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# Involutions of the Mathieu group $M_{24}$

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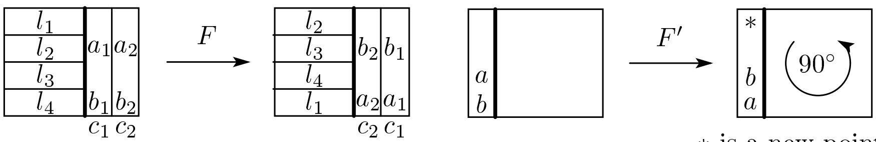
**Abstract.** We shall construct involutions, in the symmetric group $S_{24}$, which generate $M_{22}$, $M_{23}$ and $M_{24}$.

*Key words*: $M$-matrix, Golay code, Mathieu group.

## 1. Introduction

In his paper [1], N. Chigira constructed involutions, in the symmetric group $S_{12}$ by playing a $3\times3$ board game, which generate the Mathieu groups $M_{11}$ and $M_{12}$. Order three version of Chigira's theorem are proved in [4, 5]. In this article, we make a $4 \times 5$ board game and, by playing this game, we construct involutions which generate the Mathieu groups $M_{22}$, $M_{23}$ and $M_{24}$. The precise statement is as follows.

We define two actions $F$ and $F'$ on the $4 \times 5$ board by the following:




$*$ is a new point.

First of all we take the following board, which is denoted by $I$ :

| | | | | |
|---|---|---|---|---|
| 1 | 5 | 9 | 13 | 17 |
| 2 | 6 | 10 | 14 | 18 |
| 3 | 7 | 11 | 15 | 19 |
| 4 | 8 | 12 | 16 | 20 |

By applying actions $F$, $F^2$ and $F^3$ to the board $I$, we get the following three boards:

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| | | | | |
|---|---|---|---|---|
| 2 | 6 | 10 | 17 | 13 |
| 3 | 7 | 11 | 20 | 16 |
| 4 | 8 | 12 | 19 | 15 |
| 1 | 5 | 9 | 18 | 14 |

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| | | | | |
|---|---|---|---|---|
| 3 | 7 | 11 | 13 | 17 |
| 4 | 8 | 12 | 14 | 18 |
| 1 | 5 | 9 | 15 | 19 |
| 2 | 6 | 10 | 16 | 20 |

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| | | | | |
|---|---|---|---|---|
| 4 | 8 | 12 | 17 | 13 |
| 1 | 5 | 9 | 20 | 16 |
| 2 | 6 | 10 | 19 | 15 |
| 3 | 7 | 11 | 18 | 14 |

.

These three boards are denoted by $J$, $K$, $L$, respectively. Next, applying the action $F'$ to $I$, $J$, $K$ and $L$ respectively, we get the following four boards:

| | | | | |
|---|---|---|---|---|
| 21 | 17 | 18 | 19 | 20 |
| 2 | 13 | 14 | 15 | 16 |
| 4 | 9 | 10 | 11 | 12 |
| 3 | 5 | 6 | 7 | 8 |

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| | | | | |
|---|---|---|---|---|
| 22 | 13 | 16 | 15 | 14 |
| 3 | 17 | 20 | 19 | 18 |
| 1 | 10 | 11 | 12 | 9 |
| 4 | 6 | 7 | 8 | 5 |

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| | | | | |
|---|---|---|---|---|
| 23 | 17 | 18 | 19 | 20 |
| 4 | 13 | 14 | 15 | 16 |
| 2 | 11 | 12 | 9 | 10 |
| 1 | 7 | 8 | 5 | 6 |

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| | | | | |
|---|---|---|---|---|
| 24 | 13 | 16 | 15 | 14 |
| 1 | 17 | 20 | 19 | 18 |
| 3 | 12 | 9 | 10 | 11 |
| 2 | 8 | 5 | 6 | 7 |

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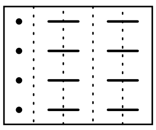
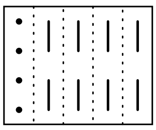
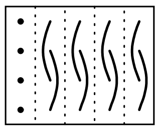
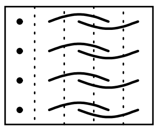
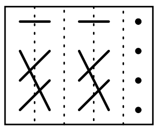
We call them $I'$, $J'$, $K'$, $L'$.

