Hokkaido Mathematical Journal Vol. 32 (2003) p. 345–353

Notes on commutators and Morrey spaces

Yasuo KOMORI and Takahiro MIZUHARA

(Received December 17, 2001; Revised May 24, 2002)

Abstract. We show that the commutator $[M_b, I_\alpha]$ of the multiplication operator M_b by b and the fractional integral operator I_α is bounded from the Morrey space $L^{p,\lambda}(R^n)$ to the Morrey space $L^{q,\lambda}(R^n)$ where $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$ and $1/q = 1/p - \alpha/(n-\lambda)$ if and only if b belongs to $BMO(R^n)$.

Key words: commutator, fractional integral, Morrey space.

1. Introduction

Let I_α, $0 < \alpha < n$, be the fractional integral operator defined by

$$I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

We consider the commutator

$$[M_b, I_\alpha] f(x) = b(x) I_\alpha f(x) - I_\alpha(bf)(x), \qquad b \in L^1_{\mathrm{loc}}(R^n).$$

Chanillo [1] and the first author [7] obtained the necessary and sufficient condition for which the commutator $[M_b, I_\alpha]$ is bounded on $L^p(R^n)$. Di Fazio and Ragusa [4] obtained the necessary and sufficient condition for which the commutator $[M_b, I_\alpha]$ is bounded on Morrey spaces for some α.

In this paper we refine their results in [4] by using the duality argument and the factorization theorem for $H^1(R^n)$ (Theorem 2). Our proof is different from the one in [4].

2. Definitions and Notations

For a set $E \subset R^n$ we denote the characteristic function of E by χ_E and $|E|$ is the Lebesgue measure of E.

We denote a ball of radius t centered at x by $B(x,t) = \{y; |x-y| < t\}$.

Definition 1 Let $1 \le p < \infty$, $\lambda \ge 0$. We define the classical Morrey space by

---

2001 Mathematics Subject Classification : Primary 42B20.

$$L^{p,\lambda}(R^n) = \left\{ f \in L^p_{\mathrm{loc}}(R^n); \right.$$
$$\left. \|f\|_{L^{p,\lambda}} = \sup_{\substack{x \in R^n \\ t>0}} \left(\frac{1}{t^\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p} < \infty \right\}.$$

The classical Morrey spaces $L^{p,\lambda}$, $0 < \lambda < n$, on some bounded region, were originally introduced by Morrey [11] in 1938, and used by himself and the others in the problems related to the calculus of variations and the theory of elliptic PDE's.

For the classical Morrey space $L^{p,\lambda}(R^n)$, the next results are well-known.

If $1 \leq p < \infty$, then we have $L^{p,0}(R^n) = L^p(R^n)$ and $L^{p,n}(R^n) = L^\infty(R^n)$ (isometrically), and if $n < \lambda$, then we have $L^{p,\lambda}(R^n) = \{0\}$. So we consider the case $0 \leq \lambda \leq n$.

Chiarenza and Frasca [2] showed that the Hardy-Littlewood maximal operator is bounded on Morrey spaces and consequently gave a proof of the boundedness of the Calderón-Zygmund singular integral operators on Morrey spaces.

We introduce the blocks and the spaces generated by blocks following Taibleson and Weiss [12] and Long [9]. See also Lu, Taibleson and Weiss [10].

Definition 2 Let $1 \leq q < r \leq \infty$. A function $b(x)$ on R^n is called a (q,r)-block, if there exists a ball $B(x_0,t)$ such that

$$\operatorname{supp} b \subset B(x_0,t), \quad \|b\|_{L^r} \leq t^{n(1/r-1/q)}.$$

Definition 3 Let $1 \leq q < r \leq \infty$. We define the space generated by blocks by

$$h_{q,r}(R^n) = \left\{ f = \sum_{j=1}^\infty m_j b_j;\ b_j \text{ are } (q,r)\text{-blocks}, \right.$$
$$\left. \|f\|_{h_{q,r}} = \inf \sum_{j=1}^\infty |m_j| < \infty \right\},$$

where the infimum extends over all representations $f = \sum_{j=1}^\infty m_j b_j$ (see [9]).

