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A note on the large time asymptotics for a system of Klein-Gordon equations

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Abstract. We study the asymptotic behavior of the solution in the large time for the system

$$\begin{cases} (\Box + m_1^2)u_1 = 0 \\ (\Box + m_2^2)u_2 = 0 \\ (\Box + m_3^2)u_3 = u_1u_2 \end{cases}$$

in two space dimensions. We prove the existence of a solution whose amplitude is modulated by the long range effect when $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some λ_1, $\lambda_2 \in \{\pm 1\}$.

Key words: Klein-Gordon equation; asymptotic behavior.

1. Introduction

We are concerned with the large time behavior of the solution to the Cauchy problem

$$\begin{cases} (\Box + m_i^2)u_i = F_i(u_1, \ldots, u_N), & t > 0, x \in \mathbf{R}^n, \\ (u, \partial_t u)|_{t=0} = (\varepsilon f_i, \varepsilon g_i), & i = 1, \ldots, N. \end{cases} \tag{1.1}$$

Here $\Box = \partial_t^2 - \Delta_x$, $m_i > 0$ is a constant, f_i and g_i are real valued functions which belong to the Schwartz class $\mathcal{S}(\mathbf{R}^n)$, and $\varepsilon > 0$ is a small parameter. We also assume that for each $i \in \{1, \ldots, N\}$, F_i is a smooth function of $u = (u_1, \ldots, u_N)$ in its argument and vanishes to p-th order at the origin, i.e.

$$\frac{\partial^\alpha F_i}{\partial u^\alpha}(0) = 0 \quad \text{for} \quad |\alpha| \leq p - 1.$$

Since we are interested in the critical nonlinear case, we restrict ourselves to the case $p = 1 + 2/n$ and $n \leq 2$, that is, $(n, p) = (1, 3)$ or $(n, p) = (2, 2)$.

In the previous work [8], it is shown that the Cauchy problem (1.1) admits a unique global smooth solution which tends to a free solution as $t \to \infty$ under the following condition, which we call *the non-resonance*

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condition ;

$$m_i = \lambda_1 m_j + \lambda_2 m_k \quad \text{for some} \quad \lambda_1,\, \lambda_2 \in \{\pm 1\} \quad \text{implies} \quad \frac{\partial^2 F_i}{\partial u_j \partial u_k}(0) = 0$$

for each $i,\, j,\, k \in \{1, \ldots, N\}$ when $(n,\, p) = (2,\, 2)$, and

$$m_i = \lambda_1 m_j + \lambda_2 m_k + \lambda_3 m_l \quad \text{for some} \quad \lambda_1,\, \lambda_2,\, \lambda_3 \in \{\pm 1\} \quad \text{implies} \quad \frac{\partial^3 F_i}{\partial u_j \partial u_k \partial u_l}(0) = 0$$

for each $i,\, j,\, k,\, l \in \{1, \ldots, N\}$ when $(n,\, p) = (1,\, 3)$. More precisely, we have

Theorem 1.1 *If the non-resonance condition is satisfied, there exists a unique solution in* $C^\infty((0,\, \infty) \times \mathbf{R}^n)$ *to the Cauchy problem* (1.1) *for sufficiently small* ε. *Furthermore, we have*

$$\|u(t) - u^L(t)\|_E \to 0 \quad as \quad t \to \infty$$

for some free solution $u^L = (u_i^L)_{1 \le i \le N}$, *i.e. the function satisfying* $(\Box + m_i^2) u_i^L = 0$.

Here and later on as well, we use the notation

$$\|u(t)\|_E^2 = \sum_{i=1}^{N} \int \left\{ \left(\partial_t u_i(t,\, x)\right)^2 + \left|\nabla_x u_i(t,\, x)\right|^2 + m_i^2 u_i(t,\, x)^2 \right\} dx.$$

Remark 1 Similar result is true when F_i contains also the first order derivatives of the unknowns, i.e. $F_i = F_i(u,\, \partial u)$, but we do not mention here for the simplicity of exposition (see [8] for detail).

Remark 2 Typical examples of the systems which satisfy the non-resonance condition are

$$\begin{cases} (\Box + m_1^2) u_1 = u_2 u_3 \\ (\Box + m_2^2) u_2 = u_3 u_1 \\ (\Box + m_3^2) u_3 = u_1 u_2 \end{cases}$$

in two space dimensions with $\sum_{i=1}^{3} \lambda_i m_i \neq 0$ for any $\lambda_1,\, \lambda_2,\, \lambda_3 \in \{\pm 1\}$, and

$$\begin{cases} (\Box + m_1^2)u_1 = u_2u_3u_4 \\ (\Box + m_2^2)u_2 = u_3u_4u_1 \\ (\Box + m_3^2)u_3 = u_4u_1u_2 \\ (\Box + m_4^2)u_4 = u_1u_2u_3 \end{cases}$$

in one space dimension with $\sum_{i=1}^{4} \lambda_i m_i \neq 0$ for any $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \{\pm 1\}$.

