L^p boundedness for singular integrals with rough kernels on product domains

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Abstract. In this paper we study a class of singular integral operators defined by polynomial mappings on product domains. We prove the L^p boundedness of such singular integral operators under very weak conditions on the singular kernels. For the corresponding maximal truncated singular integral operators we also obtained appropriate L^p bounds which were previously unavailable even in relatively simple settings.

Key words: singular integrals, product domains, rough kernels, Block spaces.

1. Introduction and statements of results

For $n, m \geq 2$ let $K(\cdot, \cdot)$ be the singular kernel on $\mathbf{R}^n \times \mathbf{R}^m$ given by

$$K(u, v) = \Omega\left(\frac{u}{|u|}, \frac{v}{|v|}\right) |u|^{-n} |v|^{-m} \quad (1.1)$$

where $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(u, \cdot) d\sigma(u) = 0 \quad (1.2)$$

and

$$\int_{\mathbf{S}^{m-1}} \Omega(\cdot, v) d\sigma(v) = 0 \quad (1.3)$$

with $d\sigma(\cdot)$ being the normalized Lebesgue measure on the unit sphere.

Define the singular integral operator T_c and the corresponding maximal truncated singular integral operator T_c^* by

$$(T_c f)(x, y) = \text{p.v.} \int \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x - u, y - v) K(u, v) du dv \quad (1.4)$$

and

$$(T_c^* f)(x, y) = \sup_{\varepsilon_1, \varepsilon_2 > 0} \left| \int_{|v| \geq \varepsilon_2} \int_{|u| \geq \varepsilon_1} f(x - u, y - v) K(u, v) du dv \right|. \quad (1.5)$$

The systematic study of singular integrals on product domains began in R. Fefferman [Fe] and R. Fefferman-E. Stein [FS]. It follows from [FS] (see also [St1], p.52) that the operators T_c and T_c^* are bounded on $L^p(\mathbf{R}^{n+m})$ for $1 < p < \infty$ if Ω satisfies certain Lipschitz conditions. Subsequently, the L^p ($1 < p < \infty$) boundedness of T_c was established under weaker conditions on Ω , first in Duoandikoetxea [Du1] for $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ with $q > 1$ and then in Fan-Guo-Pan [FGP] for $\Omega \in \bigcup_{q>1} B_q^{0,1}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ which contains $\bigcup_{q>1} L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ as a proper subspace, where $B_q^{0,1}$ represents a block space (for $p = 2$, it was proved by Jiang and Lu in [JL]). In the meantime, the question regarding the L^p boundedness of T_c^* under these weaker conditions has remained unanswered, as pointed out in [Du2].

One of the main purposes of this paper is to resolve this issue by establishing the L^p boundedness of T_c^* under the condition $\Omega \in B_q^{0,1}$ for some $q > 1$. As in the classical theory of singular integrals, our results implies that the truncated integrals

$$\int_{\varepsilon_2 \leq |v| \leq N_2} \int_{\varepsilon_1 \leq |u| \leq N_1} f(x - u, y - v) K(u, v) du dv \quad (1.6)$$

converge almost everywhere as $\varepsilon_1, \varepsilon_2 \rightarrow 0^+$ and $N_1, N_2 \rightarrow \infty$ for every $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ with $1 < p < \infty$.

The scope of our investigation extends well beyond the operators given in (1.4) and (1.5). To give a full statement of our results, we let $N, M \in \mathbf{N}$, $\mathcal{P} = (P_1, \dots, P_N)$ and $\mathcal{Q} = (Q_1, \dots, Q_M)$ where $P_1, \dots, P_N$ and $Q_1, \dots, Q_M$ are real-valued polynomials on $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. For a measurable function h on $\mathbf{R}^+ \times \mathbf{R}^+$ and an Ω satisfying (1.2)–(1.3), we define the operators $T = T_{\mathcal{P}, \mathcal{Q}, \Omega, h}$ and $T^* = T_{\mathcal{P}, \mathcal{Q}, \Omega, h}^*$ on $\mathbf{R}^N \times \mathbf{R}^M$ by

$$\begin{aligned}
& (Tf)(x, y) \\
&= \text{p.v.} \iint_{\mathbf{R}^n \times \mathbf{R}^m} f(x - \mathcal{P}(u), y - \mathcal{Q}(v)) \frac{\Omega\left(\frac{u}{|u|}, \frac{v}{|v|}\right)}{|u|^n |v|^m} h(|u|, |v|) du dv
\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}
& (T^*f)(x, y) \\
&= \sup_{\varepsilon_1, \varepsilon_2 > 0} \left| \int_{|v| \geq \varepsilon_2} \int_{|u| \geq \varepsilon_1} f(x - \mathcal{P}(u), y - \mathcal{Q}(v)) \frac{\Omega\left(\frac{u}{|u|}, \frac{v}{|v|}\right)}{|u|^n |v|^m} h(|u|, |v|) du dv \right|,
\end{aligned} \tag{1.8}$$

where $x \in \mathbf{R}^N$ and $y \in \mathbf{R}^M$. We have the following result concerning T .

Theorem A *Let T be given as in (1.7). Suppose that $\Omega \in B_q^{0,1}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$ and h satisfies*

$$\sup_{R_2, R_1 > 0} \left[\frac{1}{R_1 R_2} \int_0^{R_2} \int_0^{R_1} |h(t, s)|^\gamma dt ds \right]^{\frac{1}{\gamma}} < \infty \tag{1.9}$$

for some $\gamma > 1$.

(i) *For $\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$ there exists a constant $C_p > 0$ such that*

$$\|T(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \tag{1.10}$$

for any $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

(ii) *In addition, if $\mathcal{P}(-u) = -\mathcal{P}(u)$ and $\mathcal{Q}(-v) = -\mathcal{Q}(v)$, then (1.10) holds for all $p \in (1, \infty)$.*

The constant C_p may depend on the degrees of $P_1, \dots, P_N$ and $Q_1, \dots, Q_M$, but it is independent of their coefficients.

Notice that when $\gamma \geq 2$ the range of p given by $\left|\frac{1}{p} - \frac{1}{2}\right| < \min\left\{\frac{1}{2}, \frac{1}{\gamma'}\right\}$ in Theorem A (i) is the entire interval $(1, \infty)$. However, for the class of odd polynomials Theorem A (ii) allows p to be in the full range $(1, \infty)$ even when $1 < \gamma < 2$. Previous results on T , including those in [FS], [Du1], [JL] and [FGP], dealt with the case where $(N, M) = (n, m)$ and $(\mathcal{P}(u), \mathcal{Q}(v)) =$

(u, v) . The appearance of the function h in the kernel was also allowed in [Du1] ($\gamma \geq 2$), [JL] ($\gamma = \infty$), and [FGP] ($\gamma \geq 2$).

For T^* we have the following:

Theorem B *Let T^* be given by (1.8). Suppose that $\Omega \in B_q^{0,1}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$ and $h(t, s) = h_1(t)h_2(s)$ with $h_j(\cdot) \in L^\infty(\mathbf{R}^+)$ for $j = 1, 2$. Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that*

$$\|T^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (1.11)$$

for any $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. The constant C_p may depend on the degrees of $P_1, \dots, P_N$ and $Q_1, \dots, Q_M$, but it is independent of their coefficients.

Additional results on singular integrals and related maximal operators are also obtained. Our method is based on a combination of ideas from [DR], [Du1], [FP], and [FGP] and it also leads to solutions to other problems in this area which will appear in forthcoming papers. In Section 2, we shall recall the definition of the block spaces $B_q^{0,1}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ which were introduced in [JL] and can be traced back to [MTW] and [LTW]. In Sections 3 and 4 we shall establish some preliminary estimates for maximal functions and oscillatory integrals. The proofs of Theorems A, B and additional results will be given in Sections 5–7.

2. Classes of functions generated by blocks on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$

The block spaces originated in the work of M.H. Taibleson and G. Weiss on the convergence of the Fourier series (see [TW]) in connection with the developments of the real Hardy spaces. Below we shall recall the definition of block spaces on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$. For further background information about the theory of spaces generated by blocks and its applications to harmonic analysis, see [LTW], [MTW] and [So].

Definition 1 For $1 < q \leq \infty$ we say that a measurable function $b(x, y)$ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ is a q -block if it satisfies the following:

- (i) $\text{supp}(b) \subseteq I$ where I is an interval on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, i.e.,

$$I = \{x' \in \mathbf{S}^{n-1} : |x' - x'_0| < \alpha\} \times \{y' \in \mathbf{S}^{m-1} : |y' - y'_0| < \beta\}$$

for some $\alpha, \beta > 0$, $x'_0 \in \mathbf{S}^{n-1}$ and $y'_0 \in \mathbf{S}^{m-1}$.

- (ii) $\|b\|_{L^q} \leq |I|^{-\frac{1}{q'}}$ where $\frac{1}{q} + \frac{1}{q'} = 1$.

Jiang and Lu introduced in [JL] the special class of block spaces $B_q^{\kappa,v}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (for $\kappa \geq 0$ and $v > -1$) with respect to the study of singular integral operators on product domains.