We note that each (q,r)-block b_j belongs to $L^q(R^n)$ and $\|b_j\|_q \leq 1$. So the series of blocks $\sum_j m_j b_j$ converges in $L^q(R^n)$ and absolutely almost everywhere provided $\sum_j |m_j| < \infty$. Hence each space $h_{q,r}(R^n)$ is a function space and a Banach space (see [9], p. 17).

Definition 4 $H^1(R^n)$ is the Hardy space in the sense of Fefferman and Stein [5] and $BMO(R^n)$ is the John-Nirenberg space (see [6] or [13], p. 199). $BMO(R^n)$ is a Banach space, modulo constants, with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_{\substack{x\in R^n \\ t>0}} \frac{1}{|B(x,t)|} \int_{B(x,t)} |b(y) - b_B|\, dy \quad \text{where}$$

$$b_B = \frac{1}{|B(x,t)|} \int_{B(x,t)} b(y)\, dy.$$

Fefferman and Stein [5] showed that the Banach space dual of $H^1(R^n)$ is isomorphic to $BMO(R^n)$, that is,

$$\|b\|_* \approx \sup_{\|f\|_{H^1}\le 1} \left| \int b(x) f(x)\, dx \right|.$$

3. Theorems

The L^p theory about the commutator $[M_b, I_\alpha]$ is as follows;

Theorem A (Chanillo [1] and Komori [7]) *The commutator $[M_b, I_\alpha]$ is a bounded operator from $L^p(R^n)$ to $L^q(R^n)$ for $1/q = 1/p - \alpha/n$, $1 < p < n/\alpha$ and $0 < \alpha < n$, if and only if $b \in BMO(R^n)$.*

Theorem A says about the results for the particular Morrey spaces $L^{p,0}(R^n)$ and $L^{q,0}(R^n)$.

Recently, Di Fazio and Ragusa [4] obtained the next results corresponding to index λ, $0 < \lambda < n$.

Theorem B (Di Fazio and Ragusa [4]) *Let $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/(n-\lambda)$ and $1/q + 1/q' = 1$.*

If $b \in BMO(R^n)$ then $[M_b, I_\alpha]$ is a bounded operator from $L^{p,\lambda}(R^n)$ to $L^{q,\lambda}(R^n)$.

Conversely if $n - \alpha$ is an even integer and $[M_b, I_\alpha]$ is bounded from $L^{p,\lambda}(R^n)$ to $L^{q,\lambda}(R^n)$ for some p, q, λ as above, then $b \in BMO(R^n)$.

As we can see easily, the conditions for the converse part of Theorem B are very strong. In fact, when $n = 1, 2$ there does not exist α satisfying the conditions. When $n = 3$, the assumptions are satisfied only for $\alpha = 1$. When $n = 4$, the assumptions are satisfied for $\alpha = 1, 2$.

The aim of this paper is to remove this restriction. Our result is the following.

Theorem 1 *Let* $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/(n-\lambda)$ *and* $1/q + 1/q' = 1$.

If the commutator $[M_b, I_\alpha]$ *is bounded from* $L^{p,\lambda}(R^n)$ *to* $L^{q,\lambda}(R^n)$ *for some* p, q, λ *as above, then* $b \in BMO(R^n)$ *and* $\|b\|_*$ *is bounded by* $C_n \|[M_b, I_\alpha]\|_{L^{p,\lambda} \to L^{q,\lambda}}$ *where* C_n *is a positive constant depending only on* n.

Theorem 1 is a consequence of Theorem 2 below.

