *into* $L^q(\mathbb{R}^n)$ *which satisfies the norm estimate*

$$\|T(f_1, \ldots, f_m)\|_{L^q(\mathbb{R}^n)} \le C(m, n, p^*, p_j, \alpha)\prod_{j=1}^{m}\|f_j\|_{H^{p_j}(\mathbb{R}^n)}.$$

Proof. The theorem will be proved by the atomic decomposition of Hardy space H^p. Write f_j, $j = 1, \ldots, m$, as a sum of H^{p_j}-atoms, $f_j = \sum_{k_j}\lambda_{j,k_j} \times a_{j,k_j}$, where a_{j,k_j} are H^{p_j}-atoms and $(\sum_{k_j}|\lambda_{j,k_j}|^{p_j})^{1/p_j} \le C\|f_j\|_{H^{p_j}}$. Suppose $\operatorname{supp} a_{j,k_j} \subset Q_{j,k_j}$, where Q_{j,k_j} is a cube centered at c_{j,k_j} in $\mathbb{R}^n$ with side length $l(Q_{j,k_j})$. Then a_{j,k_j} satisfies the following properties.

(1) $\|a_{j,k_j}\|_\infty \le |Q_{j,k_j}|^{-1/p_j}$,

(2) $\int x^\gamma a_{j,k_j}(x)dx = 0$, for all $|\gamma| \le [n(1/p_j - 1)]$.

By the theory of H^p spaces [13, p. 112], the atoms a_{j,k_j} can have vanishing moments up to any large fixed specified integer. In this discussion, we will assume that all the a_{j,k_j}'s satisfy (2) for all $|\gamma| \le N$, where $N = \max_{1\le j\le m}\{[mn(1/p_j - 1)]\}$.

For the decomposition of f_j, $j = 1, \ldots, m$, we can write

$$T(f_1, \ldots, f_m)(x) = \sum_{k_1}\cdots\sum_{k_m}\lambda_{1,k_1}\cdots\lambda_{m,k_m} \times T(a_{1,k_1}, \ldots, a_{m,k_m})(x).$$

For $x \in \mathbb{R}^n$ and fixed $k_1, \ldots, k_m$, there are two cases.

1° $x \in \widetilde{Q}_{1,k_1}\cap\cdots\cap\widetilde{Q}_{m,k_m}$, 2° $x \in \widetilde{Q}^c_{1,k_1}\cup\cdots\cup\widetilde{Q}^c_{m,k_m}$.

Then

$$|T(f_1, \ldots, f_m)(x)| \le I_1(x) + I_2(x),$$

where

$$I_1(x) = \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \times |T(a_{1,k_1}, \ldots, a_{m,k_m})(x)| \chi_{\widetilde{Q}_{1,k_1} \cap \cdots \cap \widetilde{Q}_{m,k_m}}(x),$$

$$I_2(x) = \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \times |T(a_{1,k_1}, \ldots, a_{m,k_m})(x)| \chi_{\widetilde{Q}^c_{1,k_1} \cup \cdots \cup \widetilde{Q}^c_{m,k_m}}(x).$$

First, we consider the estimate of $I_1(x)$. For fixed $k_1, \ldots, k_m$, assume that $\widetilde{Q}_{1,k_1} \cap \cdots \cap \widetilde{Q}_{m,k_m} \neq \emptyset$, or else, this term in $I_1(x)$ can be cancelled.

Suppose that $Q_{j_0,k_{j_0}}$, $j_0 \in \{1, \ldots, m\}$, has the smallest size among all these cubes. We can pick a cube $G_{k_1,\ldots,k_m}$ such that

$$\widetilde{Q}_{1,k_1} \cap \cdots \cap \widetilde{Q}_{m,k_m} \subset G_{k_1,\ldots,k_m} \subset \widetilde{G}_{k_1,\ldots,k_m} \subset \widetilde{\widetilde{Q}}_{1,k_1} \cap \cdots \cap \widetilde{\widetilde{Q}}_{m,k_m},$$

and $|G_{k_1,\ldots,k_m}| \geq C|Q_{j_0,k_{j_0}}|$.

From (2.2) of the theorem, T maps $L^\infty \times \cdots \times L^\infty \times L^{p^*} \times L^\infty \times \cdots \times L^\infty$ into L^{q^*}, where $1/q^* = 1/p^* - \alpha/n$ and L^{p^*} is the j_0^{th} space of the m product of Lebesgue spaces. We have

$$\begin{aligned}
&\int_{G_{k_1,\ldots,k_m}} |T(a_{1,k_1}, \ldots, a_{m,k_m})(x)|dx \\
&\leq |G_{k_1,\ldots,k_m}|^{1-1/q^*} \|T(a_{1,k_1}, \ldots, a_{m,k_m})\|_{L^{q^*}} \\
&\leq C|\widetilde{\widetilde{Q}}_{j_0,k_{j_0}}|^{1-1/q^*} \|a_{j_0,k_{j_0}}\|_{L^{p^*}} \prod_{j=1,\, j\neq j_0}^{m} \|a_{j,k_j}\|_\infty \\
&\leq C|Q_{j_0,k_{j_0}}|^{1+\alpha/n} \prod_{j=1}^{m} |Q_{j,k_j}|^{-1/p_j}.
\end{aligned}$$

Notice that $|G_{k_1,\ldots,k_m}| \geq C|Q_{j_0,k_{j_0}}|$ and $Q_{j_0,k_{j_0}}$ has the smallest size, then

$$\begin{aligned}
&\frac{1}{|G_{k_1,\ldots,k_m}|} \int_{G_{k_1,\ldots,k_m}} |T(a_{1,k_1}, \ldots, a_{m,k_m})(x)|dx \\
&\leq C|Q_{j_0,k_{j_0}}|^{\alpha/n} \prod_{j=1}^{m} |Q_{j,k_j}|^{-1/p_j}
\end{aligned}$$

$$\leq C \prod_{j=1}^{m} |Q_{j,\,k_j}|^{\alpha/(mn)-1/p_j}. \tag{2.3}$$

It is easy to see that

$$I_1(x) \leq \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,\,k_1}| \cdots |\lambda_{m,\,k_m}| \times |T(a_{1,\,k_1}, \ldots, a_{m,\,k_m})(x)| \chi_{G_{k_1,\,\ldots,\,k_m}}(x).$$

By using Lemma 2.1 and (2.3), we have

$$\begin{aligned}
\|I_1\|_{L^q} &\leq C \bigg\| \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,\,k_1}| \cdots |\lambda_{m,\,k_m}| \times \prod_{j=1}^{m} |Q_{j,\,k_j}|^{\alpha/(mn)-1/p_j} \chi_{\widetilde{G}_{k_1,\,\ldots,\,k_m}} \bigg\|_{L^q} \\
&\leq C \bigg\| \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,\,k_1}| \cdots |\lambda_{m,\,k_m}| \times \prod_{j=1}^{m} |Q_{j,\,k_j}|^{\alpha/(mn)-1/p_j} \chi_{\widetilde{\widetilde{Q}}_{1,\,k_1} \cap \cdots \cap \widetilde{\widetilde{Q}}_{m,\,k_m}} \bigg\|_{L^q} \\
&\leq C \bigg\| \prod_{j=1}^{m} \bigg(\sum_{k_j} |\lambda_{j,\,k_j}| |Q_{j,\,k_j}|^{\alpha/(mn)-1/p_j} \chi_{\widetilde{\widetilde{Q}}_{j,\,k_j}} \bigg) \bigg\|_{L^q}.
\end{aligned}$$

Denote $1/q_j = 1/p_j - \alpha/(mn)$, $j = 1, \ldots, m$. Then $0 < q_j < \infty$ and

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}.$$

By Hölder's inequality with exponents $q_1, \ldots, q_m$ and q, we obtain

$$\|I_1\|_{L^q} \leq C \prod_{j=1}^{m} \bigg\| \sum_{k_j} |\lambda_{j,\,k_j}| |Q_{j,\,k_j}|^{-1/q_j} \chi_{\widetilde{\widetilde{Q}}_{j,\,k_j}} \bigg\|_{L^{q_j}}.$$

For every q_j, $j = 1, \ldots, m$, if $0 < q_j \leq 1$, then since $p_j < q_j$,

$$\begin{aligned}
\bigg\| \sum_{k_j} |\lambda_{j,\,k_j}| |Q_{j,\,k_j}|^{-1/q_j} \chi_{\widetilde{\widetilde{Q}}_{j,\,k_j}} \bigg\|_{L^{q_j}}^{q_j} &\leq \sum_{k_j} |\lambda_{j,\,k_j}|^{q_j} |Q_{j,\,k_j}|^{-1} \big|\widetilde{\widetilde{Q}}_{j,\,k_j}\big| \\
&\leq C \bigg(\sum_{k_j} |\lambda_{j,\,k_j}|^{p_j} \bigg)^{q_j/p_j}.
\end{aligned}$$

If $1 < q_j < \infty$, then since $0 < p_j \leq 1$,

$$\Big\| \sum_{k_j} |\lambda_{j,k_j}| \, |Q_{j,k_j}|^{-1/q_j} \chi_{\widetilde{\widetilde{Q}}_{j,k_j}} \Big\|_{L^{q_j}} \leq \sum_{k_j} |\lambda_{j,k_j}| |Q_{j,k_j}|^{-1/q_j} \big|\widetilde{\widetilde{Q}}_{j,k_j}\big|^{1/q_j} \leq C\Big(\sum_{k_j} |\lambda_{j,k_j}|^{p_j}\Big)^{1/p_j}.$$

Thus,

$$\|I_1\|_{L^q} \leq C \prod_{j=1}^{m} \Big(\sum_{k_j} |\lambda_{j,k_j}|^{p_j}\Big)^{1/p_j} \leq C \prod_{j=1}^{m} \|f_j\|_{H^{p_j}}.$$

Second, we consider the estimate of $I_2(x)$.

