On some new seminormed sequence spaces defined by modulus functions

Yavuz ALTIN and Mahmut IŞIK

(Received October 14, 2004; Revised April 25, 2005)

Abstract. In this paper we introduce the sequence spaces $\ell_\infty(f, p, \sigma, q, s)$, $c_0(f, p, \sigma, q, s)$ and $c(f, p, \sigma, q, s)$ using a modulus function f and defined over a seminormed space (X, q) , seminormed by q . We study some properties of these sequence spaces and obtain some inclusion relations.

Key words: modulus function, invariant mean, seminorm.

1. Introduction

Let ℓ_∞ and c denote the Banach spaces of real bounded and convergent sequences $x = (x_n)$ normed by $\|x\| = \sup_n |x_n|$, respectively.

Let σ be a one to one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, $m = 1, 2, \dots$. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a σ -mean if and only if

- i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
- ii) $\varphi(e) = 1$ where $e = (1, 1, 1, \dots)$ and
- iii) $\varphi(\{x_{\sigma(n)}\}) = \varphi(\{x_n\})$ for all $x \in \ell_\infty$.

For certain kinds of mappings σ , every invariant mean φ extends the limit functional on the space c of real convergent sequences, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ -means are equal.

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown (see Schaefer [16]) that

$$V_\sigma = \{x = (x_n) : \lim_m t_{mn}(x) = Le \text{ uniformly in } n, L = \sigma - \lim x\}, \quad (1)$$

where $t_{mn}(x) = (x_n + Tx_n + \dots + T^m x_n)/(m+1)$.

The special case of (1), in which $\sigma(n) = n+1$ was given by Lorentz [6].

Much literature concerning invariant means, etc., can be found in ([1],

[10, 11], [14])

The notion of a modulus function was introduced by Nakano [12] in 1953. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

- i) $f(x) = 0$ if and only if $x = 0$,
- ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,
- iii) f is increasing,
- iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$. Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$, from condition (ii) and so

$$f(x) = f\left(nx \frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right) \text{ hence}$$

$$\frac{1}{n}f(x) \leq f\left(\frac{x}{n}\right) \text{ for all } n \in \mathbb{N}.$$

A modulus may be bounded or unbounded. For example, $f(x) = x^p$, ($0 < p \leq 1$) is unbounded and $f(x) = x/(1+x)$ is bounded. Maddox [9] and Ruckle [15] used a modulus function to construct some sequence spaces.

After then some sequence spaces, defined by a modulus function, were introduced and studied by Bhardwaj [2], Bilgin [3], Pehlivan and Fisher [13], Waszak [17] and many others.

Proposition 1.1 *Let f be a modulus and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$, [13].*

Definition 1.2 Let p, q be seminorms on a vector space X . Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \rightarrow 0$, then also $q(x_n) \rightarrow 0$. If each is stronger than the others p and q are said to be equivalent (one may refer to Wilansky [18]).

Lemma 1.3 *Let p and q be seminorms on a linear space X . Then p is stronger than q if and only if there exists a constant M such that $q(x) \leq Mp(x)$ for all $x \in X$ (see for instance Wilansky [18]).*

Definition 1.4 A sequence space E is said to be *solid* or *normal* if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$, (see Ref. [5]).

Definition 1.5 A sequence space E is said to be *monotone* if it contains the canonical pre-images of all its step spaces, (see Ref. [5]).

Remark 1.6 From the two above definitions it is clear that “A sequence space E is solid implies that E is monotone” (see Ref. [4]).

Definition 1.7 A sequence space E is said to be *convergence free* if $(y_k) \in E$, whenever $(x_k) \in E$ and $x_k = 0$ implies that $y_k = 0$.

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = G$, and let $D = \max(1, 2^{G-1})$. Then for $a_k, b_k \in \mathbb{C}$, the set of complex numbers for all $k \in \mathbb{N}$, we have (see for instance Maddox [8])