Theorem 2 *If* $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/(n-\lambda)$, $1/q + 1/q' = 1$ *and* $f \in H^1(R^n)$, *then there exist* $\{\varphi_j\}_{j=1}^{\infty} \subset L^{p,\lambda}(R^n)$ *and* $\{\psi_j\}_{j=1}^{\infty} \subset h_{nq/(nq-n+\lambda),q'}(R^n)$ *such that*

$$f = \sum_{j=1}^{\infty} (\varphi_j \cdot I_\alpha \psi_j - \psi_j \cdot I_\alpha \varphi_j),$$

$$\sum_{j=1}^{\infty} \|\varphi_j\|_{L^{p,\lambda}} \|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}} \le C_n \|f\|_{H^1}.$$

Remark Uchiyama [14] showed the factorization theorem on $H^p(X)$ when X is the space of homogeneous type, in the sense of Coifman-Weiss [3]. His result is corresponding to the case $\lambda = 0$ for Morrey spaces $L^{p,\lambda}(R^n)$. Also he applied his result to the boundedness problem of the commutators of the Calderón-Zygmund singular integral operator T.

Applying Uchiyama's method, the first author [7] showed the boundedness of the commutators of the fractional integral operator I_α when $X = R^n$ and $\lambda = 0$.

4. Preliminary Lemmas

We need four lemmas in order to prove our theorems. The first lemma is proved easily from the definitions.

Lemma 1 *Let* $1 \le p < \infty$, $0 \le \lambda \le n$ *and* $1 \le q < r \le \infty$. *Then we have*

$$\|\chi_{B(x_0,t)}\|_{L^{p,\lambda}} \le C_n t^{\frac{n-\lambda}{p}}, \qquad \|\chi_{B(x_0,t)}\|_{h_{q,r}} \le C_n t^{\frac{n}{q}}$$

where C_n is a positive constant depending only on n.

The following two lemmas are proved by Long [9].

Lemma 2 *Let X be the whole space R^n or the unit cube Q^n in R^n. If $1 \leq q < p' < \infty$, $q = \frac{np}{np-n+\lambda}$ and $1/p + 1/p' = 1$, then we have*

$$\|\phi\|_{L^{p,\lambda}(X)} = \sup_{b:(q,p')\text{-blocks}} \left| \int_X \phi(x)b(x)\,dx \right|,$$

where for the definitions of $L^{p,\lambda}(Q^n)$ and $h_{q,p'}(Q^n)$, see Remark (i) *in Section* 6.

Lemma 3 (Duality between $h_{q,p'}$ and $L^{p,\lambda}$) *Let $1 \leq q < p' < \infty$, $q = \frac{np}{np-n+\lambda}$ and $1/p + 1/p' = 1$, then the Banach space dual of $h_{q,p'}(R^n)$ is isomorphic to $L^{p,\lambda}(R^n)$.*

The last lemma is obtained from the elementary properties of $H^1(R^n)$.

Lemma 4 *If $\int f(x)\,dx = 0$ and $|f(x)| \leq (\chi_{B(x_0,1)} + \chi_{B(y_0,1)})$ where $N > 10$ and $|x_0 - y_0| = N$, then we have $\|f\|_{H^1} \leq C_n \log N$.*

5. Proofs of Theorems

First now we shall prove Theorem 2.

Proof of Theorem 2. We use the atomic decomposition of H^1 (see [8] or [13], p. 347). First we consider an atom a such that

$$\operatorname{supp} a \subset B(x_0, t), \quad \|a\|_{L^\infty} \leq t^{-n} \quad \text{and} \quad \int a(x)\,dx = 0.$$

We apply the method due to Komori [7]. Let N be a large integer and take $y_0 \in R^n$ such that $|x_0 - y_0| = Nt$ and set

$$\varphi(x) = N^{n-\alpha}\chi_{B(y_0,t)}(x),$$
$$\psi(x) = -a(x)/I_\alpha\varphi(x_0).$$

By Lemma 1, We have

$$\|\varphi\|_{L^{p,\lambda}} \leq C_n N^{n-\alpha} t^{\frac{n-\lambda}{p}},$$
$$\|\psi\|_{h_{nq/(nq-n+\lambda),q'}} \leq C_n t^{-n-\alpha} t^{\frac{nq-n+\lambda}{q}},$$

and

$$\|\varphi\|_{L^{p,\lambda}}\|\psi\|_{h_{nq/(nq-n+\lambda),q'}} \le C_n N^{n-\alpha}. \tag{1}$$

