Hokkaido Mathematical Journal Vol. 36 (2007) p. 175–191

Commutators with Reisz potentials in one and several parameters

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(Received September 5, 2005)

Abstract. Let M_b be the operator of pointwise multiplication by b, that is $\mathrm{M}_b f = bf$. Set $[\mathrm{A}, \mathrm{B}] = \mathrm{AB} - \mathrm{BA}$. The Reisz potentials are the operators

$$\mathrm{R}_\alpha f(x) = \int f(x-y)\frac{dy}{|y|^\alpha}, \quad 0 < \alpha < 1.$$

They map $L^p \mapsto L^q$, for $1 - \alpha + 1/q = 1/p$, a fact we shall take for granted in this paper. A Theorem of Chanillo [6] states that one has the equivalence

$$\|[\mathrm{M}_b, \mathrm{R}_\alpha]\|_{p\to q} \simeq \|b\|_{\mathrm{BMO}}$$

with the later norm being that of the space of functions of bounded mean oscillation. We discuss a proof of this result in a discrete setting, and extend part of the equivalence above to the higher parameter setting.

Key words: Reisz potential, fractional integral, paraproduct, commutator, multiparameter, bounded mean oscillation.

1. Introduction: One parameter

Our topic is norm bounds on commutators of different operators with the operation of multiplication by a function. Chanillo [6] proved that commutators with Reisz potentials characterize the function space BMO. We are concerned with a proof of this result, and extensions to multi-parameter situations.

To set notation, let H be the Hilbert transform, that is

$$\mathrm{H}f(x) = \text{p.v.} \int f(x-y)\frac{dy}{y}.$$

Let M_b be the operator of pointwise multiplication by b, that is $\mathrm{M}_b f = bf$ and $[\mathrm{A}, \mathrm{B}] = \mathrm{AB} - \mathrm{BA}$. A classical result states that

$$\|[\mathrm{M}_b, \mathrm{H}]\|_{p\to p} \simeq \|b\|_{\mathrm{BMO}}, \quad 1 < p < \infty. \tag{1.1}$$

2000 Mathematics Subject Classification : Primary: 42B20 Secondary: 42B25, 42B35.
Research supported in part by an NSF grant and a fellowship from the Guggenheim Foundation. The hospitality of the University of British Columbia is gratefully acknowledged

The latter space is that of functions of bounded mean oscillation, the dual to the real valued Hardy space H^1. The dyadic version of this space is defined in (2.4).

The history of this result goes back to the characterization of the boundedness of Hankel operators due to Nehari [24]. In the purely harmonic analysis setting, Coifmann, Rochberg and Weiss [10] provided an extensive study of this equivalence.

Set the Reisz potentials to be

$$\mathrm{R}_\alpha f(x) = \int f(x-y)\frac{dy}{|y|^\alpha}, \quad 0 < \alpha < 1. \tag{1.2}$$

These operators map $L^p \mapsto L^q$, for $1 - \alpha + 1/q = 1/p$, a fact we shall take for granted in this paper. We are interested in the Theorem of Chanillo [6].

Theorem 1.3 *For $1 - \alpha + 1/q = 1/p$, and $1 < p < q < \infty$ we have*

$$\|[\mathrm{M}_b, \mathrm{R}_\alpha]\|_{p\to q} \simeq \|b\|_{\mathrm{BMO}} \tag{1.4}$$

The method of proof introduced by Chanillo [6] is to dominate the sharp function of the commutator, a method that has been extended by a variety of authors in different settings, see [11, 13, 19, 27, 31]. We give a new proof, showing that the commutator with the Reisz potential is a sum of paraproducts. See the next section for a definition of paraproducts.

The rationale for this new proof is an extension of part Chanillo's result to a higher parameter setting, motivated in part by an extension of the Nehari theorem to higher parameter settings in papers of Ferguson and Lacey [18] and Lacey and Terwelleger [22]. Also see the recent papers of Muscalu, Pipher, Tao and Thiele [25, 26].

Theorem 3.1 is the main result of this paper. It is a partial extension of the one parameter result above, in that the upper bound on the commutator is established. This upper bound is in terms of the product BMO norm of the symbol b. Product BMO is the one identified by S.-Y. Chang and R. Fefferman [5, 4], a definition of which we will recall below. The lower bound on the commutator norm may not be true. See Section 3.4.

We restrict attention to discrete forms of the Reisz potentials. In situations such as this one, it permits one to concentrate on the most essential parts of the proofs, see e.g. [1] which is just one reference that is closely associated with the themes of this paper.

Appropriate averaging procedures will permit one to recover the continuous analogs, but we omit this argument, as it is well represented in the literature. For different versions of this argument, see [23, 28, 29]. In addition, the Reisz potentials on higher dimensional spaces are also frequently considered. The author is not aware of any reason why the methods of this paper will not extend to this level of generality; it is not pursued as it would complicate our presentation.

2. The one parameter statement

2.1. Haar functions and paraproducts

The dyadic intervals are

$$\mathcal{D} \stackrel{\text{def}}{=} \{[j2^k,\, (j+1)2^k) : j,\, k \in \mathbb{Z}\}.$$

Each dyadic interval I is a union of its left and right halves I_-, and I_+ respectively. The Haar function h_I adapted to I is

$$h_I \stackrel{\text{def}}{=} |I|^{-1/2}(-\mathbf{1}_{I_-} + \mathbf{1}_{I+}). \tag{2.1}$$

We will also denote the Haar functions as h_I^0, setting

$$h_I^1 = |I|^{-1/2}\mathbf{1}_I. \tag{2.2}$$

Thus, h_I^0 has integral zero, while h_I^1 is a multiple of an indicator function.