Let A be a non-empty subset of $\{1, \ldots, m\}$, and we denote the cardinality of A by $|A|$. It is easy to see that $1 \leq |A| \leq m$. Let $A^c = \{1, \ldots, m\} \setminus A$, $\vec{y} = (y_1, \ldots, y_m)$ and $\vec{y}^A = (y_j)_{j \in A^c}$. If $A = \{1, \ldots, m\}$, we define

$$\Big(\bigcap_{j\in A} \widetilde{Q}^c_{j,k_j}\Big) \bigcap \Big(\bigcap_{j\in A^c} \widetilde{Q}_{j,k_j}\Big) = \bigcap_{j\in A} \widetilde{Q}^c_{j,k_j}.$$

Then we can write

$$\widetilde{Q}^c_{1,k_1} \cup \cdots \cup \widetilde{Q}^c_{m,k_m} = \bigcup_{A \subset \{1,\ldots,m\}} \bigg(\Big(\bigcap_{j\in A} \widetilde{Q}^c_{j,k_j}\Big) \bigcap \Big(\bigcap_{j\in A^c} \widetilde{Q}_{j,k_j}\Big)\bigg).$$

For fixed A, we assume that the side length of the cube $Q_{j^*,k_{j^*}}$, $j^* \in A$, is the smallest among the side lengths of the cubes Q_{j,k_j}, $j \in A$. Let $P^N_{c_{j^*,k_{j^*}}}(x, y_1, \ldots, y_m)$ be the N^{th} order Taylor polynomial of $K(x, y_1, \ldots, y_m)$ about the variable y_{j^*} at the point $c_{j^*,k_{j^*}}$.

Since $a_{j^*,k_{j^*}}$ has zero vanishing moments up to N, by (2.1), it is true that

$$|T(a_{1,k_1}, \ldots, a_{m,k_m})(x)| = \Big| \int_{(\mathbb{R}^n)^{m-1}} \prod_{j=1, j\neq j^*}^{m} a_{j,k_j}(y_j) \int_{\mathbb{R}^n} a_{j^*,k_{j^*}}(y_{j^*})$$

$$\cdot\Big(K(x, y_1, \ldots, y_m) - P^N_{c_{j^*, k_{j^*}}}(x, y_1, \ldots, y_m)\Big) d\vec{y}\Big|$$

$$\le C \int_{(\mathbb{R}^n)^{m-1}} \prod_{j=1,\, j\ne j^*}^m |a_{j, k_j}(y_j)| \int_{\mathbb{R}^n} |a_{j^*, k_{j^*}}(y_{j^*})|\, |y_{j^*} - c_{j^*, k_{j^*}}|^{N+1} \cdot\Big(|x-\xi| + \sum_{j=1,\, j\ne j^*}^m |x - y_j|\Big)^{-mn+\alpha-N-1} d\vec{y},$$

where ξ is on the line segment joining y_{j^*} to $c_{j^*, k_{j^*}}$ by Taylor's theorem.

For $x \in (\cap_{j\in A} \widetilde{Q}^c_{j, k_j}) \cap (\cap_{j\in A^c} \widetilde{Q}_{j, k_j})$,

$$|x-\xi| \ge |x - c_{j^*, k_{j^*}}| - |\xi - c_{j^*, k_{j^*}}| \ge \frac{1}{2}|x - c_{j^*, k_{j^*}}|,$$

since $x \notin \widetilde{Q}_{j^*, k_{j^*}}$.

Similarly,

$$|x - y_j| \ge \frac{1}{2}|x - c_{j, k_j}|, \quad \text{for} \quad y_j \in Q_{j, k_j},\, j \in A \setminus \{j^*\}.$$

Since

$$\int_{\mathbb{R}^n} |a_{j^*, k_{j^*}}(y_{j^*})|\, |y_{j^*} - c_{j^*, k_{j^*}}|^{N+1} dy_{j^*} \le C |Q_{j^*, k_{j^*}}|^{(N+1)/n + 1 - 1/p_{j^*}},$$

we have

$$\begin{aligned}
&|T(a_{1, k_1}, \ldots, a_{m, k_m})(x)| \\
&\le C \int_{(\mathbb{R}^n)^{m-1}} \prod_{j\in A^c} |Q_{j, k_j}|^{-1/p_j} \\
&\qquad \cdot \prod_{j\in A\setminus\{j^*\}} |a_{j, k_j}(y_j)|\, |Q_{j^*, k_{j^*}}|^{(N+1)/n+1-1/p_{j^*}} \\
&\qquad \cdot\Big(\frac{1}{2}\sum_{j\in A} |x - c_{j, k_j}| + \sum_{j\in A^c} |x - y_j|\Big)^{-mn+\alpha-N-1} d(\vec{y} \setminus \{y_{j^*}\}) \\
&\le C \int_{(\mathbb{R}^n)^{m-|A|}} \prod_{j\in A^c} |Q_{j, k_j}|^{-1/p_j} \\
&\qquad \cdot \prod_{j\in A\setminus\{j^*\}} |Q_{j, k_j}|^{1-1/p_j}\, |Q_{j^*, k_{j^*}}|^{(N+1)/n+1-1/p_{j^*}}
\end{aligned}$$

$$\cdot\biggl(\frac{1}{2}\sum_{j\in A}|x-c_{j,k_j}|+\sum_{j\in A^c}|x-y_j|\biggr)^{-mn+\alpha-N-1}d\vec{y}^A.$$

By integrating the above over $y_j \in \mathbb{R}^n$, $j \in A^c$, since the side length of the cube $Q_{j^*,k_{j^*}}$ is the smallest among the side lengths of the cubes Q_{j,k_j}, $j \in A$, and $x \in (\cap_{j\in A}\widetilde{Q}^c_{j,k_j}) \cap (\cap_{j\in A^c}\widetilde{Q}_{j,k_j})$, we obtain

$$\begin{aligned}
&|T(a_{1,k_1},\dots,a_{m,k_m})(x)|\\
&\le C\prod_{j\in A^c}|Q_{j,k_j}|^{-1/p_j}\prod_{j\in A}|Q_{j,k_j}|^{1-1/p_j}\,|Q_{j^*,k_{j^*}}|^{(N+1)/n}\\
&\qquad\qquad\cdot\biggl(\frac{1}{2}\sum_{j\in A}|x-c_{j,k_j}|\biggr)^{-mn+\alpha-N-1+n(m-|A|)}\\
&\le C\prod_{j\in A^c}|Q_{j,k_j}|^{-1/p_j}\prod_{j\in A}|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}\\
&\qquad\qquad\cdot\biggl(\frac{1}{2}\sum_{j\in A}|x-c_{j,k_j}|\biggr)^{-N-1-n|A|+\alpha}\\
&\le C\prod_{j\in A}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}}{\bigl(|x-c_{j,k_j}|+l(Q_{j,k_j})\bigr)^{n+(N+1)/|A|-\alpha/|A|}}\prod_{j\in A^c}|Q_{j,k_j}|^{-1/p_j}\\
&\le C\prod_{j\in A}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}}{\bigl(|x-c_{j,k_j}|+l(Q_{j,k_j})\bigr)^{n+(N+1)/|A|-\alpha/|A|}}\\
&\qquad\qquad\cdot\prod_{j\in A^c}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}}{\bigl(|x-c_{j,k_j}|+l(Q_{j,k_j})\bigr)^{n+(N+1)/|A|}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_2(x)\le C\sum_{k_1}\cdots\sum_{k_m}&|\lambda_{1,k_1}|\cdots|\lambda_{m,k_m}|\sum_{A\subset\{1,\dots,m\}}\\
&\cdot\prod_{j\in A}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}}{\bigl(|x-c_{j,k_j}|+l(Q_{j,k_j})\bigr)^{n+(N+1)/|A|-\alpha/|A|}}\\
&\cdot\prod_{j\in A^c}\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}}{\bigl(|x-c_{j,k_j}|+l(Q_{j,k_j})\bigr)^{n+(N+1)/|A|}}
\end{aligned}$$

