On the closure of an elementary abelian 2-subgroup of a finite group

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Abstract. Let H be a finite group and $S \in \text{Syl}_2(H)$. Let V be a $GF(2)H$ module and U be an S submodule of V . For an integer t , we write $U \in \mu_t(S)$ if $U = \langle u; \dim_{GF(2)}[u, A] \leq t \rangle$ for any elementary abelian subgroup A of S with $[U, A, A] = 0$. The purpose of this note is to give an estimation of m with $U^H \in \mu_{t+m}(S)$, whenever $U \in \mu_t(S)$.

Key words: the closure of an elementary abelian 2-subgroup, solvable group, component.

Notation All groups considered in this note will be finite. Let A and B be subgroups of a group H . We write $A \leq B$ ($A < B$) to indicate A is a (proper) subgroup of B . For a subset X of H , $\langle X \rangle$ denotes the subgroup of H generated by X . $|H|$ denotes the number of elements of H , and $H^\#$ denotes the set of all the nonidentity elements of H . Let $a^b = b^{-1}ab$ for $a \in A$ and $b \in B$, $A^B = \langle a^b; a \in A, b \in B \rangle$, and $[A, B] = \langle a^{-1}a^b; a \in A, b \in B \rangle$. $N_H(A)$ and $C_H(A)$ denote the normalizer and centralizer of A in H respectively. Let $Z(H) = C_H(H)$. For a prime p , $\text{Syl}_p(H)$ denotes the set of all Sylow p -subgroups of H , $O_p(H)$ denotes the largest normal p -subgroup of H , and $O^p(H)$ denotes the subgroup which is generated by all the elements of H whose orders are relatively prime to p . A is a component of H if A is a subnormal subgroup of H such that $[A, A] = A$ and $A/Z(A)$ is simple. $E(H)$ denotes the subgroup of H generated by all the components of H . Let $X = (x_{ij})$ and Y be matrices. Then $X \otimes Y$ denotes the matrix whose (i, j) block is $x_{ij}Y$. If $(0) \leq V_1 \leq V_2 \leq V$ be $GF(2)H$ modules, then V_2/V_1 is a constituent of V . If V_2/V_1 is irreducible it is an irreducible constituent of V .

Lemma 1 *Let A be an elementary abelian 2-group, and let $V = V_1 \oplus \cdots \oplus V_s$ be a $GF(2)A$ module. Assume that $[V, A, A] = 0$, and that A permutes transitively on $\{V_i; 1 \leq i \leq s\}$. Then $s \leq 2$, and if the equality holds, then $|A/C_A(V)| = 2$.*

Proof. Assume that $s \geq 2$. Let B be a maximal subgroup of A which acts intransitively on $\{V_i; 1 \leq i \leq s\}$. Since $(V_1^B)^a \cap V_1^B = 0$ for $a \in A - B$ and $[V, A, A] = 0$, it follows that $[V_1^B, B] = 0$. This implies that $V_1^B = V_1$ and $s \leq 2$. Moreover, if the equality holds, then $[V, B] = [V_1^A, B] = [V_1, B]^A = 0$ and $|A/C_A(V)| = 2$. $\square$

Let t be an integer. Let S be a finite 2-group and V be a $GF(2)S$ module. Then we write $V \in \mu_t(S)$ if $V = \langle v; \dim_{GF(2)}[v, A] \leq t \rangle$ for any elementary abelian subgroup A of S with $[V, A, A] = 0$.

Lemma 2 *Let H be a finite group, and T be a 2-subgroup of H . Let V be a $GF(2)H$ module, and U be a T submodule of V such that $U^H = V$ and $U \in \mu_t(T)$ for some integer t . Let A be an elementary abelian subgroup of T with $[V, A, A] = 0$. Then*

- (1) *If $|A/A \cap O_2(H)| \leq 2^n$, then $V \in \mu_{t+n}(A)$.*
- (2) *$U^{N_H(T)} \in \mu_t(T)$.*

Proof. $a \mapsto [v, a]$ is a homomorphism on A for each $v \in V$ and for $a \in A$. This implies (1). Since $U^h \in \mu_t(T)$ for all $h \in N_H(T)$, (2) follows from the definition. $\square$

