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The wave equation for the p-Laplacian

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Abstract. We consider generalized wave equations for the p-Laplacian and prove the local in time existence of solutions to the Cauchy problem. We give an estimate of the life-span of the solution, and show by a generic counter-example that global in time solutions can not be expected.

Key words: local in time Sobolev solutions, blow-up in finite time, weakly hyperbolic equations.

1. Introduction

This paper is devoted to strong solutions to the hyperbolic Cauchy problem

$$\begin{aligned} &w_{tt}(t, x) - (|w_x(t, x)|^{p-2} w_x(t, x))_x = 0, \\ &w(0, x) = \Phi(x), \quad w_t(0, x) = \Psi(x), \end{aligned} \tag{1.1}$$

where p is a positive real number, not necessarily an even integer. More generally, we shall study

$$\begin{aligned} &w_{tt}(t, x) - a(w_x(t, x)) w_{xx}(t, x) = 0, \\ &w(0, x) = \Phi(x), \quad w_t(0, x) = \Psi(x), \end{aligned} \tag{1.2}$$

where $a = a(s)\colon [-M, M] \to \mathbb{R}$ is a function with the following properties.

Condition 1 For all $s \in [-M, M] = \overline{B_M}$, the following holds:

$$a(s) \ge 0, \quad a(s) = 0 \Longleftrightarrow s = 0, \tag{1.3}$$

$$a(s) = s^2 a_0(s), \quad a_0(s) \le C_a, \tag{1.4}$$

$$0 \le s a_0'(s) \le C_a a_0(s), \quad 0 \le s a'(s) \le C_a a(s). \tag{1.5}$$

Additionally, a_0 is even and a_0, $a_1 \in C^P(\overline{B_M})$, where $a_1(s) = a'(s)/s$, and $P \in \mathbb{N}$.

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Remark 1.1 The choice $a(s) = (p-1)|s|^{p-2}$ leads to (1.1) with $p > P+4$, or $p \in 2\mathbb{N}$, $p \geq 4$, and $P \in \mathbb{N}$ is arbitrary.

The first of our main results is the following.

Theorem 1.2 *Assume that the function $a = a(s)$ satisfies* Condition 1, *and suppose that the initial data Φ, $\Psi \in C_0^{k+2}(\mathbb{R})$ with $4 \leq k \leq P+1$, $P, k \in \mathbb{N}$, are compatible to $a(s)$, i.e., they are real–valued and $\|\Phi_x\|_{L^\infty} < M$.*

Then the Cauchy problem (1.2) *has a real–valued local solution w with*

$$w \in L^\infty([0,\, T_0],\, H^k(\mathbb{R})), \quad \partial_t^2 w \in L^\infty([0,\, T_0],\, H^{k-2}(\mathbb{R})).$$

This solution vanishes outside $[0, T_0] \times \operatorname{supp}(\Phi, \Psi)$. The estimate T_0 of the life span only depends on M, $\operatorname{supp}(\Phi, \Psi)$, and the norms $\|(\Phi, \Psi)\|_{C^6(\mathbb{R})}$, $\|(a_0, a_1)\|_{C^3(B_M)}$. The solution is unique in the space of all functions w with $w \in L^\infty([0, T_0], H^4(\mathbb{R}))$, $\partial_t^2 w \in L^\infty([0, T_0], H^2(\mathbb{R}))$.

Remark 1.3 By the same arguments, we can study the more general equation

$$w_{tt} - a(w_x)w_{xx} - b(w_x) - cw = 0,$$

where $a(s)$ is as above, $b(s)$ is sufficiently smooth with $b(0) = 0$ and $|b'(s)|^2 \leq Ca(s)$, and c is a real constant. It is even possible to allow an additional dependence on time of a, b, c. However, for simplicity, we stick to (1.2).

Remark 1.4 If the function $a(s)$ is analytic, we can drop Condition 1 and follow a modified version of the proof given in [17], where equations $u_{tt} - a(u)\Delta u = 0$ with analytic function a were studied.

In the proof, we shall replace the nonsmooth coefficient $a(s)$ by a smooth approximation, preserving the other conditions.

Condition 2 The coefficient $a = a(s)$ satisfies Condition 1, and $a_0 \in C^\infty(\overline{B_M})$.

Theorem 1.5 *Let the assumptions of* Theorem 1.2 *be satisfied. Additionally, suppose that $a = a(s)$ satisfies* Condition 2, *and Φ, $\Psi \in C_0^\infty(\mathbb{R})$.*

Then the solution w to the Cauchy problem (1.2) *belongs to $C_b^\infty([0, T_0] \times \mathbb{R})$.*

The life span of the solution tends to infinity for initial data approaching zero, in the following sense. Fix some $0 < \lambda \ll 1$, and consider the Cauchy problem

$$\begin{aligned} &w_{tt}(t, x) - a(w_x(t, x))w_{xx}(t, x) = 0, \\ &w(0, x) = \lambda\Phi(x), \quad w_t(0, x) = \lambda\Psi(x). \end{aligned} \tag{1.6}$$

Theorem 1.6 *Let the assumptions of* Theorem 1.2 *be satisfied. Then the lower estimate of the life span $T_0 = T_0(\lambda)$ goes to infinity for $\lambda \to 0$. More precisely,*

$$T_0(\lambda) \geq C|\ln\lambda|^{1/3}, \quad 0 < \lambda \ll 1.$$

It is known (see [7]) that (1.2) admits a unique local solution in Sobolev spaces in the strictly hyperbolic case, ($a(s) \geq \alpha > 0$). However, this solution is never a global classical solution, except in trivial cases. In [13], the Cauchy problem

$$\begin{aligned} &w_{tt}(t, x) - a(w_x(t, x))^2 w_{xx}(t, x) = 0, \\ &w(0, x) = \Phi(x), \quad w_t(0, x) = \Psi(x) \end{aligned}$$

has been considered, where $a(w_x) > 0$, $a'(w_x) \neq 0$, and the data Φ, Ψ have compact support. It was shown that the only global solution $w \in C^2(\mathbb{R}_t \times \mathbb{R}_x)$ is $w \equiv 0$. In other words, every nontrivial solution develops a singularity in finite time, it is the second derivatives of w that become infinite. This result can not be applied to (1.2) since (1.2) is neither strictly hyperbolic nor everywhere genuinely nonlinear. However, by a different method, we show in Section 9 that global solutions to (1.1) can not exist in case of $p = 4$ provided that the initial data satisfy appropriate sign conditions.

At first glance, it seems natural to attack (1.2) by a linearisation argument, leading to a family of Cauchy problems

$$\begin{aligned} &w_{tt}^{(n+1)}(t, x) - a(w_x^{(n)}(t, x))w_{xx}^{(n+1)}(t, x) = 0, \\ &w^{(n+1)}(0, x) = \Phi(x), \quad w_t^{(n+1)}(0, x) = \Psi(x), \end{aligned}$$

and then one hopes to be able to show convergence $w^{(n)} \to w^*$ at least for small times. In general, this direct approach will not work in the weakly hyperbolic case. In fact, a Cauchy problem

$$\begin{aligned} &w_{tt}(t, x) - a(t)w_{xx}(t, x) = 0, \quad a \geq 0, \quad a \in C^\infty, \\ &w(0, x) = \Phi(x), \quad w_t(0, x) = \Psi(x), \quad \Phi, \Psi \in C^\infty \end{aligned}$$

without solution was constructed in [3]. On the other hand, (1.2) is well–posed in Gevrey spaces with Gevrey index between 1 and 2 if $a = a(s)$ is analytic. This is a special case of much more general results in [14], [15]. If one allows damping terms of the form $(-\Delta)^\alpha \partial_t w$ in (1.2), $0 < \alpha \leq 1$, then the global existence and the energy decay of weak solutions can be proved, see for instance [1], [2], [9], [11].

In [8], the Cauchy problem

$$w_{tt} - \nabla(|\nabla w|^{p-2}\nabla w) - |w|^{q-1}w = 0, \quad p, q > 1, \; q \geq p-1,$$
$$w(0, x) = \Phi_0(x), \quad w_t(0, x) = \Psi_0(x),$$

has been studied. Assuming that Φ_0 and Ψ_0 are real–valued and that $\|\Psi_0\|_{L^2}^2/2 + \|\nabla\Phi_0\|_{L^p}^p/p \leq \|\Phi_0\|_{L^{q+1}}^{q+1}/(q+1)$, it was shown that $\|w(t,\cdot)\|_{L^2}$ blows up in finite time if $\int \Phi_0(x)\Psi_0(x)dx > 0$, and that $\|w(t,\cdot)\|_{L^2}$ decays (for $t \to \infty$) if $\int \Phi_0(x)\Psi_0(x)dx < 0$.

The life span of periodic analytic solutions to the nonlinear Cauchy problem

$$w_{tt} = F(x, w, Dw, D^2w), \quad w(0, x) = \lambda\Phi(x), \quad w_t(0, x) = \lambda\Psi(x)$$

has been studied in [5]. Assuming that this equation is weakly hyperbolic at $(x, 0, 0, 0)$, the estimate $T_0(\lambda) \geq C \log|\log\lambda|$ was proved.

Our approach relies on two key ingredients. The first is a careful investigation of a so-called separating curve, a method which is represented in [6]. The second is a certain decomposition of the solution and the reduction to a hyperbolic 2×2 system of second order. This technique has been developed in [4] and [17], where certain semilinear and quasilinear cases were studied. This method consists of several steps, which are performed in the Sections 2 to 8. A more detailed description can be found at the end of Section 2. The blow–up of solutions for a variant of (1.1) is shown in Section 9.

We employ the standard notations $\partial_x = \partial/\partial x$, $\partial_t = \partial/\partial t$; $H^k(X) = W_2^k(X)$ are the usual Sobolev spaces on an open set X, and $C_b^\infty(X)$ denotes the linear space of all functions that are bounded and continuous together with all their derivatives.

2. Transformation into a system

In order to be able to derive *a priori* estimates for (1.2), we shall transform this Cauchy problem into a second order system. The main advantage is that we will have more information about the principal part available.

Set $u(t, x) = \partial_x w(t, x)$, $\phi(x) = \partial_x \Phi(x)$, $\psi(x) = \partial_x \Psi(x)$. Assuming that w is a solution to (1.2), we find that u solves

$$\begin{aligned} &u_{tt}(t, x) - \partial_x(a(u(t, x))\partial_x u(t, x)) = 0, \\ &u(0, x) = \phi(x), \quad u_t(0, x) = \psi(x). \end{aligned} \tag{2.1}$$

If $\phi(x_0) = \psi(x_0) = 0$, then $(\partial_t^k u)(0, x_0) = 0$ for all $k \in \mathbb{N}$. This suggests the educated guess

$$\begin{aligned} &u(t, x) = \phi(x)g(t, x) + \psi(x)h(t, x), \\ &g(0, x) = 1, \quad h(0, x) = 0, \quad g_t(0, x) = 0, \quad h_t(0, x) = 1. \end{aligned}$$

A direct calculation gives us $u_{tt} = \phi g_{tt} + \psi h_{tt}$ and

$$\begin{aligned} \partial_x(a(u)u_x) &= a(u)(\phi g_{xx} + \psi h_{xx}) \\ &\quad + a'(u)u_x(\phi g_x + \psi h_x) + 2a_0(u)(\phi g + \psi h)^2(\phi_x g_x + \psi_x h_x) \\ &\quad + (\phi g + \psi h)(a_0(u)u(\phi_{xx} g + \psi_{xx} h) + a_1(u)(\phi_x g + \psi_x h)^2), \end{aligned}$$

which leads us to

$$\begin{aligned} &\phi(g_{tt} - \partial_x(a(u)g_x) - 2a_0(u)ug(\phi_x g_x + \psi_x h_x) - cg) \\ &+ \psi(h_{tt} - \partial_x(a(u)h_x) - 2a_0(u)uh(\phi_x g_x + \psi_x h_x) - ch) = 0, \end{aligned}$$

where we have introduced

$$c = c(x, g, h) = a_0(u)u(\phi_{xx} g + \psi_{xx} h) + a_1(u)(\phi_x g + \psi_x h)^2.$$

Now we define the vector $U = (g, h)^T$ of unknowns and

$$A(x, U) = \begin{pmatrix} a(\phi(x)g + \psi(x)h) & 0 \\ 0 & a(\phi(x)g + \psi(x)h) \end{pmatrix}, \tag{2.2}$$

$$\begin{aligned} B(x, U) &= 2a_0(\phi(x)g + \psi(x)h) \\ &\qquad \times (\phi(x)g + \psi(x)h) \begin{pmatrix} \phi_x(x)g & \psi_x(x)g \\ \phi_x(x)h & \psi_x(x)h \end{pmatrix}, \end{aligned} \tag{2.3}$$

$$C(x, U) = \begin{pmatrix} c(x, U) & 0 \\ 0 & c(x, U) \end{pmatrix}. \tag{2.4}$$

Clearly, if we are able to find a solution $U = U(t, x)$ to the Cauchy problem

$$\partial_t^2 U - \partial_x(A(x, U)\partial_x U) - B(x, U)\partial_x U - C(x, U)U = 0, \tag{2.5}$$

$$U(0,\,x) = (1,\,0)^T, \quad U_t(0,\,x) = (0,\,1)^T,$$

then the function $u(t,\,x) = \phi(x)g(t,\,x) + \psi(x)h(t,\,x)$ solves (2.1).