$$\begin{aligned}
&\le C \sum_{A\subset\{1,\dots,m\}} \prod_{j\in A}\bigg(\sum_{k_j}|\lambda_{j,k_j}| \\
&\qquad\qquad \times \frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}}{\big(|x-c_{j,k_j}|+l(Q_{j,k_j})\big)^{n+(N+1)/|A|-\alpha/|A|}}\bigg) \\
&\qquad \cdot \prod_{j\in A^c}\bigg(\sum_{k_j}|\lambda_{j,k_j}|\frac{|Q_{j,k_j}|^{1-1/p_j+(N+1)/(n|A|)}}{\big(|x-c_{j,k_j}|+l(Q_{j,k_j})\big)^{n+(N+1)/|A|}}\bigg) \\
&:= C \sum_{A\subset\{1,\dots,m\}}\bigg(\prod_{j\in A} W_{|A|,j}(x)\bigg)\bigg(\prod_{j\in A^c} W^c_{|A|,j}(x)\bigg).
\end{aligned}$$

Denote $1/s_j = 1/p_j - \alpha/(n|A|)$, $j\in A$. Then $0 < s_j < \infty$ and

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} = \sum_{j\in A}\frac{1}{s_j} + \sum_{j\in A^c}\frac{1}{p_j}.$$

By Hölder's inequality with exponents s_j, $j\in A$, p_j, $j\in A^c$, and q, we obtain

$$\begin{aligned}
&\bigg\|\bigg(\prod_{j\in A} W_{|A|,j}\bigg)\bigg(\prod_{j\in A^c} W^c_{|A|,j}\bigg)\bigg\|_{L^q} \\
&\qquad\qquad \le \bigg(\prod_{j\in A}\|W_{|A|,j}\|_{L^{s_j}}\bigg)\bigg(\prod_{j\in A^c}\|W^c_{|A|,j}\|_{L^{p_j}}\bigg).
\end{aligned}$$

Since $0 < p_j \le 1$, $j\in A^c$, and $np_j + (N+1)p_j/|A| > n$,

$$\begin{aligned}
&\|W^c_{|A|,j}\|^{p_j}_{L^{p_j}} \\
&\le \sum_{k_j}|\lambda_{j,k_j}|^{p_j}|Q_{j,k_j}|^{p_j-1+(N+1)p_j/(n|A|)} \\
&\qquad\qquad \times \int \big(|x-c_{j,k_j}|+l(Q_{j,k_j})\big)^{-np_j-(N+1)p_j/|A|}dx \\
&\le C\sum_{k_j}|\lambda_{j,k_j}|^{p_j}|Q_{j,k_j}|^{p_j-1+(N+1)p_j/(n|A|)}|Q_{j,k_j}|^{1-p_j-(N+1)p_j/(n|A|)} \\
&= C\sum_{k_j}|\lambda_{j,k_j}|^{p_j}.
\end{aligned}$$

For $0 < s_j < \infty$, $j\in A$, if $0 < s_j \le 1$, since $ns_j + (N+1)s_j/|A| -$

$\alpha s_j/|A| > n$ and $s_j > p_j$, then

$$\begin{aligned}
&\|W_{|A|,\,j}\|_{L^{s_j}}^{s_j}\\
&\le \sum_{k_j} |\lambda_{j,\,k_j}|^{s_j} |Q_{j,\,k_j}|^{s_j - s_j/p_j + (N+1)s_j/(n|A|)}\\
&\qquad\qquad \times \int \big(|x - c_{j,\,k_j}| + l(Q_{j,\,k_j})\big)^{-ns_j - (N+1)s_j/|A| + \alpha s_j/|A|} dx\\
&\le C \sum_{k_j} |\lambda_{j,\,k_j}|^{s_j}\\
&\le C \bigg(\sum_{k_j} |\lambda_{j,\,k_j}|^{p_j}\bigg)^{s_j/p_j}.
\end{aligned}$$

If $1 < s_j < \infty$, since $ns_j + (N+1)s_j/|A| - \alpha s_j/|A| > n$ and $0 < p_j \le 1$, then

$$\begin{aligned}
&\|W_{|A|,\,j}\|_{L^{s_j}}\\
&\le \sum_{k_j} |\lambda_{j,\,k_j}| |Q_{j,\,k_j}|^{1 - 1/p_j + (N+1)/(n|A|)}\\
&\qquad \times \bigg(\int \big(|x - c_{j,\,k_j}| + l(Q_{j,\,k_j})\big)^{-ns_j - (N+1)s_j/|A| + \alpha s_j/|A|} dx\bigg)^{1/s_j}\\
&\le C \sum_{k_j} |\lambda_{j,\,k_j}|\\
&\le C \bigg(\sum_{k_j} |\lambda_{j,\,k_j}|^{p_j}\bigg)^{1/p_j}.
\end{aligned}$$

Since $0 < q \le 1$,

$$\begin{aligned}
\|I_2\|_{L^q}^q &\le C \sum_{A \subset \{1,\dots,m\}} \bigg\| \bigg(\prod_{j \in A} W_{|A|,\,j}\bigg) \bigg(\prod_{j \in A^c} W_{|A|,\,j}^c\bigg) \bigg\|_{L^q}^q\\
&\le C \bigg\{ \prod_{j=1}^m \bigg(\sum_{k_j} |\lambda_{j,\,k_j}|^{p_j}\bigg)^{1/p_j} \bigg\}^q\\
&\le C \bigg(\prod_{j=1}^m \|f_j\|_{H^{p_j}}\bigg)^q.
\end{aligned}$$

In a conclusion, we can obtain

$$\|T(f_1,\,\ldots,\,f_m)\|_{L^q} \le C \prod_{j=1}^{m} \|f_j\|_{H^{p_j}}.$$

This completes the proof of Theorem 2.1. □

It is easy to see from (1.4) and Lemma 2.2 that the multilinear fractional integral operator obviously satisfies (2.1) and (2.2), thus we can get the following corollary immediately.

Corollary 2.1 *Let $I_{\alpha,m}$ be a m-linear fractional integral operator with kernel K satisfying* (1.4)*. Suppose $0 < p_1,\,\ldots,\,p_m,\,q \le 1$ with $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$. Then $I_{\alpha,m}$ is a bounded operator from $H^{p_1}(\mathbb{R}^n) \times \cdots \times H^{p_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ which satisfies the norm estimate*

$$\|I_{\alpha,m}(f_1,\,\ldots,\,f_m)\|_{L^q(\mathbb{R}^n)} \le C(m,\,n,\,p_j,\,\alpha) \prod_{j=1}^{m} \|f_j\|_{H^{p_j}(\mathbb{R}^n)}.$$

3. Boundedness on product of Herz-type spaces

For $k \in \mathbb{Z}$ and measurable function $f(x)$ on $\mathbb{R}^n$, let $m_k(\sigma,\,f) = |\{x \in E_k\colon |f(x)| > \sigma\}|$; for $k \in \mathbb{N}$, let $\widetilde{m}_k(\sigma,\,f) = m_k(\sigma,\,f)$ and $\widetilde{m}_0(\sigma,\,f) = |\{x \in B(0,\,1)\colon |f(x)| > \sigma\}|$. In [4] the authors introduced the following weak Herz spaces.

Definition 3.1 Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, and $0 < p \le \infty$.
(1) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak Herz space $W\dot{K}_q^{\alpha,\,p}(\mathbb{R}^n)$, if

$$\|f\|_{W\dot{K}_q^{\alpha,\,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda,\,f)^{p/q} \Big\}^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.
(2) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the inhomogeneous weak Herz space $WK_q^{\alpha,\,p}(\mathbb{R}^n)$, if

$$\|f\|_{WK_q^{\alpha,\,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \Big\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \widetilde{m}_k(\lambda,\,f)^{p/q} \Big\}^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

In this section, we will study the boundedness of the multi-sublinear operator satisfying (1.8) in Theorem B when $\alpha_j = n(1-1/q_j)$, $j = 1, \ldots, m$. In fact, the condition (1.8) can be weakened as follows in this situation.

$$|T(f_1, \ldots, f_m)(x)| \le C|x|^{-mn} \prod_{j=1}^{m} \|f_j\|_{L^1(\mathbb{R}^n)}, \tag{3.1}$$

for any integrable functions f_j with $\operatorname{supp} f_j \subset B(0, r_j)$, $r_j > 0$, $j = 1, \ldots, m$, and $x \notin \cap_{j=1}^m B(0, 2r_j)$.