For the board $I$, we define five involutions $i_1$, ..., $i_5$ in $S_{24}$ by the following:

$i_1 =$  , $i_2 =$  , $i_3 =$  ,

$i_4 =$  , $i_5 =$  .

We denote the subgroup $\langle i_1, \ldots, i_5 \rangle$ by $I$, i.e., the name of the board. Similarly we define the subgroups $I'$, $J$, $J'$, $K$, $K'$, $L$ and $L'$ of $S_{24}$.

**Theorem** *Let $X_1$, $X_2$, $X_3$, $X_4$ be any permutation of the four group $I'$, $J'$, $K'$, $L'$. Then we have the following*:

(1) $\langle I, X_1 \rangle \simeq \langle J, X_1 \rangle \simeq \langle K, X_1 \rangle \simeq \langle L, X_1 \rangle \simeq \mathrm{PSL}(3, 4)$

(2) $\langle I, X_1, X_2 \rangle \simeq \langle J, X_1, X_2 \rangle \simeq \langle K, X_1, X_2 \rangle \simeq \langle L, X_1, X_2 \rangle \simeq M_{22}$

(3) $\langle I, X_1, X_2, X_3 \rangle \simeq \langle J, X_1, X_2, X_3 \rangle \simeq \langle K, X_1, X_2, X_3 \rangle$
$\simeq \langle L, X_1, X_2, X_3 \rangle \simeq M_{23}$

(4) $\langle I, X_1, X_2, X_3, X_4 \rangle \simeq \langle J, X_1, X_2, X_3, X_4 \rangle \simeq \langle K, X_1, X_2, X_3, X_4 \rangle$
$\simeq \langle L, X_1, X_2, X_3, X_4 \rangle \simeq M_{24}$

## 2. Involutions of $M_{24}$

First of all we recall the binary extended Golay code. Let $\Omega = \{1, 2, \ldots, 24\}$. We denote by $P(\Omega)$ the power set $\Omega$. It is a vector space over $\mathbb{F}_2$ via the addition operation $A + B := (A \cup B) \backslash (A \cap B)$. Consider the following $4 \times 6$ arrangement of $\Omega$:

$$\Omega := \begin{array}{|c|c|c|c|c|c|}\hline 21 & 1 & 5 & 9 & 13 & 17 \\ \hline 22 & 2 & 6 & 10 & 14 & 18 \\ \hline 23 & 3 & 7 & 11 & 15 & 19 \\ \hline 24 & 4 & 8 & 12 & 16 & 20 \\ \hline \end{array} .$$

We denote this arrangement by

$$\big((k, i)\big), \; k \in \mathbb{F}_4, \quad 1 \leq i \leq 6.$$

Namely, $(0, 1) = 21$, $(1, 1) = 22$, $(\omega, 1) = 23$, $(\bar{\omega}, 1) = 24$, $(0, 2) = 1$ and so on. Here $\mathbb{F}_4^{\times} = \langle \omega \rangle$. Therefore we have

$$P(\Omega) \simeq \sum_{(k, i)} \mathbb{F}_2(k, i).$$

For each $i$, we define a typical element $c_i$ in $P(\Omega)$ by

$$c_i = \begin{array}{|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{i} & \multicolumn{1}{c}{} \\ \hline & 1 & \\ 0 & \vdots & 0 \\ & 1 & \\ \hline \end{array} .$$

Define a map

$$s_i \colon \mathbb{F}_4 \longrightarrow \sum_{k \in \mathbb{F}_4} \mathbb{F}_2(k, i)$$

by

$$k \longmapsto (k, i),$$

and set $S = (s_1, \ldots, s_6)$. When $\mathcal{H}$ is the 4-ary hexacode [6, 3, 4], the linear code

$$\mathcal{C}_{24} = \langle c_i - c_j, \, c_i + S(t) \mid 1 \leq i, \, j \leq 6, \, t \in \mathcal{H} \rangle \subset P(\Omega)$$

is the binary extended Golay code. For details, we refer to [2], [3] and [4].