It is an essential fact that the Haar functions form an unconditional basis for L^p, in particular

$$\|f\|_p \simeq \Big\| \Big[\sum_{I\in\mathcal{D}} |\langle f,\, h_I\rangle h_I^1|^2 \Big]^{1/2} \Big\|_p \tag{2.3}$$

Define the dyadic BMO semi norm by

$$\|f\|_{\mathrm{BMO}} \stackrel{\text{def}}{=} \sup_{J\in\mathcal{D}} \Big[\frac{1}{|J|} \sum_{I\subset J} |\langle f,\, h_I\rangle|^2 \Big]^{1/2}. \tag{2.4}$$

The Haar paraproducts are

$$\mathrm{B}(f_1,\, f_2) \stackrel{\text{def}}{=} \sum_{I\in\mathcal{D}} \frac{\langle f_1,\, h_I\rangle}{\sqrt{|I|}} \langle f_2,\, h_I^1\rangle\, h_I \tag{2.5}$$

It is critical that there is exactly one function which is a multiple of the identity. We take as a given the fundmental fact about the boundedness of these operators.

Theorem 2.6 *We have*

$$\|\mathrm{B}(f_1,\,\cdot\,)\|_p \simeq \|f_1\|_{\mathrm{BMO}}, \quad 1 < p < \infty. \tag{2.7}$$

This theorem goes back to the work of Coifman and Meyer [9, 8, 7]. It plays a critical role in the T1 theorem of David and Journé [12]. See for instance the discussion in the text of E.M. Stein [30]. An analogous result in the higher parameter situation will be stated and proved in the next section.

We shall also appeal to some operators, related to, but not as central, the paraproducts. Define

$$\mathrm{C}(f_1,\, f_2) \stackrel{\mathrm{def}}{=} \sum_{I\in\mathcal{D}} |I|^{-1/2} h_I \prod_{j=1}^{2} \langle f_j,\, h_I\rangle \tag{2.8}$$

Notice that every Haar function that appears has zero integral. Therefore, we can estimate

$$\begin{aligned}
\|\mathrm{C}(f_1,\, f_2)\|_p &\simeq \Big\| \Big[\sum_{I\in\mathcal{D}} \big| |I|^{-1/2} \prod_{j=1}^{2} \langle f_j,\, h_I\rangle h_I^1 \big|^2 \Big]^{1/2} \Big\|_p \\
&\le \sup_{I\in\mathcal{D}} \frac{|\langle f_1,\, h_I\rangle|}{\sqrt{|I|}} \Big\| \Big[\sum_{I\in\mathcal{D}} |\langle f_2,\, h_I\rangle h_I^1|^2 \Big]^{1/2} \Big\|_p \\
&\lesssim \sup_{I\in\mathcal{D}} \frac{|\langle f_1,\, h_I\rangle|}{\sqrt{|I|}} \|f_2\|_p
\end{aligned}$$

We will not have recourse to these operators until the next section.

Define Haar projections to a particular scale by

$$\mathrm{P}_n f \stackrel{\mathrm{def}}{=} \sum_{|I|=2^n} \langle f,\, h_I\rangle h_I. \tag{2.9}$$

And define a related paraproduct by

$$\mathrm{D}_k(f_1,\, f_2) = \sum_{n\in\mathbb{Z}} (\mathrm{P}_n f_1)(\mathrm{P}_{n+k} f_2). \tag{2.10}$$

It is straight forward to see that

$$\|\mathrm{D}_k(f_1,\, f_2)\|_p \lesssim \sup_{I\in\mathcal{D}} \frac{|\langle f_1,\, h_I\rangle|}{\sqrt{|I|}} \|f_2\|_p \tag{2.11}$$

In particular, these paraproducts admit a bound on their operator norms that is strictly smaller than the BMO norm.

2.1.1. The Dyadic Reisz Potential and Commutator Consider a dyadic analog of the Reisz Potentials given by

$$\mathrm{I}_\alpha f \stackrel{\text{def}}{=} \sum_{I\in\mathcal{D}} \frac{\langle f,\, \mathbf{1}_I\rangle}{|I|^\alpha}\mathbf{1}_I, \quad 0<\alpha<1. \tag{2.12}$$

This operator enjoys the same mapping properties of the continuous Reisz potentials, a fact we shall take for granted.

And the continuous versions can be recovered from the dyadic models by an appropriate averaging procedure. This point of view is nicely illustrated in the article of Petermichl [29], in which the Hilbert transform is recovered from a dyadic model.

We discuss the proof of a dyadic version of the Theorem of Chanillo, Theorem 1.3

Theorem 2.13 *For $0<\alpha<1$, $1-\alpha+1/q=1/p$ and $1<p<q<\infty$ we have*

$$\|[\mathrm{M}_b,\, \mathrm{I}_\alpha]\|_{p\to q} \simeq \|b\|_{\mathrm{BMO}}. \tag{2.14}$$

Indeed concerning the upper bound on the commutator, the main point is this: The commutator $[\mathrm{I}_\alpha,\, \mathrm{M}_b]$ is a linear combination of the four terms

$$\mathrm{B}(b,\,\cdot\,)\circ \mathrm{I}_\alpha, \quad \mathrm{I}_\alpha\circ \mathrm{D}_0(b,\,\cdot\,), \tag{2.15}$$

$$\mathrm{D}_0(b,\,\cdot\,)\circ \mathrm{I}_\alpha, \quad \sum_{k=1}^{\infty} 2^{-k(1-\alpha)}\,\mathrm{D}_k(b,\,\cdot\,)\circ \mathrm{I}_\alpha \tag{2.16}$$

These operators are defined in (2.5) and (2.10). Therefore, the upper bound on the commutator is an immediate consequence of those for the Reisz potentials, and the corresponding paraproducts.