In case of (1.6), we obtain the Cauchy problem

$$\begin{aligned} &\partial_t^2 U - \partial_x(A_\lambda(x,\,U)\partial_x U) - B_\lambda(x,\,U)\partial_x U - C_\lambda(x,\,U)U = 0, \quad (2.6)\\ &U(0,\,x) = (1,\,0)^T, \quad U_t(0,\,x) = (0,\,1)^T, \end{aligned}$$

where A_λ, B_λ, C_λ are defined as in (2.2)–(2.4), with $(\phi,\,\psi)$ replaced by $(\lambda\phi,\,\lambda\psi)$.

We will consider a linearised version of (2.5),

$$\partial_t^2 V - \partial_x(A(x,\,U)\partial_x V) - B(x,\,U)\partial_x V - C(x,\,U)V = F(t,\,x), \tag{2.7}$$

with one of the following initial conditions:

$$V(0,\,x) = V_0(x), \quad V_t(0,\,x) = V_1(x), \tag{2.8}$$

$$V(t_0,\,x) = V_0(x), \quad V_t(t_0,\,x) = V_1(x), \quad 0 < t_0 < T, \tag{2.9}$$

where $U = U(t,\,x)$ is some vector valued function with

$$\left\| U(t,\,\cdot\,) - \begin{pmatrix} 1 \\ t \end{pmatrix} \right\|_{C_b^1([0,\,T]\times B_R)} < \varepsilon \ll 1. \qquad (2.10)_T$$

The paper is organised as follows. In Section 3, we study the behaviour of $A = A(x, U(t,\,x))$ under the condition $(2.10)_T$. Using results from [17], we shall derive *a priori* estimates in Sobolev spaces for a solution V to (2.7) in Section 4. Then, a regularisation argument will enable us to prove the existence of a unique C^∞ solution V to (2.7) in Section 5. By means of Nash–Moser–Hamilton theory, the existence of a local C^∞ solution U to (2.5) will be shown in Section 6. The life span of this solution is studied in Section 7, leading to a proof of Theorem 1.5. Finally, Theorem 1.2 is proved in Section 8. The proof of Theorem 1.6 relies on a careful analysis of the dependence of all constants on λ.

3. The separating curve

Assume that $U = (g, h)^T$ is defined on $[0,\,T] \times B_R$ and fulfils $(2.10)_T$. Setting $U(t,\,x) + U(-t,\,x) := 2U(0,\,x)$, we extend U as a C^1 function to $[-T,\,T] \times B_R$, and have $\left\|U(t,\,\cdot\,) - (1,\,t)^T\right\|_{C^1([-T,\,T]\times B_R)} < \varepsilon$, allowing some modification in ε. The next proposition describes the behaviour of the function $a_*(t, x) = a(\phi(x)g(t, x) + \psi(x)h(t, x))$ in a neighbourhood of the line $\{0\} \times B_R$.

Proposition 3.1 *Let $a = a(s)$ satisfy* Condition 1*, and assume that ϕ, $\psi \in C_0^1(\mathbb{R})$ are compatible data, i.e., $\|\phi\|_{L^\infty} < M$. Introduce the notation*

$$\Omega_{\phi\psi} = \{x\colon |\phi(x)| + |\psi(x)| > 0\}.$$

Then there are constants ε, α, $\tau > 0$ such that for every $U = (g, h)^T$ with $(2.10)_\tau$ there is a $\gamma \in C^1(\Omega_{\phi\psi})$ such that $a_*(t, x) = a(\phi(x)g(t, x) + \psi(x)h(t, x))$ satisfies

$$\alpha a_*(t, x) - \partial_t a_*(t, x) \geq 0 \quad : t < \gamma(x), \quad (t, x) \in [-\tau, \tau] \times \Omega_{\phi\psi}, \tag{3.1}$$

$$\alpha a_*(t, x) + \partial_t a_*(t, x) \geq 0 \quad : t > \gamma(x), \quad (t, x) \in [-\tau, \tau] \times \Omega_{\phi\psi}, \tag{3.2}$$

$$a_*(\gamma(x), x)(\gamma'(x))^2 \leq \frac{1}{4} \qquad : x \in \Omega_{\phi\psi}. \tag{3.3}$$

Moreover, the function γ has the same regularity as ϕ, ψ, and U; and the constants ε, τ, α depend only on M, C_a, $\|(\phi, \psi)\|_{C^1}$.

Remark 3.2 The curve $\{t = \gamma(x)\}$ separates the (t, x) space into two parts. In the following section, different methods will be employed in both parts in order to derive *a priori* estimates of the solution V of (2.7).

Remark 3.3 Condition (3.3) means that the curve $\{t = \gamma(x)\}$ is noncharacteristic.

Proof. This proof is based on ideas from [17].

Set $M' = \|\phi\|_{L^\infty} < M$. If $\tau \leq (M - M')/(2\,\|\psi\|_{L^\infty})$ and $|t| \leq \tau$, then $\|\phi + t\psi\|_{L^\infty} \leq (M + M')/2$. If $0 < \varepsilon \leq \varepsilon_0(M, M', \|\psi\|_{L^\infty})$, then $\|\phi g + \psi h\|_{L^\infty} \leq M$ for $|t| \leq \tau$ and $U = (g, h)^T$ satisfying $(2.10)_\tau$; and the mapping $t \mapsto \chi(t; x) = h(t, x)/g(t, x)$ is invertible for every $|x| \leq R$, $|t| \leq \tau$. Assuming $\varepsilon\tau \leq 1/6$, we get

$$|\chi(t; x) - t| \leq 2\varepsilon + \frac{|t|}{2}, \quad |\chi(t; x)| \leq 2(\varepsilon + |t|), \tag{3.4}$$

since $|\chi_t(t; x) - 1| \leq 1/2$. Then the inverse function $\chi^{-1}(s; x)$ of the mapping

$t \mapsto \chi(t;x)$ satisfies $|\chi^{-1}(s;x)| \le 2(\varepsilon + |s|)$. For every $r > 0$, we set

$$\Omega^r_{\phi\psi} = \{x \in \Omega_{\phi\psi} \colon |\phi(x)| \le r|\psi(x)|\}.$$

Clearly, if $x \in \Omega^r_{\phi\psi}$, then $\psi(x) \ne 0$. Assuming $x \in \Omega_{\phi\psi} \setminus \Omega^r_{\phi\psi}$, we have

$$\begin{aligned}
&|\phi(x)g(t,\,x) + \psi(x)h(t,\,x)| \ge |\phi(x)|g(t,\,x)\Big(1 - \frac{|\chi(t;x)|}{r}\Big),\\
&|\partial_t a_*(t,\,x)| \le C_a a_*(t,\,x)\frac{|\phi(x)g_t(t,\,x) + \psi(x)h_t(t,\,x)|}{|\phi(x)g(t,\,x) + \psi(x)h(t,\,x)|} \qquad (3.5)\\
&\quad \le C_a a_*(t,\,x)\frac{|\phi(x)g_t(t,\,x)| + |\psi(x)h_t(t,\,x)|}{|\phi(x)|g(t,\,x)}\Big(1 - \frac{|\chi(t;x)|}{r}\Big)^{-1}\\
&\quad \le C_a a_*(t,\,x)\frac{\varepsilon r + 1 + \varepsilon}{1-\varepsilon}\frac{1}{r - |\chi(t;x)|}\\
&\quad \le \frac{(2+r)C_a}{r - |\chi(t;x)|}a_*(t,\,x)
\end{aligned}$$

if $|\chi(t;x)| < r$, due to (1.5). Trivially, if $x \in \Omega^{2r}_{\phi\psi}$, then

$$\left|\partial_x \frac{\phi(x)}{\psi(x)}\right| \le \frac{\|\phi'\|_{L^\infty} + 2r\,\|\psi'\|_{L^\infty}}{|\psi(x)|}. \qquad (3.6)$$

Now choose some odd function $\beta = \beta(s) \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\beta \subset (-2,\,2)$ and $\|\beta\|_{L^\infty} \le 2$, $\|\beta'\|_{L^\infty} \le 2$, satisfying $s\beta(s) \le 0$ and $\beta(s) = -s$, $-1 \le s \le 1$. Then we define the separating curve by

$$\gamma(x) = \chi^{-1}\Big(r\beta\Big(\frac{\phi(x)}{r\psi(x)}\Big); x\Big), \quad 0 < r \ll 1.$$

We see that $|\gamma(x)| \le 4(\varepsilon + r)$. Now we check that this function $\gamma = \gamma(x)$ satisfies (3.1)–(3.3) for small r. If $x \in \Omega^r_{\phi\psi}$, then $-\phi(x)/\psi(x) = h(\gamma(x),x)/g(\gamma(x),x)$. In case $t<\gamma(x)$ we have $-\phi(x)/\psi(x)>h(t,x)/g(t,x)$. Assuming

$$\varepsilon(1+r) < 1, \qquad (3.7)$$

we then obtain

$$\frac{\phi(x)g_t(t,\,x) + \psi(x)h_t(t,\,x)}{\phi(x)g(t,\,x) + \psi(x)h(t,\,x)} < 0,$$

which implies

$$\alpha a_*(t,\,x) - \partial_t a_*(t,\,x) = \alpha a_*(t,\,x)$$
$$- a'(\phi(x)g(t,\,x) + \psi(x)h(t,\,x))(\phi(x)g(t,\,x) + \psi(x)h(t,\,x))$$
$$\times \frac{\phi(x)g_t(t,\,x) + \psi(x)h_t(t,\,x)}{\phi(x)g(t,\,x) + \psi(x)h(t,\,x)} \geq 0,$$

for any $\alpha \geq 0$, see Condition 1. The case $t > \gamma(x)$ can be considered similarly.