Definition 2 $B_q^{\kappa,v} = B_q^{\kappa,v}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) = \{\sum_{\mu=1}^{\infty} c_{\mu} b_{\mu} : c_{\mu} \in \mathbf{C}; \text{ each } b_{\mu} \text{ is a } q\text{-block supported in an interval } I_{\mu}; \text{ and } M_q^{\kappa,v}(\{c_{\mu}\}) < \infty\}$, where

$$M_q^{\kappa,v}(\{c_{\mu}\}) = \sum_{\mu=1}^{\infty} |c_{\mu}| (1 + \phi_{\kappa,v}(|I_{\mu}|)) \quad \text{with}$$

$$\phi_{\kappa,v}(t) = \chi_{(0,1)}(t) \int_t^1 u^{-1-\kappa} \log^v(u^{-1}) du.$$

One observes that

$$\phi_{\kappa,v}(t) \sim t^{-\kappa} \log^v(t^{-1}) \quad \text{as } t \rightarrow 0 \quad \text{for } \kappa > 0, v \in \mathbf{R},$$

and

$$\phi_{0,v}(t) \sim \log^{v+1}(t^{-1}) \quad \text{as } t \rightarrow 0 \quad \text{for } v > -1.$$

The following properties of $B_q^{\kappa,v}$ can be found in [JL]:

$$B_q^{\kappa,v_2} \subset B_q^{\kappa,v_1} \quad \text{if } v_2 > v_1 > -1 \quad \text{and } \kappa \geq 0; \quad (2.1)$$

$$B_q^{\kappa_2,v_2} \subset B_q^{\kappa_1,v_1} \quad \text{if } v_1, v_2 > -1 \quad \text{and } 0 \leq \kappa_1 < \kappa_2; \quad (2.2)$$

$$B_{q_2}^{\kappa,v} \subset B_{q_1}^{\kappa,v} \quad \text{if } 1 < q_1 < q_2; \quad (2.3)$$

and

$$L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subset B_q^{\kappa,v}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subset L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \quad (2.4)$$

for $v > -1$ and $\kappa \geq 0$. Also, by following the ideas of Keitoku and Sato in [KS] one can prove the following result which clarifies the relationship between the spaces $B_q^{\kappa,v}$ and the L^q -space on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$.

Theorem 1 (i) If $1 < p \leq q \leq \infty$, then for $\kappa > \frac{1}{p'}$ we have

$$B_q^{\kappa,v}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subseteq L^p(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \quad \text{for any } v > -1; \quad (2.5)$$

$$(ii) \quad B_q^{\kappa, v}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) = L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$$

$$\text{if and only if } \kappa \geq \frac{1}{q'} \text{ and } v \geq 0; \quad (2.6)$$

(iii) for any $v > -1$, we have

$$\bigcup_{q>1} L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subset \bigcup_{q>1} B_q^{0, v}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}), \quad (2.7)$$

and the above inclusion is proper.

3. Maximal functions

Let $O_{k,j} = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m : 2^k \leq |x| < 2^{k+1} \text{ and } 2^j \leq |y| < 2^{j+1}\}$. For suitable mappings $\Gamma : \mathbf{R}^n \rightarrow \mathbf{R}^N$, $\Lambda : \mathbf{R}^m \rightarrow \mathbf{R}^M$ and $\vartheta : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \rightarrow \mathbf{R}$, we define the measures $\{\sigma_{\Gamma, \Lambda, \vartheta, k, j} : k, j \in \mathbf{Z}\}$ and the maximal operator $\sigma_{\Gamma, \Lambda, \vartheta}^*$ on $\mathbf{R}^N \times \mathbf{R}^M$ by

$$\int_{\mathbf{R}^N \times \mathbf{R}^M} f d\sigma_{\Gamma, \Lambda, \vartheta, k, j} = \int_{O_{k,j}} f(\Gamma(u), \Lambda(v)) h(|u|, |v|) \frac{\vartheta(u', v')}{|u|^n |v|^m} du dv, \quad (3.1)$$

and

$$\sigma_{\Gamma, \Lambda, \vartheta}^*(f) = \sup_{k, j \in \mathbf{Z}} |\sigma_{\Gamma, \Lambda, \vartheta, k, j} * f|. \quad (3.2)$$

For $l \in \mathbf{N}$, let $\mathcal{A}_l$ denote the class of polynomials of l variables with real coefficients. For $d \in \mathbf{N}$ and $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_d) \in (\mathcal{A}_1)^d$ define the maximal function $\mathcal{M}_{\mathcal{R}}f$ on $\mathbf{R}^d$ by

$$\mathcal{M}_{\mathcal{R}}f(x) = \sup_{r>0} \frac{1}{r} \int_{-r}^r |f(x - \mathcal{R}(t))| dt.$$

We shall need the following result which can be found in [St2], pp. 476–478.

Lemma 2 For $1 < p \leq \infty$ there exists a positive constant C_p such that

$$\|\mathcal{M}_{\mathcal{R}}f\|_p \leq C_p \|f\|_p$$

for $f \in L^p(\mathbf{R}^d)$. The constant C_p may depend on the degrees of the polynomials $\mathcal{R}_1, \dots, \mathcal{R}_d$, but it is independent of their coefficients.

For $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_N) \in (\mathcal{A}_1)^N$, $\mathfrak{R} = (\mathfrak{R}_1, \dots, \mathfrak{R}_M) \in (\mathcal{A}_1)^M$ and a measurable function $h : [0, \infty) \times [0, \infty) \rightarrow \mathbf{R}$ we let

$$\begin{aligned} M_{\mathcal{L}, \mathfrak{R}, h} f(x, y) \\ = \sup_{R_1, R_2 > 0} \frac{1}{R_1 R_2} \int_{-R_2}^{R_2} \int_{-R_1}^{R_1} |f(x - \mathcal{L}(t), y - \mathfrak{R}(r))| |h(t, r)| dt dr. \end{aligned}$$

When $h \equiv 1$, we shall denote $M_{\mathcal{L}, \mathfrak{R}, 1}$ by $M_{\mathcal{L}, \mathfrak{R}}$. We also define two operators $\mathcal{M}_{\mathcal{L}}^1$ and $\mathcal{M}_{\mathfrak{R}}^2$ on $\mathbf{R}^N \times \mathbf{R}^M$ by $(\mathcal{M}_{\mathcal{L}}^1 f)(x, y) = (\mathcal{M}_{\mathcal{L}} f(\cdot, y))(x)$ and $(\mathcal{M}_{\mathfrak{R}}^2 f)(x, y) = (\mathcal{M}_{\mathfrak{R}} f(x, \cdot))(y)$.

Lemma 3 *For every $1 < p \leq \infty$, there exists a positive constant $C_p = C(p, N, M)$ such that*

$$\|M_{\mathcal{L}, \mathfrak{R}} f\|_p \leq C_p \|f\|_p$$

for $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. The constant C_p may depend on the degrees of the polynomials $\mathcal{L}_1, \dots, \mathcal{L}_N$ and $\mathfrak{R}_1, \dots, \mathfrak{R}_M$, but it is independent of their coefficients.

The proof of this lemma follows easily by using Lemma 2 and the inequality $M_{\mathcal{L}, \mathfrak{R}} f(x, y) \leq \mathcal{M}_{\mathfrak{R}}^2 \circ \mathcal{M}_{\mathcal{L}}^1 f(x, y)$ where “o” denotes the composition of operators.

For $\gamma > 1$, let $\Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ denote the set of all functions on $\mathbf{R}^+ \times \mathbf{R}^+$ satisfying (1.9). By using Lemma 3 and Hölder’s inequality we have the following.

Lemma 4 *Suppose that $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $\gamma > 1$. Then for $\gamma' < p \leq \infty$, there exists a constant $C_p = C(p, N, M, h)$ such that*

$$\|M_{\mathcal{L}, \mathfrak{R}, h} f\|_p \leq C_p \|f\|_p$$

for $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. The constant C_p may depend on the degrees of the polynomials $\{\mathcal{L}_l\}$ and $\{\mathfrak{R}_s\}$, but it is independent of the coefficients of $\{\mathcal{L}_l\}$ and $\{\mathfrak{R}_s\}$.

Theorem 5 *Let $\vartheta \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $\gamma > 1$. For $(x, y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, $t, r \in \mathbf{R}$, and two mappings $\Gamma : \mathbf{R}^n \rightarrow \mathbf{R}^N$ and $\Lambda : \mathbf{R}^m \rightarrow \mathbf{R}^M$, let $\mathcal{L}_x(t) = \Gamma(tx)$ and $\mathfrak{R}_y(r) = \Lambda(ry)$. If $\mathcal{L}_x(\cdot), \mathfrak{R}_y(\cdot) \in \mathcal{A}_1$ for every $x \in \mathbf{S}^{n-1}$ and $y \in \mathbf{S}^{m-1}$, $\mathcal{C}(\Gamma) = \sup\{\deg(\mathcal{L}_x) :$*

$x \in \mathbf{S}^{n-1}\} < \infty$ and $\mathcal{D}(\Lambda) = \sup\{\deg(\mathfrak{R}_y) : y \in \mathbf{S}^{m-1}\} < \infty$, then for $\gamma' < p \leq \infty$ there exists a constant $C_p = C(p, h, n, m, N, M, \mathcal{C}(\Gamma), \mathcal{D}(\Lambda))$ such that

$$\|\sigma_{\Gamma, \Lambda, \vartheta}^*(f)\|_p \leq C_p \|\vartheta\|_{L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \|f\|_p$$

for $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

The proof of this theorem follows easily by using Lemma 4, the inequality

$$\sigma_{\Gamma, \Lambda, \vartheta}^* f(u, v) \leq 4 \int_{\mathbf{S}^{m-1}} \int_{\mathbf{S}^{n-1}} M_{\mathcal{L}_x, \mathfrak{R}_y, h}(|f|)(u, v) |\vartheta(x, y)| d\sigma(x) d\sigma(y),$$

and the assumption on the uniform boundedness of $\{\deg(\mathcal{L}_x), \deg(\mathfrak{R}_y) : (x, y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\}$.