Observe that our Reisz potential, applied to a Haar function, has an explicit form.

$$\mathrm{I}_\alpha h_I = c_\alpha |I|^{1-\alpha} h_I, \tag{2.17}$$

for a choice of constant $c_\alpha = \sum_{n=1}^\infty 2^{-n(1-\alpha)}$. In addition, for a dyadic interval J, we have

$$\mathrm{I}_\alpha \mathbf{1}_J = \sum_{K\supset J} \frac{|J|}{|K|^\alpha}\mathbf{1}_K. \tag{2.18}$$

For later use, observe that $\mathrm{I}_\alpha \mathbf{1}_J$ equals $(1+c_\alpha)|J|^{1-\alpha}$ on the interval J.

We can then compute the commutator, with multiplying function h_I applied to another Haar function h_J

$$
\begin{aligned}
[\mathrm{M}_{h_I}, \mathrm{I}_\alpha]h_J &= c_\alpha h_I |J|^{1-\alpha} h_J - \mathrm{I}_\alpha(h_I \cdot h_J) \\
&= \begin{cases} 0 & J \subsetneq I \\ c_\alpha |I|^{-\alpha}\mathbf{1}_I - |I|^{-1}\mathrm{I}_\alpha(\mathbf{1}_I) & I = J \\ c_\alpha\, h_J(I)\{|J|^{1-\alpha} - |I|^{1-\alpha}\} h_I & I \subsetneq J. \end{cases}
\end{aligned} \tag{2.19}
$$

And in the case that $I \subsetneq J$, note that h_J takes exactly one value on I, which is denoted as $h_J(I)$.

We expand $[\mathrm{M}_b, \mathrm{I}_\alpha]f$ as a double sum over Haar functions. The leading term in the case of $I \subsetneq J$ in (2.19) gives us

$$
c_\alpha \sum_{I \in \mathcal{D}} \sum_{I \subsetneq J} \langle b, h_I\rangle \langle f, h_J\rangle |J|^{1-\alpha} h_J(I) h_I = \mathrm{B}(b, \mathrm{I}_\alpha f)
$$

which is the first term in (2.15).

For the second term in (2.19) in the case of $I \subsetneq J$, given a dyadic interval I and integer $k > 0$, let I_k denote the dyadic interval that contains I and has length $|I_k| = 2^k |I|$. Note that

$$
\sum_{I \in \mathcal{D}} \langle b, h_I\rangle \langle f, h_{I_k}\rangle |I|^{1-\alpha} h_{I_k}(I) h_I = 2^{-k(1-\alpha)} \mathrm{D}_k(b, \mathrm{I}_\alpha f).
$$

This leads to the second half of (2.16).

Consider the leading term in the case of $I = J$ in (2.19). It gives us

$$
\sum_{I \in \mathcal{D}} \langle b, h_I\rangle \langle f, h_I\rangle |I|^{-\alpha} \mathbf{1}_I = \mathrm{D}_0(b, \mathrm{I}_\alpha f)
$$

which is the first half of (2.16).

Consider the second term in the case of $I = J$ in (2.19). It gives us

$$
\sum_{I \in \mathcal{D}} \langle b, h_I\rangle \langle f, h_I\rangle |I|^{-1} \mathrm{I}_\alpha \mathbf{1}_I = \mathrm{I}_\alpha \circ \mathrm{D}_0(b, f)
$$

which is the second half of (2.15). Our proof of the upper bound on the commutator norm in Theorem 2.13 is finished.

Let us discuss the proof of the lower bound. We can take $b \in \mathrm{BMO}$ of norm one. Fix an interval J so that

$$\sum_{I\subset J} |\langle b,\, h_I\rangle|^2 \geq \frac{1}{2}|J|.$$

It is important to observe that by the John Nirenberg estimates we have

$$\begin{aligned} 1 &\lesssim |J|^{-1/p} \Big\| \sum_{I\subset J} \langle b,\, h_I\rangle h_I \Big\|_p \\ &\leq |J|^{-1/q} \Big\| \sum_{I\subset J} \langle b,\, h_I\rangle h_I \Big\|_q \\ &\lesssim 1. \end{aligned}$$

We obtain a lower bound on the L^q norm of the commutator applied to $\mathbf{1}_J$. Write the function b as $b = b' + b''$ where $b' = \sum_{|I|\leq|J|} \langle b,\, h_I\rangle h_I$.

$$\begin{aligned} [\mathrm{M}_b,\, \mathrm{I}_\alpha]\mathbf{1}_J &= b(\mathrm{I}_\alpha \mathbf{1}_J) - \mathrm{I}_\alpha(b\mathbf{1}_J) \\ &= b'\mathrm{I}_\alpha \mathbf{1}_J - \mathrm{I}_\alpha b' - b''\mathrm{I}_\alpha \mathbf{1}_J + b''(J)\mathrm{I}_\alpha \mathbf{1}_J. \end{aligned}$$

Notice that b'' takes a single value on J, and that the last two terms cancel on that interval. Thus,

$$\|[\mathrm{M}_b,\, \mathrm{I}_\alpha]\mathbf{1}_J\|_q \geq \|b'\mathrm{I}_\alpha \mathbf{1}_J - \mathrm{I}_\alpha b'\|_{L^q(J)} \tag{2.20}$$