The following question arises naturally: How does the solution behave in the large time if the non-resonance condition is not satisfied? For the one-dimensional scalar equation ($n = 1$, $N = 1$), the results due to Delort [3] or Georgiev-Yordanov [4] might answer this question, but for the case $N \geq 2$, the question seems completely open except Tsutsumi's remark [9] (see also [5], [7]). The aim of this paper is to give an answer to this question for the following special case

$$\begin{cases} (\Box + m_1^2)u_1 = 0 \\ (\Box + m_2^2)u_2 = 0 \\ (\Box + m_3^2)u_3 = u_1u_2 \end{cases} \tag{1.2}$$

in two space dimensions with $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some $\lambda_1, \lambda_2 \in \{\pm 1\}$. We will find the asymptotic profile of the solution to this system to show that the large time behavior is quite different from that of the free solution. More precisely, we will prove the existence of a solution whose *amplitude* is modulated by the long range effect. Note that the nonlinear behavior found in [3], [4] take place at the level of the *phase* of oscillation of the solution. The main result is the following:

Theorem 1.2 *For the solution $u = (u_1, u_2, u_3)$ of (1.2), we have*

$$C_1 \log t \geq \|u(t)\|_E \geq C_2 \log t \quad \textit{for} \quad t \geq T$$

when $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some $\lambda_1, \lambda_2 \in \{\pm 1\}$ and the initial data is appropriately chosen in $\mathcal{S}(\mathbf{R}^2)$. Here $T > 1$ and $C_1 \geq C_2 > 0$ are constants which depend on the initial data.

Remark 1 If u_3 had a free profile in the sense of Theorem 1.1, the energy should stay bounded, i.e., $\|u(t)\|_E \leq C$ for any $t > 0$ with some positive constant C. So u_3 does not have a free profile in the usual sense when $m_3 = \lambda_1 m_1 + \lambda_2 m_2$ for some $\lambda_1, \lambda_2 \in \{\pm 1\}$. The asymptotic profile of u_3 will be obtained in §2.3 (see Theorem 2.1 below).

Remark 2 On the other hand, when $m_3 \neq \lambda_1 m_1 + \lambda_2 m_2$ for any λ_1, $\lambda_2 \in \{\pm 1\}$, u_3 has a free profile in the sense of Theorem 1.1 since the non-resonance condition is satisfied.

Remark 3 We can obtain the similar result in the same way for the system

$$\begin{cases} (\Box + m_1^2)u_1 = 0 \\ (\Box + m_2^2)u_2 = 0 \\ (\Box + m_3^2)u_3 = 0 \\ (\Box + m_4^2)u_4 = u_1 u_2 u_3 \end{cases}$$

in one space dimension when $m_4 = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3$ for some λ_1, λ_2, $\lambda_3 \in \{\pm 1\}$.

2. Proof of the main theorem

This section is devoted to the proof of Theorem 1.2. The proof is divided into 4 steps. In the first step, we will consider the asymptotic forms in the large time of the solutions to the linear Klein-Gordon equations using the result from §7.2 of [6]. In the next step, we will prepare some lemma related to the normal form argument (cf. §2.1 of [2]). In the third step, we will obtain the asymptotic profile for u_3 in the system (1.2), and we will reach the desired conclusion in the final step. Since we are interested in the large time behavior, we always suppose that $t \gg 1$ in what follows.

2.1. Asymptotic behavior of the solutions to the free Klein-Gordon equations

We first investigate the asymptotic form of the oscillatory integral

$$I(t, x) = \int e^{i(x\xi + t\langle\xi\rangle)} h(\xi)\, d\xi,$$

where $h \in \mathcal{S}(\mathbf{R}^n)$, $t > 1$, $x \in \mathbf{R}^n$ and $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. The following lemma is due to Hörmander [6], though we state a slightly modified version here.

Lemma 2.1 *If $h \in \mathcal{S}(\mathbf{R}^n)$, then the oscillatory integral $I(t, x)$ can be written in the form*

$$I(t, x) = \frac{e^{i\phi(t, x)}}{t^{n/2}} A(t, x) + R(t, x)$$

where A, $R \in C^\infty(]1, \infty[\times\mathbf{R}^n)$ *and* $\phi(t, x) = (t^2 - |x|^2)_+^{1/2}$. *The function* A *has the asymptotic expansion*

$$A(t, x) \sim \sum_{j\geq 0} t^{-j} a_j\left(\frac{x}{t}\right),$$

where $a_j \in C^\infty(\mathbf{R}^n; \mathbf{C})$, $j = 0, 1, 2, \ldots$, *satisfy*

$$|\partial_y^\alpha a_j(y)| \leq C_{j,\alpha,N}(1 - |y|^2)_+^N \tag{2.1}$$

with some positive constant $C_{j,\alpha,N}$ *for any multi-indices* α *and* $N \in \mathbf{N}$. *In particular, the leading term* a_0 *is given by*

$$a_0(y) = \begin{cases} \dfrac{e^{i\pi/4}}{(2\pi)^{n/2}}\left(1 - |y|^2\right)^{-(n+2)/4} h\left(\dfrac{-y}{\sqrt{1-|y|^2}}\right) & \text{if} \quad |y| < 1, \\ 0 & \text{if} \quad |y| \geq 1. \end{cases}$$