Now assume that $x \in \Omega_{\phi\psi} \setminus \Omega^r_{\phi\psi}$, $|\chi(t;x)| \leq r/2$. According to (3.5),

$$|\partial_t a_*(t,\,x)| \leq \frac{(2+r)C_a}{r - |\chi(t;x)|} a_*(t,\,x) \leq \frac{(4+2r)C_a}{r} a_*(t,\,x),$$

which proves (3.1) and (3.2) with

$$2\varepsilon \leq \frac{r}{4}, \quad 2\tau \leq \frac{r}{4}, \quad \alpha = \frac{(4+2r)C_a}{r}, \tag{3.8}$$

see (3.4). It remains to check (3.3). This holds true for $x \in \Omega^r_{\phi\psi}$, since then the left–hand side vanishes. Now let $x \in \Omega_{\phi\psi} \setminus \Omega^r_{\phi\psi}$, but $x \in \Omega^{2r}_{\phi\psi}$, which implies $r|\psi(x)| < |\phi(x)| \leq 2r|\psi(x)|$. By elementary computation,

$$\gamma'(x) = \left.\frac{\beta'(\phi(x)/(r\psi(x)))\partial_x(\phi(x)/\psi(x))}{\partial_t(h(t,\,x)/g(t,\,x))}\right|_{t=\gamma(x)}$$
$$- \left.\frac{\partial_x(h(t,\,x)/g(t,\,x))}{\partial_t(h(t,\,x)/g(t,\,x))}\right|_{t=\gamma(x)}.$$

From $(2.10)_\tau$ we obtain $\|\partial_x(h/g)\|_{L^\infty} \leq (2+r)\varepsilon \leq 2$ and $|\partial_t(h/g)| = |\chi_t| \geq 1/2$. Consequently, according to (3.6) and (1.4),

$$|\gamma'(x)| \leq 4\frac{\|\phi'\|_{L^\infty} + 2r\,\|\psi'\|_{L^\infty}}{|\psi(x)|} + 4,$$

$$\begin{aligned} &a_*(\gamma(x),\,x)(\gamma'(x))^2 \\ &\quad\leq 32C_a(\phi(x)g(\gamma(x),\,x) + \psi(x)h(\gamma(x),\,x))^2 \\ &\qquad\qquad \times \left(\frac{(\|\phi'\|_{L^\infty} + 2r\,\|\psi'\|_{L^\infty})^2}{|\psi(x)|^2} + 1\right) \\ &\quad\leq 32C_a r^2\left(2g(\gamma(x),\,x) + \frac{5(\varepsilon+r)}{r}\right)^2 \\ &\qquad\qquad \times \left((\|\phi'\|_{L^\infty} + 2r\|\psi'\|_{L^\infty})^2 + \|\psi\|^2_{L^\infty}\right) \\ &\quad\leq \frac{1}{4} \end{aligned} \tag{3.9}$$

if r is sufficiently small, compare (3.8). It remains to consider $x \in \Omega_{\phi\psi} \setminus \Omega^{2r}_{\phi\psi}$. Then $\gamma(x) = \chi^{-1}(0; x)$; hence $|\gamma'(x)| \leq 4\varepsilon$. Then we need

$$a_*(\gamma(x),\, x)(\gamma'(x))^2 \leq 32C_a(\|\phi\|_{L^\infty} + \tau\, \|\psi\|_{L^\infty})^2\varepsilon^2 \leq \frac{1}{4}. \tag{3.10}$$

We choose r according to (3.9), and then ε, τ, α as in (3.7), (3.8) and (3.10). □

Remark 3.4 In the case of (2.6), ε, τ, α will depend on λ. Careful checking of the proof shows $r = \mathscr{O}(\lambda^{-1/2})$, $\tau = \mathscr{O}(\lambda^{-1/2})$, $\alpha = \mathscr{O}(1)$, $\varepsilon = \mathscr{O}(\lambda^{1/2})$.

Remark 3.5 Consider (2.6) and choose ε, τ as given in Remark 3.4. Suppose that $U = (g, h)^T$ satisfies (2.10) with that τ and that ε. Then we have, for all λ,

$$\sum_{|\alpha|+|\beta|\leq k} |\partial_x^\alpha \partial_U^\beta A_\lambda(x,\, U)| + |\partial_x^\alpha \partial_U^\beta B_\lambda(x,\, U)| + |\partial_x^\alpha \partial_U^\beta C_\lambda(x,\, U)| \leq C_k\lambda.$$

From Lemma 10.1, we conclude that

$$\begin{aligned} &\|A_\lambda(\,\cdot\,, U(t,\,\cdot\,))\|_{H^k(B_R)} + \|B_\lambda(\,\cdot\,, U(t,\,\cdot\,))\|_{H^k(B_R)} \\ &+ \|C_\lambda(\,\cdot\,, U(t,\,\cdot\,))\|_{H^k(B_R)} \\ &\qquad \leq C_k\lambda(1 + \|U(t,\,\cdot\,)\|^k_{L^\infty})(1 + \|U(t,\,\cdot\,)\|_{H^k(B_R)}) \end{aligned}$$

for $k \geq 1$. By computation,

$$\left\|\partial_x^2 a_{*,\lambda}(t,\,\cdot\,)\right\|_{L^\infty} \leq C\lambda(1 + \left\|\partial_x^2 U(t,\,\cdot\,)\right\|_{L^\infty}).$$

4. A priori estimates for (2.7)

The system (2.7) can be written in the form

$$\begin{aligned} &\partial_t^2 V - a_*(t,\, x)\partial_x^2 V - \tilde{B}(t,\, x)\partial_x V - \tilde{C}(t,\, x)V = F(t,\, x), \\ &V(0,\, x) = V_0(x), \quad V_t(0,\, x) = V_1(x), \end{aligned} \tag{4.1}$$

where $\tilde{B}(t,\, x) = B(x, U(t,\, x)) + \partial_x a_*(t,\, x)I$, $\tilde{C}(t,\, x) = C(x, U(t,\, x))$. More generally, we consider the Cauchy problem

$$\partial_t^2 V - a_*(t,\, x)\partial_x^2 V - B_*(t,\, x)\partial_x V - C_*(t,\, x)V = F(t,\, x), \tag{4.2}$$

$$V(t_0,\, x) = V_0(x), \quad V_t(t_0,\, x) = V_1(x), \tag{4.3}$$

where a_*, B_*, C_* are functions satisfying the following hypothesis.

Hypothesis 1

(a) $a_*(t, x) = a(\phi(x)g(t, x) + \psi(x)h(t, x))$, and $a = a(s)$ satisfies Condition 1,

(b) $|B_*(t, x)|^2 \leq La_*(t, x)$ for some $L \geq 0$ (Levi Condition),

(c) $\phi,\, \psi \in C_0^2(\mathbb{R})$ with $\operatorname{supp}(\phi, \psi) \subset B_R = \{|x| < R\}$, and $\|\phi\|_{L^\infty} < M$,

(d) the coefficient a_* admits a separating curve in the sense of Proposition 3.1,

(e) the numbers ε and τ from $(2.10)_\tau$, (3.1), (3.2) are chosen as in Proposition 3.1.

For the proof of (b) we only recall Condition 1 and Glaeser's inequality [10],

$$|e'(x)|^2 \leq 2\,\|e\|_{C^2(\mathbb{R})}\, e(x),$$

for every function $e = e(x) \in C^2(\mathbb{R})$ with $e(x) \geq 0$ for all x.

Now we give estimates of $|V(t, x)|$ separately in the both zones $\{x\colon \gamma(x) > t\}$ and $\{x\colon \gamma(x) < t\}$. Our approach is based on a work of Manfrin, we only list the results and refer the reader to [17] for the proofs. See also [18].

We introduce the sets

$$D(t) = \{(t', x)\colon x \in \Omega_{\phi\psi},\ 0 < t' < \min\{\gamma(x), t\}\},$$
$$G(t) = \{(t', x)\colon x \in \Omega_{\phi\psi},\ \max\{\gamma(x), 0\} < t' < t\},$$

and define the energies

$$\mathscr{E}(t, x) = |V_t(t, x)|^2 + a_*(t, x)|V_x(t, x)|^2 + |V(t, x)|^2,$$
$$E_1(t) = \int_{\{x\colon \gamma(x) > t\}} e^{\theta_1 t}\mathscr{E}(t, x)dx,$$
$$E_2(t) = e^{-\beta_2 t}\iint_{G(t)} e^{\theta_2 t'}|V(t', x)|^2 dxdt'.$$

The following results have been proved in [17], Lemmas 5.1 and 5.2.

Lemma 4.1 *Let* $V(t, x)$ *be a solution of* (4.1), (4.2) *and assume* Hypothesis 1*. Then there is a* $\theta_{1,0} \in \mathbb{R}$*,*

$$\theta_{1,0} = -\,\text{const.}(1+\alpha+L+\sup_{[0,\tau]}\left\|\partial_x^2 a_*(t,\,\cdot)\right\|_{L^\infty} + \|C_*(t,\,\cdot)\|_{L^\infty(B_R)}), \tag{4.4}$$

such that if we define $E_1(t)$ *with* $\theta_1 \le \theta_{1,0}$*, the following estimate holds*:

$$E_1(t) + \frac{1}{2}\int_{\{x:\ 0<\gamma(x)\le t\}} e^{\theta_1\gamma(x)}\mathscr{E}(\gamma(x),\,x)dx \le E_1(0) + \iint_{D(t)} e^{\theta_1 t'}|F(t',\,x)|^2 dxdt', \quad 0 \le t \le \tau. \tag{4.5}$$

Lemma 4.2 *Let* $V(t,\,x)$ *be a solution of* (4.1), (4.2) *and assume* Hypothesis 1*. Then there is a* $\theta_{2,0}$*,*

$$\theta_{2,0} = \text{const.}(\alpha+L+\sup_{[0,\tau]}\left\|\partial_x^2 a_*(t,\,\cdot)\right\|_{L^\infty}), \tag{4.6}$$

such that if we define $E_2(t)$ *with* $\theta_2 \ge \theta_{2,0}$*, there is a* $\beta_{2,0} > 0$*,*

$$\beta_{2,0} = \text{const.}(1+\tau^2) \times \sup_{[0,\tau]}\Big(1+\theta_2^2+L+\left\|\partial_x^2 a_*(t,\,\cdot)\right\|_{L^\infty} + \|B_*(t,\,\cdot)\|_{C^1} + \|C_*(t,\,\cdot)\|_{L^\infty(B_R)}\Big), \tag{4.7}$$

such that for $\beta_2 \ge \beta_{2,0}$ *and* $t \in [0,\,\tau]$ *we have*

$$E_2(t) \le \int_0^t e^{-\beta_2 s}\int_{\{x:\ 0<\gamma(x)<s\}} e^{\theta_2\gamma(x)}\mathscr{E}(\gamma(x),\,x)dxds + \int_0^t e^{-\beta_2 s}\iint_{G(s)} e^{\theta_2 t'}|F(t',\,x)|^2 dxdt'ds + \frac{1-e^{-\beta_2 t}}{\beta_2}\int_{\{x:\ \gamma(x)\le 0\}} |V(0,\,x)|^2 + |V_t(0,\,x)|^2 dx.$$

Moreover, almost everywhere in $[0,\,\tau]$ *we have*

$$\int_{\{x:\ \gamma(x)<t\}} e^{\theta_2 t}|V(t,\,x)|^2 dx \le \beta_2 e^{\beta_2 t}E_2(t) + \int_{\{x:\ 0<\gamma(x)<t\}} e^{\theta_2\gamma(x)}\mathscr{E}(\gamma(x),\,x)dx + \int_{\{x:\ \gamma(x)\le 0\}} |V(0,\,x)|^2 + |V_t(0,\,x)|^2\, dx \tag{4.8}$$

$$+ \iint_{G(t)} e^{\theta_2 t'} |F(t', x)|^2 \, dx \, dt'.$$

Remark 4.3 The above two estimates have been proved in [17] in case of

$$a_*(t, x) = a_0(t, x)(\phi(x)g(t, x) + \psi(x)h(t, x))^{2q}, \quad q \in \mathbb{N}_+,$$

where $a_0 \geq \delta > 0$ is some C^2 function. However, in the proofs of Lemmas 5.1 and 5.2 in [17] this special form of the coefficient a_* was never used. Actually, it suffices to assume that a_* admits a separating curve in the sense of Proposition 3.1.

Now we are in a position to estimate the $L^2(B_R)$ norm of $V(t, x)$.