Hypothesis A *Let p be an odd prime, H be a finite $\{2, p\}$ -group, $S \in \text{Syl}_2(H)$ and $P \in \text{Syl}_p(H)$. Let V be a $GF(2)H$ module and U be an S submodule of V with $U^H = V$. Let A be an elementary abelian subgroup of S with $[V, A, A] = 0$. Assume that $H = O_{2,p,2}(H)$.*

Lemma 3 *Assume Hypothesis A. Further assume that*

- (a) *$O_2(H) = 1$ and $P = [P, A]$; and*
- (b) *$S = A$ and $|A| \geq 2^2$.*

Then (1) There are generators $\{a_i; i \in I\}$ of A such that $[P, a_i]A < H$ for all $i \in I$.

Let $H_i = [P, a_i]A$ for $i \in I$. Then

- (2) *$V = \sum_{i \in I} U^{H_i}$.*

Proof. (1) is obvious. To prove (2), let W be an irreducible H submodule of V with $[W, P] \neq 0$ and let s be the number of Wedderburn components of W w.r.t. P . Let $\bar{H} = H/C_H(W)$. We will show that $|\bar{A}| = 2$. By Lemma 1, $s \leq 2$, and if the equality holds, then $|\bar{A}| = 2$. Assume that $s = 1$. Then $|\Omega_1 Z(\bar{P})| = p$ because W is the sum of isomorphic P submodules. Then

we have that $1 \neq \Omega_1 Z(\overline{P}) \leq \bigcap_{\overline{a} \in \overline{A}^\#} [\overline{P}, \overline{a}]$. By [1], we have that $|\overline{A}| \leq 2$. By (a), $|\overline{A}| = 2$. Thus we obtain $|\overline{A}| = 2$. This implies that $[W, H_i] \neq 0$ if and only if $[W, a_i] \neq 0$, that is, $\langle \overline{a}_i \rangle = \overline{A}$ for $i \in I$. Let $I = \{1, \dots, r\}$. Define the mapping ρ from the set of all irreducible H submodules W of V with $[W, P] \neq 0$ to I as follows: $\rho(W) = i$ if and only if $[W, a_j] = 0$ for all $j < i$ and $[W, a_i] \neq 0$, $1 \leq i \leq r$. Let W_i be the sum of all irreducible H submodules W of V with $\rho(W) = i$, $1 \leq i \leq r$, and $W_0 = C_V(P)$. Then we have that $V = \bigoplus_{i=0}^r W_i$. Let $\pi_i : V \rightarrow W_i$ be the natural projection, $0 \leq i \leq r$. Now we will show that

$$\bigoplus_{j=0}^i W_j \leq \sum_{j=1}^i U^{H_j}, \quad 1 \leq i \leq r,$$

using induction on i . The assumption $V = U^H$ implies $W_i = \pi_i(U)^{H_i}$ for all i . Since

$$\left[\bigoplus_{j=2}^r \pi_j(U), O^2(H_1) \right] \leq \left[\bigoplus_{j=2}^r W_j, O^2(H_1) \right] = 0,$$

we have that $W_1 \leq U^{H_1}$. Assume that $\bigoplus_{j=0}^{i-1} W_j \leq \sum_{j=1}^{i-1} U^{H_j}$ for some $i \geq 2$. Since $\left[\bigoplus_{j=i+1}^r \pi_j(U), O^2(H_i) \right] = 0$, it follows that

$$W_i \leq \sum_{j=0}^i \pi_j(U)^{H_j} \leq \bigoplus_{j=0}^{i-1} W_j + U^{H_i} \leq \sum_{j=1}^i U^{H_j}.$$

Therefore, $V = \bigoplus_{i=0}^r W_i = \sum_{i=1}^r U^{H_i}$, as required. $\square$

Lemma 4 Assume Hypothesis A. Let $\overline{H} = H/O_2(H)$. Further assume that

- (a) $\overline{P} = [\overline{P}, \overline{A}]$; and
- (b) $\overline{S} = \overline{A}$, and $|\overline{A}| \geq 2^2$.