Concerning the function R, *for any multi-indices* α *and* $N \in \mathbf{N}$, *there exists a positive constant* $C_{\alpha,N}$ *such that*

$$\left|\partial_{t,x}^\alpha R(t, x)\right| \leq C_{\alpha,N}(t + |x|)^{-N}.$$

Remark In the above statement, we have used the notation $\sim$ in the following sense: We write

$$p(t, x) \sim \sum_{j\geq 0} p_j(t, x)$$

if for any multi-indices α and $N \in \mathbf{N}$ there exists a positive constant $C_{\alpha,N}$ such that

$$\left|\partial_{t,x}^\alpha \left\{p(t, x) - \sum_{j=0}^{N-1} p_j(t, x)\right\}\right| \leq C_{\alpha,N}\left(t + |x|\right)^{-N-|\alpha|}.$$

The proof of Lemma 2.1 may be found in Section 7.2 of [6] except the estimate (2.1), so we shall only check it. According to Hörmander's original argument, $a_j(y)$ is given of the form

$$a_j(y) = \begin{cases} b_j\left(\dfrac{-y}{\sqrt{1-|y|^2}}\right) & \text{if} \quad |y| < 1, \\ 0 & \text{if} \quad |y| \geq 1 \end{cases}$$

with appropriate $b_j \in \mathcal{S}(\mathbf{R}^n)$. So it is easy to see that

$$|a_j(y)| \le C_{j,0,N}(1-|y|^2)_+^N.$$

Here we note that $1-|y|^2 = \langle\xi\rangle^{-2}$ when $\xi = -y/\sqrt{1-|y|^2}$. Also, $\partial a_j/\partial y_k$ can be written

$$\frac{\partial a_j}{\partial y_k}(y) = \begin{cases} (\mathcal{L}_k b_j)\left(\dfrac{-y}{\sqrt{1-|y|^2}}\right) & \text{if} \quad |y|<1, \\ 0 & \text{if} \quad |y| \ge 1 \end{cases}$$

with

$$\mathcal{L}_k = -\langle\xi\rangle \sum_{l=1}^{n} (\delta_{kl} + \xi_k \xi_l) \frac{\partial}{\partial \xi_l}.$$

Since $\mathcal{L}_k b_j \in \mathcal{S}(\mathbf{R}^n)$, we have

$$\max_{|\alpha|=1} |\partial_y^\alpha a_j(y)| \le C_{j,1,N}(1-|y|^2)_+^N.$$

By induction we obtain (2.1).

Now, we put

$$h(\xi) = (2\pi)^{-n}\left(\widehat{f}(\xi) - i\frac{\widehat{g}(\xi)}{\langle\xi\rangle}\right),$$

where $\widehat{f}$ stands for the Fourier transform of f. Since the solution v of the Cauchy problem

$$(\Box+1)v = 0, \quad v(0,x) = f(x), \quad \partial_t v(0,x) = g(x) \tag{2.2}$$

is given by

$$\begin{aligned} v(t,x) &= \frac{1}{2}\sum_{\lambda\in\{\pm1\}} \int e^{i(x\xi+\lambda t\langle\xi\rangle)}\left(\widehat{f}(\xi) - i\lambda\frac{\widehat{g}(\xi)}{\langle\xi\rangle}\right)\frac{d\xi}{(2\pi)^n} \\ &= \mathrm{Re}\left[\int e^{i(x\xi+t\langle\xi\rangle)} h(\xi)\, d\xi\right], \end{aligned}$$

we can apply Lemma 2.1 to obtain

$$v(t,x) = \mathrm{Re}\left[\frac{e^{i\phi(t,x)}}{t^{n/2}} A(t,x) + R(t,x)\right]$$

$$=\mathrm{Re}\left[\frac{e^{i\phi(t,\,x)}}{t^{n/2}}a_0\left(\frac{x}{t}\right)\right]$$
$$+\,\mathrm{Re}\left[\frac{e^{i\phi(t,\,x)}}{t^{n/2+1}}t\left\{A(t,\,x)-a_0\left(\frac{x}{t}\right)\right\}+R(t,\,x)\right],$$

where A, R, a_0 are as in Lemma 2.1 with above h. Summing up, we have the following:

Corollary 2.1 *For the solution v of (2.2), we have*

$$|v(t,\,x)|\le C_{N_1,N_2}\left\{t^{-n/2}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{N_1}+(t+|x|)^{-N_2}\right\}$$

and

$$\left|v(t,\,x)-\sum_{\lambda\in\{\pm1\}}\frac{e^{i\lambda\phi(t,\,x)}}{t^{n/2}}a^{(\lambda)}\left(\frac{x}{t}\right)\right|$$
$$\le C_{N_1,N_2}\left\{t^{-(n/2+1)}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{N_1}+(t+|x|)^{-N_2}\right\}$$

with some constant C_{N_1,N_2} for any N_1, $N_2\in\mathbf{N}$. Here $a^{(\lambda)}:y\in\mathbf{R}^n\mapsto\mathbf{C}$ is given by $a^{(+1)}(y)=a_0(y)/2$ and $a^{(-1)}(y)=\overline{a_0(y)}/2$.