Taking the explicit formulas (2.17) and (2.18) into account, we see that the last term above is at least a constant times

$$\begin{aligned} |J|^{1-\alpha} \Big\| \sum_{I\subset J} \langle b,\, h_I\rangle h_I \Big\|_q &\gtrsim |J|^{1-\alpha+1/q-1/p} \Big\| \sum_{I\subset J} \langle b,\, h_I\rangle h_I \Big\|_p \\ &\gtrsim |J|^{1-\alpha+1/q} \\ &= |J|^{1/p}. \end{aligned}$$

It follows that this commutator admits a universal lower bound on its $L^p \mapsto L^q$ norm, assuming that the BMO of the function b is one. The proof is complete. □

It is of interest to provide another proof of the lower bound. Let us begin by establishing the lower bound

$$\|[\mathrm{M}_b,\, \mathrm{I}_\alpha]\|_{p\to q} \gtrsim \sup_{I\in\mathcal{D}} \frac{|\langle b,\, h_I\rangle|}{\sqrt{|I|}}. \tag{2.21}$$

Indeed, apply the commutator to the Haar function h_I,

$$\begin{aligned}[\mathrm{M}_b,\, \mathrm{I}_\alpha]h_I &= c_\alpha b \cdot h_I - \mathrm{I}_\alpha(bh_I) \\ &= -\langle b,\, h_I\rangle |I|^{-1/2+1-\alpha}\mathbf{1}_I - c_\alpha \sum_{J\subsetneq I} h_I(J)\langle b,\, h_I\rangle \frac{|I|}{|J|}\end{aligned}$$

By the Littlewood Paley inequality for Haar functions, the latter term can be ignored in providing a lower bound on the L^q norm. We can then estimate

$$\begin{aligned}\|[\mathrm{M}_b,\, \mathrm{I}_\alpha]h_I\|_q &\gtrsim \frac{|\langle b,\, h_I\rangle|}{\sqrt{|I|}} |I|^{1-\alpha} \|\mathbf{1}_I\|_q \\ &\gtrsim \frac{|\langle b,\, h_I\rangle|}{\sqrt{|I|}} |I|^{1/p}.\end{aligned}$$

This proves (2.21).

Now, in seeking to prove the lower bound, we can assume that $\|b\|_{\mathrm{BMO}} = 1$, while

$$\sup_{I\in\mathcal{D}} \frac{|\langle b,\, h_I\rangle|}{\sqrt{|I|}} < \eta$$

where $\eta > 0$ is a small absolute constant to be chosen.

Recall that the paraproducts D_k have an upper bound on their norm given in (2.11). As well, we have shown that the commutator $[\mathrm{M}_b,\, \mathrm{I}_\alpha]$ as a sum of the terms in (2.15)—(2.16). Notice that for all of these terms, save one, we have an upper bound on their norm of an absolute constant times η.

The one term that this does not apply to is $\mathrm{B}(b,\,\cdot\,)\circ \mathrm{I}_\alpha$. But, it is very easy to see that

$$\|\mathrm{B}(b,\,\cdot\,)\circ \mathrm{I}_\alpha\|_{p\to q} \gtrsim c > 0.$$

Indeed, just apply the commutator to $\mathbf{1}_J$ for dyadic intervals J. And so, for $\eta > 0$ sufficiently small, we see that $\|[\mathrm{M}_b,\, \mathrm{I}_\alpha]\|_{p\to q} > c/2$.

3. Higher parameter commutators and paraproducts

We work in the setting of more variables, so that functions f are defined on $\mathbb{R}^d$. Set $\mathrm{I}_{\alpha,j}$ to be the Reisz potential as defined in (2.12), applied in the jth coordinate. For a sequence of choices of $0 < \alpha_j < 1$, observe that the operator

$$\mathrm{I}_{\alpha_1} \circ \cdots \circ \mathrm{I}_{\alpha_d}$$

will map L^p to L^q provided $1 - \sum_{j=1}^{d} \alpha_j + 1/q = 1/p$, and $1 < p < q < \infty$. One uses the one parameter result in each coordinate seperately.

Our main result is

Theorem 3.1 *Let* $0 < \alpha_j < 1$, $1 - \sum_{j=1}^{d} \alpha_j + 1/q = 1/p$, *and* $1 < p < q < \infty$ *we have*

$$\|[\cdots[\mathrm{M}_b, \mathrm{I}_{\alpha_1,1}], \ldots, \mathrm{I}_{\alpha_d,d}]\|_{p \to q} \lesssim \|b\|_{\mathrm{BMO}_d} \tag{3.2}$$

The strategy of appealing to sharp function estimates has well known difficulties in the higher parameter setting, * and so we adopt the strategy given in the previous section in the one parameter setting. We recall the necessary results for the paraproducts in the higher parameter setting, and then detail the proof of the Theorem above.

Notice that this is not a full extension of Chanillo's result as we do no claim that the two norms are comparable. We comment on this in more detail in Section 3.4 below.

3.1. Higher parameter paraproducts

Let $\mathcal{R} \stackrel{\text{def}}{=} \otimes_{j=1}^{d} \mathcal{D}$ denote the dyadic rectangles in $\mathbb{R}^d$. When needed, we will write such a rectangle as $R = \otimes_{j=1}^{d} R_j$.

$$h_R(x_1, \ldots, x_d) = \prod_{j=1}^{d} h_{R_j}(x_j). \tag{3.3}$$

And, by h_R^0 we mean h_R. The other distinguished function of this type is $h_R^1 = |h_R|$.