Take $\{a_i; i \in I\}$ so that $[\overline{P}, \overline{a}_i] \overline{A} < \overline{H}$ and let H_i be the pre-image of $\overline{H}_i$ for $i \in I$. Then $V = \sum_{i \in I} U^{H_i}$.

Proof. We proceed using induction on $|U|$. Let U_0 be a maximal S submodule of U , and $W = U_0^H$. By induction, $W = \sum_{i \in I} U_0^{H_i}$. It follows from the maximality of U_0 that $[(U + W)/W, S] = 0$ and $[V/W, O_2(H)] = 0$. By Lemma 3, we have that $V = \sum_{i \in I} U^{H_i} + W = \sum_{i \in I} U^{H_i}$. $\square$

Theorem 1 *Assume Hypothesis A. If $U \in \mu_t(S)$ for some integer t , then $V \in \mu_{t+1}(S)$.*

Proof. We proceed using induction on $|H|$. Let $\bar{H} = H/O_2(H)$. It suffices to show that $V^H \in \mu_{t+1}(A)$. By way of a contradiction, assume that $V \notin \mu_{t+1}(A)$. If $\bar{A} < \bar{S}$, then we have by induction that $V = U^H = U^P = U^{O_2(H)AP} \in \mu_{t+1}(A)$, a contradiction. Hence, $\bar{S} = \bar{A}$. Let R be the pre-image of $[\bar{P}, \bar{A}]$. Assume that $R < P$. By the Frattini argument, $H = RN_H(S)$. Since $U^R \in \mu_{t+1}(S)$ by induction, we have by Lemma 2 (2) that $V = U^H = (U^R)^{N_H(S)} \in \mu_{t+1}(S)$, a contradiction. Therefore, $\bar{P} = [\bar{P}, \bar{A}]$. By Lemma 2 (1), $|\bar{A}| \geq 2^2$. Then we can choose generators $\{a_i, i \in I\}$ of A such that $[\bar{P}, \bar{a}_i]\bar{A} < \bar{H}$ for all $i \in I$. Let H_i be the pre-image of $[\bar{P}, \bar{a}_i]\bar{A}$, $i \in I$. Now applying Lemma 4, we have by induction that $V = U^H = \sum_{i \in I} U^{H_i} \in \mu_{t+1}(A)$, a contradiction. This completes the proof of the theorem. $\square$

Hypothesis B *Let H be a finite group, and $S \in \text{Syl}_2(H)$. Let V be a faithful $GF(2)H$ module. Let $\bar{H} = H/O_2(H)$. Let $\{\bar{K}_i; 1 \leq i \leq r\}$ be the set of all components of $\bar{H}$, E and K_i be the pre-images of $E(\bar{H})$ and $\bar{K}_i$ respectively, $1 \leq i \leq r$. Let U be an S submodule of V with $U^H = V$. Let A be an elementary abelian subgroup of S with $[V, A, A] = 0$. Assume that $H = ES$.*

If $O_2(H) = 1$, we identify $\bar{X}$ with X for a subset X of H .

Lemma 5 *Assume Hypothesis B. Let L_1 and L_2 be normal subgroup of H , enjoying the following properties:*

- (a) $L_1 L_2 = E$ and $[\bar{L}_1, \bar{L}_2] = 1$; and
- (b) Either $[V_2/V_1, L_1] = 0$ or $[V_2/V_1, L_2] = 0$ for any irreducible constituent V_2/V_1 of V .

Then $V = U^{L_1} + U^{L_2}$.

Proof. Assume first that $O_2(H) = 1$. Let $W = C_V(L_1) + C_V(L_2)$. We may assume that $L_1 \not\leq L_2$ and $L_2 \not\leq L_1$. Then $O^2(L_i) = L_i$, $i = 1, 2$. Then [2, Theorem 3.15 p.187] implies that $C_{V/W}(L_i) = 0$, $i = 1, 2$. By (b), $V/W = 0$, that is, $V = C_V(L_1) + C_V(L_2)$. Let $V_0 = C_V(E)$, and let V_i be a complement submodule of V_0 in $C_V(L_i)$, $i = 1, 2$. Let $u = \sum_{i=0}^2 u_i \in U$, $h = h_1 h_2 \in E$, where $u_i \in V_i$, $0 \leq i \leq 2$, and $h_i \in L_i$, $i = 1, 2$. Then it follows that $u^h = \sum_{i=0}^2 u_i^{h_1 h_2} = u_0 + u_1^{h_2} + u_2^{h_1} = (u_0 + u_1 + u_2) + (u_0 +$