Theorem 6 Let $\vartheta \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $\gamma \in (1, 2]$. Let $\Gamma : \mathbf{R}^n \rightarrow \mathbf{R}^N$ and $\Lambda : \mathbf{R}^m \rightarrow \mathbf{R}^M$ be given as in Theorem 5. Then for $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{\gamma'}$ there exists a positive constant $C_p = C(p, h, n, m, N, M, \mathcal{C}(\Gamma), \mathcal{D}(\Lambda))$ such that the vector-valued inequality

$$\begin{aligned} & \left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{\Gamma, \Lambda, \vartheta, k, j} * g_{k, j}|^2 \right)^{\frac{1}{2}} \right\|_p \\ & \leq C_p \|\vartheta\|_{L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k, j}|^2 \right)^{\frac{1}{2}} \right\|_p \end{aligned} \quad (3.3)$$

holds for arbitrary functions $\{g_{k, j}\}$ on $\mathbf{R}^N \times \mathbf{R}^M$.

The proof of this theorem follows by using Theorem 5 and arguments similar to the proof of Theorem 7.5 (in the one-parameter setting) in ([FP], p.824). We omit the details.

4. Oscillatory estimates

For $n, d \in \mathbf{N}$ we let $\mathcal{V}_d^n$ denote the space of real-valued homogeneous polynomials of degree d on $\mathbf{R}^n$ and $Z_d^n : \mathcal{V}_d^n \rightarrow \mathcal{V}_d^n$ be the linear transformation defined as in [FP], p.807. For $P(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$ we let $\|P\| = \sum_{|\alpha|=d} |a_\alpha|$. Let $\psi \in C_0^\infty(\mathbf{R})$, $\lambda > 0$ and $F : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}$ be a function

given by

$$F(t, x) = t^d P(x) + W_1(t, x) + W_2(t) \quad (4.1)$$

where $P \in \mathcal{V}_d^n$, $W_1(\cdot, \cdot)$ is a function on $\mathbf{R} \times \mathbf{R}^n$ satisfying

$$\frac{\partial^d W_1(t, x)}{\partial t^d} \equiv 0$$

and $W_2(\cdot)$ is an arbitrary function on $\mathbf{R}$. Define the operator $S_\lambda = S_{\lambda, F}$ by

$$S_\lambda(g)(t) = \psi(t) \int_{\mathbf{S}^{n-1}} e^{i\lambda F(t, x)} g(x) d\sigma(x). \quad (4.2)$$

The following two results follow immediately from Corollaries 4.3 and 4.4 in [FP].

Lemma 7 *Let S_λ be given by (4.1)–(4.2). Then for any $p \in (1, 2]$ there exists a constant $C_{p,d} > 0$ such that*

$$\|S_\lambda(g)\|_{L^{p'}(\mathbf{R})} \leq C_{p,d} \lambda^{-\frac{1}{4dp'}} \|Z_d^n(P)\|^{-\frac{1}{4dp'}} \|g\|_{L^p(\mathbf{S}^{n-1})}. \quad (4.3)$$

Lemma 8 *Let U be a subspace of $\mathcal{V}_d^n$ satisfying $|x|^d \notin U$. Suppose S_λ is given by (4.1)–(4.2) with $P(\cdot) \in U$. Then for any $p \in (1, 2]$ there exists a constant $C_{p,d} > 0$ such that*

$$\|S_\lambda(g)\|_{L^{p'}(\mathbf{R})} \leq C_{p,d} \lambda^{-\frac{1}{4dp'}} \|P\|^{-\frac{1}{4dp'}} \|g\|_{L^p(\mathbf{S}^{n-1})}. \quad (4.4)$$

The constant $C_{p,d}$ may depend on U if d is even, but it is independent of U if d is odd.

Proposition 9 *Let $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$, $\tilde{\Omega} \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $\gamma, q > 1$ and set $\omega = \min\{2, \gamma, q\}$. Suppose $F: \mathbf{R}^n \rightarrow \mathbf{R}$ and $G: \mathbf{R}^m \rightarrow \mathbf{R}$ are functions given by*

$$F(x) = \sum_{\nu=0}^l P_\nu(x) + W_1(|x|) \quad \text{and} \quad G(y) = \sum_{d=0}^s Q_d(y) + W_2(|y|)$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are arbitrary functions on $\mathbf{R}$, $P_\nu \in \mathcal{V}_\nu^n$ for $0 \leq \nu \leq l$, and $Q_d \in \mathcal{V}_d^m$ for $0 \leq d \leq s$. Let

$$J_{k,j}(\tilde{\Omega}) = \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{m-1}} e^{i(F(tx)+G(ry))} \tilde{\Omega}(x, y) h(t, r) d\sigma(y) d\sigma(x) \frac{dt}{t} \frac{dr}{r}$$

for $k, j \in \mathbf{Z}$. Then we have

$$\begin{aligned} & \left| J_{k,j}(\tilde{\Omega}) \right| \\ & \leq C \|\tilde{\Omega}\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} (2^{kl} \|Z_l^n(P_l)\|)^{-\frac{1}{8l\omega'}} (2^{js} \|Z_s^m(Q_s)\|)^{-\frac{1}{8s\omega'}}. \end{aligned} \quad (4.5)$$

The constant C is independent of $\tilde{\Omega}(\cdot, \cdot)$, k , j , $W_1(\cdot)$, $W_2(\cdot)$ and the coefficients of $\{P_\nu(\cdot)\}$ and $\{Q_d(\cdot)\}$.

Furthermore, if $U^{(n,1)}$ is a subspace of $\mathcal{V}_l^n$ and $U^{(m,2)}$ is a subspace $\mathcal{V}_s^m$ with $|x|^l \notin U^{(n,1)}$ and $|y|^s \notin U^{(m,2)}$, then there exists a constant C' such that

$$\left| J_{k,j}(\tilde{\Omega}) \right| \leq C' \|\tilde{\Omega}\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} (2^{kl} \|P_l\|)^{-\frac{1}{8l\omega'}} (2^{js} \|Q_s\|)^{-\frac{1}{8s\omega'}} \quad (4.6)$$

holds for all $k, j \in \mathbf{Z}$ and F and G with $P_l(\cdot) \in U^{(n,1)}$ and $Q_s(\cdot) \in U^{(m,2)}$. The constant C' may depend on the subspaces $U^{(n,1)}$ or $U^{(m,2)}$ if either l or s is even, but it is independent of these subspaces if both l and s are odd.

Proof. By Hölder's inequality we get

$$\begin{aligned} \left| J_{k,j}(\tilde{\Omega}) \right| & \leq \int_{\mathbf{S}^{m-1}} \left(\frac{1}{2^k 2^j} \int_0^{2^{j+1}} \int_0^{2^{k+1}} |h(t, r)|^\gamma dt dr \right)^{\frac{1}{\gamma}} \\ & \quad \times \left(\int_1^2 \int_1^2 \left| \int_{\mathbf{S}^{n-1}} e^{iF(2^k tx)} \tilde{\Omega}(x, y) d\sigma(x) \right|^{\gamma'} dt dr \right)^{\frac{1}{\gamma'}} d\sigma(y) \\ & \leq C \int_{\mathbf{S}^{m-1}} \left(\int_1^2 \left| \int_{\mathbf{S}^{n-1}} e^{iF(2^k tx)} \tilde{\Omega}(x, y) d\sigma(x) \right|^{\omega'} dt \right)^{\frac{1}{\omega'}} d\sigma(y). \end{aligned}$$

Then by using Lemma 7 and Hölder's inequality we get

$$\left| J_{k,j}(\tilde{\Omega}) \right| \leq C (2^{kl} \|Z_l^n(P_l)\|)^{-\frac{1}{4l\omega'}} \int_{\mathbf{S}^{m-1}} \left\| \tilde{\Omega}(\cdot, y) \right\|_{L^\omega(\mathbf{S}^{n-1})} d\sigma(y)$$

and hence

$$\left| J_{k,j}(\tilde{\Omega}) \right| \leq C(2^{kl} \|Z_l^n(P_l)\|)^{-\frac{1}{4l\omega'}} \|\tilde{\Omega}\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})}. \quad (4.7)$$

Similarly, we have

$$\left| J_{k,j}(\tilde{\Omega}) \right| \leq C(2^{js} \|Z_s^m(Q_s)\|)^{-\frac{1}{4s\omega'}} \|\tilde{\Omega}\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})}. \quad (4.8)$$

Putting together the above estimates in (4.7)–(4.8) gives the desired estimate in (4.5). By using Lemma 8 and the argument employed in the proof of (4.5) we get (4.6). This ends the proof of our proposition. $\square$

Let $q > 1$ and b be a q -block function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ supported in an interval I with $\|b\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq |I|^{-\frac{1}{q}}$. To such function b we associate a function $\tilde{b}$ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ defined by

$$\begin{aligned} \tilde{b}(x, y) = & b(x, y) - \int_{\mathbf{S}^{n-1}} b(u, y) d\sigma(u) - \int_{\mathbf{S}^{m-1}} b(x, v) d\sigma(v) \\ & + \int_{\mathbf{S}^{m-1}} \int_{\mathbf{S}^{n-1}} b(u, v) d\sigma(u) d\sigma(v). \end{aligned} \quad (4.9)$$

It is easy to see that the function $\tilde{b}$ enjoys the following properties:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}(u, \cdot) d\sigma(u) = \int_{\mathbf{S}^{m-1}} \tilde{b}(\cdot, v) d\sigma(v) = 0; \quad (4.10)$$

$$\|\tilde{b}\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq 4|I|^{-\frac{1}{q}}; \quad (4.11)$$

and

$$\|\tilde{b}\|_{L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq 4. \quad (4.12)$$

We shall call the function $\tilde{b}$ the q -blocklike function corresponding to the block function b . Occasionally we shall also call the function $\tilde{b}$ the q -blocklike function associated to I .