2.2. A lemma related to the normal form argument

The goal of the second step is to show the following:

Lemma 2.2 *Let $a:y\in\mathbf{R}^n\mapsto\mathbf{C}$ be a smooth function supported on the unit ball $\{y\in\mathbf{R}^n\mid|y|\le1\}$, and m, μ be real constants. Also let $\phi(t,\,x)=(t^2-|x|^2)_+^{1/2}$. Then we have*

$$\frac{e^{i\mu\phi(t,\,x)}}{t^{1+n/2}}a\left(\frac{x}{t}\right)-(\Box+m^2)\left[\frac{e^{i\mu\phi(t,\,x)}}{(m^2-\mu^2)t^{1+n/2}}a\left(\frac{x}{t}\right)\right]\in L^1(1,\,\infty;L^2(\mathbf{R}^n))$$

if $|m|\ne|\mu|$, while

$$\frac{e^{i\mu\phi(t,x)}}{t^{1+n/2}}a\left(\frac{x}{t}\right)-(\Box+m^2)\left[\frac{e^{i\lambda\{m\phi(t,x)-\pi/2\}}\log t}{2mt^{n/2}}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{1/2}a\left(\frac{x}{t}\right)\right]\in L^1(1,\infty;L^2(\mathbf{R}^n))$$

if $m=\lambda\mu$, $\lambda\in\{\pm1\}$.

Remark For non-trivial $a \in L^2(\mathbf{R}^n)$, the function $e^{i\mu\phi(t,x)}t^{-\nu}a(x/t)$ belongs to $L^1(1, \infty; L^2(\mathbf{R}^n))$ if and only if $\nu > 1 + n/2$ because

$$\left\| \frac{e^{i\mu\phi(t,\cdot)}}{t^\nu} a\Big(\frac{\cdot}{t}\Big) \right\|_{L^2_x} = \frac{1}{t^{\nu - n/2}} \left\{ \int \left| a\Big(\frac{x}{t}\Big) \right|^2 \frac{dx}{t^n} \right\}^{1/2} = t^{n/2-\nu} \|a\|_{L^2_y}.$$

Lemma 2.2 is a consequence of the following lemma, which appears in [2] in somewhat different form.

Lemma 2.3 *Let m, μ, ϕ be as in the Lemma 2.2 and let $A : (s, y) \in]0, \infty[\times \mathbf{R}^n \mapsto \mathbf{C}$ be a smooth function which vanishes when $|y| \geq 1$. Then, for $\nu \in \mathbf{R}$, we have*

$$\begin{aligned}
(\Box + m^2) &\left[\frac{e^{i\mu\phi(t,x)}}{t^\nu} A\Big(\log t, \frac{x}{t}\Big) \right] \\
&= \frac{e^{i\mu\phi(t,x)}}{t^\nu}(m^2 - \mu^2) A\Big(\log t, \frac{x}{t}\Big) \\
&\quad + \frac{e^{i\mu\phi(t,x)}}{t^{\nu+1}} 2i\mu \left[(1 - |y|^2)_+^{-1/2} \Big(\frac{\partial}{\partial s} + \frac{n}{2} - \nu \Big) A \right] \Big(\log t, \frac{x}{t}\Big) \\
&\quad + \frac{e^{i\mu\phi(t,x)}}{t^{\nu+2}} (P_\nu A) \Big(\log t, \frac{x}{t}\Big),
\end{aligned}$$

where

$$P_\nu = \Big(\frac{\partial}{\partial s} - \sum_{j=1}^n y_j \frac{\partial}{\partial y_j} - \nu - 1 \Big) \Big(\frac{\partial}{\partial s} - \sum_{j=1}^n y_j \frac{\partial}{\partial y_j} - \nu \Big) - \Delta_y.$$

Remark The auxiliary variable $s = \log t$ in this lemma is often called the *slow time*, which is familiar in the theory of blowup for quasilinear hyperbolic equations (see [1], [6]).

Proof of Lemma 2.2 via Lemma 2.3. When we choose $A(s, y) = (m^2 - \mu^2)^{-1} a(y)$ and $\nu = 1 + n/2$ in Lemma 2.3, we have

$$\begin{aligned}
&\frac{e^{i\mu\phi(t,x)}}{t^{1+n/2}} a\Big(\frac{x}{t}\Big) - (\Box + m^2) \left[\frac{e^{i\mu\phi(t,x)}}{(m^2 - \mu^2) t^{1+n/2}} a\Big(\frac{x}{t}\Big) \right] \\
&= \frac{2i\mu e^{i\mu\phi(t,x)}}{(m^2 - \mu^2) t^{2+n/2}} \left(1 - \left| \frac{x}{t} \right|^2 \right)_+^{-1/2} a\Big(\frac{x}{t}\Big) - \frac{e^{i\mu\phi(t,x)}}{(m^2 - \mu^2) t^{3+n/2}} (P_\nu a) \Big(\frac{x}{t}\Big)
\end{aligned}$$

if $|m| \neq |\mu|$. Since both $(1-|y|^2)_+^{-1/2}a(y)$ and $(P_\nu a)(y)$ are smooth functions with their support on the unit ball if $a(y)$ is so, we obtain the first half of Lemma 2.2. When $m = \lambda\mu$, $\lambda \in \{\pm 1\}$, we have