The simplest higher parameter paraproduct, and the only one needed for this paper, is

$$\mathrm{B}(f_1, f_2) \stackrel{\text{def}}{=} \sum_{R \in \mathcal{R}} \frac{\langle f_1, h_R \rangle}{\sqrt{|R|}} \langle f_2, h_R^1 \rangle \, h_R. \tag{3.4}$$

The principal fact about these paraproducts is this.

*R. Fefferman [15] has found a partial substitute for the sharp function in two parameters.

Theorem 3.5 *We have*

$$\|\mathrm{B}(b,\,\cdot\,)\|_p \simeq \|b\|_{\mathrm{BMO}_d}, \quad 1 < p < \infty. \tag{3.6}$$

In these inequalities, the BMO_d space is the dual to product H^1, as identified by S.-Y. Chang and R. Fefferman. Specifically,

$$\|b\|_{\mathrm{BMO}_d} = \sup\Big[\frac{1}{|U|}\sum_{R\subset U} |\langle f,\, h_R\rangle|^2\Big]^{1/2} \tag{3.7}$$

It is essential that in this definition, the supremum be formed over all open sets $U \subset \mathbb{R}^d$ of finite measure.

We caution the reader that the Theorem above does not include the full range of multiparameters paraproducts. † For more infomation about this theorem, see Journé [20]. More recently, see Muscalu, Pipher, Tao and Thiele, [25, 26] for certain extensions of the Theorem above. Also see Lacey and Metcalfe [21].

Proof. The proof we will give will rely upon the structure of the Hardy and BMO space, and the interpolation theory for this pair of spaces.

It is efficient to establish appropriate end point estimates for the dual to this operator. Fix $b \in \mathrm{BMO}_d$ of norm one. We establish that the dual operator B^* maps $H^1 \mapsto L^1$ and $L^\infty \mapsto \mathrm{BMO}_d$. An interpolation argument will complete the proof.

For the H^1 estimate, we use the atomic theory, as given in [4]. Recall that an H^1 atom is a function α with Haar support in a set A of finite measure, that is

$$\alpha = \sum_{R\subset A} \langle \alpha,\, h_R\rangle h_R. \tag{3.8}$$

Moreover, it satisfies the size condition $\|\alpha\|_2 \le |A|^{-1/2}$. Every element $f \in H^1$ admits a representation $f = \sum_j c_j \alpha_j$ where each α_j is an atom, c_j is a scalar, and

$$\|f\|_{H^1} \simeq \sum_j |c_j|. \tag{3.9}$$

†The presence of the full range of paraproducts is the source of part of the difficulties in Ferguson and Lacey [18] and Lacey and Terwelleger [22].

Observe that

$$\begin{aligned}\|\mathrm{B}^*(b,\,\alpha)\|_1 &= \sum_{R\subset A} |\langle b,\,h_R\rangle\langle \alpha,\,h_R\rangle| \\ &\le \Big[\sum_{R\subset A}|\langle b,\,h_R\rangle|^2 \sum_{R\subset A}|\langle \alpha,\,h_R\rangle|^2\Big]^{1/2} \\ &\le [|A||A|^{-1}]^{1/2} = 1.\end{aligned}$$

Thus, by (3.9), it is clear that we have the $H^1 \mapsto L^1$ estimate.

For the other estimate, fix a function $f \in L^\infty$ of norm one. And take a set $U \subset \mathbb{R}^d$ of finite measure. Let us set

$$F_U \stackrel{\text{def}}{=} \sum_{R\subset U} \frac{\langle b,\,h_R\rangle}{\sqrt{|R|}}\langle f,\,h_R\rangle h_R^1.$$

Then, appealing to the definition of BMO, it is the case that

$$\begin{aligned}\|F_U\|_1 &\le \sum_{R\subset U} |\langle b,\,h_R\rangle\langle f,\,h_R\rangle| \\ &\le \Big[\sum_{R\subset U}|\langle b,\,h_R\rangle|^2 \sum_{R\subset U}|\langle f,\,h_R\rangle|^2\Big]^{1/2} \\ &\le |U|.\end{aligned}$$

As this estimate is uniform over all choices of U, it is a reflection of the John Nirenberg estimate in the multiparameter setting [5] that this implies that

$$\|F_U\|_2 \lesssim |U|^{1/2}.$$

Let us observe that for rectangles $R \subset U$ and $S \not\subset U$, we necessarily have $\langle h_R,\,h_S^1\rangle = 0$. Therefore, we have

$$\begin{aligned}\sum_{R\subset U}|\langle \mathrm{B}^*(b,\,f),\,h_R\rangle|^2 &= \sum_{R\subset U}|\langle F_U,\,h_R\rangle|^2 \\ &\le \|F_U\|_2^2 \\ &\lesssim |U|.\end{aligned}$$

This proves the $L^\infty \mapsto \mathrm{BMO}_d$ bound. $\square$

3.2. A secondary result on paraproducts

For the proof of our main theorem, an estimate on certain paraproducts is needed. These paraproducts are of a secondary nature. In particular, we can give an upper bound on their norm that is strictly smaller, in general, than the BMO_d norm. The paraproducts are easiest to define in terms of tensor products of operators. Let $\mathbb{B}$ and $\mathbb{D}$ be a partition of the coordinates $\{1, \ldots, d\}$, and define

$$\mathrm{E}_{\mathbb{B}}(b,\,\cdot\,) \stackrel{\text{def}}{=} \bigotimes_{j\in\mathbb{B}} \mathrm{B}_j(b,\,\cdot\,) \bigotimes_{j\in\mathbb{D}} \mathrm{D}_{v(j),j}(b,\,\cdot\,) \tag{3.10}$$

Here, $v(j)$ is a non negative integer, and $\mathrm{D}_{v(j),j}$ is the operator $\mathrm{D}_{v(j)}$ as defined in (2.10) acting in the jth coordinate.