It is worthwhile to point out that the function $\tilde{b}$ enjoys the advantage, over b , of having the mean zero property (4.10) which is indispensable in many parts of our proofs.

We have the following oscillatory estimates related to blocklike functions:

Proposition 10 *Let $\tilde{b}$ be a q -blocklike function associated with an interval I in $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ and $J_{k,j} = J_{k,j}(\tilde{b})$ be given as in Proposition 9. Then if $|I| < e^{-2}$, we have*

$$|J_{k,j}| \leq C(2^{kl} \|Z_l^n(P_l)\|)^{\frac{1}{8l\omega' \log |I|}} (2^{js} \|Z_s^m(Q_s)\|)^{\frac{1}{8s\omega' \log |I|}} \quad (4.13)$$

and if $|I| \geq e^{-2}$, we have

$$|J_{k,j}| \leq C(2^{kl} \|Z_l^n(P_l)\|)^{-\frac{1}{8l\omega'}} (2^{js} \|Z_s^m(Q_s)\|)^{-\frac{1}{8s\omega'}}. \quad (4.14)$$

The constant C is independent of $\tilde{b}(\cdot, \cdot)$, k, j , $W_1(\cdot)$, $W_2(\cdot)$ and the coefficients of $\{P_\nu(\cdot)\}$ and $\{Q_d(\cdot)\}$.

Proof. By Proposition 9 and (4.11) we have

$$|J_{k,j}| \leq C|I|^{-\frac{1}{q'}} (2^{kl} \|Z_l^n(P_l)\|)^{-\frac{1}{8l\omega'}} (2^{js} \|Z_s^m(Q_s)\|)^{-\frac{1}{8s\omega'}}. \quad (4.15)$$

Also, by (4.12) we have

$$|J_{k,j}| \leq C \|\tilde{b}\|_{L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \left[\sup_{R_1, R_2 > 0} \frac{1}{R_1 R_2} \int_0^{R_2} \int_0^{R_1} |h(t, r)| dt dr \right] \leq C.$$

Now, if $|I| < e^{-2}$, then (4.13) follows by interpolating between the two preceding estimates of $|J_{k,j}|$. On the other hand, if $|I| \geq e^{-2}$, (4.14) follows easily from (4.15). $\square$

5. Boundedness of singular integrals

5.1. General results

We shall start off with an extension of a result due to J. Duoandikoetxea ([Du1], Theorem 1).

Theorem 11 *Let $\{\sigma_{k,j} : k, j \in \mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$. Let $L : \mathbf{R}^n \rightarrow \mathbf{R}^{d_1}$ and $Q : \mathbf{R}^m \rightarrow \mathbf{R}^{d_2}$ be linear transformations. Suppose that for some $a, b \in \mathbf{R}^+ \setminus \{1\}$, $\alpha, \beta, C > 0$, $p_0 \in (2, \infty)$, and $B > 1$, the following hold for $k, j \in \mathbf{Z}$, $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$ and arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$:*

$$(i) \quad |\hat{\sigma}_{k,j}(\xi, \eta)| \leq C(a^k |L(\xi)|)^{\pm \frac{\alpha}{B}} (b^j |Q(\eta)|)^{\pm \frac{\beta}{B}};$$

(ii)

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq C \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0}. \quad (5.1)$$

Then for $p'_0 < p < p_0$ there exists a positive constant C_p such that

$$\left\| \sum_{k,j \in \mathbf{Z}} \sigma_{k,j} * f \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (5.2)$$

and

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (5.3)$$

hold for all f in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p is independent of B and the linear transformations L and Q .

Proof. Without loss of generality, we may assume that $a > 1$ and $b > 1$. Let θ be a fixed positive integer. For $1 \leq u \leq \theta$ we define the projection operator $\pi_u^\theta : \mathbf{R}^\theta \rightarrow \mathbf{R}^u$ by $\pi_u^\theta(\xi) = (\xi_1, \dots, \xi_u)$. By an argument similar to the one in the proof of Lemma 6.2 in [FP], we may assume that $d_1 \leq n$, $d_2 \leq m$, $L = \pi_{d_1}^n$ and $Q = \pi_{d_2}^m$. The rest now follows by using the proof of Theorem 1 in [Du1] with appropriate modifications. Details will be omitted.

The following theorem will be used in the proof of Theorem A (i) and it may be useful in some other situations as well. $\square$

Theorem 12 Let $M, N \in \mathbf{N}$ and $\{\sigma_{k,j}^{(l,s)} : k, j \in \mathbf{Z}, 0 \leq l \leq N, 0 \leq s \leq M\}$ be a family of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$ with $\sigma_{k,j}^{(l,0)} = 0$ and $\sigma_{k,j}^{(0,s)} = 0$ for $k, j \in \mathbf{Z}$. Let $\{a_l, b_s : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{R}^+ \setminus \{1\}$, $\{B(l), D(s) : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{N}$, $\{\alpha_l, \beta_s : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{R}^+$, and let $L_l : \mathbf{R}^n \rightarrow \mathbf{R}^{B(l)}$ and $Q_s : \mathbf{R}^m \rightarrow \mathbf{R}^{D(s)}$ be linear transformations for $1 \leq l \leq N, 1 \leq s \leq M$. Suppose that for some $C > 0$, $p_0 \in (2, \infty)$ and $B > 1$, the following hold for $k, j \in \mathbf{Z}, 1 \leq l \leq N, 1 \leq s \leq M, (\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$ and arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$:

$$(i) \quad \|\sigma_{k,j}^{(l,s)}\| \leq C;$$

$$(ii) \quad |\hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta)| \leq C |a_l^k L_l(\xi)|^{-\frac{\alpha_l}{B}} |b_s^j Q_s(\eta)|^{-\frac{\beta_s}{B}};$$

$$\begin{aligned}
\text{(iii)} \quad & \left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) \right| \leq C |a_l^k L_l(\xi)|^{\frac{\alpha_l}{B}} \left| b_s^j Q_s(\eta) \right|^{-\frac{\beta_s}{B}}; \\
\text{(iv)} \quad & \left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) \right| \leq C |a_l^k L_l(\xi)|^{-\frac{\alpha_l}{B}} \left| b_s^j Q_s(\eta) \right|^{\frac{\beta_s}{B}}; \\
\text{(v)} \quad & \left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) + \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\
& \leq C |a_l^k L_l(\xi)|^{\frac{\alpha_l}{B}} \left| b_s^j Q_s(\eta) \right|^{\frac{\beta_s}{B}}; \\
\text{(vi)} \quad & \left| \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq C |a_l^k L_l(\xi)|^{\frac{\alpha_l}{B}}; \\
\text{(vii)} \quad & \left| \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq C \left| b_s^j Q_s(\eta) \right|^{\frac{\beta_s}{B}}; \\
\text{(viii)} \quad & \left\| \left(\sum_{k,j \in \mathbf{Z}} \left| \sigma_{k,j}^{(l,s)} * g_{k,j} \right|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq C \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0}. \quad (5.4)
\end{aligned}$$

Then for $p'_0 < p < p_0$ there exists a positive constant C_p such that

$$\left\| \sum_{k,j \in \mathbf{Z}} \sigma_{k,j}^{(N,M)} * f \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (5.5)$$

and

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} \left| \sigma_{k,j}^{(N,M)} * f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (5.6)$$

hold for all f in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p is independent of the linear transformations $\{L_l\}_{l=1}^N$ and $\{Q_s\}_{s=1}^M$.