Proposition 4.4 *Let* $V = V(t, x)$ *with* $\partial_t^j V \in L^\infty([t_0, \tau], H^{2-j}(B_R))$, $j = 0, 1, 2$, *be a solution of* (4.2), (4.3) *and assume that* Hypothesis 1 *holds. Then there is a constant* C_0 *such that for all* $t \in [t_0, \tau]$ *we have*

$$\begin{aligned} &\|V(t, \cdot)\|^2_{L^2(B_R)} \\ &\leq C_0 \left(\|V_0(\cdot)\|^2_{H^1(B_R)} + \|V_1(\cdot)\|^2_{L^2(B_R)} + \int_{t_0}^t \|F(s, \cdot)\|^2_{L^2(B_R)} \, ds \right). \end{aligned} \tag{4.9}$$

The constant C_0 *depends only on* τ, α, L, *and the norms* $\sup_{[0,\tau]} \|a_*(t, \cdot)\|_{C^2(B_R)}$, $\sup_{[0,\tau]} \|B_*(t, \cdot)\|_{C^1(B_R)}$, $\|C_*(\cdot, \cdot)\|_{L^\infty([0,\tau]\times B_R)}$.

Proof. Assume for a moment that $t_0 = 0$. If $x \in B_R \setminus \Omega_{\phi\psi}$, the Cauchy problem (4.2) degenerates into

$$\partial_t^2 V - C_*(t, x)V = F(t, x),$$

which directly leads to an estimate of $\|V\|_{L^2(B_R \setminus \Omega_{\phi\psi})}$ in terms of $\|V_0\|_{L^2(B_R \setminus \Omega_{\phi\psi})}$, $\|V_1\|_{L^2(B_R \setminus \Omega_{\phi\psi})}$, and $\|F(s, \cdot)\|_{L^2(B_R \setminus \Omega_{\phi\psi})}$. Therefore we may restrict ourselves to the case $x \in \Omega_{\phi\psi}$. Then we can apply the Lemmas 4.1 and 4.2. We set $\theta_1 = \theta_{1,0}$, $\theta_2 = \theta_{2,0}$, and $\beta_2 = \beta_{2,0}(\theta_2)$. Let $t \in [0, \tau]$ be a number such that (4.8) holds. By Sard's Lemma, the set of all t with

$$\text{meas}\{x \in \Omega_{\phi\psi} \colon \gamma(x) = t\} > 0$$

has Lebesgue measure 0. Assume that t is not from that set. Then we have

$$\int_{\Omega_{\phi\psi}} |V(t, x)|^2 dx$$

$$
\begin{aligned}
&= \int_{\{x:\, \gamma(x)>t\}} |V(t,\, x)|^2 dx + \int_{\{x:\, \gamma(x)<t\}} |V(t,\, x)|^2 dx \\
&\le e^{-\theta_1 t} E_1(t) + \beta_2 e^{(\beta_2-\theta_2)t} E_2(t) \\
&\quad + e^{-\theta_2 t} \int_{\{x:\, 0<\gamma(x)<t\}} e^{\theta_2\gamma(x)} \mathscr{E}(\gamma(x),\, x) dx \\
&\quad + e^{-\theta_2 t} (\|V_0(\,\cdot\,)\|^2_{L^2(\Omega_{\phi\psi})} + \|V_1(\,\cdot\,)\|^2_{L^2(\Omega_{\phi\psi})}) \\
&\quad + e^{-\theta_2 t} \iint_{G(t)} e^{\theta_2 t'} |F(t',\, x)|^2 dx dt',
\end{aligned}
$$

due to Lemmas 4.1 and 4.2. Applying these lemmas once more, we get

$$
\begin{aligned}
&\|V(t,\,\cdot\,)\|^2_{L^2(\Omega_{\phi\psi})} \\
&\le C_0 \left(\|V_0(\,\cdot\,)\|^2_{H^1(\Omega_{\phi\psi})} + \|V_1(\,\cdot\,)\|^2_{L^2(\Omega_{\phi\psi})} + \int_0^t \|F(s,\,\cdot\,)\|^2_{L^2(\Omega_{\phi\psi})}\, ds \right).
\end{aligned}
$$

This gives us the desired estimate for a.e. $t \in [0,\, \tau]$. Since $\partial_t V$ belongs to the space $L^\infty([0,\, \tau],\, H^1(B_R))$, we have shown (4.9) for all values of t.

Now let $t_0 > 0$. We set $\tilde{V}(t,\, x) = V(t + t_0,\, x)$. Since Hypothesis 1 is invariant under the translation $t \mapsto t + t_0$, we get from (4.9) an estimate for $\tilde{V}(t,\, x)$. □

Remark 4.5 Consider (2.6) and suppose $\left\|\partial_x^2 U(t,\,\cdot\,)\right\|_{L^\infty} \le C$, uniformly in λ. Then $C_0 = C_0(\lambda) \le \exp(C(1 + \tau(\lambda)^3))$, for all λ, see Remark 3.5 and (4.4), (4.6), (4.7).

By standard arguments, we can estimate derivatives $\partial_x^k V(t,\, x)$.

Proposition 4.6 *Let ε, τ be determined as in* Proposition 3.1, *and suppose that U satisfies* $(2.10)_\tau$. *Let $k \in \mathbb{N}$, and V with $\partial_t^j V \in L^\infty\big([t_0,\, \tau],\, H^{k+2-j}(B_R)\big)$, $j = 0, 1, 2$, be a solution to* (2.7), (2.9). *Then the estimate*

$$
\begin{aligned}
&\|V(t,\,\cdot\,)\|^2_{H^k(B_R)} \le C_k (1 + \sup_{[t_0,t]} \|U(s,\,\cdot\,)\|^2_{H^{k+2}(B_R)}) \qquad (4.10) \\
&\times \left(\|V_0(\,\cdot\,)\|^2_{H^{k+1}(B_R)} + \|V_1(\,\cdot\,)\|^2_{H^k(B_R)} + \int_{t_0}^t \|F(s,\,\cdot\,)\|^2_{H^k(B_R)}\, ds \right)
\end{aligned}
$$

holds for $0 \le t_0 \le t \le \tau$, where C_k depends only on τ, α, L, and the norms

$$
\begin{aligned}
&\sup_{[0,\tau]} \|U(t,\,\cdot\,)\|_{H^3(B_R)}, \quad \|A(\,\cdot\,,\,\cdot\,)\|_{C^{k+2}(B_R\times[1-\varepsilon,1+\varepsilon]\times[\tau-\varepsilon,\tau+\varepsilon])}, \\
&\|B(\,\cdot\,,\,\cdot\,)\|_{C^k(B_R\times[1-\varepsilon,1+\varepsilon]\times[\tau-\varepsilon,\tau+\varepsilon])},
\end{aligned}
$$

$$\|C(\,\cdot\,,\,\cdot\,)\|_{C^k(B_R\times[1-\varepsilon,1+\varepsilon]\times[\tau-\varepsilon,\,\tau+\varepsilon])}\,.$$

Proof. The estimate (4.10) holds for $k=0$, see Proposition 4.4. Assume that (4.10) is true for k replaced by $k-1$. We set $V^k(t,\,x)=\partial_x^k V(t,\,x)$ and obtain

$$\begin{aligned}
&\partial_t^2 V^k - A(x,\,U)\partial_x^2 V^k - ((k+1)(\partial_x A(x,\,U(t,\,x))) + B(x,\,U))\,\partial_x V^k\\
&-\left(\frac{k(k+1)}{2}(\partial_x^2 A(x,\,U(t,\,x))) + k(\partial_x B(x,\,U(t,\,x))) + C(x,\,U)\right)V^k\\
&= F^k = \partial_x^k F + I_1 + I_2 + I_3 + I_4\\
&= \partial_x^k F + \sum_{l=3}^{k}\binom{k}{l}(\partial_x^l A(x,\,U(t,\,x)))V^{k+2-l}\\
&\quad + \sum_{l=2}^{k}\binom{k}{l}(\partial_x^{l+1} A(x,\,U(t,\,x)))V^{k+1-l}\\
&\quad + \sum_{l=2}^{k}\binom{k}{l}(\partial_x^l B(x,\,U(t,\,x)))V^{k+1-l}\\
&\quad + \sum_{l=1}^{k}\binom{k}{l}(\partial_x^l C(x,\,U(t,\,x)))V^{k-l}.
\end{aligned}$$

By Proposition 4.4, we deduce that

$$\begin{aligned}
&\left\|V^k(t,\,\cdot\,)\right\|_{L^2}^2\\
&\le C_0\left(\|V_0(\,\cdot\,)\|_{H^{k+1}}^2 + \|V_1(\,\cdot\,)\|_{H^k}^2 + \int_{t_0}^{t}\left\|F^k(s,\,\cdot\,)\right\|_{L^2}^2 ds\right).
\end{aligned}$$

For the estimate of I_1 and I_2, we have to consider terms of the form $(\partial_x^m A)V^{k+2-m}$ with $m=3,\,\ldots,\,k+1$. From Lemma 10.1 and Sobolev's embedding theorem,

$$\begin{aligned}
&\left\|(\partial_x^m A(\,\cdot\,,\,U(t,\,\cdot\,)))V^{k+2-m}(t,\,\cdot\,)\right\|_{L^2}\\
&\le \|\partial_x^m A(\,\cdot\,,U(t,\,\cdot\,))\|_{L^\infty}\left\|V^{k+2-m}(t,\,\cdot\,)\right\|_{L^2}\\
&\le C(\|U(t,\,\cdot\,)\|_{L^\infty})(1+\|U(t,\,\cdot\,)\|_{H^{m+1}})\,\|V(t,\,\cdot\,)\|_{H^{k+2-m}}\,,
\end{aligned}$$

Similarly, we get

$$I_3 + I_4 \le C(\|U(t,\,\cdot\,)\|_{C^2})\,\|V(t,\,\cdot\,)\|_{H^{k-1}}$$

$$+ C(\|U(t,\cdot)\|_{L^\infty}) \sum_{m=3}^{k} (1 + \|U(t,\cdot)\|_{H^{m+1}}) \|V(t,\cdot)\|_{H^{k+1-m}}.$$

Then it follows that

$$\begin{aligned}
\|V(t,\cdot)\|^2_{H^k(B_R)} &\le C_0 \left(\|V_0(\cdot)\|^2_{H^{k+1}} + \|V_1(\cdot)\|^2_{H^k} \right) \\
&+ C_0 \int_{t_0}^{t} \|F(s,\cdot)\|^2_{H^k(B_R)} + \|V(s,\cdot)\|^2_{H^{k-1}(B_R)}\, ds \\
&+ C(\sup_{[t_0,t]} \|U(s,\cdot)\|_{C^2(B_R)}) \\
&\times \sum_{m=3}^{k+1} \sup_{[t_0,t]} (1 + \|U(s,\cdot)\|^2_{H^{m+1}(B_R)}) \int_{t_0}^{t} \|V(s,\cdot)\|^2_{H^{k+2-m}(B_R)}\, ds.
\end{aligned}$$

From the induction assumption,

$$\begin{aligned}
&\sup_{[t_0,t]} \|U(s,\cdot)\|^2_{H^{m+1}(B_R)} \int_{t_0}^{t} \|V(s,\cdot)\|^2_{H^{k+2-m}(B_R)}\, ds \\
&\le C_k \sup_{[t_0,t]} \|U(s,\cdot)\|^2_{H^{m+1}(B_R)} \left(1 + \sup_{[t_0,t]} \|U(s,\cdot)\|^2_{H^{k+4-m}(B_R)}\right) \\
&\times \left(\|V_0(\cdot)\|^2_{H^k(B_R)} + \|V_1(\cdot)\|^2_{H^{k-1}(B_R)} + \int_{t_0}^{t} \|F(s,\cdot)\|^2_{H^{k-1}(B_R)}\, ds \right).
\end{aligned}$$

By Nirenberg–Gagliardo interpolation,

$$\begin{aligned}
&\|U(s,\cdot)\|_{H^{m+1}(B_R)} \\
&\qquad \le C \|U(s,\cdot)\|^{(m-2)/(k-1)}_{H^{k+2}(B_R)} \|U(s,\cdot)\|^{1-(m-2)/(k-1)}_{H^3(B_R)},
\end{aligned}$$

$$\begin{aligned}
&\|U(s,\cdot)\|_{H^{k+4-m}(B_R)} \\
&\qquad \le C \|U(s,\cdot)\|^{(k+1-m)/(k-1)}_{H^{k+2}(B_R)} \|U(s,\cdot)\|^{1-(k+1-m)/(k-1)}_{H^3(B_R)},
\end{aligned}$$

for $k \ge 2$. This completes the proof. □

5. Existence of solutions to (2.7)

Proposition 5.1 *Let $a = a(s)$ satisfy* Condition 2*, and let $\phi, \psi \in C_0^\infty(\mathbb{R})$ be to $a(s)$ compatible data, i.e., $\|\phi\|_{L^\infty} < M$. Assume* $\operatorname{supp}(\phi, \psi) \subset B_R = \{|x| < R\}$*. Choose ε, τ as in* Proposition 3.1*, and suppose that $U \in C^2([0,\tau], C_b^\infty(B_R))$ satisfies* $(2.10)_\tau$*. Finally, assume that*

$F \in C([t_0, \tau], C_b^\infty(B_R))$, V_0, $V_1 \in C_b^\infty(B_R)$. *Then the problem* (2.7), (2.9) *has a unique solution* $V \in C^2([t_0, \tau], C_b^\infty(B_R))$.