Proposition 3.11 *The paraproducts* $\mathrm{E}_{\mathbb{B}}$ *admit the bound*

$$\|\mathrm{E}_{\mathbb{B}}(b,\,\cdot\,)\|_{p\to p} \lesssim \begin{cases} \|b\|_{\mathrm{BMO}_{d,\mathbb{B}}} & 1 < p \le 2 \\ \|b\|_{\mathrm{BMO}_{d,\mathbb{B}}}^{2/p} \|b\|_{\mathrm{BMO}_d}^{1-2/p} & 2 < p < \infty. \end{cases} \tag{3.12}$$

In this Proposition, the norm $\|\cdot\|_{\mathrm{BMO}_{d,\mathbb{B}}}$ is defined in terms of collections of rectangles $\mathcal{S}$ which are restricted in the following way. Say that $\mathcal{S}$ is of type $\mathbb{B}$ if the rectangles in $\mathcal{S}$ have a union with finite measure, and for all $R, R' \in \mathcal{S}$, and coordinates $j \notin \mathbb{B}$ we have $R_j = R'_j$. Thus, only the coordinates in $\mathbb{B}$ are permitted to vary. Then define

$$\|b\|_{\mathrm{BMO}_{d,\mathbb{B}}} \stackrel{\text{def}}{=} \sup_{\mathcal{S}} \Big[\big| \bigcup_{R\in\mathcal{S}} R \big|^{-1} \sum_{R\in\mathcal{S}} |\langle f, h_R\rangle|^2 \Big]^{1/2} \tag{3.13}$$

where the supremum is over all collections of rectangles of type $\mathbb{B}$. Clearly, this norm is strictly smaller than that of BMO_d. And an example of Carleson [3], and published in [15], shows that these norms are essentially smaller than the BMO norm. (The use of norms of these types are illustrated in [22] and [2].)

Proof. We proceed to the proof. The case of the cardinality of $\mathbb{B}$ is full, that is equal to d, is contained in Theorem 3.5.

Now assume that the cardinality of $\mathbb{B}$ is not full. The L^2 case of (3.12) follows immediately, as we are forming the tensor product of operators D_v that act on a family of orthogonal spaces.

We should take care to consider the form of the operator $\mathrm{E}_{\mathbb{B}}(b, f)$. For a dyadic rectangle R, and coordinate j, let $\widetilde{R}_j = R_j$ if $j \notin \mathbb{D}$, and otherwise take this to be the dyadic interval that contains R_j and has length $2^{v(j)}|R_j|$. Let $\widetilde{R} = \otimes \widetilde{R}_j$. Then, the operator in question is

$$\mathrm{E}_{\mathbb{B}}(b,\, f) = \sum_R \varepsilon_R \frac{\langle b,\, h_R\rangle}{\sqrt{|\widetilde{R}|}} \langle f,\, h^{\epsilon}_{\widetilde{R}}\rangle h^{\epsilon}_{\widetilde{R}}$$

Here, $\epsilon \in \{0,\, 1\}^d$ is equal to 1 for those coordinates in $\mathbb{B}$, and is zero otherwise, and

$$h^{\epsilon}_{\widetilde{R}} = \prod_{j=1}^{d} h^{\epsilon_j}_{\widetilde{R}_j}.$$

The coefficient ε_R is a choice of sign. (To be specific, the value of ε_R is the product of the signs $\operatorname{sgn} h_{\widetilde{R}_j}(R_j)$ over those $j \in \mathbb{D}$ such that $v(j) > 0$.)

It is natural to exploit the availible L^2 estimate by establishing the boundedness of $\mathrm{E}_{\mathbb{B}}$ as a map from $H^1 \mapsto H^1$. Recall the definition of an H^1 atom in (3.8) and (3.9)

Since $R \subset \widetilde{R}$, it is clear that $\mathrm{E}_{\mathbb{B}}$ applied to an atom has the same Haar support. And by the L^2 bound,

$$\|\mathrm{E}_{\mathbb{B}}\alpha\|_2 \lesssim \|b\|_{\mathrm{BMO}_{d,\mathbb{B}}} |A|^{-1/2}$$

This proves the bound at H^1. And by interpolation, we deduce the result for $1 < p < 2$.

At the other endpoint, we prove that

$$\|\mathrm{E}_{\mathbb{B}}(b,\, \cdot\,)\|_{L^\infty \mapsto \mathrm{BMO}_d} \lesssim \|b\|_{\mathrm{BMO}_d}$$

Indeed, take $f \in L^\infty$ of norm one, and a set $U \subset \mathbb{R}^d$ of finite measure. Then,

$$\sum_{\widetilde{R} \subset U} |\langle b,\, h_R\rangle|^2 \frac{|\langle f,\, h^{\varepsilon}_{\widetilde{R}}\rangle|^2}{|\widetilde{R}|} \le \|f\|_\infty^2 \sum_{\widetilde{R} \subset U} |\langle b,\, h_R\rangle|^2 \le \|f\|_\infty^2 \|b\|^2_{\mathrm{BMO}_d}$$

This proves the inequality at L^∞, and interpolation will prove the bound for $2 < p < \infty$. □

3.3. The proof of Theorem 3.1

The same proof strategy as in one dimension is used. We expand the commutator as a double sum over Haar functions. In so doing, we use

(2.19). Observe that for two rectangles R and S, we have

$$\begin{aligned}[\dots[\mathrm{M}_{h_R}, \mathrm{I}_{\alpha,1}], \dots, \mathrm{I}_{\alpha,d}]h_S &= \prod_{j=1}^{d}[\mathrm{M}_{h_{R_j}}, \mathrm{I}_{\alpha_j}]h_{S_j} \\ &= 0\end{aligned}$$

if for any coordinate j we have $S_j \subsetneq R_j$. Assuming that this is not the case, we see that one of two terms can arise in each coordinate, depending upon $S_j = R_j$ or $R_j \subsetneq S_j$, as described in (2.19). In this way, we expand the commutator as sum of paraproduct operators.