$u_1 + u_2^{h_1}) + (u_0 + u_1^{h_2} + u_2) = u^1 + u^{h_1} + u^{h_2} \in U^{L_1} + U^{L_2}$. Since u and h are arbitrary elements of U and E respectively, we have that $V = U^H = U^E = U^{L_1} + U^{L_2}$. Then we can handle the case $O_2(H) \neq 1$ by the similar way to the proof of Lemma 4. $\square$

Assume Hypothesis B. For $\bar{x} \in \bar{A}^\sharp$, define $\Theta(\bar{x}) = \{i; [\bar{K}_i, \bar{x}] = 1, 1 \leq i \leq r\}$. Define $\Theta(\bar{A}) = \{\bar{x} \in \bar{A}^\sharp; \Theta(\bar{x}) \neq \emptyset\}$. Note that if $\bar{x} \in \Theta(\bar{A})$, then $[\bar{E}, \bar{x}] \neq 1$ because $\bar{H} = \bar{E}\bar{S}$ and $\bar{x} \notin O_2(\bar{H}) = 1$.

Lemma 6 *Assume Hypothesis B. Further assume that V is irreducible, and that $\bar{H} = [\bar{E}, \bar{A}]\bar{A}$. Then $\Theta(\bar{A}) = \emptyset$.*

Proof. Note that $O_2(H) = 1$, since V is irreducible. By way of a contradiction, assume that $\Theta(\bar{A}) \neq \emptyset$. Let $x \in \Theta(\bar{A})$, $L_1 = \langle K_i; i \in \Theta(x) \rangle$ and $L_2 = \langle K_i; i \notin \Theta(x) \rangle$. Note that L_i is A invariant and normal in H , $i = 1, 2$ because $H = [E, A]A$. We may assume that V has the only one Wedderburn component w.r.t. E . Otherwise, by Lemma 1 $|A| = 2$ and $\Theta(\bar{A}) = \emptyset$ because $[E, A] = E$. Then V is the sum of isomorphic E submodules. Let F be a splitting field for L_1 and L_2 with characteristic 2 and $V^* = V \otimes F$. By [2, Theorem 3.7.1 and 2 pp. 100–102], V^* is the sum of irreducible E submodules of type $U_1 \otimes U_2$, where U_i is a faithful and irreducible L_i module, $i = 1, 2$. Hence, either $(U_1 \otimes U_2)^h = U_1 \otimes U_2$ or $(U_1 \otimes U_2)^h \cap U_1 \otimes U_2 = 0$ for $h \in H$. Since $[E, A] = E$, we have that $[U_1 \otimes U_2, A] \neq 0$. Then Lemma 1 implies $U_1 \otimes U_2$ is A invariant. Since $[L_1, x] = 1$, we have that $[U_1, x] = 0$. Let X be the representation matrix of x of the action on U_2 . Since $[E, A] = E$, we have that $[L_1, A] \neq 1$. So, we can take $y \in A - C_A(L_1)$. Let $Y_1 = (y_{ij})$ and Y_2 be the representation matrices of y of the actions on U_1 and U_2 respectively. Then the representation matrices of x and y on $U_1 \otimes U_2$ are $I \otimes X$ and $Y_1 \otimes Y_2$ respectively, where I is the identity matrix. Note that $[V, x, y] = 0$ if and only if $(X - I)(Y_2 y_{ii} - I) = O$, $1 \leq i \leq n$, and $(X - I)Y_2 y_{ij} = O$, $1 \leq i \neq j \leq n$, where $n = \dim_F U_1$ and O is the zero matrix. Since Y_2 is non-singular, the latter equation yields $(X - I)y_{ij} = O$. Since $X \neq I$, we have that $y_{ij} = 0$, $1 \leq i \neq j \leq n$. Therefore, Y_1 is a diagonal matrix. Then the former equation yields $y_{ii}(X - I) = (X - I)Y_2^{-1} = y_{jj}(X - I)$, $1 \leq i, j \leq n$. Since $X \neq I$, this implies that Y_1 is a scalar matrix. Since L_1 acts faithfully on U_1 , it follows that $[L_1, y] = 1$, contrary to the choice of y . This proves the assertion. $\square$