$$\frac{e^{i\mu\phi(t,x)}}{t^{1+n/2}}a\left(\frac{x}{t}\right) - (\Box + m^2)\left[\frac{e^{i\lambda\{m\phi(t,x)-\pi/2\}}\log t}{2mt^{n/2}}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{1/2} a\left(\frac{x}{t}\right)\right]$$
$$= -\frac{e^{i\lambda m\phi(t,x)}}{t^{2+n/2}}(P_\nu A)\left(\log t, \frac{x}{t}\right)$$

by choosing $A(s, y) = (2i\mu)^{-1}s(1-|y|^2)_+^{1/2}a(y)$ and $\nu = n/2$ in Lemma 2.3 and using the relation $i\lambda = e^{i\lambda\pi/2}$. Since $(P_\nu A)(s, y)$ is majorized by $(1+s)b(y)$ with appropriate $b \in L^2(\mathbf{R}^n)$, we have

$$\left\|\frac{e^{i\lambda m\phi(t,\cdot)}}{t^{2+n/2}}(P_\nu A)\left(\log t, \frac{\cdot}{t}\right)\right\|_{L_x^2} \leq \frac{(1+\log t)}{t^2}\|b\|_{L_y^2} \in L^1(1, \infty),$$

which yields the latter half of Lemma 2.2.

Proof of Lemma 2.3. First, we note that we may assume $|x| < t$ since the both left and right hand sides are 0 when $|x| \geq t$. Now, let us introduce

$$X_\nu = \frac{\partial}{\partial s} - \sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j} - \nu$$

so that

$$\frac{\partial}{\partial t}\left[\frac{1}{t^\nu}A\left(\log t, \frac{x}{t}\right)\right] = \frac{1}{t^{\nu+1}}(X_\nu A)\left(\log t, \frac{x}{t}\right).$$

Then it follows that

$$\frac{\partial^2}{\partial t^2}\left[\frac{e^{i\mu\phi(t,x)}}{t^\nu}A\left(\log t, \frac{x}{t}\right)\right] = \frac{e^{i\mu\phi}}{t^\nu}\left\{-\mu^2\left(\frac{\partial\phi}{\partial t}\right)^2 + i\mu\frac{\partial^2\phi}{\partial t^2}\right\}A\left(\log t, \frac{x}{t}\right)$$
$$+ \frac{e^{i\mu\phi}}{t^{\nu+1}}2i\mu\left(\frac{\partial\phi}{\partial t}\right)(X_\nu A)\left(\log t, \frac{x}{t}\right)$$
$$+ \frac{e^{i\mu\phi}}{t^{\nu+2}}(X_{\nu+1}X_\nu A)\left(\log t, \frac{x}{t}\right).$$

Similarly, straightforward calculation yields

$$\frac{\partial^2}{\partial x_j^2}\left[\frac{e^{i\mu\phi(t,x)}}{t^\nu}A\left(\log t,\frac{x}{t}\right)\right]=\frac{e^{i\mu\phi}}{t^\nu}\left\{-\mu^2\left(\frac{\partial\phi}{\partial x_j}\right)^2+i\mu\frac{\partial^2\phi}{\partial x_j^2}\right\}A\left(\log t,\frac{x}{t}\right)$$
$$+\frac{e^{i\mu\phi}}{t^{\nu+1}}2i\mu\left(\frac{\partial\phi}{\partial x_j}\right)\left(\frac{\partial A}{\partial y_j}\right)\left(\log t,\frac{x}{t}\right)$$
$$+\frac{e^{i\mu\phi}}{t^{\nu+2}}\left(\frac{\partial^2 A}{\partial y_j^2}\right)\left(\log t,\frac{x}{t}\right).$$

Thus we have

$$\begin{aligned}
&(\Box+m^2)\left[\frac{e^{i\mu\phi(t,x)}}{t^\nu}A\left(\log t,\,\frac{x}{t}\right)\right]\\
&=\frac{e^{i\mu\phi}}{t^\nu}\left\{m^2-\mu^2\left[\left(\partial_t\phi\right)^2-\left|\nabla_x\phi\right|^2\right]+i\mu(\Box\phi)\right\}A\left(\log t,\,\frac{x}{t}\right)\\
&\quad+\frac{e^{i\mu\phi}}{t^{\nu+1}}2i\mu\left[\left(\partial_t\phi\right)X_\nu-\left(\nabla_x\phi\right)\cdot\nabla_y\right]A\left(\log t,\,\frac{x}{t}\right)\\
&\quad+\frac{e^{i\mu\phi}}{t^{\nu+2}}\left(X_{\nu+1}X_\nu-\Delta_y\right)A\left(\log t,\,\frac{x}{t}\right)\\
&=\frac{e^{i\mu\phi}}{t^\nu}(m^2-\mu^2)A\left(\log t,\,\frac{x}{t}\right)\\
&\quad+\frac{e^{i\mu\phi}}{t^{\nu+1}}2i\mu\left[\left(\partial_t\phi\right)X_\nu-\left(\nabla_x\phi\right)\cdot\nabla_y+\frac{t}{2}(\Box\phi)\right]A\left(\log t,\,\frac{x}{t}\right)\\
&\quad+\frac{e^{i\mu\phi}}{t^{\nu+2}}\left(P_\nu A\right)\left(\log t,\,\frac{x}{t}\right).
\end{aligned}$$