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}. \quad (2)$$

Throughout the article $p = (p_k)$ will represent a sequence of strictly positive real numbers and (X, q) a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q . The symbol $\ell_\infty(X)$ denotes the spaces of all bounded sequences defined over X . We define the following sequence spaces:

$$\begin{aligned} \ell_\infty(f, p, \sigma, q, s) &= \left\{ x \in \ell_\infty(X) : \sup_{n,k} k^{-s} [f(q(x_{\sigma^k(n)}))]^{p_k} < \infty, s \geq 0 \right\}, \\ c(f, p, \sigma, q, s) &= \left\{ x \in \ell_\infty(X) : \lim_{k \rightarrow \infty} k^{-s} [f(q(x_{\sigma^k(n)} - \ell))]^{p_k} = 0, \right. \\ &\quad \left. \ell \in X \text{ and } s \geq 0, \text{ uniformly in } n \right\}, \\ c_0(f, p, \sigma, q, s) &= \left\{ x \in \ell_\infty(X) : \lim_{k \rightarrow \infty} k^{-s} [f(q(x_{\sigma^k(n)}))]^{p_k} = 0, \right. \\ &\quad \left. s \geq 0, \text{ uniformly in } n \right\}. \end{aligned}$$

2. Main results

Theorem 2.1 Let $p = (p_k)$ be a bounded sequence, then $Z(f, p, \sigma, q, s)$ are linear spaces over the set of complex numbers, for $Z = \ell_\infty, c$ and c_0 .

Proof. We give the proof for $Z = c_0$ only. The others cases can be treated similarly. Let $x, y \in c_0(f, p, \sigma, q, s)$. For $\lambda, \mu \in \mathbb{C}$, there exist positive integers M_λ and N_μ such that $|\lambda| \leq M_\lambda$ and $|\mu| \leq N_\mu$. Since f is subadditive, q is a seminorm,

$$k^{-s} [f(q(\lambda x_{\sigma^k(n)} + \mu y_{\sigma^k(n)}))]^{p_k} \leq D(M_\lambda)^H k^{-s} [f(q(\lambda x_{\sigma^k(n)}))]^{p_k}$$

$$\begin{aligned}
& + D(N_\lambda)^H k^{-s} [f(q(\mu y_{\sigma^k(n)}))]^{p_k} \\
& \rightarrow 0
\end{aligned}$$

as $k \rightarrow \infty$, $s \geq 0$, uniformly in n .

This proves that $c_0(M, p, \sigma, q, s)$ is linear. $\square$

Theorem 2.2 *The spaces $Z(M, p, \sigma, q, s)$ are paranormed space (not necessarily totally paranormed), paranormed by*

$$g(x) = \sup_{n,k} \{k^{-s} [f(q(x_{\sigma^k(n)}))]^{p_k}\}^{1/H} \quad \text{uniformly in } n,$$

where $H = \max(1, \sup_k p_k)$.

The proof is a routine verification by using standard arguments and therefore omitted.

Theorem 2.3 *Let f be modulus function, then*

- i) $c_0(f, p, \sigma, q, s) \subseteq \ell_\infty(f, p, \sigma, q, s)$,
- ii) $c(f, p, \sigma, q, s) \subseteq \ell_\infty(f, p, \sigma, q, s)$.

Proof. We prove the second inclusion, the first inclusion is obvious. Let $x \in c(f, p, \sigma, q, s)$, by definition of a modulus function and the inequality (2), we have

$$k^{-s} [f(q(x_{\sigma^k(n)}))]^{p_k} \leq Dk^{-s} [f(q(x_{\sigma^k(n)} - \ell))]^{p_k} + Dk^{-s} [f(q(\ell))]^{p_k}.$$

Then there exists an integer K_ℓ such that $q(\ell) \leq K_\ell$. Hence, we have

$$k^{-s} [f(q(x_{\sigma^k(n)}))]^{p_k} \leq Dk^{-s} [f(q(x_{\sigma^k(n)} - \ell))]^{p_k} + k^{-s} [(K_\ell)f(1)]^H.$$

Hence $x \in \ell_\infty(f, p, \sigma, q, s)$. $\square$

Theorem 2.4 *Let f , f_1 , f_2 be modulus functions, q , q_1 , q_2 seminorms and s , s_1 , $s_2 \geq 0$ real numbers. Then*

- i) $Z(f_1, p, \sigma, q, s) \subseteq Z(f \circ f_1, p, \sigma, q, s)$,
- ii) $Z(f_1, p, \sigma, q, s) \cap Z(f_2, p, \sigma, q, s) \subseteq Z(f_1 + f_2, p, \sigma, q, s)$,
- iii) $Z(f, p, \sigma, q_1, s) \cap Z(f, p, \sigma, q_2, s) \subseteq Z(f, p, \sigma, q_1 + q_2, s)$,
- iv) If q_1 is stronger than q_2 then $Z(f, p, \sigma, q_1, s) \subseteq Z(f, p, \sigma, q_2, s)$,
- v) If $s_1 \leq s_2$, then $Z(f, p, \sigma, q, s_1) \subseteq Z(f, p, \sigma, q, s_2)$

where $Z = \ell_\infty$, c and c_0 .