Lemma 7 Assume Hypothesis B. Further assume that $\overline{H} = [\overline{E}, \overline{A}]\overline{A}$, and that $\Theta(\overline{A}) \neq \emptyset$. Let $\overline{x} \in \Theta(\overline{A})$, $\overline{L}_1 = \langle \overline{K}_i; i \in \Theta(\overline{x}) \rangle$ and $\overline{L}_2 = \langle \overline{K}_i; i \notin \Theta(\overline{x}) \rangle$. Let L_i be the pre-image of $\overline{L}_i$, $i = 1, 2$. Then $V = U^{L_1} + U^{L_2}$.

Proof. Let V_2/V_1 be an irreducible constituent of V and $C = C_H(V_2/V_1)$. Replacing H by H/C and V by V_2/V_1 if necessary, we can apply Lemma 6, and have that either $L_1 \leq C$ or $L_2 \leq C$. Then the assertion follows from Lemma 5. $\square$

Let H be a finite group, let L be a subgroup of $E(H)$ which is generated by some components of H . Let V be an irreducible $GF(2)H$ module. Then we say that V is a non-trivial product L submodule, if $C_V(K) = 0$ for all the components K of L .

Lemma 8 Assume Hypothesis B. Further assume that V is irreducible, and that $K_1^A \neq K_1$. Then $|A/N_A(K_1)| = 2$. Moreover, one of the following holds:

$$(i) \quad [\overline{K}_1, N_{\overline{A}}(\overline{K}_1)] = 1; \text{ or}$$

$$(ii) \quad [\overline{K}_1, N_{\overline{A}}(\overline{K}_1)] = \overline{K}_1 \simeq SL(2, 2^n) \text{ for some } n \geq 2, [\overline{E}, \overline{A}] = \overline{K}_1^{\overline{A}},$$

and V is a non-trivial product $\overline{K}_1^{\overline{A}}$ submodule.

Proof. Note that $O_2(H) = 1$ because V is irreducible. Throughout the proof, assume that (i) does not occur. Then V has the only one Wedderburn component w.r.t. E , otherwise by Lemma 1 $|A| = 2$ and (i) occurs, a contradiction. So, V is the sum of isomorphic E submodules. Thus, $C_V(K_i) = 0$, $1 \leq i \leq r$. In particular, V is a non-trivial product K_1^A submodule. Note that V contains a faithful and irreducible $[E, A]A$ submodule. Let F be a splitting field for all K_i , $1 \leq i \leq r$, with characteristic 2, and $V^* = V \otimes F$. By [2, Theorem 3.7.1 and 2], V^* is the sum of irreducible E submodules of type $\bigotimes_{i=1}^r U_i$, where U_i is a faithful and irreducible K_i module, $1 \leq i \leq r$. Since $[\bigotimes_{i=1}^r U_i, A] \neq 0$, Lemma 1 implies $\bigotimes_{i=1}^r U_i$ is A invariant. Let $y \in A - N_A(K_1)$. Renumbering if necessary, we may assume that $U_i^y = U_{\ell+i}$, $1 \leq i \leq \ell$ and $U_i^y = U_i$, $2\ell + 1 \leq i \leq r$. Let

$$W_1 = \bigotimes_{i=1}^{\ell} U_i \quad \text{and} \quad W_2 = \bigotimes_{i=\ell+1}^{2\ell} U_i.$$

Let $x \in N_A(K_1) - C_A(K_1)$. Note that $[K_1^A, x] = K_1^A$ because A is abelian and $[K_1, x] \neq 1$ if and only if $[K_1/Z(K_1), x] \neq 1$. Assume that