Here we have used the relations $\left(\partial_t\phi\right)^2-\left|\nabla_x\phi\right|^2=1$ and $P_\nu=X_{\nu+1}X_\nu-\Delta_y$. Finally, since ϕ satisfies

$$\Box\phi=\frac{n}{(t^2-|x|^2)^{1/2}}$$

and

$$\left(t\frac{\partial}{\partial x_j}+x_j\frac{\partial}{\partial t}\right)\phi=0,$$

we have

$$\begin{aligned}
&(\partial_t\phi)X_\nu - (\nabla_x\phi)\cdot\nabla_y + \frac{t}{2}(\Box\phi)\\
&= (\partial_t\phi)\Big[X_\nu + \frac{x}{t}\cdot\nabla_y\Big] + \frac{n}{2}\Big(1-\Big|\frac{x}{t}\Big|^2\Big)_+^{-1/2}\\
&= \big(1-|y|^2\big)_+^{-1/2}\Big(\frac{\partial}{\partial s} + \frac{n}{2} - \nu\Big)\Big|_{(s,y)=(\log t,\, x/t)},
\end{aligned}$$

which completes the proof. □

2.3. Construction of the asymptotic profile for u_3

In this step, we are going to find the large time asymptotic profile for u_3 in the system (1.2). The goal is Theorem 2.1 below.

First, we put $v_j(t, x) = u_j(t/m_j, x/m_j)$ and

$$r_j(t, x) = u_j(t, x) - \sum_{\lambda\in\{\pm1\}} \frac{e^{i\lambda m_j\phi(t,x)}}{m_j t} a_j^{(\lambda)}\Big(\frac{x}{t}\Big)$$

for u_j in the system (1.2) so that $(\Box+1)v_j = 0$, $v_j(0, x) = u_j(0, x/m_j)$, $(\partial_t v_j)(0, x) = m_j^{-1}(\partial_t u_j)(0, x/m_j)$ and

$$r_j(t, x) = \Big[v_j(t', x') - \sum_{\lambda\in\{\pm1\}} \frac{e^{i\phi(t',x')}}{t'} a_j^{(\lambda)}\Big(\frac{x'}{t'}\Big)\Big]_{(t',x')=(m_j t,\, m_j x)}.$$

Here $j = 1, 2$ and $a_j^{(\lambda)} : y \in \mathbf{R}^2 \mapsto \mathbf{C}$ is given by

$$\begin{aligned}
a_j^{(+1)}(y) =& \frac{m_j^2 e^{i\pi/4}}{16\pi^3}\big(1-|y|^2\big)_+^{-1}\widehat{u}_j\Big(0, \frac{-m_j y}{\sqrt{1-|y|^2}}\Big)\\
&- \frac{m_j e^{i3\pi/4}}{16\pi^3}\big(1-|y|^2\big)_+^{-1/2}\big(\widehat{\partial_t u_j}\big)\Big(0, \frac{-m_j y}{\sqrt{1-|y|^2}}\Big),\\
a_j^{(-1)}(y) =& \overline{a_j^{(+1)}(y)}.
\end{aligned}$$

Then it follows from Corollary 2.1 that

$$|u_j(t, x)| \le C_{N_1,N_2}\Big\{t^{-1}\Big(1-\Big|\frac{x}{t}\Big|^2\Big)_+^{N_1} + (t+|x|)^{-N_2}\Big\}$$

and

$$|r_j(t,\,x)| \le C_{N_1,\,N_2}\left\{t^{-2}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{N_1} + (t+|x|)^{-N_2}\right\}$$

for $j = 1,\,2$ and any $N_1,\,N_2 \in \mathbf{N}$ with some positive constant $C_{N_1,\,N_2}$. Thus, putting

$$R_1 := u_1u_2 - (u_1 - r_1)(u_2 - r_2),$$

we have

$$\begin{aligned}|R_1(t,\,x)| &\le |u_1||r_2| + |u_2||r_1| + |r_1||r_2| \\ &\le C'_{N_1,\,N_2}\left\{t^{-3}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{N_1} + (t+|x|)^{-N_2}\right\}\end{aligned}$$

for any $N_1,\,N_2 \in \mathbf{N}$ with some positive constant $C'_{N_1,\,N_2}$. In particular we have $R_1 \in L^1(1,\,\infty;L^2(\mathbf{R}^2))$.