Theorem 3.1 *Let T be a multi-sublinear operator satisfying* (3.1)*. Suppose $0 < p_j \le 1$, $1 < q_j < \infty$, $j = 1, \ldots, m$, $1/p = \sum_{j=1}^m 1/p_j$, $1/q = \sum_{j=1}^m 1/q_j$. If T is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$, then T is bounded from $\dot{K}_{q_1}^{n(1-1/q_1),\,p_1}(\mathbb{R}^n) \times \cdots \times \dot{K}_{q_m}^{n(1-1/q_m),\,p_m}(\mathbb{R}^n)$ into $W\dot{K}_q^{n(m-1/q),\,p}(\mathbb{R}^n)$.*

Proof. In order to simplify the proof, we only consider the situation when $m = 2$. Actually, the similar procedure works for all $m \in \mathbb{N}$.

Let f_1, f_2 be functions in $\dot{K}_{q_1}^{n(1-1/q_1),\,p_1}(\mathbb{R}^n)$ and $\dot{K}_{q_2}^{n(1-1/q_1),\,p_2}(\mathbb{R}^n)$, respectively.

Since $f_j(x) = \sum_{l_j=-\infty}^{\infty} f_j(x)\chi_{l_j}(x)$, $j = 1, 2$, and T is a multi-sublinear operator, we have

$$\begin{aligned}
&\|T(f_1, f_2)\|_{W\dot{K}_q^{n(2-1/q),\,p}(\mathbb{R}^n)} \\
&\le C \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p} \\
&\qquad \times \Big| \Big\{ x \in E_k \colon \Big| T\Big(\sum_{l_1=k-2}^{\infty} f_1\chi_{l_1}, \sum_{l_2=k-2}^{\infty} f_2\chi_{l_2} \Big)(x) \Big| > \frac{\lambda}{4} \Big\} \Big|^{p/q} \Big\}^{1/p} \\
&+ C \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p} \\
&\qquad \times \Big| \Big\{ x \in E_k \colon \sum_{l_1=-\infty}^{k-3} \sum_{l_2=k-2}^{\infty} |T(f_1\chi_{l_1}, f_2\chi_{l_2})(x)| > \frac{\lambda}{4} \Big\} \Big|^{p/q} \Big\}^{1/p} \\
&+ C \sup_{\lambda>0} \lambda \Big\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p}
\end{aligned}$$

$$\times \left| \left\{ x \in E_k \colon \sum_{l_1=-\infty}^{k-3} \sum_{l_2=-\infty}^{k-3} |T(f_1\chi_{l_1},\, f_2\chi_{l_2})(x)| > \frac{\lambda}{4} \right\} \right|^{p/q} \Bigg\}^{1/p}$$

$$+C \sup_{\lambda>0} \lambda \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p}$$

$$\times \left| \left\{ x \in E_k \colon \sum_{l_1=k-2}^{\infty} \sum_{l_2=-\infty}^{k-3} |T(f_1\chi_{l_1},\, f_2\chi_{l_2})(x)| > \frac{\lambda}{4} \right\} \right|^{p/q} \Bigg\}^{1/p}$$

$$:= I_1 + I_2 + I_3 + I_4.$$

Using the fact that T is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$ and $0 < p_j \le 1$, $1 < q_j < \infty$, $j = 1, 2$, by Hölder's inequality, we see that

$$\begin{aligned}
I_1 &\le C \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p} \Bigg\| \sum_{l_1=k-2}^{\infty} f_1\chi_{l_1} \Bigg\|_{L^{q_1}}^{p} \Bigg\| \sum_{l_2=k-2}^{\infty} f_2\chi_{l_2} \Bigg\|_{L^{q_2}}^{p} \Bigg\}^{1/p} \\
&\le C \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{kn(1-1/q_1)p_1} \Bigg\| \sum_{l_1=k-2}^{\infty} f_1\chi_{l_1} \Bigg\|_{L^{q_1}}^{p_1} \Bigg\}^{1/p_1} \\
&\quad \times \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{kn(1-1/q_2)p_2} \Bigg\| \sum_{l_2=k-2}^{\infty} f_2\chi_{l_2} \Bigg\|_{L^{q_2}}^{p_2} \Bigg\}^{1/p_2} \\
&\le C \Bigg\{ \sum_{l_1=-\infty}^{\infty} \|f_1\chi_{l_1}\|_{L^{q_1}}^{p_1} \sum_{k=-\infty}^{l_1+2} 2^{kn(1-1/q_1)p_1} \Bigg\}^{1/p_1} \\
&\quad \times \Bigg\{ \sum_{l_2=-\infty}^{\infty} \|f_2\chi_{l_2}\|_{L^{q_2}}^{p_2} \sum_{k=-\infty}^{l_2+2} 2^{kn(1-1/q_2)p_2} \Bigg\}^{1/p_2} \\
&= C \Bigg\{ \sum_{l_1=-\infty}^{\infty} 2^{l_1 n(1-1/q_1)p_1} \|f_1\chi_{l_1}\|_{L^{q_1}}^{p_1} \Bigg\}^{1/p_1} \\
&\quad \times \Bigg\{ \sum_{l_2=-\infty}^{\infty} 2^{l_2 n(1-1/q_2)p_2} \|f_2\chi_{l_2}\|_{L^{q_2}}^{p_2} \Bigg\}^{1/p_2} \\
&= C \|f_1\|_{\dot K_{q_1}^{n(1-1/q_1),\,p_1}(\mathbb{R}^n)} \|f_2\|_{\dot K_{q_2}^{n(1-1/q_2),\,p_2}(\mathbb{R}^n)}.
\end{aligned}$$

For I_2, noticing that $x \in E_k$, $\operatorname{supp} f_j\chi_{l_j} \subset \{x \in \mathbb{R}^n \colon |x| \le 2^{l_j}\}$, $j = 1, 2$, and $l_1 \le k-3$, by (3.1) we have

$$\begin{aligned}
&\sum_{l_1=-\infty}^{k-3}\sum_{l_2=k-2}^{\infty}|T(f_1\chi_{l_1},f_2\chi_{l_2})(x)|\\
&\le C|x|^{-2n}\Big(\sum_{l_1=-\infty}^{k-3}\|f_1\chi_{l_1}\|_{L^1}\Big)\Big(\sum_{l_2=k-2}^{\infty}\|f_2\chi_{l_2}\|_{L^1}\Big)\\
&\le C_0 2^{-2kn}\|f_1\chi_{\{|x|\le 2^{k-3}\}}\|_{L^1}\|f_2\chi_{\{|x|>2^{k-3}\}}\|_{L^1}\\
&\le C_0 2^{-2kn}\|f_1\|_{L^1}\|f_2\|_{L^1}. \qquad (3.2)
\end{aligned}$$

It is easy to see from the definition of Herz space that $\dot{K}_{q_j}^{n(1-1/q_j),\,p_j}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and $\|f_j\|_{L^1(\mathbb{R}^n)} \le \|f_j\|_{\dot{K}_{q_j}^{n(1-1/q_j),\,p_j}(\mathbb{R}^n)}$ for $0 < p_j \le 1 < q_j < \infty$, $j = 1, 2$.

Given any fixed $\lambda > 0$, if

$$\Big|\Big\{x \in E_k\colon \sum_{l_1=-\infty}^{k-3}\sum_{l_2=k-2}^{\infty}|T(f_1\chi_{l_1},\, f_2\chi_{l_2})(x)| > \frac{\lambda}{4}\Big\}\Big| \ne 0,$$

then by (3.2),

$$\frac{\lambda}{4} \le C_0 2^{-2kn}\|f_1\|_{\dot{K}_{q_1}^{n(1-1/q_1),\,p_1}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2}^{n(1-1/q_2),\,p_2}(\mathbb{R}^n)}.$$

That is

$$k \le 2^{-1}n^{-1}\log_2\big(4C_0\lambda^{-1}\|f_1\|_{\dot{K}_{q_1}^{n(1-1/q_1),\,p_1}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2}^{n(1-1/q_2),\,p_2}(\mathbb{R}^n)}\big) := N_\lambda.$$

From this, we can obtain

$$\begin{aligned}
I_2 &\le C\sup_{\lambda>0}\lambda\Big(\sum_{k=-\infty}^{[N_\lambda]}2^{kn(2-1/q)p}2^{knp/q}\Big)^{1/p}\\
&= C\sup_{\lambda>0}\lambda 2^{2n[N_\lambda]}\\
&\le C\sup_{\lambda>0}\Big(\lambda\cdot 4C_0\lambda^{-1}\|f_1\|_{\dot{K}_{q_1}^{n(1-1/q_1),\,p_1}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2}^{n(1-1/q_2),\,p_2}(\mathbb{R}^n)}\Big)\\
&= C\|f_1\|_{\dot{K}_{q_1}^{n(1-1/q_1),\,p_1}(\mathbb{R}^n)}\|f_2\|_{\dot{K}_{q_2}^{n(1-1/q_2),\,p_2}(\mathbb{R}^n)}.
\end{aligned}$$

From the analogous argumentation of I_2, the similar estimates of I_3 and I_4 can be deduced. Combining all the estimates on I_1, I_2, I_3 and I_4, the desired result can be established. This completes the proof of Theorem 3.1. $\square$

Remark 3.1 It should be pointed out that there is a counterpart of the above result on inhomogeneous Herz space, but we omit the details for their similarity.