A $4 \times 6$ arrangement $(v_{(k,i)})$ of $\Omega$ is called an $M$-matrix if the map $(k, i) \longmapsto v_{(k,i)}$ induces an automorphism of $\mathcal{C}_{24}$. As for the $M$-matrices, we refer to [4] and [6]. The matrix $\big((k,\, i)\big)$ itself is apparently an $M$-matrix, and we have the following seven $M$-matrices:

| | | | | | |
|---|---|---|---|---|---|
| 22 | 2 | 6 | 10 | 17 | 13 |
| 23 | 3 | 7 | 11 | 20 | 16 |
| 24 | 4 | 8 | 12 | 19 | 15 |
| 21 | 1 | 5 | 9 | 18 | 14 |

,

| | | | | | |
|---|---|---|---|---|---|
| 23 | 3 | 7 | 11 | 13 | 17 |
| 24 | 4 | 8 | 12 | 14 | 18 |
| 21 | 1 | 5 | 9 | 15 | 19 |
| 22 | 2 | 6 | 10 | 16 | 20 |

,

| | | | | | |
|---|---|---|---|---|---|
| 24 | 4 | 8 | 12 | 17 | 13 |
| 21 | 1 | 5 | 9 | 20 | 16 |
| 22 | 2 | 6 | 10 | 19 | 15 |
| 23 | 3 | 7 | 11 | 18 | 14 |

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| | | | | | |
|---|---|---|---|---|---|
| 1 | 21 | 17 | 18 | 19 | 20 |
| 22 | 2 | 13 | 14 | 15 | 16 |
| 23 | 4 | 9 | 10 | 11 | 12 |
| 24 | 3 | 5 | 6 | 7 | 8 |

,

| | | | | | |
|---|---|---|---|---|---|
| 2 | 22 | 13 | 16 | 15 | 14 |
| 23 | 3 | 17 | 20 | 19 | 18 |
| 24 | 1 | 10 | 11 | 12 | 9 |
| 21 | 4 | 6 | 7 | 8 | 5 |

,

| | | | | | |
|---|---|---|---|---|---|
| 3 | 23 | 17 | 18 | 19 | 20 |
| 24 | 4 | 13 | 14 | 15 | 16 |
| 21 | 2 | 11 | 12 | 9 | 10 |
| 22 | 1 | 7 | 8 | 5 | 6 |

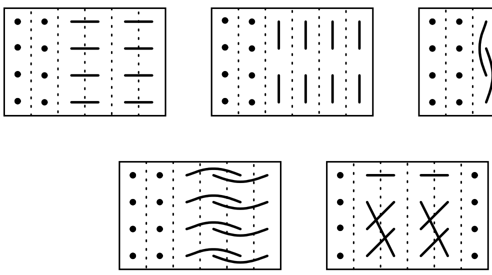
,

| | | | | | |
|---|---|---|---|---|---|
| 4 | 24 | 13 | 16 | 15 | 14 |
| 21 | 1 | 17 | 20 | 19 | 18 |
| 22 | 3 | 12 | 9 | 10 | 11 |
| 23 | 2 | 8 | 5 | 6 | 7 |

.

The table in [7] of octads, i.e., 8-set elements in $\mathcal{C}_{24}$ is helpful in determining $M$-matrices.