Next, let us introduce

$$\begin{aligned}\Lambda_+ &= \left\{(\lambda_1,\,\lambda_2) \in \{\pm 1\}^2 \;\middle|\; \lambda_1m_1 + \lambda_2m_2 = m_3\right\},\\ \Lambda_- &= \left\{(\lambda_1,\,\lambda_2) \in \{\pm 1\}^2 \;\middle|\; \lambda_1m_1 + \lambda_2m_2 = -m_3\right\},\\ \Lambda_0 &= \left\{(\lambda_1,\,\lambda_2) \in \{\pm 1\}^2 \;\middle|\; |\lambda_1m_1 + \lambda_2m_2| \ne m_3\right\}\end{aligned}$$

and

$$\begin{aligned}w_+(t,\,x) &= \frac{e^{i\{m_3\phi(t,\,x)-\pi/2\}}\log t}{2m_1m_2m_3t}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{1/2} \\ &\qquad\times \sum_{(\lambda_1,\,\lambda_2)\in\Lambda_+} a_1^{(\lambda_1)}\left(\frac{x}{t}\right)a_2^{(\lambda_2)}\left(\frac{x}{t}\right),\\ w_-(t,\,x) &= \frac{e^{-i\{m_3\phi(t,\,x)-\pi/2\}}\log t}{2m_1m_2m_3t}\left(1-\left|\frac{x}{t}\right|^2\right)_+^{1/2} \\ &\qquad\times \sum_{(\lambda_1,\,\lambda_2)\in\Lambda_-} a_1^{(\lambda_1)}\left(\frac{x}{t}\right)a_2^{(\lambda_2)}\left(\frac{x}{t}\right),\\ w_0(t,\,x) &= \sum_{(\lambda_1,\,\lambda_2)\in\Lambda_0} \frac{e^{i(\lambda_1m_1+\lambda_2m_2)\phi(t,\,x)}}{m_1m_2\{m_3^2-(\lambda_1m_1+\lambda_2m_2)^2\}t^2} \\ &\qquad\times a_1^{(\lambda_1)}\left(\frac{x}{t}\right)a_2^{(\lambda_2)}\left(\frac{x}{t}\right).\end{aligned}$$

Then it follows from Lemma 2.2 that

$$R_2(t, x) := \sum_{(\lambda_1, \lambda_2)\in\{\pm 1\}^2} \frac{e^{i(\lambda_1 m_1 + \lambda_2 m_2)\phi(t, x)}}{m_1 m_2 t^2} a_1^{(\lambda_1)}\Big(\frac{x}{t}\Big) a_2^{(\lambda_2)}\Big(\frac{x}{t}\Big) - (\Box + m_3^2)\big[w_+(t, x) + w_-(t, x) + w_0(t, x)\big] \in L^1(1, \infty; L^2(\mathbf{R}^2)).$$

Now, we put $w_1(t, x) = w_+(t, x) + w_-(t, x) = 2\,\mathrm{Re}\big[w_+(t, x)\big]$. Since

$$R_1 + R_2 = u_1 u_2 - (\Box + m_3^2)\big(w_0 + w_1\big),$$

we have

$$(\Box + m_3^2)\big(u_3 - w_0 - w_1\big) = R_1 + R_2 \in L^1(1, \infty; L^2(\mathbf{R}^2)).$$

Therefore there exists a solution $w_2(t, x)$ of the linear Klein-Gordon equation $(\Box + m_3^2)w = 0$ such that

$$\big\|\{u_3(t) - w_0(t) - w_1(t)\} - w_2(t)\big\|_e \to 0 \quad \text{as} \quad t \to \infty,$$

where the norm $\|\cdot\|_e$ is defined by

$$\|w(t)\|_e = \left[\int \Big\{\big(\partial_t w(t, x)\big)^2 + \big|\nabla_x w(t, x)\big|^2 + m_3^2 w(t, x)^2\Big\}\, dx\right]^{1/2}.$$

Furthermore, since

$$\|w_0(t)\|_e \to 0 \quad \text{as} \quad t \to \infty,$$

$w_3 := u_3 - w_1 - w_2$ has the same property. Summing up, we obtain

Theorem 2.1 *For u_3 in the system* (1.2), *we have*

$$u_3 = w_1 + w_2 + w_3,$$

where

$$w_1 = \frac{\log t}{t}\,\mathrm{Re}\left[e^{i\{m_3\phi(t, x) - \pi/2\}}\mathcal{A}\Big(\frac{x}{t}\Big)\right]$$

with

$$\mathcal{A}(y) = \begin{cases} \dfrac{(1 - |y|^2)_+^{1/2}}{m_1 m_2 m_3} \displaystyle\sum_{(\lambda_1, \lambda_2)\in\Lambda_+} a_1^{(\lambda_1)}(y) a_2^{(\lambda_2)}(y) & \text{if} \quad \Lambda_+ \neq \emptyset, \\ 0 & \text{if} \quad \Lambda_+ = \emptyset, \end{cases}$$

w_2 is a solution of the free Klein-Gordon equation $(\Box + m_3^2)w = 0$ and w_3 satisfies $\|w_3(t)\|_e \to 0$ as $t \to \infty$.

2.4. The end of the proof

Now, we are in position to show Theorem 1.2. Since the upper bound follows immediately from Theorem 2.1, we omit the proof of it and we prove the lower bound here.