Proof. We shall use some of the ideas employed in the proof of Theorem 7.6 in [FP]. As in the proof of Theorem 11, we may assume without any loss of generality that $0 < \alpha_l, \beta_s \leq 1$, $B(l) \leq n$, $D(s) \leq m$, $L_l = \pi_{B(l)}^n$, $Q_s = \pi_{D(s)}^m$ for each $1 \leq l \leq N$ and $1 \leq s \leq M$. Choose and fix a Schwartz function $\phi \in \mathcal{S}(\mathbf{R})$ such that $(\hat{\phi})(t) \equiv 1$ if $|t| \leq \frac{1}{2}$ and $(\hat{\phi})(t) \equiv 0$ if $|t| \geq 1$. For $1 \leq l \leq N$ and $1 \leq s \leq M$ define $\{\varphi_k^{(l)}\}$ on $\mathbf{R}^{B(l)}$ and $\{\psi_j^{(s)}\}$ on $\mathbf{R}^{D(s)}$ by

$$(\varphi_k^{(l)})^\wedge(w) = (\hat{\phi})(|a_l^k w|^2) \quad \text{and} \quad (\psi_j^{(s)})^\wedge(z) = (\hat{\phi})(|b_s^j z|^2).$$

Define the family of measures $\{\lambda_{k,j}^{(l,s)} : k, j \in \mathbf{Z}, 1 \leq l \leq N, 1 \leq s \leq M\}$ by

$$\begin{aligned} \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) &= \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) \prod_{l < i \leq N} (\varphi_k^{(i)})^\wedge(\pi_{B(i)}^n \xi) \prod_{s < d \leq M} (\psi_j^{(d)})^\wedge(\pi_{D(d)}^m \eta) \\ &\quad - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) \prod_{l-1 < i \leq N} (\varphi_k^{(i)})^\wedge(\pi_{B(i)}^n \xi) \prod_{s < d \leq M} (\psi_j^{(d)})^\wedge(\pi_{D(d)}^m \eta) \\ &\quad - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) \prod_{l < i \leq N} (\varphi_k^{(i)})^\wedge(\pi_{B(i)}^n \xi) \prod_{s-1 < d \leq M} (\psi_j^{(d)})^\wedge(\pi_{D(d)}^m \eta) \\ &\quad + \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \prod_{l-1 < i \leq N} (\varphi_k^{(i)})^\wedge(\pi_{B(i)}^n \xi) \\ &\quad \times \prod_{s-1 < d \leq M} (\psi_j^{(d)})^\wedge(\pi_{D(d)}^m \eta) \end{aligned} \quad (5.7)$$

with the convention that $\prod_{i \in \emptyset} E_i = 1$ where $\emptyset$ is the empty set.

We now show that

$$\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| \leq C(a_l^k |\pi_{B(N)}^n \xi|)^{\pm \frac{\alpha_l}{2B}} (b_s^j |\pi_{D(M)}^m \eta|)^{\pm \frac{\beta_s}{2B}} \quad (5.8)$$

for $1 \leq l \leq N$ and $1 \leq s \leq M$.

First, if $|a_l^k \pi_{B(l)}^n \xi| \geq 1$ and $|b_s^j \pi_{D(s)}^m \eta| \geq 1$ then by conditions (i) and (ii) we get

$$\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| \leq C(a_l^k |\pi_{B(l)}^n \xi|)^{-\frac{\alpha_l}{2B}} (b_s^j |\pi_{D(s)}^m \eta|)^{-\frac{\beta_s}{2B}}. \quad (5.9)$$

Next, if $|a_l^k \pi_{B(l)}^n \xi| < 1$ and $|b_s^j \pi_{D(s)}^m \eta| \geq 1$ we have

$$\begin{aligned} \left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| &\leq \left\{ \left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) \right| \prod_{l-1 < i \leq N} (\varphi_k^{(i)})^\wedge(\pi_{B(i)}^n \xi) \right. \\ &\quad \left. + \left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) \right| \left| 1 - (\varphi_k^{(l)})^\wedge(\pi_{B(l)}^n \xi) \right| \prod_{l < i \leq N} (\varphi_k^{(i)})^\wedge(\pi_{B(i)}^n \xi) \right\} \\ &\quad \times \prod_{s < d \leq M} (\psi_j^{(d)})^\wedge(\pi_{D(d)}^m \eta). \end{aligned} \quad (5.10)$$

By condition (i) we get

$$\left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) \right| \left| 1 - (\varphi_k^{(l)})^\wedge(\pi_{B(l)}^n \xi) \right| \leq C(a_l^k |\pi_{B(l)}^n \xi|)^{\frac{\alpha_l}{B}}. \quad (5.11)$$

On the other hand, by condition (ii) we have

$$\begin{aligned} & \left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) \right| \left| 1 - (\varphi_k^{(l)})^\wedge(\pi_{B(l)}^n \xi) \right| \\ & \leq C(a_l^k |\pi_{B(l)}^n \xi|)^{\frac{\alpha_l}{B}} (a_l^k |\pi_{B(l)}^n \xi|)^{-\frac{\alpha_l}{B}} (b_s^j |\pi_{D(s)}^m \eta|)^{-\frac{\beta_s}{B}} \end{aligned} \quad (5.12)$$

and hence

$$\left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) \right| \left| 1 - (\varphi_k^{(l)})^\wedge(\pi_{B(l)}^n \xi) \right| \leq C(b_s^j |\pi_{D(s)}^m \eta|)^{-\frac{\beta_s}{B}}. \quad (5.13)$$

By combining (5.11) and (5.13) we get

$$\left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) \right| \left| 1 - (\varphi_k^{(l)})^\wedge(\pi_{B(l)}^n \xi) \right| \leq C(a_l^k |\pi_{B(l)}^n \xi|)^{\frac{\alpha_l}{2B}} (b_s^j |\pi_{D(s)}^m \eta|)^{-\frac{\beta_s}{2B}}. \quad (5.14)$$

By (5.10), (5.14) and conditions (i) & (iii) we get

$$\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| \leq C(a_l^k |\pi_{B(l)}^n \xi|)^{\frac{\alpha_l}{2B}} (b_s^j |\pi_{D(s)}^m \eta|)^{-\frac{\beta_s}{2B}}. \quad (5.15)$$

Similarly, if $|a_l^k \pi_{B(l)}^n \xi| \geq 1$ and $|b_s^j \pi_{D(s)}^m \eta| < 1$ we obtain by conditions (i), (ii) and (iv)

$$\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| \leq C(a_l^k |\pi_{B(l)}^n \xi|)^{-\frac{\alpha_l}{2B}} (b_s^j |\pi_{D(s)}^m \eta|)^{\frac{\beta_s}{2B}}. \quad (5.16)$$

Finally, if $|a_l^k \pi_{B(l)}^n \xi| < 1$ and $|b_s^j \pi_{D(s)}^m \eta| < 1$ we get

$$\begin{aligned} & \left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| \\ & \leq \left\{ \left| \hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) + \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \right. \\ & \quad + \left| 1 - (\varphi_k^{(l)})^\wedge(\pi_{B(l)}^n \xi) \right| \left| \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\ & \quad + \left| 1 - (\psi_j^{(s)})^\wedge(\pi_{D(s)}^m \eta) \right| \left| \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\ & \quad + \left| 1 - (\varphi_k^{(l)})^\wedge(\pi_{B(l)}^n \xi) \right| \left| 1 - (\psi_j^{(s)})^\wedge(\pi_{D(s)}^m \eta) \right| \left| \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta) \right| \Big\} \\ & \quad \times \left| \prod_{l < i \leq N} (\varphi_k^{(i)})^\wedge(\pi_{B(i)}^n \xi) \prod_{s < d \leq M} (\psi_j^{(d)})^\wedge(\pi_{D(d)}^m \eta) \right|. \end{aligned} \quad (5.17)$$

By the conditions (i), (v)–(vii) and the choice of $\varphi_k^{(l)}$ and $\psi_j^{(s)}$ we have

$$\left| \hat{\lambda}_{k,j}^{(l,s)}(\xi, \eta) \right| \leq C(a_l^k |\pi_{B(l)}^n \xi|)^{\frac{\alpha_l}{2B}} (b_s^j |\pi_{D(s)}^m \eta|)^{\frac{\beta_s}{2B}}. \quad (5.18)$$

By (5.9), (5.15)–(5.16) and (5.18) we get (5.8). Now, by (5.4) and (5.7) we have

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\lambda_{k,j}^{(l,s)} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq C \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \quad (5.19)$$

for $1 \leq l \leq N$, $1 \leq s \leq M$ and arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$. By (5.8), (5.19) and Theorem 11 we have

$$\left\| \sum_{k,j \in \mathbf{Z}} \lambda_{k,j}^{(l,s)} * f \right\|_p \leq C_p B^2 \|f\|_p \quad (5.20)$$

and

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\lambda_{k,j}^{(l,s)} * f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p B^2 \|f\|_p \quad (5.21)$$

for all $p \in (p'_0, p_0)$, $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$, $1 \leq l \leq N$, and $1 \leq s \leq M$.

By the definition of $\lambda_{k,j}^{(l,s)}$ and the assumptions that $\sigma_{k,j}^{(l,0)} = 0$ and $\sigma_{k,j}^{(0,s)} = 0$ we have

$$\sigma_{k,j}^{(N,M)} = \sum_{s=1}^M \sum_{l=1}^N \lambda_{k,j}^{(l,s)}. \quad (5.22)$$

By (5.20)–(5.22) we obtain (5.5) and (5.6). This completes the proof of our theorem. $\square$

Remark 1 The conclusion in Theorem 12 remains valid when we replace the sequences $\{a_l^k\}$ and $\{b_s^j\}$ by any arbitrary lacunary sequences $\{a_{k,l}\}$ and $\{b_{j,s}\}$ of positive numbers satisfying $\frac{a_{k+1,l}}{a_{k,l}} \geq \bar{a}_l$ and $\frac{b_{j+1,s}}{b_{j,s}} \geq \bar{b}_s$ for some constants $\bar{a}_l, \bar{b}_s > 1$ and for $1 \leq l \leq N$ and $1 \leq s \leq M$.