If T is a m-linear Calderón-Zygmund operator, then from (1.2) and Theorem A, we can obtain a corollary of Theorem 3.1.

Corollary 3.1 *Let T be a m-linear Calderón-Zygmund operator. Suppose $0 < p_j \le 1$, $1 < q_j < \infty$, $j = 1, \ldots, m$, $1/p = \sum_{j=1}^m 1/p_j$, $1/q = \sum_{j=1}^m 1/q_j$. Then T is a bounded operator from $\dot{K}_{q_1}^{n(1-1/q_1), p_1}(\mathbb{R}^n) \times \cdots \times \dot{K}_{q_m}^{n(1-1/q_m), p_m}(\mathbb{R}^n)$ into $W\dot{K}_q^{n(m-1/q), p}(\mathbb{R}^n)$ and also from $K_{q_1}^{n(1-1/q_1), p_1}(\mathbb{R}^n) \times \cdots \times K_{q_m}^{n(1-1/q_m), p_m}(\mathbb{R}^n)$ into $WK_q^{n(m-1/q), p}(\mathbb{R}^n)$.*

Remark 3.2 If an operator T satisfies the hypotheses of Theorem 3.1, then T may be unbounded from $\dot{K}_{q_1}^{n(1-1/q_1), p_1}(\mathbb{R}^n) \times \cdots \times \dot{K}_{q_m}^{n(1-1/q_m), p_m}(\mathbb{R}^n)$ into $\dot{K}_q^{n(m-1/q), p}(\mathbb{R}^n)$ and from $K_{q_1}^{n(1-1/q_1), p_1}(\mathbb{R}^n) \times \cdots \times K_{q_m}^{n(1-1/q_m), p_m}(\mathbb{R}^n)$ into $K_q^{n(m-1/q), p}(\mathbb{R}^n)$; see [9] for an example when $m = 1$.

Remark 3.3 The restriction $0 < p_j \le 1$ in Theorem 3.1 can not be removed; see [5] for an example when $m = 1$.

In order to study the behaviors of the multilinear operators when $\alpha_j > n(1 - 1/q_j)$, $j = 1, \ldots, m$, the appropriate substitute spaces for the Herz spaces are the Herz-type Hardy spaces.

Definition 3.2 Let $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$.

(1) The homogeneous Herz-type Hardy space $H\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha, p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \colon G(f) \in \dot{K}_q^{\alpha, p}(\mathbb{R}^n)\},$$

and

$$\|f\|_{H\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} = \|G(f)\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)}.$$

(2) The inhomogeneous Herz-type Hardy space $HK_q^{\alpha, p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha, p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \colon G(f) \in K_q^{\alpha, p}(\mathbb{R}^n)\},$$

and

$$\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|G(f)\|_{K_q^{\alpha,p}(\mathbb{R}^n)}.$$

Here, $\mathcal{S}'(\mathbb{R}^n)$ is the space of the tempered distributions on $\mathbb{R}^n$ and $G(f)$ is the grand maximal function of f defined by

$$G(f)(x) = \sup_{\varphi\in\mathcal{A}_N} |\varphi_\nabla^*(f)(x)|,$$

where $\mathcal{A}_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n)\colon \sup_{|\alpha|,\,|\beta|\le N} |x^\alpha D^\beta \varphi(x)| \le 1\}$ and $N \ge n+1$.

We now consider the boundedness of a class of multilinear operators defined by (1.1) on product of Herz-type Hardy spaces.

Theorem 3.2 *Suppose* $0 < p_j < \infty$, $1 < q_j < \infty$, $n(1-1/q_j) < \alpha_j < n(1-1/q_j)+\delta_j$, $\delta_j > 0$, $j = 1, \dots, m$, $\alpha = \sum_{j=1}^m \alpha_j$, $1/p = \sum_{j=1}^m 1/p_j$, $1/q = \sum_{j=1}^m 1/q_j$, $q \ge 1$, $\delta = \sum_{j=1}^m \delta_j$. *Let T be a m-linear operator defined by* (1.1) *with kernel $K(x, y_1, \dots, y_m)$ satisfying*

$$\sup_{y_j\in B_{l_j}} \left(\int_{E_k} |K(x, y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_m) - K(x, y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_m)|^q dx \right)^{1/q} \le C 2^{kn(1/q-m)+(l_j-k)\delta}, \tag{3.3}$$

for $k, l_j \in \mathbb{Z}$ and $l_j \le k-2$, where $C > 0$ and C is independent of $(x, y_1, \dots, y_m)$, l_j and k, $j = 1, \dots, m$.

If T is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, then T is bounded from $H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n) \times \cdots \times H\dot{K}_{q_m}^{\alpha_m,p_m}(\mathbb{R}^n)$ into $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

Proof. In order to simplify the proof, we only consider the situation when $m = 2$. Actually, the similar procedure works for all $m \in \mathbb{N}$.

For $f_j \in H\dot{K}_{q_j}^{\alpha_j,p_j}(\mathbb{R}^n)$, by the atomic decomposition of the Herz-type Hardy space [10], we can write $f_j = \sum_{l_j=-\infty}^{\infty} \lambda_{l_j} a_{l_j}$, where $(\sum_{l_j=-\infty}^{\infty} |\lambda_{l_j}|^{p_j})^{1/p_j} \le C\|f_j\|_{H\dot{K}_{q_j}^{\alpha_j,p_j}(\mathbb{R}^n)}$, each a_{l_j} is a central (α_j, q_j)-atom supported on B_{l_j} and satisfies

(i) $\|a_{l_j}\|_{L^{q_j}(\mathbb{R}^n)} \le |B_{l_j}|^{-\alpha_j/n}$,

(ii) $\int a_{l_j}(x) x^\beta dx = 0$ for $|\beta| \le [\alpha_j - n(1-1/q_j)]$, $j = 1, 2$.

$$
\begin{aligned}
&\|T(f_1, f_2)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}\\
&\le C_p\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\bigg(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2} |\lambda_{l_1}||\lambda_{l_2}|\|T(a_{l_1}, a_{l_2})\chi_k\|_{L^q}\bigg)^p\bigg\}^{1/p}\\
&\quad + C_p\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\bigg(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{\infty} |\lambda_{l_1}||\lambda_{l_2}|\|T(a_{l_1}, a_{l_2})\chi_k\|_{L^q}\bigg)^p\bigg\}^{1/p}\\
&\quad + C_p\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\bigg(\sum_{l_1=k-1}^{\infty}\sum_{l_2=-\infty}^{k-2} |\lambda_{l_1}||\lambda_{l_2}|\|T(a_{l_1}, a_{l_2})\chi_k\|_{L^q}\bigg)^p\bigg\}^{1/p}\\
&\quad + C_p\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\bigg(\sum_{l_1=k-1}^{\infty}\sum_{l_2=k-1}^{\infty} |\lambda_{l_1}||\lambda_{l_2}|\|T(a_{l_1}, a_{l_2})\chi_k\|_{L^q}\bigg)^p\bigg\}^{1/p}\\
&:= C_p(I_1 + I_2 + I_3 + I_4).
\end{aligned}
$$

For I_1, we use the fact that $l_1 \le k-2$, $l_2 \le k-2$ and $x \in E_k$. Without loss of generality, for fixed l_1 and l_2, we assume $l_1 \le l_2$.