We write

$$a-(\varphi\cdot I_\alpha\psi-\psi\cdot I_\alpha\varphi)=\frac{a\cdot(I_\alpha\varphi(x_0)-I_\alpha\varphi)}{I_\alpha\varphi(x_0)}-\varphi\cdot I_\alpha\psi,$$

and we have

$$\int\{a-(\varphi\cdot I_\alpha\psi-\psi\cdot I_\alpha\varphi)\}\,dx=0,$$
$$|a-(\varphi\cdot I_\alpha\psi-\psi\cdot I_\alpha\varphi)|\le C_n N^{-1}t^{-n}(\chi_{B(x_0,t)}+\chi_{B(y_0,t)}).$$

By Lemma 4, we have

$$\|a-(\varphi\cdot I_\alpha\psi-\psi\cdot I_\alpha\varphi)\|_{H^1}\le C_n N^{-1}\log N. \tag{2}$$

Next for any $f\in H^1$ such that $\|f\|_{H^1}\le 1$, we can write $f=\sum_j m_j a_j$ where $\{a_j\}$ are atoms and $\sum_j |m_j|\le C_n$ by the atomic decomposition. Then there exist

$$\{\varphi_j\}_{j=1}^\infty\subset L^{p,\lambda}\quad\text{and}\quad\{\psi_j\}_{j=1}^\infty\subset h_{nq/(nq-n+\lambda),q'}$$

such that

$$\|\varphi_j\|_{L^{p,\lambda}}\|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}}\le C_n N^{n-\alpha}$$

and

$$\|a_j-(\varphi_j I_\alpha\psi_j-\psi_j I_\alpha\varphi_j)\|_{H^1}\le C_n N^{-1}\log N$$

by (1) and (2). So we have

$$\Big\|f-\sum_j\{(m_j\varphi_j)I_\alpha\psi_j-\psi_j I_\alpha(m_j\varphi_j)\}\Big\|_{H^1}$$
$$\le C_n N^{-1}\log N\sum_j|m_j|\le 1/2$$

if N is sufficiently large and

$$\sum_j\|m_j\varphi_j\|_{L^{p,\lambda}}\|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}}\le C_n N^{n-\alpha}\sum_j|m_j|\le C_{n,N}$$

Repeating this process, we get the desired result. □

Lastly we shall prove Theorem 1.

Proof of Theorem 1. We assume that the commutator $[M_b, I_\alpha]$ is bounded from $L^{p,\lambda}(R^n)$ to $L^{q,\lambda}(R^n)$ for some p, q, λ in Theorem 1. Let $f \in H^1(R^n)$. Then, by Theorem 2 and Lemma 3, we have

$$\begin{aligned} |\langle b, f\rangle| &\leq \sum_j \left| \int_{R^n} b(x)[\varphi_j(x) I_\alpha \psi_j(x) - \psi_j(x) I_\alpha \varphi_j(x)]\, dx \right| \\ &= \sum_j \left| \int_{R^n} \psi_j(x)[b(x) I_\alpha \varphi_j(x) - I_\alpha(b\varphi_j)(x)]\, dx \right| \\ &\leq C_n \sum_j \|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}} \|[M_b, I_\alpha]\varphi_j\|_{L^{q,\lambda}}. \end{aligned}$$

From the assumption and Theorem 2 again, this is bounded by

$$\begin{aligned} C_n \sum_j \|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}} \|\varphi_j\|_{L^{p,\lambda}} \|[M_b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\lambda}} \\ \leq C_n \|[M_b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\lambda}} \|f\|_{H^1}. \end{aligned}$$

By the duality for $H^1(R^n)$ and $BMO(R^n)$, we have that $b \in BMO(R^n)$ and $\|b\|_*$ is bounded by $C_n \|[M_b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\lambda}}$. Thus we complete the proof.

□

6. Some Remarks

(i) In Definitions 1, 2 and 4, we can replace a ball $B(x,t)$ by a cube $Q(x,t)$ centered at x with sides parallel to coordinates and sidelength t.