Remark 5.2 Fix $0 < R' < R$ with $\text{supp}(\phi, \psi) \subset B_{R'}$. Then the functions $A(x, U)$, $B(x, U)$, $C(x, U)$ vanish for $R' \leq |x| \leq R$; and the existence of a solution $V \in C^2([t_0, \tau], C_b^\infty(\{R' \leq |x| \leq R\}))$ is clear. Hence, we assume in the sequel $|x| \leq R'$.

The proof of Proposition 5.1 is based on an approximation argument.

Definition 5.3 Let $\varrho = \varrho(s)$ be an even function from the Gevrey space $G_0^d(\mathbb{R})$,

$$|\partial_s^k \varrho(s)| \leq C^{k+1} k!^d, \quad k \in \mathbb{N}, \quad s \in \mathbb{R}, \quad 1 < d < 2,$$

and $\text{supp}\, \varrho \subset (-1, 1)$. Additionally, suppose that $s\varrho'(s) \leq 0 \leq \varrho(s)$, $\int_{-\infty}^{\infty} \varrho(s) ds = 1$, and write $\varrho_m(s) = m\varrho(ms)$ for $1 \ll m \in \mathbb{R}$. Then we define for large m

$$\begin{aligned}
&a_{0,m}(s) = (a_0 * \varrho_m)(s), \quad a_m(s) = s^2 a_{0,m}(s), \quad a_{1,m}(s) = a_m'(s)/s, \\
&\phi_m(x) = (\phi * \varrho_m)(x), \quad \psi_m(x) = (\psi * \varrho_m)(x), \\
&U_m(t, x) = (U * \varrho_m)(t, x), \quad F_m(t, x) = (F * \varrho_m)(t, x), \\
&V_{0,m}(x) = (V_0 * \varrho_m)(x), \quad V_{1,m}(x) = (V_1 * \varrho_m)(x),
\end{aligned}$$

where $*$ denotes the usual convolution.

Lemma 5.4 *Replace the interval* $\overline{B_M} = [-M, M]$ *of* Condition 1 *by some shrinked interval* $[-M', M']$, $0 < M' < M$. *If* m *is large enough, then the coefficient* $a_m(s)$ *satisfies* Condition 1 *with* C_a *replaced by* $C_a + 3$.

Proof. Suppose that m is so large that $a_m(s)$ is well defined on $[-M', M']$. The properties of the convolution imply $0 < a_{0,m}(s) \leq C_a$ for all $|s| \leq M'$. We have

$$0 \leq s \int a_0'(s - r) m\varrho(mr) dr = s\partial_s a_{0,m}(s),$$

since a_0 and ϱ are even functions. From $r\varrho'(mr) \leq 0$ we deduce that

$$\begin{aligned}
s\partial_s a_{0,m}(s) &= s \int a_0'(s - r) m\varrho(mr) dr \\
&= s \int a_0(s - r) m^2 \varrho'(mr) dr \leq \int (s - r) a_0(s - r) m^2 \varrho'(mr) dr
\end{aligned}$$

$$= \int (a_0(s-r) + (s-r)a_0'(s-r))m\varrho(mr)dr \le (C_a+1)a_{0,m}(s).$$

Clearly, $0 \le sa_m'(s) \le (C_a+3)a_m(s)$. This completes the proof. □

Proof of Proposition 5.1. We consider the linear system

$$\begin{aligned} &\partial_t^2 V_m - \partial_x(A_m(x, U_m)\partial_x V_m) - B_m(x, U_m)\partial_x V_m \\ &\qquad - C_m(x, U_m)V_m = F_m(t, x), \\ &V_m(0, x) = V_{0,m}(x), \quad \partial_t V_m(0, x) = V_{1,m}(x), \end{aligned} \tag{5.1}$$

where A_m, B_m, C_m are defined as in (2.2)–(2.4) with $a(s)$ replaced by $a_m(s)$. According to [16], the problem (5.1) has a unique solution $V_m \in C^2([t_0, \tau], G^d(B_{R'}))$. Similarly to Section 4, we set

$$\begin{aligned} a_{*,m}(t, x) &= a_m(\phi_m(x)g_m(t, x) + \psi_m(x)h_m(t, x)), \\ B_{*,m}(t, x) &= B_m(x, U_m(t, x)) + \partial_x a_{*,m}(t, x)I, \\ C_{*,m}(t, x) &= C_m(x, U_m(t, x)). \end{aligned}$$

Obviously, $a_{*,m} \to a_*$, $B_{*,m} \to B_*$, $C_{*,m} \to C_*$ in the topology of the space $C([t_0, \tau], C_b^\infty(B_{R'}))$. Due to Proposition 4.6, we have uniform estimates

$$\sup_{[t_0,\tau]} \|V_m(t, \cdot)\|_{H^k(B_{R'})} \le C_k, \quad m \ge m_0, \quad k \in \mathbb{N}.$$

Then (5.1) yields $\|V_m(\cdot, \cdot)\|_{C^2([t_0,\tau], H^k(B_{R'}))} \le C_k$. By the Arzela–Ascoli theorem, there is a subsequence $\{V_{m'}\}$ converging in $C^1([t_0, \tau], H^{k-1}(B_{R'}))$ to some limit $V^{(k)}$ which solves (2.7). By Proposition 4.4, solutions to (2.7) are unique. Therefore, $V^{(k)} = V^{(l)}$ for all k, l; hence we have a solution $V \in C^2([t_0, \tau], C_b^\infty(B_{R'}))$. □

6. Existence of solutions to (2.5)

Now we prove the existence of C^∞ solutions U to (2.5) for small times. In the next section, more attention will be paid to a better description of the life span of this solution. We shall show that, under suitable assumptions, a solution U to (2.5) can be extended to some longer interval. Therefore, we now discuss the equation (2.5) with slightly more general initial conditions.

Define A, B, C as in (2.2)–(2.4), and consider the Cauchy problem

$$\begin{aligned} &\partial_t^2 U - \partial_x(A(x, U)\partial_x U) - B(x, U)\partial_x U - C(x, U)U = 0, \\ &U(t_0, x) = U_0(x), \quad U_t(t_0, x) = U_1(x), \end{aligned} \tag{6.1}$$

$$\left\|U_0(\,\cdot\,) - (1, t_0)^T\right\|_{C^1(B_R)} < \varepsilon_0, \quad \left\|U_1(\,\cdot\,) - (0,\,1)^T\right\|_{L^\infty(B_R)} < \varepsilon_0, \tag{6.2}$$

Proposition 6.1 *Let* $a = a(s)$ *satisfy* Condition 2, *and let* $(\phi,\,\psi) \in C_0^\infty(\mathbb{R})$ *with* $\operatorname{supp}(\phi,\,\psi) \subset B_R$ *be to* $a(s)$ *compatible data, i.e.,* $\|\phi\|_{L^\infty} < M$.

Then there is an ε_0*, depending only on* M*,* C_a*,* $\|\phi\|_{C^1(B_R)}$*,* $\|\psi\|_{C^1(B_R)}$*, such that*:

For every $U_0,\,U_1 \in C_b^\infty(B_R)$ *with* (6.2) *there is some* $T_1 > t_0$ *and a unique local solution* $U \in C_b^\infty([t_0,\,T_1] \times B_R)$ *to the Cauchy problem* (6.1).

The proof bases on the Nash–Moser–Hamilton theory. We recall the main results of that theory and refer the reader to [12] for the details.

Definition 6.2

(a) A *graded* (*Fréchet*) *space* E is a Fréchet space whose topology is induced by a grading, that is a sequence of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ such that $\|e\|_n \le \|e\|_{n+1}$ for all $e \in E$ and all $n \in \mathbb{N}$.

(b) A *tame linear map* is a linear map $L \in L(E_1,\,E_2)$ between two graded spaces $E_1,\,E_2$ such that constants $r,\,b \in \mathbb{N}$ exist with

$$\|Le\|_{E_2,n} \le C_n \|e\|_{E_1,n+r}\,, \quad e \in E_1, \quad n \ge b,$$

where the C_n do not depend on $e \in E_1$.

(c) For a Banach space B, we define the graded space $\sum(B)$ of *exponentially decreasing sequences* by

$$\sum(B) = \left\{ \{v_k\}_{k=0}^\infty \colon v_k \in B,\, \|\{v_k\}\|_n = \sum_{k=0}^\infty e^{nk} \|v_k\|_B < \infty,\, n \in \mathbb{N} \right\}.$$

(d) The graded space E is a *tame space* if some Banach space B and linear tame maps $L_1 \in L(E, \sum(B))$, $L_2 \in L(\sum(B), E)$ exist with the property that L_2L_1 is the identity on E.

Example 6.3 Spaces of C_b^∞ functions on smooth compact manifolds X (with or without boundary) are tame (see [12], pp. 135–138), when we define the seminorms $\|v\|_n = \|v(\,\cdot\,)\|_{W_p^n(X)}$, $1 \le p \le \infty$.

Definition 6.4 Let $P \colon \mathscr{M} \subset E_1 \to E_2$ be a (nonlinear) mapping between the graded spaces E_1, E_2, and be defined on the open set $\mathscr{M}$. The map P is called *tame* if for each point $e^* \in \mathscr{M}$ there is a neighbourhood $e^* \in \Omega \subset \mathscr{M}$

and constants r, $b \in \mathbb{N}$ such that

$$\|P(e)\|_{E_2,n} \leq C_n(1 + \|e\|_{E_1,n+r}), \quad e \in \Omega, \quad n \geq b.$$

Remark 6.5 A map is a tame linear map if and only if it is linear and tame.

Definition 6.6 Let $P\colon \mathscr{M} \subset E_1 \to E_2$ be a tame map. Then, P is called *smooth tame* if it is C^∞ and $D^n P$ is tame for all $n \in \mathbb{N}$.

Example 6.7 Nonlinear partial differential operators acting on the tame space $C_b^\infty(X)$ are smooth tame. Sums and compositions of smooth tame maps are smooth tame (see [12], p. 146).

The following implicit function theorem is the crucial tool in the following.