**Lemma 1** *For each $M$-matrix, we have five involutions in* $\mathrm{Aut}(\mathcal{C}_{24})$:



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## 3. Proof of Theorem

We begin with the following lemma:

**Lemma 1** *Let*

$$c := (i_1 i_5 i_4)^{i_5} = (1)(17)(5\ 9\ 13)(8\ 10\ 15)$$
$$(6\ 11\ 16)(7\ 12\ 14)(3\ 2\ 4)(18\ 19\ 20).$$

*Then we have*

$$\langle i_1,\, i_2,\, i_3,\, i_4,\, c\rangle \simeq 2^4 \rtimes Z_3.$$

*Moreover $I$ is transitive on the set $\bar{\Omega} = \{1, 2, \ldots, 20\}$.*

*Proof.* We have the following elements in $\langle I\rangle$:

$i_1$:=(1)(2)(3)(4)(5 9)(6 10)(7 11)(8 12)(13 17)(14 18)(15 19)(16 20)
$i_2$:=(1)(2)(3)(4)(5 6)(7 8)(9 10)(11 12)(13 14)(15 16)(17 18)(19 20)
$i_3$:=(1)(2)(3)(4)(5 7)(6 8)(9 11)(10 12)(13 15)(14 16)(17 19)(18 20)
$i_4$:=(1)(2)(3)(4)(5 13)(9 17)(6 14)(10 18)(7 15)(11 19)(8 16)(12 20)
$i_5$:=(17)(18)(19)(20)(5 1)(2 8)(3 6)(4 7)(13 9)(14 11)(15 12)(10 16)
$b_1$:=$i_2$=(1)(2)(3)(4)(5 6)(7 8)(9 10)(11 12)(13 14)(15 16)(17 18)(19 20)
$b_2$:=$i_3$=(1)(2)(3)(4)(5 7)(6 8)(9 11)(10 12)(13 15)(14 16)(17 19)(18 20)
$b_3$:=$i_2i_3$=(1)(2)(3)(4)(5 8)(7 6)(9 12)(11 10)(13 16)(15 14)(17 20)(19 18)
$b_4$:=$i_1$=(1)(2)(3)(4)(5 9)(6 10)(7 11)(8 12)(13 17)(14 18)(15 19)(16 20)
$b_5$:=$i_1i_2$=(1)(2)(3)(4)(5 10)(6 9)(7 12)(8 11)(13 18)(14 17)(15 20)(16 19)
$b_6$:=$i_1i_3$=(1)(2)(3)(4)(5 11)(6 12)(7 9)(8 10)(13 19)(14 20)(15 17)(16 18)
$b_7$:=$i_1i_2i_3$=(1)(2)(3)(4)(5 12)(6 11)(7 10)(8 9)(13 20)(14 19)(15 18)(16 17)
$b_8$:=$i_4$=(1)(2)(3)(4)(5 13)(9 17)(6 14)(10 18)(7 15)(11 19)(8 16)(12 20)
$b_9$:=$i_2i_4$=(1)(2)(3)(4)(5 14)(7 16)(9 18)(11 20)(13 6)(15 8)(17 10)(19 12)
$b_{10}$:=$i_3i_4$=(1)(2)(3)(4)(5 15)(6 16)(9 19)(10 20)(13 7)(14 8)(17 11)(18 12)
$b_{11}$:=$i_2i_3i_4$=(1)(2)(3)(4)(5 16)(7 14)(9 20)(11 18)(13 8)(15 6)(17 12)(19 10)
$b_{12}$:=$i_1i_4$=(1)(2)(3)(4)(5 17)(6 18)(7 19)(8 20)(13 9)(14 10)(15 11)(16 12)
$b_{13}$:=$i_1i_2i_4$=(1)(2)(3)(4)(5 18)(6 17)(7 20)(8 19)(13 10)(14 9)(15 12)(16 11)
$b_{14}$:=$i_1i_3i_4$=(1)(2)(3)(4)(5 19)(6 20)(7 17)(8 18)(13 11)(14 12)(15 9)(16 10)
$b_{15}$:=$i_1i_2i_3i_4$=(1)(2)(3)(4)(5 20)(6 19)(7 18)(8 17)(13 12)(14 11)(15 10)(16 9)