Proof. **i)** We prove this part for $Z = c$ and the rest of the cases will follow similarly. Let $x \in c(f_1, p, \sigma, q, s)$, so that

$$S_k = k^{-s} [f(q(x_{\sigma^k(n)} - \ell))]^{p_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \text{ uniformly in } n.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Now we write

$$I_1 = \{k \in \mathbb{N} : f_1(q(x_{\sigma^k(n)} - \ell)) \leq \delta\},$$

$$I_2 = \{k \in \mathbb{N} : f_1(q(x_{\sigma^k(n)} - \ell)) > \delta\}.$$

For $f_1(q(x_{\sigma^k(n)} - \ell)) > \delta$,

$$\begin{aligned} f_1(q(x_{\sigma^k(n)} - \ell)) &< f_1(q(x_{\sigma^k(n)} - \ell))\delta^{-1} \\ &< 1 + [f_1(q(x_{\sigma^k(n)} - \ell))\delta^{-1}] \end{aligned}$$

where $k \in I_2$ and $[|u|]$ denotes the integer part of u . By the definition of f we have for $f_1(q(x_{\sigma^k(n)} - \ell)) > \delta$,

$$\begin{aligned} f(f_1(q(x_{\sigma^k(n)} - \ell))) &\leq (1 + [f_1(q(x_{\sigma^k(n)} - \ell))\delta^{-1}])f(1) \\ &\leq 2f(1)(f_1(q(x_{\sigma^k(n)} - \ell)))\delta^{-1}. \end{aligned} \quad (3)$$

For $f_1(q(x_{\sigma^k(n)} - \ell)) \leq \delta$,

$$f(f_1(q(x_{\sigma^k(n)} - \ell))) < \varepsilon \quad (4)$$

where $k \in I_1$. By (3) and (4) we have

$$\begin{aligned} k^{-s} [f(f_1(q(x_{\sigma^k(n)} - \ell)))]^{p_k} &\leq k^{-s} \varepsilon^H + [2f(1)\delta^{-1}]^H S_k \rightarrow 0, \\ &\text{as } k \rightarrow \infty, \text{ uniformly in } n. \end{aligned}$$

Hence $c(f_1, p, \sigma, q, s) \subseteq c(f \circ f_1, p, \sigma, q, s)$.

ii) It follows from the following inequality

$$\begin{aligned} k^{-s} [(f_1 + f_2)q(x_{\sigma^k(n)})]^{p_k} &\leq Dk^{-s} [f_1(q(x_{\sigma^k(n)}))]^{p_k} \\ &\quad + Dk^{-s} [f_2(q(x_{\sigma^k(n)}))]^{p_k}. \end{aligned}$$

(iii) (iv), and (v) and follows easily. $\square$

The proof of the following result is a routine work in view of the above Theorem.

Proposition 2.5

- i) $Z(p, \sigma, q, s) \subseteq Z(f, p, \sigma, q, s)$,
- ii) $Z(f, p, \sigma, q) \subseteq Z(f, p, \sigma, q, s)$,
- iii) $Z(p, \sigma, q) \subseteq Z(p, \sigma, q, s)$.

Proposition 2.6 If $q_1 \cong$ (equivalent to) q_2 , then $Z(f, p, \sigma, q_1, s) = Z(f, p, \sigma, q_2, s)$ for $Z = c, c_0$ and ℓ_∞ .

Theorem 2.7 For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and for any two seminorms q_1 and q_2 on X we have

$$Z(f, p, \sigma, q_1, s) \cap Z(f, t, \sigma, q_2, s) \neq \emptyset.$$

Proof. The proof follows from the fact that the zero element $\bar{\theta}$ belongs to each of the classes of sequences involved in the intersection.

(Where θ is the zero element of X and $\bar{\theta} = (\theta, \theta, \theta \dots)$.) □

Theorem 2.8 For any two sequences (p_k) and (r_k) we have $c_0(f, r, \sigma, q) \subseteq c_0(f, p, \sigma, q)$ if and only if $\liminf p_k/r_k > 0$.

Proof. If we take $y_{kn} = f(q(x_{\sigma^k(n)}))$ for all $k \in \mathbb{N}$, then using the same technique of Lemma 1 of Maddox [7], it is easy to prove the theorem. □

The following results is a consequence of the above Theorem.