$\dim_F C_{W_1}(x) \geq 2$. Then we can choose $v_1 \in C_{W_1}(x) - \{0\}$ and $v \in W_1$ such that $v + v^x$ is not contained in the subspace spanned by v_1 . Let $v_2 = v + v^x$. Since $W_1^y = W_2$, we can choose bases $\{v_i; 1 \leq i \leq t\}$ of W_1 and $\{w_i; 1 \leq i \leq t\}$ of W_2 such that $v_i^y = w_i$, $1 \leq i \leq t$. Note that $[v_i, x] = 0$ if and only if $[w_i, x] = 0$, $1 \leq i \leq t$. Let $w = v^y$. Then we have that $w^x + w = (v^x + v)^y = v_2^y = w_2$. It follows that

$$[v \otimes w_1, y] = (v \otimes w_1)^y + v \otimes w_1 = v_1 \otimes w + v \otimes w_1$$

and

$$\begin{aligned} [v \otimes w_1, y, x] &= (v_1 \otimes w + v \otimes w_1)^x + (v_1 \otimes w + v \otimes w_1) \\ &= v_1 \otimes w^x + v^x \otimes w_1 + (v_1 \otimes w + v \otimes w_1) \\ &= v_1 \otimes (w + w^x) + (v + v^x) \otimes w_1 = v_1 \otimes w_2 + v_2 \otimes w_1 \neq 0. \end{aligned}$$

This yields $[V^*, y, x] \neq 0$ and $[V, A, A] \neq 0$, a contradiction. Hence, $\dim_F C_{W_1}(x) = 1$. Since $\dim_F W_1 = \dim_F C_{W_1}(x) + \dim_F [W_1, x]$ and $\dim_F [W_1, x] \leq \dim_F C_{W_1}(x) = 1$, we have that $\dim_F W_1 = 2$. Since K_1 is nonsolvable, we have that $\dim_{GF(2)} U_1 \geq 2$. It must be that $W_i = U_i$, $i = 1, 2$. This implies $K_1 \simeq SL(2, 2^n)$ for some $n \geq 2$. Moreover, $|A/N_A(K_1)| = 2$ because $\ell = 1$ and y is an arbitrary element of $A - N_A(K_1)$. Now we will show that $[E, A] = K_1^A$. Assume that $E > K_1^A$. Let $W_3 = \bigotimes_{i=3}^r U_i$. Let X_i be the representation matrix of x on W_i , $1 \leq i \leq 3$. Let Y_1 and Y_2 be the representation matrices of y on $W_1 \otimes W_2$ and on W_3 , respectively.

We may assume that $X_1 = X_2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ under bases $\{v_i; 1 \leq i \leq 2\}$ and $\{w_i; 1 \leq i \leq 2\}$ respectively, where a is a nonzero element of F , and

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ under a basis } \{v_i \otimes w_j; 1 \leq i, j \leq 2\}. \text{ Furthermore,}$$

we may set $X_3 = \begin{pmatrix} 1 & A_2 \\ O & A_3 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} 1 & B_2 \\ O & B_3 \end{pmatrix}$ since the characteristic

of F is 2. Let $X = X_1 \otimes X_2 \otimes X_3$ and $Y = Y_1 \otimes Y_2$. Since $[V, x, y] = 0$, we have that $(X - I)(Y - I) = O$. Then (4,6) and (5,4) blocks of $(X - I)(Y - I)$ are $(A_3 - I)B_3$ and A_2B_3 respectively. Hence, $(A_3 - I)B_3 = O$ and $A_2B_3 = O$. Since B_3 is non-singular, it follows that $A_3 = I$ and $A_2 = O$, that is, $X_3 = I$ and x acts trivially on W_3 . Since K_i acts faithfully on W_3

for $i \geq 3$, we have that $[K_i, x] = 1$ for all $i \geq 3$. Hence, $[E, x] = K_1^A$. Since V contains a faithful and irreducible $[E, A]A$ submodule, Lemma 6 implies that $[K_i, A] = 1$ if and only if $[K_i, x] = 1$, $1 \leq i \leq r$. Therefore, $[E, A] = [E, x] = K_1^A$. $\square$

Lemma 9 *Assume Hypothesis B. Further assume that $K_1^A \neq K_1$, and that $[\overline{K}_1, N_{\overline{A}}(\overline{K}_1)] = \overline{K}_1$. Let $M_1 = K_1^A$ and $M_2 = \langle K_i; \overline{K}_i \leq [\overline{E}, \overline{A}] \text{ and } K_i \not\leq K_1^A, 1 \leq i \leq r \rangle$ (possibly, $M_2 = O_2(H)$). Then*