Using these elements, we see that $\langle I\rangle$ is transitive on $\bar{\Omega}$. By an easy but tedious calculation, we see that $\langle i_1, i_2, i_3, i_4\rangle = \langle i_1\rangle \times \langle i_2\rangle \times \langle i_3\rangle \times \langle i_4\rangle \simeq 2^4$. Since $(i_1)^c = b_{12}$, $(i_2)^c = b_2$, $(i_3)^c = b_3$ and $(i_4)^c = b_4$, it follows that $\langle i_1, i_2, i_3, i_4\rangle$ is a normal subgroup of $\langle i_1, i_2, i_3, i_4, c\rangle$. Hence we get

$$\langle i_1,\, i_2,\, i_3,\, i_4,\, c\rangle = (\langle i_1\rangle \times \langle i_2\rangle \times \langle i_3\rangle \times \langle i_4\rangle) \rtimes \langle c\rangle \simeq 2^4 \rtimes Z_3.$$

□

Now we shall prove the theorem. Corresponding each component of the $4 \times 6$ array of $\Omega$ defined in §2 to one of the following:

$$\begin{array}{cccccc} (0,1,0) & (1,0,0) & (0,0,1) & (1,0,1) & (\omega,0,1) & (\bar{\omega},0,1) \\ I & (1,1,0) & (0,1,1) & (1,1,1) & (\omega,1,1) & (\bar{\omega},1,1) \\ II & (1,\omega,0) & (0,\omega,1) & (1,\omega,1) & (\omega,\omega,1) & (\bar{\omega},\omega,1) \\ III & (1,\bar{\omega},0) & (0,\bar{\omega},1) & (1,\bar{\omega},1) & (\omega,\bar{\omega},1) & (\bar{\omega},\bar{\omega},1) \end{array},$$

we obtain a bijection between the set $\{1, 2, \ldots, 24\}$ and $\mathbb{P}^3(\mathbb{F}_4) \cup \{I, II, III\}$ (cf. [2], Ch. 11. § 11). Then the involutions $\{i_1, i_2, \ldots, i_5\}$ act on the space $\mathbb{P}^3(\mathbb{F}_4)$ as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \bar{\omega} & 0 & 0 \end{pmatrix},$$

respectively. It is easily seen that $i'_1 = i_2$, $i'_2 = i_1$, $i'_3 = i_4$, $i'_4 = i_3$, and that $i'_5$ corresponds to

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & \omega \\ \bar{\omega} & \bar{\omega} & 0 \end{pmatrix}.$$

By Lemma 1, we see that $\langle I, I' \rangle$ is 2-transitive on the set $\bar{\Omega} \cup \{21\} \simeq \mathbb{P}^3(\mathbb{F}_4)$; thus $|\langle I, I' \rangle| \geq 21 \cdot 20 \cdot 2^4 \cdot 3$. On the other hand, we see that $\langle I, I' \rangle$ can be considered as a subgroup of PSL(3, 4); hence we have $|\langle I, I' \rangle| \leq 21 \cdot 20 \cdot 2^4 \cdot 3$. Therefore $\langle I, I' \rangle \simeq \mathrm{PSL}(3, 4)$.

Since $\langle I, I', J' \rangle$ is transitive on the set $\bar{\Omega} \cup \{21, 22\}$ and $\langle I, I', J' \rangle \subset (M_{24})_{24\ 23} \simeq M_{22}$, it follows that

$$\begin{aligned} 22 \cdot 21 \cdot 20 \cdot 2^4 \cdot 3 = |M_{22}| &\geq |\langle I, I', J' \rangle| \\ &\geq 22 \cdot |PSL(3, 4)| = 22 \cdot 21 \cdot 20 \cdot 2^4 \cdot 3. \end{aligned}$$

Therefore $\langle I, I', J' \rangle \simeq M_{22}$. Similarly, we have

$$\langle I, I', J', K' \rangle \simeq M_{23}, \quad \langle I, I', J', K', L' \rangle \simeq M_{24}.$$

By symmetry, we have (1), (2), (3) and (4).

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