Theorem 6.8 (Nash–Moser–Hamilton) *Let E_1, E_2 be tame spaces, $\mathscr{M} \subset E_1$ be an open set, and $P\colon \mathscr{M} \subset E_1 \to E_2$ be a smooth tame map. Suppose that the derivative $DP(u) \in L(E_1, E_2)$ has a right inverse $VP(u) \in L(E_2, E_1)$ for each $u \in \mathscr{M}$, which is smooth tame as a mapping $VP(u)\colon \mathscr{M} \times E_2 \to E_1$. Then P is in $\mathscr{M}$ locally invertible, and each inverse is smooth tame.*

Proof of Proposition 6.1. We show $U \in C^2([t_0, T_1], C_b^\infty(B_R))$. The smoothness in time then follows from (2.5). We fix the tame spaces

$$\begin{aligned}
&E_1 = (C^2([t_0, T], C_b^\infty(B_R)))^2,\\
&E_2 = (C([t_0, T], C_b^\infty(B_R)))^2 \times (C_b^\infty(B_R))^2 \times (C_b^\infty(B_R))^2,\\
&\|e\|_{E_1,n}\\
&= \sup_{[t_0, T]} \Big(\|e(t, \cdot)\|_{H^n(B_R)} + \|e_t(t, \cdot)\|_{H^n(B_R)} + \|e_{tt}(t, \cdot)\|_{H^n(B_R)} \Big),\\
&\|(e_I, e_{II}, e_{III})\|_{E_2,n}\\
&= \sup_{[t_0, T]} \|e_I(t, \cdot)\|_{H^n(B_R)} + \|e_{II}(\cdot)\|_{H^n(B_R)} + \|e_{III}(\cdot)\|_{H^n(B_R)},
\end{aligned}$$

where T with $0 < T - t_0 \ll 1$ will be chosen later. The map $P\colon E_1 \to E_2$ is

$$\begin{aligned}
P(U) = \Big(&\partial_t^2 U - \partial_x(A(x, U)\partial_x U) - B(x, U)\partial_x U - C(x, U)U,\\
&U(t_0, x) - U_0(x),\ U_t(t_0, x) - U_1(x)\Big),
\end{aligned}$$

which is a smooth tame map. To fix the open set $\mathscr{M}$, we introduce

$$U_*(t, x) = U_0(x) + (t - t_0)U_1(x) + \frac{1}{2}(t - t_0)^2 \partial_x(A(x, U_0(x))U_{0,x}(x)) + \frac{1}{2}(t - t_0)^2 B(x, U_0(x))U_{0,x}(x) + \frac{1}{2}(t - t_0)^2 C(x, U_0(x))U_0(x),$$

and define

$$\mathscr{M} = \{U \in E_1 \colon \|U - U_*\|_{C^1([t_0, T]\times B_R)} < \varepsilon_0, \sup_{[t_0, T]} \|U(t, \cdot)\|_{H^3(B_R)} < C\}$$

with some constant $C > 0$. If we fix $\varepsilon_0 = \varepsilon/10$ and choose $T = T(\varepsilon)$ with $0 < T - t_0 \ll 1$ appropriately, then each element of $\mathscr{M}$ can be extended to $[0, T] \times B_R$ in such a way that $(2.10)_T$ holds, with ε chosen as in Proposition 3.1. Obviously,

$$P(U_*)(t, x) = \big((t - t_0)Z(t, x), 0, 0\big)$$

with some $Z \in C([t_0, T], C_b^\infty(B_R))$. Choose some function $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 0$ for $t \le 1$ and $\chi(t) = 1$ for $t \ge 2$. Then $\big((t - t_0)\chi(m(t - t_0))Z(t, x), 0, 0\big)$ converges to $\big((t - t_0)Z(t, x), 0, 0\big)$ in the topology of E_2 if m tends to infinity. Therefore, every neighbourhood of $P(U_*)$ contains elements of the form $(\tilde{Z}(t, x), 0, 0)$ where $\tilde{Z}(t, x) = 0$ for $t_0 \le t \le T_1$; and $T_1 - t_0 > 0$ is small. If we are able to show that the image $P(\mathscr{M})$ contains a neighbourhood of $P(U_*)$ in E_2, then we have proved the existence of a solution U to (2.5) in $[t_0, T_1] \times B_R$. More precisely, we show that P is locally invertible in the neighbourhood $\mathscr{M}$.

The Fréchet derivative $DP(U)$ is a linear map $V \mapsto (F, V_0, V_1)$ with

$$\begin{aligned} F &= \partial_t^2 V - \partial_x(A(x, U)\partial_x V) - \partial_x((A_U(x, U)V)\partial_x U) \qquad (6.3)\\ &\quad - B(x, U)\partial_x V - B_U(x, U)V\partial_x U - C(x, U)V - C_U(x, U)VU,\\ V_0(x) &= V(t_0, x), \quad V_1(x) = V_t(t_0, x). \end{aligned}$$

Here we have introduced the notation

$$A_U(x, U)V = a'(\phi g + \psi h)(\phi, \psi)VI, \quad U = (g, h)^T, \quad V = (v_1, v_2)^T,$$

where $(\phi, \psi)V = \phi v_1 + \psi v_2$ is the usual $\mathbb{R}^2$ scalar product. This Cauchy problem is of the form (2.7); and Hypothesis 1 is satisfied if $U \in \mathscr{M}$. We note that the Levi condition (b) follows from $|a'(s)|^2 \leq C_a^3 a(s)$, see (1.5). Then the Propositions 4.6 and 5.1 imply the existence of an inverse map

$$VP\colon (U,\,F,\,V_0,\,V_1) \mapsto V, \quad \mathscr{M} \times E_2 \to E_1$$

which satisfies

$$\sup_{[t_0,\,T]} \|V(t,\,\cdot)\|_{H^k(B_R)} \leq C_k(1 + \|U\|_{E_1,k+2})\,\|(F,\,V_0,\,V_1)\|_{E_2,k}\,.$$

From the equation (6.3),

$$\|V\|_{E_1,k} \leq C_k(1 + \|U\|_{E_1,k+4})\,\|(F,\,V_0,\,V_1)\|_{E_2,k+2}\,.$$

Hence $VP\colon \mathscr{M} \times E_2 \to E_1$ is tame, see [12]. The proof is complete if we show that VP is smooth tame. We proceed by induction and only show that D^1VP is tame; the higher derivatives D^kVP can be considered in the same way. We find that

$$V^{(1)} = D^1VP(U,\,F,\,V_0,\,V_1)\{\delta U,\,\delta F,\,\delta V_0,\,\delta V_1\},$$

where $V^{(1)} \in E_1$ depends linearly on $(\delta U,\,\delta F,\,\delta V_0,\,\delta V_1) \in E_1 \times E_2$ and nonlinearly on $(U,\,F,\,V_0,\,V_1) \in \mathscr{M} \times E_2$. More precisely,

$$\begin{aligned}
&\partial_t^2 V^{(1)} - \partial_x(A(x,\,U)\partial_x V^{(1)}) - \partial_x((A_U(x,\,U)V^{(1)})\partial_x U) \\
&- B(x,\,U)\partial_x V^{(1)} - B_U(x,\,U)V^{(1)}\partial_x U \\
&- C(x,\,U)V^{(1)} - C_U(x,\,U)V^{(1)}U \\
&= \delta F + R\delta U, \\
&V^{(1)}(t_0,\,x) = \delta V_0(x), \quad V_t^{(1)}(t_0,\,x) = \delta V_1(x),
\end{aligned}$$

where R is a linear differential operator depending on U and $V = VP(U,\,F,\,V_0,\,V_1)$. By Proposition 4.6, D^1VP is tame. This completes the proof. □

7. A life span criterion

In this section, we describe the life span of the C^∞ solution U to (2.5) mentioned in Proposition 6.1.

Proposition 7.1 *Let the assumptions of* Proposition 6.1 *be satisfied. Then there is a constant* $T_0 > 0$ *depending only on* M*,* R*,* $\|(a_0, a_1)\|_{C^3(B_M)}$*,*

$\|(\phi,\,\psi)\|_{C^5(B_R)}$; *and there is a unique solution* $U \in C_b^\infty([0,\,T_0]\times B_R)$ *to* (2.5).

The proof is split into the Lemmas 7.2 and 7.5.

Lemma 7.2 *Let the assumptions of* Proposition 6.1 *be satisfied, and let* $\varepsilon,\,\tau$ *be the numbers determined in* Proposition 3.1. *Finally, let* $U \in C^2([0,\,T),\,C_b^\infty(B_R))$, $0 < T < \tau$, *be a solution to* (2.5) *which satisfies* (2.10). *Then the estimates*

$$\|U(t,\,\cdot\,)\|^2_{H^k(B_R)} \tag{7.1}$$

$$\leq C_R(1+t^2)C_k\int_0^t \varrho_k(\|U(s,\,\cdot\,)\|_{H^3(B_R)})(1+\|U(s,\,\cdot\,)\|^2_{H^k(B_R)})ds,$$

$$\sup_{[0,t]}\left\|U(s,\,\cdot\,)-(1,\,s)^T\right\|^2_{H^3(B_R)} \leq tC_3\tilde{\varrho}_3(\sup_{[0,t]}\|U(s,\,\cdot\,)\|^2_{H^3(B_R)}) \tag{7.2}$$

hold for $0 \leq t < T$, *where* $\varrho_k,\,\tilde{\varrho}_k\colon \mathbb{R}_+ \to \mathbb{R}_+$ *are certain continuous and increasing functions, and* C_k *depend on* $\|(a_0,a_1)\|_{C^k(B_M)}$, $\|(\phi,\,\psi)\|_{C^{k+2}(B_R)}$, *and* R.

The proof is based on an *a priori* estimate similar to that of Proposition 4.6 for the Cauchy problem (2.7), but now we take advantage from the fact $U \equiv V$.

Lemma 7.3 *Let* $m,\,n \in \mathbb{N}$ *with* $m \geq 2$, $n \geq 3$, *and* $X \subset \mathbb{R}$ *be a bounded domain. Then*

$$\|w\|_{C^m(X)}\,\|w\|_{H^n(X)} \leq C\,\|w\|_{H^3(X)}\,\|w\|_{H^{m+n-2}(X)}\,, \qquad w \in H^{m+n-2}(X).$$

Proof. By Sobolev's embedding theorem,

$$\begin{aligned}\|w\|_{C^m(X)}\,\|w\|_{H^n(X)} &\leq C\,\|w\|_{H^{m+1}(X)}\,\|w\|_{H^n(X)}\\ &\leq C\,\|w\|_{H^3(X)}\,\|w\|_{H^{m+n-2}(X)}\,,\end{aligned}$$

where we have used the complex interpolation method,

$$\begin{aligned}H^{m+1}(X) &= \left[H^3(X),\,H^{m+n-2}(X)\right]_{\theta_1},\\ H^n(X) &= \left[H^3(X),\,H^{m+n-2}(X)\right]_{\theta_2},\end{aligned}$$

with $\theta_1+\theta_2 = 1$. □

Proof of Lemma 7.2. We write (2.5) in the form

$$\partial_t^2 U - A(x, U)\partial_x^2 U \\ - A_x(x, U)U_x - A_U(x, U)U_xU_x - B(x, U)U_x - C(x, U)U = 0,$$

where $A_x(x, U) = a'(\phi g + \psi h)(\phi_x, \psi_x)UI$, and $(\phi_x, \psi_x)U$ is the $\mathbb{R}^2$ scalar product $\phi_x g + \psi_x h$. Similarly, $A_U(x, U)U_x = a'(\phi g + \psi h)(\phi, \psi)U_x I$. We apply ∂_x^k, set $U^k = \partial_x^k U$, and obtain

$$\begin{aligned} &\partial_t^2 U^k - A(x, U)\partial_x^2 U^k - (k+1)(\partial_x A(x, U))\partial_x U^k \\ &\quad - A_U(x, U)(\partial_x U^k)U_x - B(x, U)\partial_x U^k \\ &= F^k = I_1 + I_2 + I_3 + I_4 \\ &= \sum_{l=2}^{k} \binom{k}{l} (\partial_x^l A(x, U))U^{k+2-l} \\ &\quad + \sum_{l=1}^{k} \binom{k}{l} (\partial_x^l A_x(x, U) + \partial_x^l B(x, U))U^{k+1-l} \\ &\quad + \sum_{l+m=0}^{k-1} \frac{k!}{l!m!(k-l-m)!} (\partial_x^{k-l-m} A_U(x, U))U^{l+1}U^{m+1} \\ &\quad + \partial_x^k (C(x, U)U). \end{aligned}$$

From $U^k(0, \cdot) = (\partial_t U^k)(0, \cdot) = 0$ for $k \geq 1$ and Proposition 4.4,

$$\left\|U^k(t, \cdot)\right\|_{L^2(B_R)}^2 \leq C_0 \int_0^t \left\|F^k(s, \cdot)\right\|_{L^2(B_R)}^2 ds.$$