5.2. Proof of Theorem A (i)

Let $\mathcal{P}$, $\mathcal{Q}$ be given as in Theorem A, $\deg(\mathcal{P}) = \max\{\deg(P_\nu) : 1 \leq \nu \leq N\}$, and $\deg(\mathcal{Q}) = \max\{\deg(Q_j) : 1 \leq j \leq M\}$. Also, let $0 < n_1 < n_2 < \dots < n_{\tilde{N}} = \deg(\mathcal{P})$, $0 < m_1 < m_2 < \dots < m_{\tilde{M}} = \deg(\mathcal{Q})$ be non-negative

integers, and polynomials $\{P_\nu^l : 1 \leq \nu \leq N, 1 \leq l \leq \tilde{N}\}$ and $\{Q_j^s : 1 \leq j \leq M, 1 \leq s \leq \tilde{M}\}$ such that for $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$,

$$\mathcal{P}(x) = \sum_{l=1}^{\tilde{N}} \mathcal{P}^l(x) + \mathcal{R}^{(n,1)}(|x|) \quad \text{and} \quad \mathcal{Q}(y) = \sum_{s=1}^{\tilde{M}} \mathcal{Q}^s(y) + \mathcal{R}^{(m,2)}(|y|)$$

where

$$\begin{aligned} \mathcal{P}^l(x) &= (P_1^l(x), \dots, P_N^l(x)) \in (\mathcal{V}_{n_l}^n)^N, \\ \mathcal{Q}^s(y) &= (Q_1^s(y), \dots, Q_M^s(y)) \in (\mathcal{V}_{m_s}^m)^M, \\ \mathcal{R}^{(n,1)}(t) &= (R_1^{(n,1)}(t), \dots, R_N^{(n,1)}(t)), \\ \mathcal{R}^{(m,2)}(t) &= (R_1^{(m,2)}(t), \dots, R_M^{(m,2)}(t)) \end{aligned}$$

with $t \in \mathbf{R}$, $Z_{n_l}^n(P_\nu^l) = P_\nu^l$, $Z_{m_s}^m(Q_j^s) = Q_j^s$, and $R_\nu^{(n,1)}, R_j^{(m,2)} \in \mathcal{A}_1$ for $1 \leq \nu \leq N, 1 \leq j \leq M, 1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$.

For $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$, let ρ_l, ϱ_s denote the number of elements of

$$\{\alpha' \in (\mathbf{N} \cup \{0\})^n : |\alpha'| = n_l\} \quad \text{and} \quad \{\beta' \in (\mathbf{N} \cup \{0\})^m : |\beta'| = m_s\},$$

respectively. For $1 \leq i \leq N, 1 \leq l \leq \tilde{N}, 1 \leq j \leq M$, and $1 \leq s \leq \tilde{M}$, write

$$\begin{aligned} \{\alpha' \in (\mathbf{N} \cup \{0\})^n : |\alpha'| = n_l\} &= \{\alpha'(1), \dots, \alpha'(\rho_l)\}, \\ \{\beta' \in (\mathbf{N} \cup \{0\})^m : |\beta'| = m_s\} &= \{\beta'(1), \dots, \beta'(\varrho_s)\}, \end{aligned}$$

$$P_i^l(x) = \sum_{k=1}^{\rho_l} a_{k,i}^l x^{\alpha'(k)}, \quad Q_j^s(y) = \sum_{u=1}^{\varrho_s} b_{u,j}^s y^{\beta'(u)},$$

and define the linear mappings $L_l : \mathbf{R}^N \rightarrow \mathbf{R}^{\rho_l}$ and $G_s : \mathbf{R}^M \rightarrow \mathbf{R}^{\varrho_s}$ by

$$\begin{aligned} L_l(\xi) &= \left(\sum_{i=1}^N a_{1,i}^l \xi_i, \dots, \sum_{i=1}^N a_{\rho_l,i}^l \xi_i \right) \quad \text{and} \\ G_s(\eta) &= \left(\sum_{u=1}^M b_{1,u}^s \eta_u, \dots, \sum_{u=1}^M b_{\varrho_s,u}^s \eta_u \right). \end{aligned} \tag{5.23}$$

Let

$$\begin{aligned}\Gamma_l(x) &= \sum_{i=1}^l \mathcal{P}^i(x) + \mathcal{R}^{(n,1)}(|x|) \quad \text{for } 1 \leq l \leq \tilde{N} \quad \text{and} \\ \Gamma_0(x) &= \mathcal{R}^{(n,1)}(|x|)\end{aligned}\tag{5.24}$$

and

$$\begin{aligned}\Lambda_s(y) &= \sum_{u=1}^s \mathcal{Q}^u(y) + \mathcal{R}^{(m,2)}(|y|) \quad \text{for } 1 \leq s \leq \tilde{M} \quad \text{and} \\ \Lambda_0(y) &= \mathcal{R}^{(m,2)}(|y|).\end{aligned}\tag{5.25}$$

Also, for $\vartheta \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, $0 \leq l \leq \tilde{N}$ and $0 \leq s \leq \tilde{M}$, let

$$\sigma_{\vartheta, k, j}^{(l, s)} = \sigma_{\Gamma_l, \Lambda_s, \vartheta, k, j} \quad \text{and} \quad \sigma_{\vartheta}^{*(l, s)} = \sigma_{\Gamma_l, \Lambda_s, \vartheta}^*.\tag{5.26}$$

We have the following:

Lemma 13 *Let $\gamma, q > 1$, $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ and $b(\cdot, \cdot)$ be a q -block on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ supported on an interval I . Let $\omega = \min\{2, \gamma, q\}$ and $\tilde{b}$ be the blocklike function corresponding to b given in (4.9). Then for $1 \leq l \leq \tilde{N}, 1 \leq s \leq \tilde{M}$ we have*

- (i) $\left\| \sigma_{\tilde{b}, k, j}^{(l, s)} \right\| \leq C;$
- (ii) $\left| \hat{\sigma}_{\tilde{b}, k, j}^{(l, s)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|)^{\frac{1}{8n_l \omega' \log |I|}} (2^{jm_s} |G_s(\eta)|)^{\frac{1}{8m_s \omega' \log |I|}}$
if $|I| < e^{-2}$ and

$$\left| \hat{\sigma}_{\tilde{b}, k, j}^{(l, s)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|)^{-\frac{1}{8n_l \omega'}} (2^{jm_s} |G_s(\eta)|)^{-\frac{1}{8m_s \omega'}} \text{ if } |I| \geq e^{-2};$$
- (iii) $\left| \hat{\sigma}_{\tilde{b}, k, j}^{(l, s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b}, k, j}^{(l-1, s)}(\xi, \eta) \right|$

$$\leq C(2^{kn_l} |L_l(\xi)|)^{-\frac{1}{4n_l \omega' \log |I|}} (2^{jm_s} |G_s(\eta)|)^{\frac{1}{4m_s \omega' \log |I|}}$$

if $|I| < e^{-2}$ and

$$\left| \hat{\sigma}_{\tilde{b}, k, j}^{(l, s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b}, k, j}^{(l-1, s)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|)(2^{jm_s} |G_s(\eta)|)^{-\frac{1}{4m_s \omega'}}$$

if $|I| \geq e^{-2};$

$$(iv) \quad \left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) \right| \\ \leq C(2^{kn_l} |L_l(\xi)|)^{\frac{1}{4n_l\omega' \log |I|}} (2^{jm_s} |G_s(\eta)|)^{-\frac{1}{4m_s\omega' \log |I|}}$$

if $|I| < e^{-2}$ and

$$\left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|)^{-\frac{1}{4n_l\omega'}} (2^{jm_s} |G_s(\eta)|)$$

if $|I| \geq e^{-2}$;

$$(v) \quad \left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|);$$

$$(vi) \quad \left| \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq C(2^{jm_s} |G_s(\eta)|);$$

$$(vii) \quad \left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) + \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\ \leq C(2^{kn_l} |L_l(\xi)|)(2^{jm_s} |G_s(\eta)|)$$

where C is independent of $k, j \in \mathbf{Z}$, $(\xi, \eta) \in \mathbf{R}^N \times \mathbf{R}^M$ and the blocklike function $\tilde{b}(\cdot, \cdot)$.

Proof of Lemma 13. First, the proof of (i) is trivial and (ii) follows directly from (5.23)–(5.25) and Proposition 10. Next,

$$\left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) \right| \\ \leq \int_1^2 \int_{\mathbf{S}^{n-1}} \left| \int_1^2 \int_{\mathbf{S}^{m-1}} \tilde{b}(x, y) h(2^k t, 2^j r) e^{-i\eta \cdot \Lambda_s(2^j r y)} d\sigma(y) \frac{dr}{r} \right| \\ \times \left| e^{-i\xi \cdot \Gamma_l(2^k t x)} - e^{-i\xi \cdot \Gamma_{l-1}(2^k t x)} \right| d\sigma(x) \frac{dt}{t}.$$

The inner integral can be estimated as in the proof of Proposition 9 to yield the bound

$$C \int_1^2 \int_{\mathbf{S}^{n-1}} \left(\int_1^2 |h(2^k t, 2^j r)|^\gamma \frac{dr}{r} \right)^{\frac{1}{\gamma}} \\ \times (2^{jm_s} |G_s(\eta)|)^{-\frac{1}{4m_s\omega'}} \left\| \tilde{b}(x, \cdot) \right\|_{L^q(\mathbf{S}^{m-1})} (2^{kn_l} |L_l(\xi)|) d\sigma(x) \frac{dt}{t} \\ \leq C(2^{kn_l} |L_l(\xi)|)(2^{jm_s} |G_s(\eta)|)^{-\frac{1}{4m_s\omega'}} |I|^{-\frac{1}{q'}}.$$