Also the Morrey space $L^{p,\lambda}(Q^n)$ on the unit cube Q^n in R^n is defined by

$$L^{p,\lambda}(Q^n) = \Bigg\{ f \in L^p(Q^n); \quad \|f\|_{L^{p,\lambda}} = \sup_{\substack{x\in Q^n \\ 0<t<1}} \left(\frac{1}{t^\lambda} \int_{Q(x,t)} |f(y)|^p dy \right)^{1/p} < \infty \Bigg\}.$$

Similarly we can define the spaces $h_{p,q}(Q^n)$ generated by blocks.

(ii) We obtain the local version of Lemma 3;

Lemma 3′ [Duality between $h_{q,p'}(Q^n)$ and $L^{p,\lambda}(Q^n)$] *Let Q^n be the unit cube in R^n. Let $1 \leq q < p' < \infty$, $q = \frac{np}{np-n+\lambda}$ and $1/p + 1/p' = 1$, then the*

Banach space dual of $h_{q,p'}(Q^n)$ *is isomorphic to* $L^{p,\lambda}(Q^n)$.

(iii) Some problems are open.

Problem 1 Can we get the boundedness or the compactness of the commutators $[M_b, I_\alpha]$ from $L^{p,\lambda}(R^n)$ to $L^{q,\mu}(R^n)$ for some p, q, λ, μ?

Problem 2 Can we get the $H^p(R^n)$ $(0 < p < 1)$ version of Theorem 2?

Problem 3 In the setting of spaces of homogenous type, can we get any results corresponding to Theorems 1 and 2?

References

[1] Chanillo S., *A note on commutators*. Indiana Univ. Math. J. **31** (1982), 7–16.

[2] Chiarenza F. and Frasca M., *Morrey spaces and Hardy Littlewood maximal function*. Rend. Mat. (7) **7** 5-A (1987), 273–279.

[3] Coifman R. and Weiss G., *Extensions of Hardy spaces and their use in analysis*. Bull. Amer. Math. Soc. **83** (1977), 569–645.

[4] Di Fazio G. and Ragusa M.A., *Commutators and Morrey spaces*. Boll. U.M.I **7** 5-A (1991), 323–332.

[5] Fefferman C. and Stein E.M., *H^p spaces of several variables*. Acta Math. **129** (1972), 137–193.

[6] John F. and Nirenberg L., *On functions of bounded mean oscillation*. Comm. Pure Appl. Math. **14** (1961), 415–426.

[7] Komori Y., *The factorization of H^p and the commutators*. Tokyo J. Math. **6** (1983), 435–445.

[8] Latter R.H., *A decomposition of $H^p(R^n)$ in terms of atoms*. Studia Math. **62** (1978), 92–101.

[9] Long R., *The spaces generated by blocks*. Scientia Sinica Ser. A **27**, No.1 (1984), 16–26.

[10] Lu S., Taibleson M. and Weiss G., *Spaces generated by blocks*. Beijing Normal Univ. Press, Beijing (1989).

[11] Morrey C.B. Jr., *On the solutions of quasi-linear elliptic partial differential equations*. Trans. Amer. Math. Soc. **43** (1938), 126–166.

[12] Taibleson M.H. and Weiss G., *Certain function spaces connected with almost everywhere convergence of Fourier series*. Conference on Harmonic Analysis in Honor of A. Zygmund, Vol. I, Wadsworth, 1983, 95–113.

[13] Torchinsky A., *Real-variable methods in harmonic analysis*. Academic Press, 1986.

[14] Uchiyama A., *The factorization of H^p on the space of homogeneous type*. Pacific J. Math. **92** (1981), 453–468.

Yasuo Komori
School of High Technology and Human Welfare
Tokai University
317 Nishino Numazu Shizuoka 410-0395
Japan
E-mail: komori@wing.ncc.u-tokai.ac.jp

Takahiro Mizuhara
Department of Mathematical Sciences
Faculty of Science
Yamagata University
Yamagata 990-8560, Japan
E-mail: mizuhara@sci.kj.yamagata-u.ac.jp