Since a_{l_1} is a central (α_1, q_1)-atom, by the Minkowski inequality twice, (3.3), Hölder's inequality and $l_1 \le l_2$, we have

$$
\begin{aligned}
&\|T(a_{l_1}, a_{l_2})(x)\chi_k(x)\|_{L^q}\\
&= \bigg\{\int_{E_k}\bigg|\int_{B_{l_2}} a_{l_2}(y_2)\int_{B_{l_1}} a_{l_1}(y_1)\\
&\qquad\qquad \times\Big(K(x, y_1, y_2) - K(x, 0, y_2)\Big)dy_1dy_2\bigg|^q dx\bigg\}^{1/q}\\
&\le \int_{B_{l_2}} |a_{l_2}(y_2)|\int_{B_{l_1}} |a_{l_1}(y_1)|\\
&\qquad\qquad \times\bigg(\int_{E_k} |K(x, y_1, y_2) - K(x, 0, y_2)|^q dx\bigg)^{1/q} dy_1dy_2\\
&\le C2^{kn(1/q-2)+(l_1-k)\delta}\|a_{l_1}\|_{L^1}\|a_{l_2}\|_{L^1}\\
&\le C2^{kn(1/q_1-1)}2^{kn(1/q_2-1)}2^{(l_1-k)\delta_1}2^{(l_2-k)\delta_2}\\
&\qquad\qquad \times\|a_{l_1}\|_{L^{q_1}}2^{l_1n(1-1/q_1)}\|a_{l_2}\|_{L^{q_2}}2^{l_2n(1-1/q_2)}\\
&\le C2^{(l_1-k)\big(\delta_1+n(1-1/q_1)\big)}2^{-l_1\alpha_1}2^{(l_2-k)\big(\delta_2+n(1-1/q_2)\big)}2^{-l_2\alpha_2}. \qquad (3.4)
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
I_1 \le C\Big\{ \sum_{k=-\infty}^{\infty} \Big(\sum_{l_1=-\infty}^{k-2} |\lambda_{l_1}| 2^{(l_1-k)\left(\delta_1+n(1-1/q_1)-\alpha_1\right)} \Big)^{p} \\
\times \Big(\sum_{l_2=-\infty}^{k-2} |\lambda_{l_2}| 2^{(l_2-k)\left(\delta_2+n(1-1/q_2)-\alpha_2\right)} \Big)^{p} \Big\}^{1/p} \\
\le C\Big\{ \sum_{k=-\infty}^{\infty} \Big(\sum_{l_1=-\infty}^{k-2} |\lambda_{l_1}| 2^{(l_1-k)\left(\delta_1+n(1-1/q_1)-\alpha_1\right)} \Big)^{p_1} \Big\}^{1/p_1} \\
\times \Big\{ \sum_{k=-\infty}^{\infty} \Big(\sum_{l_2=-\infty}^{k-2} |\lambda_{l_2}| 2^{(l_2-k)\left(\delta_2+n(1-1/q_2)-\alpha_2\right)} \Big)^{p_2} \Big\}^{1/p_2} \\
:= C I_{11} \times I_{12}.
\end{aligned}
$$

Notice that $\delta_1 + n(1-1/q_1) - \alpha_1 > 0$. If $0 < p_1 \le 1$, then

$$
\begin{aligned}
I_{11} &\le \Big\{ \sum_{k=-\infty}^{\infty} \sum_{l_1=-\infty}^{k-2} |\lambda_{l_1}|^{p_1} 2^{(l_1-k)p_1\left(\delta_1+n(1-1/q_1)-\alpha_1\right)} \Big\}^{1/p_1} \\
&= \Big\{ \sum_{l_1=-\infty}^{\infty} |\lambda_{l_1}|^{p_1} \sum_{k=l_1+2}^{\infty} 2^{(l_1-k)p_1\left(\delta_1+n(1-1/q_1)-\alpha_1\right)} \Big\}^{1/p_1} \\
&= C\Big(\sum_{l_1=-\infty}^{\infty} |\lambda_{l_1}|^{p_1} \Big)^{1/p_1} \\
&\le C\|f_1\|_{H\dot{K}_{q_1}^{\alpha_1,\,p_1}(\mathbb{R}^n)}.
\end{aligned}
$$

If $1 < p_1 < \infty$, denote $1/p_1 + 1/p_1' = 1$, then by Hölder's inequality, we obtain

$$
\begin{aligned}
I_{11} &\le \Big\{ \sum_{k=-\infty}^{\infty} \Big(\sum_{l_1=-\infty}^{k-2} |\lambda_{l_1}|^{p_1} 2^{(l_1-k)\left(\delta_1+n(1-1/q_1)-\alpha_1\right)} \Big) \\
&\qquad \times \Big(\sum_{l_1=-\infty}^{k-2} 2^{(l_1-k)\left(\delta_1+n(1-1/q_1)-\alpha_1\right)} \Big)^{p_1/p_1'} \Big\}^{1/p_1} \\
&\le C\Big\{ \sum_{l_1=-\infty}^{\infty} |\lambda_{l_1}|^{p_1} \sum_{k=l_1+2}^{\infty} 2^{(l_1-k)\left(\delta_1+n(1-1/q_1)-\alpha_1\right)} \Big\}^{1/p_1} \\
&= C\Big(\sum_{l_1=-\infty}^{\infty} |\lambda_{l_1}|^{p_1} \Big)^{1/p_1}
\end{aligned}
$$

$$\le C\|f_1\|_{H\dot{K}^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)}.$$

The estimate of I_{11} can be obtained from the two cases above. With the same discussion, we get the estimate of I_{12}. Thus,

$$I_1 \le C\|f_1\|_{H\dot{K}^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)}\|f_2\|_{H\dot{K}^{\alpha_2,p_2}_{q_2}(\mathbb{R}^n)}.$$

For I_2, observing that $l_1 \le k-2$, $l_2 \ge k-1$ and $x \in E_k$, similar to (3.4), we also have

$$\begin{aligned}
&\|T(a_{l_1}, a_{l_2})(x)\chi_k(x)\|_{L^q}\\
&\le C2^{kn(1/q-2)+(l_1-k)\delta}\|a_{l_1}\|_{L^1}\|a_{l_2}\|_{L^1}\\
&\le C2^{kn(1/q_1-1)+(l_1-k)\delta}2^{kn(1/q_2-1)}\\
&\qquad\times\|a_{l_1}\|_{L^{q_1}}2^{l_1n(1-1/q_1)}\|a_{l_2}\|_{L^{q_2}}2^{l_2n(1-1/q_2)}\\
&\le C2^{(l_1-k)\left(\delta+n(1-1/q_1)\right)}2^{-l_1\alpha_1}2^{(l_2-k)n(1-1/q_2)}2^{-l_2\alpha_2}.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
I_2 &\le C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{l_1=-\infty}^{k-2}|\lambda_{l_1}|2^{(l_1-k)\left(\delta+n(1-1/q_1)-\alpha_1\right)}\Big)^p\\
&\qquad\times\Big(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|2^{(l_2-k)\left(n(1-1/q_2)-\alpha_2\right)}\Big)^p\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{l_1=-\infty}^{k-2}|\lambda_{l_1}|2^{(l_1-k)\left(\delta+n(1-1/q_1)-\alpha_1\right)}\Big)^{p_1}\Big\}^{1/p_1}\\
&\qquad\times\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|2^{(l_2-k)\left(n(1-1/q_2)-\alpha_2\right)}\Big)^{p_2}\Big\}^{1/p_2}\\
&:= CI_{21}\times I_{22}.
\end{aligned}$$

Since $\delta+n(1-1/q_1)-\alpha_1 > \delta_1+n(1-1/q_1)-\alpha_1 > 0$, similarly to I_{11}, we get $I_{21} \le C\|f_1\|_{H\dot{K}^{\alpha_1,p_1}_{q_1}(\mathbb{R}^n)}$.

Notice that $n(1-1/q_2)-\alpha_2 < 0$. If $0 < p_2 \le 1$, then

$$I_{22} \le \Big\{\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\sum_{k=-\infty}^{l_2+1}2^{(l_2-k)p_2\left(n(1-1/q_2)-\alpha_2\right)}\Big\}^{1/p_2}$$

$$
\begin{aligned}
&= C\Big(\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\Big)^{1/p_2}\\
&\le C\|f_2\|_{H\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.
\end{aligned}
$$

If $1<p_2<\infty$, denote $1/p_2+1/p_2'=1$, then by Hölder's inequality, we obtain

$$
\begin{aligned}
I_{22}\le&\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|^{p_2}2^{(l_2-k)\big(n(1-1/q_2)-\alpha_2\big)}\bigg)\\
&\qquad\times\bigg(\sum_{l_2=k-1}^{\infty}2^{(l_2-k)\big(n(1-1/q_2)-\alpha_2\big)}\bigg)^{p_2/p_2'}\bigg\}^{1/p_2}\\
\le& C\bigg\{\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\sum_{k=-\infty}^{l_2+1}2^{(l_2-k)\big(n(1-1/q_2)-\alpha_2\big)}\bigg\}^{1/p_2}\\
=& C\Big(\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\Big)^{1/p_2}\\
\le& C\|f_2\|_{H\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.
\end{aligned}
$$

The estimate of I_{22} can be obtained from the two cases above. Thus,

$$
I_2\le C\|f_1\|_{H\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)}.
$$

From the analogous argumentation of I_2, we get the estimate of I_3.