By Theorem 2.1, we have

$$\|u_3(t)\|_e \geq \|w_1(t)\|_e - \|w_2(t)\|_e - \|w_3(t)\|_e$$

and $\|w_2(t)\|_e = \text{const.}$, $\|w_3(t)\|_e \to 0$ as $t \to \infty$. Also, since

$$\begin{aligned}\partial_t w_1(t, x) =& m_3\big(\partial_t \phi(t, x)\big)\frac{\log t}{t}\operatorname{Im}\left[e^{im_3(\phi(t,x)-\pi/2)}\mathcal{A}\left(\frac{x}{t}\right)\right]\\ &+ \frac{1+\log t}{t^2}\operatorname{Re}\left[e^{im_3(\phi(t,x)-\pi/2)}\mathcal{A}\left(\frac{x}{t}\right)\right]\\ &+ \frac{\log t}{t^2}\operatorname{Re}\left[e^{im_3(\phi(t,x)-\pi/2)}\big(y\cdot\nabla_y\mathcal{A}\big)_{y=x/t}\right]\end{aligned}$$

and

$$|\partial_t\phi(t, x)|^2 = \frac{t^2}{t^2-|x|^2} \geq 1 \quad \text{for} \quad |x| < t,$$

we have

$$\begin{aligned}|\partial_t w_1(t, x)|^2 \geq& m_3^2\left|\frac{\log t}{t}\operatorname{Im}\left[e^{im_3(\phi(t,x)-\pi/2)}\mathcal{A}\left(\frac{x}{t}\right)\right]\right|^2\\ &- \frac{(\log t)^2}{t^3}\left|\mathcal{B}\left(\frac{x}{t}\right)\right|^2\end{aligned}$$

with some smooth function $\mathcal{B}(y)$ which vanishes when $|y| \geq 1$. Thus, we have

$$\begin{aligned}\|w_1(t)\|_e^2 \geq& \int |\partial_t w_1(t, x)|^2 + m_3^2|w_1(t, x)|^2\,dx\\ \geq& m_3^2\frac{(\log t)^2}{t^2}\int\left|\mathcal{A}\left(\frac{x}{t}\right)\right|^2 dx - \frac{(\log t)^2}{t^3}\int\left|\mathcal{B}\left(\frac{x}{t}\right)\right|^2 dx\\ =& \Big(\|\mathcal{A}\|_{L^2} m_3\log t\Big)^2 - \frac{(\log t)^2}{t}\|\mathcal{B}\|_{L^2}^2.\end{aligned}$$

Therefore we can choose $T > 1$ such that

$$\|u(t)\|_E \geq \|u_3(t)\|_e \geq \frac{1}{2}\|\mathcal{A}\|_{L^2} m_3 \log t \quad \text{for} \quad t \geq T,$$

provided that $\|\mathcal{A}\|_{L^2}$ is strictly positive. In order that $\|\mathcal{A}\|_{L^2}$ be strictly positive, it is sufficient to choose the initial data so that

$$\widehat{u}_j(0,\,0) = \int u_j(0,\,x)\,dx \neq 0 \quad \text{and}$$
$$(\widehat{\partial_t u_j})(0,\,0) = \int (\partial_t u_j)(0,\,x)\,dx = 0$$

for $j = 1, 2$. Indeed we can choose $\delta \in]0,\,1[$ so that $|\mathcal{A}(y)| \geq |\mathcal{A}(0)|/2 > 0$ for $|y| \leq \delta$ since $\mathcal{A}(0) = c\widehat{u}_1(0,\,0)\widehat{u}_2(0,\,0)$ with some $c \in \mathbf{C}\backslash\{0\}$ then. Therefore we have

$$\|\mathcal{A}\|_{L^2} \geq \left\{ \int_{|y|\leq\delta} \frac{|\mathcal{A}(0)|^2}{4}\,dy \right\}^{1/2} = \delta\pi^{1/2}\frac{|\mathcal{A}(0)|}{2} > 0,$$

which completes the proof. □

3. Concluding Remark

Though the system treated in this paper is nothing but an inhomogeneous linear equation, this example suggests several kinds of long range effect may appear in much wider class of resonant systems, in particular,

$$\begin{cases} (\square + m_1^2)u_1 = u_2u_3u_4 \\ (\square + m_2^2)u_2 = u_3u_4u_1 \\ (\square + m_3^2)u_3 = u_4u_1u_2 \\ (\square + m_4^2)u_4 = u_1u_2u_3 \end{cases}$$

in one space dimension with $\sum_{i=1}^4 \lambda_i m_i = 0$ for some $(\lambda_i)_{1\leq i\leq 4} \in \{\pm 1\}^4$. For sufficiently small and smooth data, the global existence of the solution to this system follows from the inclusion $H^1(\mathbf{R}^1) \hookrightarrow L^\infty(\mathbf{R}^1)$ and

$$\frac{d}{dt}\int \sum_{i=1}^4 \left\{(\partial_t u_i)^2 + (\partial_x u_i)^2 + m_i^2 u_i^2\right\} - 2u_1u_2u_3u_4\,dx = 0$$

(see e.g. [6, Theorem 7.5.2]). However, it seems difficult to obtain the asymptotic profile of the solution by this approach. In two dimensional quadratic nonlinear case, even the small data global existence is open without the non-resonance condition. It seems an interesting problem to clear up the long range effect in such resonant systems.

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