We recall that Hypothesis 1 is satisfied because of $|a'(s)|^2 \leq C_a^3 a(s)$, see (1.5). Employing Lemmas 7.3 and 10.1, we estimate $I_1, \ldots, I_4$. For $l = 2$ in I_1, we find

$$\left\|(\partial_x^2 A(x, U))U^k\right\|_{L^2}^2 \leq C(\|a\|_{C^2}, \|(\phi, \psi)\|_{C^2})(1 + \|U\|_{C^2}^4)\left\|U^k\right\|_{L^2}^2.$$

For $3 \leq l \leq k$, we have

$$\begin{aligned} &\left\|(\partial_x^l A(x, U))U^{k+2-l}\right\|_{L^2}^2 \\ &\leq C(\|a\|_{C^l}, \|U\|_{L^\infty}, \|(\phi, \psi)\|_{C^l})(1 + \|U\|_{H^l}^2)\|U\|_{C^{k+2-l}}^2 \\ &\leq C(\|a\|_{C^l}, \|U\|_{L^\infty}, \|(\phi, \psi)\|_{C^l})(1 + \|U\|_{H^3}^2)\|U\|_{H^k}^2. \end{aligned}$$

The term I_2 can be discussed similarly. Concerning I_3, it is enough to discuss the case $m \leq l$. Suppose $k-1 \geq l+m \geq k-2$ and $l \geq 2$ ($l \leq 1$ is trivial). Then

$$\begin{aligned}&\big\|(\partial_x^{k-l-m} A_U(x, U))U^{l+1}U^{m+1}\big\|_{L^2}^2\\&\leq C(\|a\|_{C^3}, \|(\phi, \psi)\|_{C^2})(1+\|U\|_{C^2}^4)\big\|U^{l+1}\big\|_{L^2}^2\big\|U^{m+1}\big\|_{L^\infty}^2\\&\leq C(\|a\|_{C^3}, \|(\phi, \psi)\|_{C^2})(1+\|U\|_{C^2}^4)\|U\|_{H^3}^2\|U\|_{H^k}^2.\end{aligned}$$

Now let $1 \leq l+m \leq k-3$. Then we have

$$\begin{aligned}&\big\|(\partial_x^{k-l-m} A_U(x, U))U^{l+1}U^{m+1}\big\|_{L^2}^2\\&\leq C(\|a\|_{C^k}, \|U\|_{L^\infty}, \|(\phi, \psi)\|_{C^k})\\&\qquad\times(1+\|U\|_{H^{k-l-m}}^2)\big\|U^{l+1}\big\|_{L^\infty}^2\big\|U^{m+1}\big\|_{L^\infty}^2.\end{aligned}$$

By Lemma 7.3,

$$\begin{aligned}&\|U\|_{H^{k-l-m}}^2\big\|U^{l+1}\big\|_{L^\infty}^2\big\|U^{m+1}\big\|_{L^\infty}^2 \leq C\|U\|_{H^3}^2\|U\|_{H^{k-m-1}}^2\|U\|_{C^{m+1}}^2\\&\leq C\|U\|_{H^3}^4\|U\|_{H^{k-1}}^2.\end{aligned}$$

In case $l=m=0$ we apply Lemma 10.1 and find

$$\begin{aligned}&\big\|(\partial_x^k A_U(x, U))U^1U^1\big\|_{L^2}^2\\&\leq C(\|a\|_{C^{k+1}}, \|U\|_{L^\infty}, \|(\phi, \psi)\|_{C^k})(1+\|U\|_{H^k}^2)\|U\|_{C^1}^4.\end{aligned}$$

The term I_4 is left to the reader, see Lemma 10.1. From $a'(s) = sa_1(s)$ we derive $\|a\|_{C^{k+1}} \leq C\|a_1\|_{C^k}$. Then we obtain the estimate

$$\begin{aligned}&\big\|\partial_x^k U(t, \cdot)\big\|_{L^2(B_R)}^2\\&\qquad\leq C_k\int_0^t \varrho_k(\|U(s, \cdot)\|_{H^3(B_R)})(1+\|U(s, \cdot)\|_{H^k(B_R)}^2)ds\end{aligned}$$

for $k \geq 1$. Since $\operatorname{supp}(\phi, \psi) \subset B_R$, there is some $0 < R' < R$ such that $\phi(x) = \psi(x) = 0$ for $R' \leq |x| \leq R$. For such x, the Cauchy problem (2.5) degenerates to $\partial_t^2 U = 0$; hence $U(t, x) = (1, t)^T$. Then Poincaré's inequality implies

$$\|U(t, \cdot) - (1, t)^T\|_{L^2(B_R)}^2 \leq C_R\|\partial_x U(t, \cdot)\|_{L^2(B_R)}^2.$$

The desired estimates (7.1), (7.2) are then obtained easily. □

Remark 7.4 Consider (2.6). Remarks 3.5, 4.5 and Lemma 10.1 give the refinement

$$\begin{aligned}
&\left\|\partial_x^k U(t,\cdot)\right\|^2_{L^2(B_R)}\\
&\le C_k e^{C(1+\tau^3)} \int_0^t \lambda^2(1+\tau^k)\\
&\qquad\qquad \times (1+\|U(s,\cdot)\|^4_{H^3(B_R)})(1+\|U(s,\cdot)\|^2_{H^k(B_R)})ds
\end{aligned}$$

for $k \ge 1$. From this we conclude that

$$\begin{aligned}
&\sup_{[0,t]} \|U(s,\cdot)-(1,s)^T\|^2_{H^3(B_R)}\\
&\qquad\qquad \le \lambda^2 t C_3 e^{C'(1+\tau^3)}(1+\sup_{[0,t]}\|U(s,\cdot)\|^6_{H^3(B_R)}), \quad (7.3)
\end{aligned}$$

for all λ and all $0 \le t < T$. Obviously,

$$\begin{aligned}
&\|U_{tt}(t,\cdot)\|_{L^\infty}\\
&\le \|A_\lambda(x,U)U_x\|_{C^1} + \|B_\lambda(x,U)U_x\|_{L^\infty} + \|C_\lambda(x,U)U\|_{L^\infty}\\
&\le C\|A_\lambda(x,U)\|_{H^2(B_R)}\|(U(t,\cdot)-(1,t)^T)_x\|_{H^2(B_R)}\\
&\qquad + \|B_\lambda(x,U)\|_{L^\infty}\|(U(t,\cdot)-(1,t)^T)_x\|_{L^\infty} + \|C_\lambda(x,U)U\|_{L^\infty}\\
&\le C\lambda(1+\|U(t,\cdot)\|^3_{H^2(B_R)})\|U(t,\cdot)-(1,t)^T\|_{H^3(B_R)} + C\lambda(1+\tau).
\end{aligned}$$

Supposing that the right–hand side of (7.3) were less than 1, we find

$$\left\|U_t(t,\cdot)-(0,1)^T\right\|_{L^\infty(B_R)} \le C'\lambda\tau(1+\tau^3). \quad (7.4)$$

Lemma 7.5 *Let the assumptions of* Proposition 6.1 *be satisfied. Assume that* $U \in C^2([0,T), C_b^\infty(B_R))$*,* $0 < T < \tau$*, is a solution to* (2.5) *which fulfils*

$$\left\|U(t,\cdot)-(1,t)^T\right\|_{C_b^1([0,T)\times B_R)} < \varepsilon_0, \quad (7.5)$$

$$\sup_{[0,T)} \|U(t,\cdot)\|_{H^3(B_R)} < \infty, \quad (7.6)$$

where ε_0 *is from* Proposition 6.1*. Then* U *can be extended to some function* $\tilde U \in C^2([0,T'], C_b^\infty(B_R))$*,* $T < T' < \tau$*, which solves* (2.5) *for* $(t,x) \in [0,T'] \times B_R$*.*

Proof. According to Lemma 7.2, $\|U(t,\cdot)\|_{H^k(B_R)} \le C_k$ for $0 \le t < T$ and all $k \in \mathbb{N}$. The equation (2.5) then gives $\|\partial_t^2 U(t,\cdot)\|_{H^k(B_R)} \le C_k$ for $0 \le$

$t < T$ and all k. Therefore, U can be smoothly extended in a unique way up to $t = T$. Now we consider the Cauchy problem

$$\partial_t^2 W - \partial_x(A(x, W)\partial_x W) - B(x, W)\partial_x W - C(x, W)W = 0,$$
$$W(t, x) = U(t, x), \quad W_t(t, x) = U_t(t, x).$$

By Proposition 6.1, this problem has a solution $W \in C^2([T, T_1], C_b^\infty(B_R))$. We set

$$\tilde{U}(t, x) = \begin{cases} U(t, x) & : 0 \le t < T, \\ W(t, x) & : T \le t \le T_1 = T', \end{cases}$$

and the proof is complete. □

Proof of Proposition 7.1. From Proposition 6.1 we conclude that there is a local solution $U \in C_b^\infty([0, T_1] \times B_R)$ to (2.5) which satisfies (7.2). By Lemma 7.5, this solution can be extended as long as (7.5) and (7.6) are satisfied. A lower estimate $T_0 > 0$ of the life span of U can then be derived from (7.2). □

Proof of Theorem 1.5. The problem (1.2) can be transformed into the system (2.5) by means of the reduction presented in Section 2. According to Proposition 7.1, this system has a unique local solution $U \in C_b^\infty([0, T_0] \times B_R)$. For $x \notin \operatorname{supp}(\phi, \psi)$, the system (2.5) degenerates into $\partial_t^2 U(t, x) = 0$, hence $u(t, x) = 0$. Therefore, we have found a solution $u \in C_b^\infty([0, T_0] \times \mathbb{R})$ to (2.1), which vanishes outside $[0, T_0] \times \operatorname{supp}(\phi, \psi)$. Then the solution w to (1.2) is given by

$$w(t, x) = \int_{-R}^{x} u(t, y)dy,$$

and it is easy to show that w vanishes outside $[0, T_0] \times \operatorname{supp}(\Phi, \Psi)$. □

Proof of Theorem 1.6. For $0 < \lambda \ll 1$, choose $\varepsilon(\lambda) = \mathscr{O}(\lambda^{1/2})$ as in Remark 3.4, and set $\varepsilon_0 = \varepsilon/10$, see the proof of Proposition 6.1. Now choose $\tau = \tau(\lambda)$ with

$$\lambda^2 \tau C_3 e^{C'(1+\tau^3)}(1 + (\varepsilon_0 + \|(1, \tau)\|_{H^3(B_R)})^6) < C_{\mathrm{sob}}^{-2}\varepsilon_0^2, \quad C'\lambda\tau(1 + \tau^3) < \varepsilon_0,$$

see (7.3), (7.4). Here C_{sob} is the norm of the embedding $H^3(B_R) \subset C^1(B_R)$.

Due to Remark 7.4, we then have

$$\left\|U(t,\,\cdot\,)-(1,\,t)^T\right\|_{C^1(B_R)}<\varepsilon_0,\quad \left\|U_t(t,\,\cdot\,)-(0,\,1)^T\right\|_{L^\infty(B_R)}<\varepsilon_0$$

provided that $t<\tau$. According to Lemma 7.5, the solution U persists in the interval $[0,\,\tau)$. Finally, $\tau(\lambda)>C|\ln\lambda|^{1/3}$. □

8. The case of non smooth $a(s)$

Proof of Theorem 1.2. We transform (1.2) into the system (2.5), where A, B, C are given by (2.2)–(2.4), and $(\phi,\,\psi)=(\Phi_x,\,\Psi_x)\in C_0^{k+1}(\mathbb{R})$. We approximate $a_0(s)$, $\phi(x)$, $\psi(x)$ by $a_{0,m}$, ϕ_m, ψ_m as in Definition 5.3, and obtain uniform estimates

$$\|(\phi_m,\,\psi_m)\|_{C^{k+1}},\quad \|a_{0,m}\|_{C^P(B_{M'})}\le C,\quad m\ge m_0(M'),\quad M'<M.$$

We set $a_m(s)=s^2a_{0,m}(s)$, $a_{1,m}(s)=a_m'(s)/s=2a_{0,m}(s)+sa_{0,m}'(s)$. Clearly,

$$\begin{aligned}sa_{0,m}'(s)&=s\int a_0'(s-r)m\varrho(mr)dr=\int(s-r)a_0'(s-r)m\varrho(rm)dr\\&\quad+\int a_0(s-r)m\varrho(rm)dr+\int a_0(s-r)rm^2\varrho'(rm)dr\\&=I_{1,m}(s)+I_{2,m}(s)+I_{3,m}(s).\end{aligned}$$

We see that $\|I_{1,m}\|_{C^P}+\|I_{2,m}\|_{C^P}\le C(\|a_0\|_{C^P}+\|a_1\|_{C^P})$, since $sa_0'(s)=a_1(s)-2a_0(s)$. Due to $|mr|\le 1$ on $\operatorname{supp}\varrho'(mr)$,

$$|\partial_s^P I_{3,m}(s)|\le\|a_0\|_{C^P}\int|m\varrho'(mr)|dr\le C\|a_0\|_{C^P}.$$

As a consequence, $\|a_{1,m}\|_{C^P}\le C$ for all m.