For I_4, using the fact that T is bounded from $L^{q_1}(\mathbb{R}^n)\times L^{q_2}(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, by Hölder's inequality we get

$$
\begin{aligned}
I_4\le& C\bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{l_1=k-1}^{\infty}|\lambda_{l_1}|\|a_{l_1}\|_{L^{q_1}}\Big)^{p}\\
&\qquad\qquad\times\Big(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|\|a_{l_2}\|_{L^{q_2}}\Big)^{p}\bigg\}^{1/p}\\
\le& C\bigg\{\sum_{k=-\infty}^{\infty}\Big(\sum_{l_1=k-1}^{\infty}|\lambda_{l_1}|2^{(k-l_1)\alpha_1}\Big)^{p_1}\bigg\}^{1/p_1}\\
&\qquad\qquad\times\bigg\{\sum_{k=-\infty}^{\infty}\Big(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|2^{(k-l_2)\alpha_2}\Big)^{p_2}\bigg\}^{1/p_2}\\
:=& CI_{41}\times I_{42}.
\end{aligned}
$$

Observing $\alpha_j > 0$, we also consider two cases $0 < p_j \le 1$ and $1 < p_j < \infty$, $j = 1, 2$, respectively. Then we have

$$I_4 \le C\|f_1\|_{H\dot{K}_{q_1}^{\alpha_1, p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot{K}_{q_2}^{\alpha_2, p_2}(\mathbb{R}^n)}.$$

The estimates for I_1, I_2, I_3 and I_4 lead to the desired result of the theorem. □

Remark 3.4 If (3.3) is true only for $k \in \mathbb{N}$, $l_j \in \mathbb{N} \cup \{0\}$ and $l_j \le k - 2$, $j = 1, \dots, m$, then the operator T in Theorem (3.2) is bounded from $HK_{q_1}^{\alpha_1, p_1}(\mathbb{R}^n) \times \cdots \times HK_{q_m}^{\alpha_m, p_m}(\mathbb{R}^n)$ into $K_q^{\alpha, p}(\mathbb{R}^n)$.

It is easy to see that the multilinear Calderón-Zygmund operator obviously satisfies the condition (3.3) by taking $\varepsilon = \delta$ in (1.3), however, Theorem 3.2 does not assume any pointwise smoothness condition of the kernel, so (3.3) is weaker than (1.3). Uniting Theorem A, we can obtain a corollary of Theorem 3.2.

Corollary 3.2 *Let T be a m-linear Calderón-Zygmund operator and ε be as in* (1.3). *Then for any $\varepsilon_1, \dots, \varepsilon_m > 0$ with $\varepsilon = \sum_{j=1}^m \varepsilon_j$, T is bounded from $H\dot{K}_{q_1}^{\alpha_1, p_1}(\mathbb{R}^n) \times \cdots \times H\dot{K}_{q_m}^{\alpha_m, p_m}(\mathbb{R}^n)$ into $\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$ and from $HK_{q_1}^{\alpha_1, p_1}(\mathbb{R}^n) \times \cdots \times HK_{q_m}^{\alpha_m, p_m}(\mathbb{R}^n)$ into $K_q^{\alpha, p}(\mathbb{R}^n)$, where $0 < p_j < \infty$, $1 < q_j < \infty$, $n(1 - 1/q_j) < \alpha_j < n(1 - 1/q_j) + \varepsilon_j$, $j = 1, \dots, m$, $\alpha = \sum_{j=1}^m \alpha_j$, $1/p = \sum_{j=1}^m 1/p_j$, $1/q = \sum_{j=1}^m 1/q_j$ and $q \ge 1$.*

When $\alpha_j = n(1 - 1/q_j) + \delta_j$, $j = 1, \dots, m$, we can get the following result.

Theorem 3.3 *Let T be a m-linear operator defined by* (1.1) *with kernel $K(x, y_1, \dots, y_m)$ satisfying*

$$\big|K(x, y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_m) - K(x, y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_m)\big| \le C\frac{|y_j|^\delta}{|x|^{mn+\delta}}, \quad (3.5)$$

if $2|y_j| < |x|$, for some $\delta > 0$, $j = 1, \dots, m$.

Suppose $0 < p_j \le 1$, $1 < q_j < \infty$, $j = 1, \dots, m$, $1/p = \sum_{j=1}^m 1/p_j$, $1/q = \sum_{j=1}^m 1/q_j$. If T is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$, then for any $\delta_1, \dots, \delta_m > 0$ with $\delta = \sum_{j=1}^m \delta_j$, T is bounded from $H\dot{K}_{q_1}^{n(1-1/q_1)+\delta_1, p_1}(\mathbb{R}^n) \times \cdots \times H\dot{K}_{q_m}^{n(1-1/q_m)+\delta_m, p_m}(\mathbb{R}^n)$ into

$W\dot{K}_q^{n(m-1/q)+\delta,\,p}(\mathbb{R}^n)$.

Proof. In order to simplify the proof, we only consider the situation when $m=2$. Actually, the similar procedure works for all $m \in \mathbb{N}$.

For $f_j \in H\dot{K}_{q_j}^{n(1-1/q_j)+\delta_j,\,p_j}(\mathbb{R}^n)$, by the atomic decomposition of the Herz-type Hardy space [10], we can write $f_j = \sum_{l_j=-\infty}^{\infty} \lambda_{l_j} a_{l_j}$, where

$$\left(\sum_{l_j=-\infty}^{\infty} |\lambda_{l_j}|^{p_j} \right)^{1/p_j} \le C \|f_j\|_{H\dot{K}_{q_j}^{n(1-1/q_j)+\delta_j,\,p_j}(\mathbb{R}^n)},$$

each a_{l_j} is a central $(n(1-1/q_j)+\delta_j,\, q_j)$-atom supported on B_{l_j} and satisfies

(i) $\|a_{l_j}\|_{L^{q_j}(\mathbb{R}^n)} \le 2^{-l_j(n(1-1/q_j)+\delta_j)}$,

(ii) $\int a_{l_j}(x) x^\beta dx = 0$ for $|\beta| \le [\delta_j]$, $j=1,\,2$.

Then

$$\|T(f_1,\, f_2)\|_{W\dot{K}_q^{n(2-1/q)+\delta,\,p}(\mathbb{R}^n)} \le C(I_1+I_2+I_3+I_4),$$

where

$$I_1 = \sup_{\lambda>0} \lambda \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{k\left(n(2-1/q)+\delta\right)p} \times \left| \left\{ x \in E_k : \left| T\left(\sum_{l_1=k-1}^{\infty} \lambda_1 a_{l_1},\, \sum_{l_2=k-1}^{\infty} \lambda_2 a_{l_2} \right)(x) \right| > \frac{\lambda}{4} \right\} \right|^{p/q} \Bigg\}^{1/p},$$

$$I_2 = \sup_{\lambda>0} \lambda \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{k\left(n(2-1/q)+\delta\right)p} \times \left| \left\{ x \in E_k : \sum_{l_1=-\infty}^{k-2} \sum_{l_2=-\infty}^{k-2} |\lambda_{l_1}||\lambda_{l_2}||T(a_{l_1}, a_{l_2})(x)| > \frac{\lambda}{4} \right\} \right|^{p/q} \Bigg\}^{1/p},$$

$$I_3 = \sup_{\lambda>0} \lambda \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{k\left(n(2-1/q)+\delta\right)p} \times \left| \left\{ x \in E_k : \sum_{l_1=-\infty}^{k-2} \sum_{l_2=k-1}^{\infty} |\lambda_{l_1}||\lambda_{l_2}||T(a_{l_1}, a_{l_2})(x)| > \frac{\lambda}{4} \right\} \right|^{p/q} \Bigg\}^{1/p},$$

$$I_4 = \sup_{\lambda>0} \lambda \Bigg\{ \sum_{k=-\infty}^{\infty} 2^{k\left(n(2-1/q)+\delta\right)p}$$

$$\times\bigg|\bigg\{x\in E_k\colon \sum_{l_1=k-1}^{\infty}\sum_{l_2=-\infty}^{k-2}|\lambda_{l_1}||\lambda_{l_2}||T(a_{l_1},a_{l_2})(x)|>\frac{\lambda}{4}\bigg\}\bigg|^{p/q}\bigg\}^{1/p}.$$