- (1) $\overline{K}_1 \simeq SL(2, 2^n)$ for some $n \geq 2$ and $|A/N_A(K_1)| = 2$.
- (2) Either $[V_2/V_1, M_1] = 0$ or $[V_2/V_1, M_2] = 0$ for any irreducible constituent V_2/V_1 of V .
- (3) If $\overline{H} = [\overline{E}, \overline{A}]\overline{A}$ and $U \in \mu_t(S)$ for some integer t , then $V = U^{M_1} + U^{M_2}$ and $U^{M_1} \in \mu_{t+n+1}(A)$. Moreover, if $U^{M_1} \notin \mu_{t+n}(A)$, then $|A/A \cap O_2(M_1A)| = 2^{n+1}$.

Proof. Since V is faithful, we can take an irreducible constituent V_2/V_1 of V with $[V_2/V_1, K_1] \neq 0$. Let $C = C_H(V_2/V_1)$. By Lemma 8, $|A/N_A(K_1)| = 2$ and $K_1/K_1 \cap C \simeq SL(2, 2^n)$ for some $n \geq 2$. Since V_2/V_1 is irreducible, we have $O_2(H) \leq K_1 \cap C$. Since $O_2(\overline{H}) = 1$, $\overline{K}_1 \cap \overline{C}$ is a subgroup of $Z(\overline{K}_1)$ of odd order. Since $K_1/K_1 \cap C \simeq SL(2, 2^n)$, we have $\overline{K}_1 \cap \overline{C} = 1$ and obtain (1). Moreover, Lemma 8 shows that $[E, A]C/C = K_1^A C/C$ and $M_2 \leq C$. This implies (2) because V_2/V_1 is an arbitrary constituent of V with $[V_2/V_1, K_1] \neq 0$. Assume that $\overline{H} = [\overline{E}, \overline{A}]\overline{A}$, and that $U \in \mu_t(S)$. Then by (2) and Lemma 5, we have that $V = U^{M_1} + U^{M_2}$. Since $M_1/O_2(M_1) \simeq SL(2, 2^n) \times SL(2, 2^n)$ and $A \not\leq N_H(K_1)$, it is easy to see that $|A/A \cap O_2(M_1A)| \leq 2^{n+1}$. Then it follows from Lemma 2 (1) that $U^{M_1} \in \mu_{t+n+1}(S)$, and if $U^{M_1} \notin \mu_{t+n}(A)$, then $|A/A \cap O_2(M_1A)| = 2^{n+1}$. $\square$

Theorem 2 *Assume Hypothesis B. Let $m = \max_{1 \leq i \leq r} \{\log_2 |N_{\overline{B}}(\overline{K}_i)/C_{\overline{B}}(\overline{K}_i)|\}$, where B ranges over all the subgroups of S with $[V, B, B] = 0$. If $U \in \mu_t(S)$ for some integer t , then one of the following holds:*

- (i) $V \in \mu_{t+m}(S)$; or
- (ii) $V \in \mu_{t+m+1}(S)$, and there is a pair of an elementary abelian subgroup A of S and a component $\overline{K}_i$ of $\overline{H}$ such that $[V, A, A] = 0$, $|A/N_A(K_i)| = 2$, $|A/A \cap O_2(K_i^A A)| = 2^{m+1}$ and $\overline{K}_i \simeq SL(2, 2^m)$. Moreover, V_2/V_1 is a non-trivial product $\overline{K}_i^{\overline{A}}$ submodule and $[\overline{E}, \overline{A}] = \overline{K}_i^{\overline{A}} C_{[\overline{E}, \overline{A}]}(V_2/V_1)$ for every irreducible constituent V_2/V_1 of V with $[V_2/V_1, K_i] \neq 0$.