Theorem 2.9 For any two sequences (p_k) and (r_k) we have $c_0(f, r, \sigma, q) = c_0(f, p, \sigma, q)$ if and only if $\liminf p_k/r_k > 0$ and $\liminf r_k/p_k > 0$.

Theorem 2.10 The spaces $\ell_\infty(f, p, \sigma, q, s)$ and $c_0(f, p, \sigma, q, s)$ are solid and as such are monotone, but $c(f, p, \sigma, q, s)$ is not monotone and hence is not solid.

Proof. Omitted (see Remark.1.6). □

Theorem 2.11 The spaces $Z(f, p, \sigma, q, s)$, for $Z = \ell_\infty, c$ and c_0 are not convergence free.

Proof. The proof follows from the following example. □

Example 1 Let $f(x) = x$, for $x \in [0, \infty)$; $s = 0$; $p_k = 1$ for k even and $p_k = 2$ for k odd. Let $X = \mathbb{C}$, $q(x) = |x|$ and $\sigma(n) = n + 1$ for all $n \in \mathbb{N}$. Then we have $\sigma^k(n) = n + k$ for all $n, k \in \mathbb{N}$. Consider the sequences (x_k) and (y_k) defined as $x_k = 1/k$ and $y_k = k$, for all $k \in \mathbb{N}$. Then $(x_k) \in Z(p, \sigma)$, for $Z = \ell_\infty, c$ and c_0 , but $(y_k) \notin Z(p, \sigma)$ for $Z = \ell_\infty, c$, and c_0 .

Hence the spaces are not convergence free.

Acknowledgment The authors wish to thank the referee for their careful reading of the manuscript and valuable of the manuscript and valuable suggestions.

References

- [1] Ahmad Z.U. and Mursaleen M., *An application of Banach limits*. Proc. Amer. Math. Soc. **103** (1983), 244–246.
- [2] Bhardwaj V.K., *A generalization of a sequence space of Ruckle*. Bull. Calcutta Math. Soc. (5) **95** (2003), 411–420.
- [3] Bilgin T., *The sequence space $\ell(p, f, q, s)$ on seminormed spaces*. Bull. Calcutta Math. Soc. **86** (1994), 295–304.
- [4] Boos J., *Classical and Modern Methods in Summability*. Oxford Univ. Press, 2000.
- [5] Kamthan P.K. and Gupta M., *Sequence Spaces and Series*. Marcel Dekker Inc., New York, 1981.
- [6] Lorentz G.G., *A contribution the theory of divergent series*. Acta Math. **80** (1948), 167–190.
- [7] Maddox I.J., *Spaces of strongly summable sequences*. Quart. J. Math. Oxford (2) **18** (1967), 345–355.
- [8] Maddox I.J., *Elements of Functional Analysis*. Camb. Univ. Press, 1970.
- [9] Maddox I.J., *Sequence spaces defined by a modulus*. Math. Proc. Camb. Phil. Soc. **100** (1986), 161–166.
- [10] Mursaleen M., *On some new invariant matrix methods of summability*. Quart. J. Math. Oxford (2) **34** (1983), 77–86.
- [11] Mursaleen M., *Matrix transformations between some new sequence spaces*. Houston J. Math. **9** (1983), 505–509.
- [12] Nakano H., *Concave modulars*. J. Math. Soc. Japon **5** (1953), 29–49.
- [13] Pehlivan S. and Fisher B., *On some sequence spaces*. Indian J. Pure and Appl. Math. (10) **25** (1994), 1067–1071.
- [14] Raimi R.A., *Invariant means and invariant matrix method of summability*. Duke Math. J. **30** (1963), 81–94.
- [15] Ruckle W.H., *FK spaces in which the sequence of coordinate vectors is bounded*. Canad. J. Math. **25** (1973), 973–978.
- [16] Schaefer P., *Infinite matrices and invariant means*. Proc. Amer. Math. Soc. **36** (1972), 104–110.
- [17] Waszak A., *On the strong convergence in some sequence spaces*. Fasciculi Mathematici **3** (2002), 125–137.
- [18] Wilansky A., *Functional Analysis*. Blaisdell Publishing Company, New York, 1964.

Y. Altın
Department of Mathematics
Firat University
23119, Elazığ-TURKEY
E-mail: yaltin23@yahoo.com

M. Işık
Department of Mathematics
Firat University
23119, Elazığ-TURKEY
E-mail: misik63@yahoo.com