Using the fact that T is bounded from $L^{q_1}(\mathbb{R}^n)\times L^{q_2}(\mathbb{R}^n)$ into $L^{q,\infty}(\mathbb{R}^n)$ and $0<p_j\le 1$, $1<q_j<\infty$, $j=1,\,2$, by Hölder's inequality, we see that

$$\begin{aligned}
I_1 &\le C\bigg\{\sum_{k=-\infty}^{\infty}2^{k\left(n(2-1/q)+\delta\right)p}\bigg\|\sum_{l_1=k-1}^{\infty}\lambda_{l_1}a_{l_1}\bigg\|_{L^{q_1}}^{p}\bigg\|\sum_{l_2=k-1}^{\infty}\lambda_{l_2}a_{l_2}\bigg\|_{L^{q_2}}^{p}\bigg\}^{1/p}\\
&\le C\bigg\{\sum_{k=-\infty}^{\infty}2^{k\left(n(1-1/q_1)+\delta_1\right)p_1}\bigg\|\sum_{l_1=k-1}^{\infty}\lambda_{l_1}a_{l_1}\bigg\|_{L^{q_1}}^{p_1}\bigg\}^{1/p_1}\\
&\qquad\times\bigg\{\sum_{k=-\infty}^{\infty}2^{k\left(n(1-1/q_2)+\delta_2\right)p_2}\bigg\|\sum_{l_2=k-1}^{\infty}\lambda_{l_2}a_{l_2}\bigg\|_{L^{q_2}}^{p_2}\bigg\}^{1/p_2}\\
&\le C\bigg\{\sum_{l_1=-\infty}^{\infty}|\lambda_{l_1}|^{p_1}\sum_{k=-\infty}^{l_1+1}2^{(k-l_1)\left(n(1-1/q_1)+\delta_1\right)p_1}\bigg\}^{1/p_1}\\
&\qquad\times\bigg\{\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\sum_{k=-\infty}^{l_2+1}2^{(k-l_2)\left(n(1-1/q_2)+\delta_2\right)p_2}\bigg\}^{1/p_2}\\
&= C\bigg(\sum_{l_1=-\infty}^{\infty}|\lambda_{l_1}|^{p_1}\bigg)^{1/p_1}\bigg(\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\bigg)^{1/p_2}\\
&\le C\|f_1\|_{H\dot K_{q_1}^{n(1-1/q_1)+\delta_1,\,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot K_{q_2}^{n(1-1/q_2)+\delta_2,\,p_2}(\mathbb{R}^n)}.
\end{aligned}$$

For I_2, noticing that $l_1\le k-2$, $l_2\le k-2$ and $x\in E_k$. Without loss of generality, for fixed l_1 and l_2, we assume $l_1\le l_2$.

Since a_{l_1} is a central $(n(1-1/q_1)+\delta_1,\,q_1)$-atom, by (3.5), Hölder's inequality and $l_1\le l_2$, we have

$$\begin{aligned}
&|T(a_{l_1},a_{l_2})(x)|\\
&=\bigg|\int_{B_{l_2}}a_{l_2}(y_2)\int_{B_{l_1}}a_{l_1}(y_1)\Big(K(x,\,y_1,\,y_2)-K(x,\,0,\,y_2)\Big)dy_1dy_2\bigg|\\
&\le C\int_{B_{l_2}}|a_{l_2}(y_2)|\int_{B_{l_1}}|a_{l_1}(y_1)|\frac{|y_1|^{\delta}}{|x|^{2n+\delta}}dy_1dy_2\\
&\le C2^{l_1\delta}2^{-k(2n+\delta)}\|a_{l_1}\|_{L^{q_1}}2^{l_1n(1-1/q_1)}\|a_{l_2}\|_{L^{q_2}}2^{l_2n(1-1/q_2)}\\
&\le C2^{(l_1-l_2)\delta_2}2^{-k(2n+\delta)}
\end{aligned}$$

$$\leq C2^{-k(2n+\delta)}.$$

Therefore,

$$\begin{aligned}&\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}|\lambda_{l_1}||\lambda_{l_2}||T(a_{l_1},\, a_{l_2})(x)|\\&\leq C2^{-k(2n+\delta)}\Big(\sum_{l_1=-\infty}^{k-2}|\lambda_{l_1}|^{p_1}\Big)^{1/p_1}\Big(\sum_{l_2=-\infty}^{k-2}|\lambda_{l_2}|^{p_2}\Big)^{1/p_2}\\&\leq C_0 2^{-k(2n+\delta)}\|f_1\|_{H\dot K_{q_1}^{n(1-1/q_1)+\delta_1,\,p_1}(\mathbb{R}^n)}\\&\qquad\times\|f_2\|_{H\dot K_{q_2}^{n(1-1/q_2)+\delta_2,\,p_2}(\mathbb{R}^n)}.\end{aligned}\tag{3.6}$$

Given any fixed $\lambda > 0$, if

$$\Big|\Big\{x\in E_k\colon \sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}|\lambda_{l_1}||\lambda_{l_2}||T(a_{l_1},\, a_{l_2})(x)|>\frac{\lambda}{4}\Big\}\Big|\neq 0,$$

then by (3.6),

$$\frac{\lambda}{4}\leq C_0 2^{-k(2n+\delta)}\|f_1\|_{H\dot K_{q_1}^{n(1-1/q_1)+\delta_1,\,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot K_{q_2}^{n(1-1/q_2)+\delta_2,\,p_2}(\mathbb{R}^n)}.$$

That is

$$\begin{aligned}&k\leq (2n+\delta)^{-1}\\&\times\log_2\big(4C_0\lambda^{-1}\|f_1\|_{H\dot K_{q_1}^{n(1-1/q_1)+\delta_1,\,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot K_{q_2}^{n(1-1/q_2)+\delta_2,\,p_2}(\mathbb{R}^n)}\big)\\&\qquad:= N_\lambda.\end{aligned}$$

From this, we can obtain

$$\begin{aligned}I_2&\leq C\sup_{\lambda>0}\lambda\Big(\sum_{k=-\infty}^{[N_\lambda]}2^{k\big(n(2-1/q)+\delta\big)p}2^{knp/q}\Big)^{1/p}\\&=C\sup_{\lambda>0}\lambda 2^{(2n+\delta)[N_\lambda]}\\&\leq C\sup_{\lambda>0}\big(\lambda\cdot 4C_0\lambda^{-1}\|f_1\|_{H\dot K_{q_1}^{n(1-1/q_1)+\delta_1,\,p_1}(\mathbb{R}^n)}\\&\qquad\times\|f_2\|_{H\dot K_{q_2}^{n(1-1/q_2)+\delta_2,\,p_2}(\mathbb{R}^n)}\big)\\&=C\|f_1\|_{H\dot K_{q_1}^{n(1-1/q_1)+\delta_1,\,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot K_{q_2}^{n(1-1/q_2)+\delta_2,\,p_2}(\mathbb{R}^n)}.\end{aligned}$$

From the analogous argumentation of I_2, the similar estimates of I_3 and I_4 can be deduced. Combining all the estimates on I_1, I_2, I_3 and I_4, the desired result can be established. This completes the proof of Theorem 3.3. $\square$

Remark 3.5 It should be pointed out that there is a counterpart of the above result on inhomogeneous Herz space, but we omit the details for their similarity.

It is easy to see that the multilinear Calderón-Zygmund operator obviously satisfies the condition (3.5) by taking $\varepsilon = \delta$ in (1.3), so (3.5) is weaker than (1.3). Uniting Theorem A, we can obtain a corollary of Theorem 3.3.

Corollary 3.3 *Let T be a m-linear Calderón-Zygmund operator and ε be as in* (1.3). *Then for any $\varepsilon_1, \ldots, \varepsilon_m > 0$ with $\varepsilon = \sum_{j=1}^m \varepsilon_j$, T is bounded from $H\dot{K}_{q_1}^{n(1-1/q_1)+\varepsilon_1, p_1}(\mathbb{R}^n) \times \cdots \times H\dot{K}_{q_m}^{n(1-1/q_m)+\varepsilon_m, p_m}(\mathbb{R}^n)$ into $W\dot{K}_q^{n(m-1/q)+\varepsilon, p}(\mathbb{R}^n)$ and from*

$$HK_{q_1}^{n(1-1/q_1)+\varepsilon_1, p_1}(\mathbb{R}^n) \times \cdots \times HK_{q_m}^{n(1-1/q_m)+\varepsilon_m, p_m}(\mathbb{R}^n)$$

into $WK_q^{n(m-1/q)+\varepsilon, p}(\mathbb{R}^n)$, where $0 < p_j \leq 1$, $1 < q_j < \infty$, $j = 1, \ldots, m$, $1/p = \sum_{j=1}^m 1/p_j$, $1/q = \sum_{j=1}^m 1/q_j$.

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