Then by following the proof of Proposition 10 we get (iii). The estimates (iv), (v) and (vi) can be established similarly. Finally, we have

$$\begin{aligned}
& \left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) + \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\
&= \left| \int_{2^k \leq |x| < 2^{k+1}} (e^{-i\xi \cdot \Gamma_l(x)} - e^{-i\xi \cdot \Gamma_{l-1}(x)}) \int_{2^j \leq |y| < 2^{j+1}} (e^{-i\eta \cdot \Lambda_s(y)} - e^{-i\eta \cdot \Lambda_{s-1}(y)}) \right. \\
&\quad \left. \times h(|x|, |y|) \frac{\tilde{b}(x', y')}{|x|^n |y|^m} dx dy \right| \\
&\leq C \|\tilde{b}\|_{L^1} \left(\sup_{R_1, R_2 > 0} \frac{1}{R_1 R_2} \int_0^{R_1} \int_0^{R_2} |h(t, r)| dt dr \right) (2^{kn_l} |L_l(\xi)|) (2^{jm_s} |G_s(\eta)|)
\end{aligned}$$

which proves (vii). $\square$

Proof of Theorem A (i). Since $\Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+) \subseteq \Delta_2(\mathbf{R}^+ \times \mathbf{R}^+)$ when $\gamma \geq 2$, we may assume that $1 < \gamma \leq 2$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{\gamma'}$. By assumption Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbf{C}$, b_μ is a q -block supported on an interval $I_\mu \subseteq \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ and satisfies $\|b_\mu\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq |I_\mu|^{-\frac{1}{q'}}$. In addition

$$M_q^{0,1}(\{c_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu| \left[1 + \left(\log^+ \frac{1}{|I_\mu|} \right)^2 \right] < \infty. \quad (5.27)$$

For each $\mu \in \mathbf{N}$ let $\tilde{b}_\mu$ be the blocklike function corresponding to b_μ . Then by the cancellation condition of Ω we can decompose Ω into

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu, \quad (5.28)$$

which in turn allows us to decompose our operator T in (1.7) into

$$Tf(x, y) = \sum_{\mu=1}^{\infty} c_\mu T_{\tilde{b}_\mu} f(x, y) \quad (5.29)$$

where

$$\begin{aligned}
& T_{\tilde{b}_\mu} f(x, y) \\
&= \text{p.v.} \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} f(x - \mathcal{P}(u), y - \mathcal{Q}(v)) \frac{\tilde{b}_\mu(u', v')}{|u|^n |v|^m} h(|u|, |v|) du dv.
\end{aligned} \tag{5.30}$$

By (4.12) and Theorem 6 we have

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} \left| \sigma_{\tilde{b}_\mu, k, j}^{(l, s)} * g_{k, j} \right|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k, j}|^2 \right)^{\frac{1}{2}} \right\|_p \tag{5.31}$$

for any p satisfying $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{\gamma'}$ and arbitrary functions $\{g_{k, j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$ with C_p independent of the blocklike functions $\tilde{b}_\mu$, $\mu = 1, 2, \dots, 1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$.

Now by (5.29)–(5.31), Lemma 13 and invoking Theorem 12 we get

$$\begin{aligned}
\|T_{\tilde{b}_\mu} f\|_p &= \left\| \sum_{k, j \in \mathbf{Z}} \sigma_{\tilde{b}_\mu, k, j}^{(N, M)} * f \right\|_p \leq C_p \left[\log \left(\frac{1}{|I_\mu|} \right) \right]^2 \|f\|_p \\
&\quad \text{if } |I_\mu| < e^{-2}, \tag{5.32}
\end{aligned}$$

and

$$\|T_{\tilde{b}_\mu} f\|_p = \left\| \sum_{k, j \in \mathbf{Z}} \sigma_{\tilde{b}_\mu, k, j}^{(N, M)} * f \right\|_p \leq C_p \|f\|_p \quad \text{if } |I_\mu| \geq e^{-2}, \tag{5.33}$$

for any p satisfying $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{\gamma'}$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. Therefore, by (5.27), (5.29) and (5.32)–(5.33) we get (1.10). This completes the proof of Theorem A (i). $\square$

6. Maximal functions and singular integrals defined by a special class of polynomial mappings

Our main purpose in this section is to prove a result that contains Theorem A (ii) as a special case. To prove our result we need to take a somewhat different approach from the one used in the proof of Theorem A (i). For n, m, N, M, l and $s \in \mathbf{N}$, we let

$$\mathcal{F}_{n,N,0} = \mathbf{R}^N \quad \text{and} \\ \mathcal{F}_{n,N,l} = \left\{ (P_1, \dots, P_N) \in (\mathcal{V}_l^n)^N : |x|^l \notin \text{span} \{P_1, \dots, P_N\} \right\}$$

where $\mathcal{V}_l^n$ represent the linear space of real-valued homogeneous polynomials of degree l on $\mathbf{R}^n$. Also, let us define the following classes of polynomial mappings

$$\mathcal{F}(n, N) = \left\{ \sum_{l=0}^d \mathcal{P}^l : d \geq 0, \mathcal{P}^l \in \mathcal{F}_{n,N,l} \quad \text{for } 0 \leq l \leq d \right\}$$

and

$$\mathcal{F}(n, m, N, M) = \mathcal{F}(n, N) \times \mathcal{F}(m, M).$$

In this section we shall consider singular integrals and maximal functions associated with polynomial mappings $(\mathcal{P}, \mathcal{Q})$ which belong to the special class $\mathcal{F}(n, m, N, M)$.

We should point out that $\mathcal{F}_{n,N,l}$ and $\mathcal{F}(n, N)$ were first defined and studied by Fan and Pan in ([FP], p.833) with respect to the study of homogeneous singular integrals of Calderón-Zygmund type.

It is clear $\mathcal{F}_{n,N,l} = (\mathcal{V}_l^n)^N$ if l is odd. Also, notice that if $\mathcal{P} = (P_1, \dots, P_N)$ and $\mathcal{Q} = (Q_1, \dots, Q_M)$ with $P_i \in \mathcal{A}_n$, $Q_j \in \mathcal{A}_m$, $\mathcal{P}(-x) = -\mathcal{P}(x)$ and $\mathcal{Q}(-y) = -\mathcal{Q}(y)$, then $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$.

We shall begin with the following lemma due to J. Duoandikoetxea and Rubio de Francia.

Lemma 14 *Let $\nu_{k,j}$ be a sequence of Borel measures in $\mathbf{R}^n \times \mathbf{R}^m$ and let $\nu^*(f) = \sup_{k,j \in \mathbf{Z}} |\nu_{k,j} * f|$. Suppose that for some $q > 1$ and $A > 0$*

$$\|\nu^*(f)\|_q \leq A\|f\|_q \tag{6.1}$$

holds for every f in $L^q(\mathbf{R}^n \times \mathbf{R}^m)$. Then the following vector-valued inequality

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\nu_{k,j} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq \left(A \sup_{k,j \in \mathbf{Z}} \|\nu_{k,j}\| \right)^{\frac{1}{2}} \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \tag{6.2}$$

holds for $\left| \frac{1}{p_0} - \frac{1}{2} \right| = \frac{1}{2q}$ and arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$.

Remark 2 A proof of Lemma 14 can be found in [Du1]. The constant “ $(A \sup_{k,j \in \mathbf{Z}} \|\nu_{k,j}\|)^{\frac{1}{2}}$ ” in (6.2) was not explicitly displayed there partly because it was not needed in any of the previous applications. However, it will play a significant role in some of the applications given in this section and Section 7.

Theorem 15 Let $L : \mathbf{R}^n \rightarrow \mathbf{R}^{d_1}$ and $Q : \mathbf{R}^m \rightarrow \mathbf{R}^{d_2}$ be linear transformations. Let $\{\sigma_{k,j} : k, j \in \mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$. Suppose that for some $a \geq 2$, $b \geq 2$, $\alpha, \beta, C > 0$, $B > 1$ and $p_0 \in (2, \infty)$ the following hold for $k, j \in \mathbf{Z}$, $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$ and arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$:

- (i) $|\hat{\sigma}_{k,j}(\xi, \eta)| \leq CB^2(a^{kB} |L(\xi)|)^{\pm \frac{\alpha}{B}} (b^{jB} |Q(\eta)|)^{\pm \frac{\beta}{B}}$
- (ii)

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq CB^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0}. \quad (6.3)$$

Then for $p'_0 < p < p_0$ there exists a positive constant C_p such that

$$\left\| \sum_{k,j \in \mathbf{Z}} \sigma_{k,j} * f \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (6.4)$$

and

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (6.5)$$

hold for all f in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p is independent of B and the linear transformations L and Q .

Proof. As in the proof of Theorem 11, we may assume without loss of generality that $d_1 \leq n$, $d_2 \leq m$, $L = \pi_{d_1}^n$ and $Q_s = \pi_{d_2}^m$. The proof can then be completed by using an argument in [Du1] and keeping track of various constants at each step.