Proof. Fix an elementary abelian subgroup A of S with $[V, A, A] = 0$. Let $\Delta(A)$ be the set of all components $\overline{K}_i$ of $\overline{H}$ such that $K_i^A \neq K_i$, $|A/A \cap O_2(K_i^A A)| = 2^{m+1}$ and $\overline{K}_i \simeq SL(2, 2^m)$. Let $\ell = t + m$ if $\Delta(A) = \emptyset$ and $\ell = t + m + 1$ if $\Delta(A) \neq \emptyset$. To prove $V \in \mu_\ell(S)$, we may only show that $V \in \mu_\ell(A)$ because A is an arbitrary elementary abelian subgroup of S with $[V, A, A] = 0$. We will show that $V \in \mu_\ell(A)$ using induction on $|H|$. By way of a contradiction, assume that $V \notin \mu_\ell(A)$. If $EA < H$, then we have by induction that $V = U^E = U^{EA} \in \mu_\ell(A)$, a contradiction. Hence, $H = EA$. Let L be the pre-image of $[\overline{E}, \overline{A}]$ and $T = L \cap S$. Since L is normal in H , the Frattini argument shows that $H = LN_H(T)$. If $LA < H$, then we have by induction that $U^L \in \mu_\ell(T)$ and by Lemma 2(2) that $V = (U^L)^{N_H(T)} \in \mu_\ell(T) \subseteq \mu_\ell(A)$, a contradiction. Hence, $[\overline{E}, \overline{A}] = \overline{E}$. Assume that there is a component $\overline{K}_i$ of $\overline{H}$ such that $K_i^A \neq K_i$ and $[\overline{K}_i, N_{\overline{A}}(\overline{K}_i)] = \overline{K}_i$. Let $M_1 = K_1^A$ and $M_2 = \langle K_i; K_i \not\leq K_1^A, 1 \leq i \leq r \rangle$. By Lemma 9, we have that $\overline{K}_i \simeq SL(2, 2^n)$ for some $n \geq 2$, $V = U^{M_1} + U^{M_2}$ and $U^{M_1} \in \mu_{t+n+1}(A)$. Since $n \leq m$ and $M_2 A < H$, we have that $V = U^{M_1} + U^{M_2} \in \mu_\ell(A)$, a contradiction. Therefore, we have that $N_{\overline{A}}(\overline{K}_i) = C_{\overline{A}}(\overline{K}_i)$, whenever $K_i^A \neq K_i$, $1 \leq i \leq r$. Now, we will show that $\Theta(\overline{A}) \neq \emptyset$. By Lemma 2(1), we have that $|\overline{A}| > 2^\ell \geq 2$. Assume that there is a component $\overline{K}_i$ of $\overline{H}$ such that $K_i^A \neq K_i$. By Lemma 9(1), $|\overline{A}/C_{\overline{A}}(\overline{K}_i)| = |\overline{A}/N_{\overline{A}}(\overline{K}_i)| = 2$. Since $|\overline{A}| \geq 2^2$, we have that $\Theta(\overline{A}) \neq \emptyset$. Assume next that K_i is A invariant for all i . Let π_i be the natural homomorphism from $\overline{A}$ to $\overline{A}/C_{\overline{A}}(\overline{K}_i)$, $1 \leq i \leq r$. If $\text{Ker } \pi_i = 1$ for all i , then it follows from the definition of m that $|\overline{A}| \leq 2^m$, contrary to our assumption. Hence, $\text{Ker } \pi_i \neq 1$ for some i and $\Theta(\overline{A}) \neq \emptyset$. It is shown that $\Theta(\overline{A}) \neq \emptyset$. Let $\overline{x} \in \Theta(\overline{A})$, $\overline{L}_1 = \langle \overline{K}_i; i \in \Theta(\overline{x}) \rangle$ and $\overline{L}_2 = \langle \overline{K}_i; i \notin \Theta(\overline{x}) \rangle$. Let L_i be the pre-image of $\overline{L}_i$, $i = 1, 2$. Now we can apply Lemma 7, and obtain that $V = U^{L_1} + U^{L_2}$. Since $L_i A < H$, $i = 1, 2$, we have that $U^{L_i} \in \mu_\ell(A)$, $i = 1, 2$ and $V = U^{L_1} + U^{L_2} \in \mu_\ell(A)$, a contradiction. This proves $V \in \mu_\ell(A)$ and $V \in \mu_\ell(S)$. The remaining part of our assertion follows from Lemma 8 and 9. This completes the proof of the theorem. $\square$

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