These operators are as in (3.10):

$$2^{-v(1-\alpha)}\mathrm{E}_{\mathbb{B}}(b, \cdot)\circ\otimes_{j=1}^{d}\mathrm{I}_{\alpha_j,j}, \quad 2^{-v(1-\alpha)}\otimes_{j=1}^{d}\mathrm{I}_{\alpha_j,j}\circ\mathrm{E}_{\mathbb{B}}(b, \cdot). \tag{3.14}$$

We permit the subset $\mathbb{B} \subset \{1, \dots, d\}$ to vary over all possible subsets. Associated to the complementary set $\mathbb{D} = \{1, \dots, d\} - \mathbb{B}$ is a vector $v = \{v(j)\} \in \mathbb{N}^{\mathbb{D}}$, and we set $v = \sum_{j\in\mathbb{D}} v(j)$.

According to Proposition 3.11 and the obvious bound on the Reisz potential, each of these terms has $L^p \mapsto L^q$ norm of at most $2^{-v(1-\alpha)}\|b\|_{\mathrm{BMO}_d}$. And these estimates are summable over all choices of $\mathbb{B}$, $\mathbb{D}$, and choices of integers $\{v(j) \colon j \in \mathbb{D}\}$. This completes the proof of the upper bound.

3.4. Concerning lower bounds on the commutator norm

In the one parameter case, to provide a lower bound on the norm of the paraproduct $\mathrm{B}(b, \cdot)$, as defined in (2.5), it suffices to test it against an indicator of a dyadic interval. Moreover, one trivially has

$$\|\mathrm{B}(b, \mathrm{I}_\alpha \mathbf{1}_J)\|_q \gtrsim c_\alpha |J|^{1-\alpha}\|\mathrm{B}(b, \mathbf{1}_J)\|_q.$$

In higher parameters, the situation is far less obvious. We can establish

Proposition 3.15 *We have the inequality*

$$\|[\dots[\mathrm{M}_b, \mathrm{I}_{\alpha,1}], \dots, \mathrm{I}_{\alpha,d}]\|_{p\to q} \gtrsim \sup_S \Big[|S|^{-1}\sum_{R\subset S}|\langle b, h_R\rangle|^2\Big]^{1/2}.$$

Here, the supremum is formed over all dyadic rectangles S.

We omit the proof, which depends upon an iteration of the argument that lead to (2.20).

This norm sometimes referred to as "rectangular BMO," denoted as $\|\cdot\|_{\mathrm{BMO(rec)}}$. But as is well known, this norm is essentially smaller than the

BMO norm, and so this proposition is not enough to prove the complete analog of Chanillo's theorem.

Continuing this line of thought, in two dimensions (and only two dimensions), it is easy to see that we have

$$\|b\|_{\mathrm{BMO(rec)}} = \sup_{\mathbb{B}=\{1\},\{2\}} \|b\|_{\mathrm{BMO}_{2,\mathbb{B}}}$$

From this, our expansion of the commutator, and Proposition 3.11, we see that we have the estimate

$$\begin{aligned}&\|[[\mathrm{M}_b, \mathrm{I}_{\alpha_1,1}], \mathrm{I}_{\alpha_2,2}]\|_{p\to q} \gtrsim \\ &\max(\|b\|^{\varepsilon(\alpha,p)}_{\mathrm{BMO(rec)}}, \|\mathrm{B}(b,\,\cdot\,)\circ\mathrm{I}_{\alpha_1,1}\otimes\mathrm{I}_{\alpha_2,2}+\mathrm{I}_{\alpha_1,1}\otimes\mathrm{I}_{\alpha_2,2}\circ\mathrm{B}(b,\,\cdot\,)\|_{p\to q})\end{aligned}$$

Here, $\varepsilon(\alpha, p)$ is a positive exponent. It is natural to then assume that the rectangular norm is small, and argue that the other norm is big, but there is a problem with continuing this line of thought.

To provide a lower bound on the norms of the paraproducts $\mathrm{B}(b,\,\cdot\,)$ as defined in (3.4), we need to apply this operator to $\mathbf{1}_U$, for arbitrary sets $U \subset \mathbb{R}^d$ of finite measure. We would then like for an inequality of this form to be true:

$$\|\mathrm{I}_{\alpha_1,1}\circ\cdots\circ\mathrm{I}_{\alpha_d,d}\mathbf{1}_U\|_q \gtrsim |U|^{1/p}, \quad U \subset \mathbb{R}^d.$$

This holds for U a rectangle, but not in general. Indeed, consider $U = \bigcup_{n=1}^{n} R_n$, where R_n are rectangles that are of the same dimension, but very widely seperated. So seperated that we can estimate

$$\begin{aligned}\|\mathrm{I}_{\alpha_1,1}\circ\cdots\circ\mathrm{I}_{\alpha_d,d}\mathbf{1}_U\|_q &\simeq \left[\sum_{n=1}^{N}\|\mathrm{I}_{\alpha_1,1}\circ\cdots\circ\mathrm{I}_{\alpha_d,d}\mathbf{1}_{R_n}\|_q^q\right]^{1/q} \\ &\simeq N^{1/q}|R|^{1/p} \\ &\simeq N^{1/q-1/p}|U|^{1/p}.\end{aligned}$$

Since $1 < p < q < \infty$, this last term is substantially smaller than $|U|^{1/p}$ for N large.