Now we consider the Cauchy problem

$$\begin{aligned}&\partial_t^2U_m-\partial_x(A_m(x,U_m)\partial_xU_m)\\&\qquad\qquad-B_m(x,U_m)\partial_xU_m-C_m(x,U_m)U_m=0,\\&U_m(0,\,x)=(1,\,0)^T,\quad U_{m,t}(0,\,x)=(0,\,1)^T,\end{aligned}$$

where A_m, B_m, C_m are defined as in (2.2)–(2.4), but with a_0, a_1, a, ϕ, ψ replaced by $a_{0,m}$, $a_{1,m}$, a_m, ϕ_m, ψ_m. According to Proposition 7.1, there is a unique local solution $U_m\in C_b^\infty([0,\,T_0]\times B_R)$ for large m, where T_0 only depends on $\|(a_{0,m},\,a_{1,m})\|_{C^3}$, $\|(\phi_m,\,\psi_m)\|_{C^5}$. These norms are uniformly in

m bounded. Taking into account that $k \geq 4$, we apply Lemma 7.2 with k replaced by $k-1$. Then we find

$$\sup_{[0,\, T_0]} \|U_m(t,\,\cdot\,)\|_{H^{k-1}(B_R)} \leq C < \infty$$

for all $m \geq m_0$. By the differential equation, it can be deduced that $\{U_m\}$ is a bounded sequence in $C([0,\, T_0],\, H^{k-1}(B_R)) \cap C^2([0,\, T_0],\, H^{k-3}(B_R))$. The Arzela–Ascoli theorem gives us a subsequence $\{U_{m'}\}$ converging in $C^1([0,\, T_0],\, H^{k-4}(B_R))$ to some limit U^*. Interpolating between the spaces $C([0,\, T_0],\, H^{k-4}(B_R))$ and $C([0,\, T_0],\, H^{k-1}(B_R))$ shows $U_{m'} \to U^*$ in $C([0,T_0],H^{k-1-\varepsilon}(B_R))$. Especially, we have convergence in $C([0,T_0],C^2(B_R))$, since $k \geq 4$. Then the limit U^* is a solution to (2.5). From the weak precompactness of bounded sets in H^{k-1} we deduce that $U_{m'} \to U^*$ in $L^\infty([0,\, T_0],\, H^{k-1}(B_R))$. The differential equation then yields

$$\partial_t^2 U^* \in L^\infty([0,\, T_0],\, H^{k-3}(B_R)).$$

The uniqueness of U^* can be shown by standard arguments, Proposition 4.4 and Gronwall's lemma.

Then we find a solution $u \in L^\infty([0,\, T_0],\, H^{k-1}(B_R))$ to (2.1), which satisfies $\partial_t^2 u \in L^\infty([0,\, T_0],\, H^{k-3}(B_R))$. A solution w to (1.2) then is given by $w(t,\, x) = \int_{-R}^{x} u(t, y)dy$, compare the proof of Theorem 1.5.

Finally, we discuss the uniqueness of this solution w. It suffices to consider the reduced problem (2.1). Let $v = v(t,\, x)$ be a second solution to (2.1) with

$$\partial_t^j v \in L^\infty([0,\, T_0],\, H^{3-j}(\mathbb{R})), \quad j = 0,\, 2.$$

Then the difference $z(t,\, x) = u(t,\, x) - v(t,\, x)$ solves

$$\partial_t^2 z - \partial_x(a_*(t,\, x)\partial_x z) - b(t,\, x)\partial_x z - c(t,\, x)z = 0$$

with the coefficients $a_*(t,\, x) = a(u(t,\, x))$, $b(t,\, x) = a'(u(t,\, x))\partial_x v(t,\, x)$, and $c(t,\, x)$ is given implicitly by $c(t,\, x)z = (a(u) - a(v))\partial_x^2 v + (a'(u) - a'(v))(\partial_x v)^2$. We see that $c(t,\, x)$ is bounded; and by Condition 1, $|b(t,\, x)|^2 \leq L a_*(t,\, x)$. From Proposition 4.4 we get $\|z(t,\,\cdot\,)\|_{L^2(B_R)} = 0$. On the other hand, $u(t,\, x) \equiv 0$ for $x \notin B_R$, which implies $\partial_t^2 z - c(t,\, x)z = 0$. Consequently, $z(t,\, x)$ vanishes everywhere. $\square$

9. A blow-up result

We consider the Cauchy problem 1.2 and describe a class of coefficients $a = a(s)$, and initial data Φ, Ψ for which the solution blows up in finite time.

Proposition 9.1 *Suppose* Condition 2 *with* $a_0(0) > 0$. *We assume that* $\Phi,\ \Psi \in C_0^\infty(\mathbb{R})$ *are even functions, and*

$$\Phi''(0) > 0, \quad \Psi''(0) > 0 \quad \textit{or} \quad \Phi''(0) < 0, \quad \Psi''(0) < 0.$$

Then the Cauchy problem (1.2) *has no global* C^∞ *solution* w.

Proof. According to Theorem 1.5, there is a unique solution $w \in C_b^\infty([0, T_0] \times \mathbb{R})$, for some $T_0 > 0$. Now we show that T_0 is bounded from above.

Since a, Φ, Ψ are even functions, the solution $w = w(t, x)$ is also even, hence $w_x(t, 0) = 0$ for $0 \le t \le T_0$, which implies $w(t, 0) = \Phi(0) + t\Psi(0)$. For $0 \le t \le T_0$, $-\varepsilon < x < \varepsilon$, we have the Taylor expansion

$$\begin{aligned} w(t, x) &= \sum_{k=0}^{2} \frac{1}{k!}(\partial_x^k w)(t, 0) + \mathscr{O}(|x|^3) \\ &= \big(\Phi(0) + t\Psi(0)\big) + \xi(t)x^2 + \mathscr{O}(|x|^3), \\ \xi(0) &= \frac{1}{2}\Phi''(0), \quad \xi'(0) = \frac{1}{2}\Psi''(0). \end{aligned}$$

Plugging this into (1.2) and collecting the terms with x^2 gives

$$\begin{aligned} &\xi_{tt}(t)x^2 - a(2\xi(t)x) \cdot 2\xi(t) + \mathscr{O}(|x|^3) = 0, \\ &\xi_{tt}(t) - (2\xi(t))^3 a_0(0) = 0, \quad 0 \le t \le T_0. \end{aligned}$$

Since $\xi(0)$ and $\xi'(0)$ have the same sign, and $a_0(0) > 0$, this ODE has no global solution, as can be seen from the equivalent formulation

$$\big((\xi_t)^2\big)_t = 4a_0(0)(\xi^4)_t, \quad 0 \le t \le T_0.$$

□

10. Appendix

The following technical lemma is proved by Nirenberg-Gagliardo interpolation.

Lemma 10.1 *Let $f = f(x, u)\colon \Omega \times \mathscr{M} \to \mathbb{R}$ be some C^k function, where $\Omega \subset \mathbb{R}^n$, $\mathscr{M} \subset \mathbb{R}^N$ are domains with smooth boundary, and Ω is bounded. Assume $k > n/2$. Then there is some continuous function $\varrho_k\colon \mathbb{R}_+ \to \mathbb{R}_+$ depending on $\|f(\,\cdot\,,\,\cdot\,)\|_{C^k(\Omega\times\mathscr{M})}$ such that*

$$\|f(x, u(x))\|_{H^k(\Omega)} \leq \varrho_k(\|u(\,\cdot\,)\|_{L^\infty(\Omega)})(1 + \|u(\,\cdot\,)\|_{H^k(\Omega)})$$

for all functions $u \in H^k(\Omega)$ taking values in $\mathscr{M}$. The function ϱ_k satisfies

$$\varrho_k(s) \leq C_k \sup_{x\in\Omega, |u|\leq s} \sum_{|\alpha|+|\beta|\leq k} |\partial_x^\alpha \partial_u^\beta f(x, u)|(1 + s^k).$$

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References

[1] Ang D.D. and Dinh A.P.N., *Strong solutions of a quasilinear wave equation with nonlinear damping.* SIAM J. Math. Anal. **19** (1988), 337–347.

[2] Biazutti A.C., *On a nonlinear evolution equation and its applications.* Nonlinear Analysis: Theory, Methods & Applications **24** (1995), 1221–1234.

[3] Colombini F. and Spagnolo S., *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞.* Acta Math. **148** (1982), 243–253.

[4] D'Ancona P. and Manfrin R., *Sufficient conditions for the local solvability of some quasilinear equations of weakly hyperbolic type.* Boll. Un. Mat. Ital. **8** (1994), 731–754.

[5] D'Ancona P. and Spagnolo S., *On the life span of the analytic solutions to quasilinear weakly hyperbolic equations.* Indiana Univ. Math. J. **40** (1991), 71–99.

[6] D'Ancona P. and Trebeschi P., *On the local solvability for a nonlinear weakly hyperbolic equation with analytic coefficients.* Comm. PDE **26** (2001), 779–811.

[7] Dionne Ph.A., *Sur les problèmes hyperboliques bien posés.* J. Analyse Math. **10** (1962), 1–90.

[8] Galaktionov V.A. and Pohozaev S.I., *Blow–up, critical exponents and asymptotic spectra for nonlinear hyperbolic equations.* Preprint 00/10, University of Bath, May 2000.

[9] Gao H. and Ma T.F., *Global solutions for a nonlinear wave equation with the p–Laplacian operator.* Electronic J. Qualitative Theory Differ. Equ. (1999), no. 11, 1–13.

[10] Glaeser G., *Racine carrée d'une fonction différentiable.* Ann. Inst. Fourier (Grenoble) **13** (1963), 203–210.

[11] Greenberg J.M., MacCamy R.C. and Mizel V.J., *On the existence, uniqueness and stability of solutions of the equation* $\sigma'(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$. J. Math. Mech. **17** (1968), 707–728.

[12] Hamilton R.S., *The inverse function theorem of Nash and Moser*. Bull. Amer. Math. Soc. (N.S.) **7** (1982), 65–222.

[13] John F., *Nonlinear wave equations, formation of singularities*. University lecture series, American Mathematical Society, Lehigh University, 1989.

[14] Kajitani K., *Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes*. Hokkaido Math. J. **12** (1983), 434–460.

[15] Kajitani K., *Propagation of analyticity of solutions to the Cauchy problem for Kirchhoff type equations*. Journées Équations aux dérivées partielles (Nantes), no. VIII, Univ. Nantes, 5-9 juin 2000, 1–14.

[16] Kajitani K., *Cauchy problem for nonstrictly hyperbolic systems in Gevrey classes*. J. Math. Kyoto Univ. **23** (1983), 599–616.

[17] Manfrin R., *Well posedness in the* C^∞ *class for* $u_{tt} = a(u)\Delta u$. Nonlinear Analysis: Theory, Methods & Applications **36** (1999), 177–212.

[18] Oleinik O.A., *On the Cauchy problem for weakly hyperbolic equations*. Comm. Pure Appl. Math. **23** (1970), 569–586.

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