Acknowledgment We thank the referee for a careful reading of the manuscript.

References

[1] Auschqueer P., Hofmann S., Muscalu C., Tao T. and Thiele C., *Carleson measures, trees, extrapolation, and $T(b)$ theorems*. Publ. Mat. (2) **46** (2002), 257–325.

[2] Cabrelli C., Lacey M.T., Molter U. and Pipher J.C., *Variations on the theme of Journé's lemma*. Houston J. Math. (3) **32** (2006), 833–861.

[3] Carleson L., *A counterexample for measures bounded on H^p spaces for the bidisk*. Mittag-Leffler Rep. no. 7, Inst. Mittag-Leffler, 1974.

[4] Sun-Yung A, C. and Fefferman R., *Some recent developments in Fourier analysis and H^p-theory on product domains*. Bull. Amer. Math. Soc. (N.S.). (1) **12** (1985), 1–43.

[5] Sun-Yung A, C. and Fefferman R., *A continuous version of duality of H^1 with* BMO *on the bidisc*. Ann. of Math. (1) **112** (1980), 179–201.

[6] Chanillo S., *A note on commutators*. Indiana Univ. Math. J. (1) **31** (1982), 7–16.

[7] Coifman R.R. and Meyer Y., *On commutators of singular integrals and bilinear singular integrals*. Trans. Amer. Math. Soc. **212** (1975), 315–331.

[8] Coifman R.R. and Meyer Y., *Commutateurs d'intégrales singulières et opérateurs multilinéaires*. Ann. Inst. Fourier (Grenoble), (3) **28** (1978), 177–202 (French, with English summary).

[9] Coifman R.R. and Meyer Y., *Au delà des opérateurs pseudo-différentiels*. Astérisque vol. 57, Société Mathématique de France, Paris, 1978 (French).

[10] Coifman R.R., Rochberg R. and Weiss G., *Factorization theorems for Hardy spaces in several variables*. Ann. of Math. (3) **103** (1976), 611–635.

[11] Cruz-Uribe D. and Fiorenza A., *Endpoint estimates and weighted norm inequalities for commutators of fractional integrals*. Publ. Mat. (1) **47** (2003), 103–131.

[12] David G. and Journé J.-L., *A boundedness criterion for generalized Calderón-Zygmund operators*. Ann. of Math. (2) **120** (1984), 371–397.

[13] Ding Y., Lu S. and Zhang P., *Weak estimates for commutators of fractional integral operators*. Sci. China Ser. A (7) **44** (2001), 877–888.

[14] Duong X.T. and Yan L.X., *On commutators of fractional integrals*. Proc. Amer. Math. Soc. (12) **132** (2004), 3549–3557 (electronic).

[15] Fefferman R., *Harmonic analysis on product spaces*. Ann. of Math. (1) **126** (1987), 109–130.

[16] Fefferman R., *A note on Carleson measures in product spaces*. Proc. Amer. Math. Soc. (3) **93** (1985), 509–511.

[17] Fefferman R., *Bounded mean oscillation on the polydisk*. Ann. of Math. (2) **110** (1979), 395–406.

[18] Ferguson S.H. and Lacey M.T., *A characterization of product* BMO *by commutators*. Acta Math. (2) **189** (2002), 143–160.

[19] Janson S., *Mean oscillation and commutators of singular integral operators*. Ark. Mat. (2) **16** (1978), 263–270.

[20] Journé J.-L., *Calderón-Zygmund operators on product spaces*. Rev. Mat. Iberoamericana (3) **1** (1985), 55–91.

[21] Lacey M.T. and Metcalfe J., *Paraproducts in One and Several Parameters*. 2005.

[22] Lacey M.T. and Terwilleger E., *Little Hankel Operators and Product* BMO. 2004.

[23] Lacey M. and Thiele C., *A proof of boundedness of the Carleson operator*. Math. Res. Lett. (4) **7** (2000), 361–370.

[24] Nehari Z., *On bounded bilinear forms*. Ann. of Math. (2) **65** (1957), 153–162.

[25] Mucalu C., Pipher J., Tao T. and Thiele C., *Bi-parameter paraproducts*. Acta Math. (2) **193** (2004), 269–296.

[26] Mucalu C., Pipher J., Tao T. and Thiele C., *Multi-parameter paraproducts*. `arXiv: math. CA/0411607`.

[27] Paluszyński M., *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*. Indiana Univ. Math. J. (1) **44** (1995), 1–17.

[28] Petermichl S., Treil S. and Volberg A., *Why the Riesz transforms are averages of the dyadic shifts?*. Publ. Mat. Vol. Extra (2002), 209–228.

[29] Petermichl S., *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*. C. R. Acad. Sci. Paris Sér. I Math. (6) **330** (2000), 455–460 (English, with English and French summaries).

[30] Stein E.M., *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993.

[31] Youssfi A., *Regularity properties of commutators and* BMO-*Triebel-Lizorkin spaces*. Ann. Inst. Fourier (Grenoble), (3) **45** (1995), 795–807 (English, with English and French summaries).

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