As in [AqP], let $\{\psi_k^1\}_{-\infty}^{\infty}$ and $\{\psi_j^2\}_{-\infty}^{\infty}$ be two smooth partition of unities in $(0, \infty)$ adapted to the intervals $[a^{-(k+1)B}, a^{-(k-1)B}]$ and $[b^{-(j+1)B}, b^{-(j-1)B}]$, respectively. To be precise, we require the following:

$$\psi_j^i \in C^\infty, \quad 0 \leq \psi_j^i \leq 1, \quad \sum_j (\psi_j^i(t))^2 = 1,$$

$$\begin{aligned}\operatorname{supp} \psi_k^1 &\subseteq \left\{ t : a^{-(k+1)B} < t < a^{-(k-1)B} \right\}, \\ \operatorname{supp} \psi_j^2 &\subseteq \left\{ t : b^{-(j+1)B} < t < b^{-(j-1)B} \right\},\end{aligned}$$

and

$$\left| \frac{d^s \psi_j^i(t)}{dt^s} \right| \leq \frac{C_s}{t^s}$$

where $i = 1, 2$ and C_s is independent of the lacunary sequences $\{a^{kB}\}$ and $\{b^{jB}\}$.

Let $X(f) = \sum_{k,j \in \mathbf{Z}} \sigma_{k,j} * f$. Define the multiplier operators $S_{k,j}$ in $\mathbf{R}^n \times \mathbf{R}^m$ by

$$(\widehat{S_{k,j}f})(\xi, \eta) = \psi_k^1(|\pi_{d_1}^n \xi|) \psi_j^2(|\pi_{d_2}^m \eta|) \hat{f}(\xi, \eta).$$

Then for $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$ we have

$$\begin{aligned}X(f) &= \sum_{k,j \in \mathbf{Z}} \sigma_{k,j} * \left(\sum_{l,s \in \mathbf{Z}} S_{k+l,j+s} S_{k+l,j+s} f \right) \\ &= \sum_{l,s \in \mathbf{Z}} \sum_{k,j \in \mathbf{Z}} S_{k+l,j+s} (\sigma_{k,j} * S_{k+l,j+s} f) = \sum_{l,s \in \mathbf{Z}} X_{l,s}(f).\end{aligned}$$

Now

$$\begin{aligned}\|X_{l,s}(f)\|_{p_0} &\leq C \left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j} * S_{k+l,j+s} f|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \\ &\leq CB^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |S_{k+l,j+s} f|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq CB^2 \|f\|_{p_0} \quad (6.6)\end{aligned}$$

for all $l, s \in \mathbf{Z}$ and $f \in L^{p_0}(\mathbf{R}^n \times \mathbf{R}^m)$ with C independent of the essential variables. The first and the last inequalities follow from Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [St1], p.96 whereas the second inequality is a consequence of (6.3).

On the other hand, by Plancherel's theorem we have

$$\|X_{l,s}(f)\|_{L^2}^2 \leq \sum_{k,j \in \mathbf{Z}} \int_{\Delta_{k,j,l,s}} |\hat{f}(\xi, \eta)|^2 |\hat{\sigma}_{k,j}(\xi, \eta)|^2 d\xi d\eta$$

where

$$\Delta_{k,j,l,s} = \left\{ (\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m : (a^B)^{-k-l-1} \leq |\pi_{d_1}^n \xi| < (a^B)^{-k-l+1}, \right. \\ \left. (b^B)^{-j-s-1} \leq |\pi_{d_2}^m \eta| < (b^B)^{-j-s+1} \right\}.$$

Then by (i) we get easily

$$\|X_{l,s}(f)\|_{L^2} \leq CB^2 a^{-\alpha|l|} b^{-|s|\beta} \|f\|_{L^2}. \quad (6.7)$$

Now, if $p \in (p'_0, p_0)$, we have $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0}$ for some θ , $0 < \theta \leq 1$. By interpolating between (6.6) and (6.7) we get

$$\begin{aligned} \|X(f)\|_p &\leq \sum_{l,s} \|X_{l,s}(f)\|_p \\ &\leq CB^2 \sum_{l,s} a^{-\alpha\theta|l|} b^{-\beta\theta|s|} \|f\|_p = C_p B^2 \|f\|_p, \end{aligned}$$

which ends the proof of (6.4).

The proof of (6.5) is similar. Alternatively, we can deduce it from the proof of the previous result by observing that for every sequence $\varepsilon_{k,j} = +1$ or -1 the linear operator

$$X_\varepsilon f = \sum_{k,j \in \mathbf{Z}} \varepsilon_{k,j} \sigma_{k,j}^{(l,s)} * f$$

has a bound in L^p independent of the sequence $\{\varepsilon_{k,j}\}$ and this bound is the same as that of $X(f)$. Consequently, the inequality (6.5) follows by the usual argument with Rademacher functions. This concludes the proof of our theorem. $\square$

Theorem 16 Let $M, N \in \mathbf{N}$ and $\{\sigma_{k,j}^{(l,s)} : k, j \in \mathbf{Z}, 0 \leq l \leq N, 0 \leq s \leq M\}$ be a family of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$ with $\sigma_{k,j}^{(l,0)} = 0$ and $\sigma_{k,j}^{(0,s)} = 0$ for every $k, j \in \mathbf{Z}$. Let $\{a_l, b_s : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{R}^+ \setminus (0, 2)$, $\{B(l), D(s) : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{N}$, $\{\alpha_l, \beta_s : 1 \leq l \leq N, 1 \leq s \leq M\} \subseteq \mathbf{R}^+$, and let $L_l : \mathbf{R}^n \rightarrow \mathbf{R}^{B(l)}$ and $Q_s : \mathbf{R}^m \rightarrow \mathbf{R}^{D(s)}$ be linear transformations for $1 \leq l \leq N, 1 \leq s \leq M$. Suppose that for some $C > 0$, $B > 1$, and $p_0 \in (2, \infty)$ the following hold for $k, j \in \mathbf{Z}, 1 \leq l \leq N, 1 \leq s \leq M, (\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$ and arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$:

- (i) $\|\sigma_{k,j}^{(l,s)}\| \leq CB^2;$
- (ii) $|\hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta)| \leq CB^2 |a_l^{kB} L_l(\xi)|^{-\frac{\alpha_l}{B}} |b_s^{jB} Q_s(\eta)|^{-\frac{\beta_s}{B}};$
- (iii) $|\hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta)| \leq CB^2 |a_l^{kB} L_l(\xi)|^{\frac{\alpha_l}{B}} |b_s^{jB} Q_s(\eta)|^{-\frac{\beta_s}{B}};$
- (iv) $|\hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta)| \leq CB^2 |a_l^{kB} L_l(\xi)|^{-\frac{\alpha_l}{B}} |b_s^{jB} Q_s(\eta)|^{\frac{\beta_s}{B}};$
- (v) $|\hat{\sigma}_{k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) + \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta)|$
 $\leq CB^2 |a_l^{kB} L_l(\xi)|^{\frac{\alpha_l}{B}} |b_s^{jB} Q_s(\eta)|^{\frac{\beta_s}{B}};$
- (vi) $|\hat{\sigma}_{k,j}^{(l,s-1)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta)| \leq CB^2 |a_l^{kB} L_l(\xi)|^{\frac{\alpha_l}{B}};$
- (vii) $|\hat{\sigma}_{k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{k,j}^{(l-1,s-1)}(\xi, \eta)| \leq CB^2 |b_s^{jB} Q_s(\eta)|^{\frac{\beta_s}{B}};$
- (viii)

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j}^{(l,s)} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq CB^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0}. \quad (6.8)$$

Then for $p'_0 < p < p_0$ there exists a positive constant C_p such that

$$\left\| \sum_{k,j \in \mathbf{Z}} \sigma_{k,j}^{(N,M)} * f \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (6.9)$$

and

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j}^{(N,M)} * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \leq C_p B^2 \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \quad (6.10)$$

hold for all f in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. The constant C_p is independent of the linear transformations $\{L_l\}_{l=1}^N$ and $\{Q_s\}_{s=1}^M$.

The proof of this theorem follows by using Theorem 15 and an argument (with minor modifications) employed in the proof of Theorem 12. We omit the details.

Let b be a q -block function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ supported in an interval I with $q > 1$ and $\|b\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq |I|^{-\frac{1}{q}}$. Let $\tilde{b}$ be the corresponding blocklike function given by (4.9). Our aim now is to study the L^p bound-

edness of those operators $T_{\tilde{b}}f(\cdot, \cdot)$ associated with $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$ given by

$$T_{\tilde{b}}f(x, y) = \text{p.v.} \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} f(x - \mathcal{P}(u), y - \mathcal{Q}(v)) \frac{\tilde{b}(u', v')}{|u|^n |v|^m} h(|u|, |v|) du dv.$$

Definition 3 For suitable mappings $\Gamma : \mathbf{R}^n \rightarrow \mathbf{R}^N$, $\Lambda : \mathbf{R}^m \rightarrow \mathbf{R}^M$ we define the measures $\{\tau_{\Gamma, \Lambda, \tilde{b}, k, j} : k, j \in \mathbf{Z}\}$ and the maximal operators $\tau_{\Gamma, \Lambda, \tilde{b}}^*$ on $\mathbf{R}^N \times \mathbf{R}^M$ by

$$\int_{\mathbf{R}^N \times \mathbf{R}^M} f d\tau_{\Gamma, \Lambda, \tilde{b}, k, j} = \int_{\tilde{O}_{k, j}} f(\Gamma(u), \Lambda(v)) h(|u|, |v|) \frac{\tilde{b}(u', v')}{|u|^n |v|^m} du dv \quad (6.11)$$

and

$$\tau_{\Gamma, \Lambda, \tilde{b}}^* f(x, y) = \sup_{k, j \in \mathbf{Z}} \left| \tau_{\Gamma, \Lambda, \tilde{b}, k, j} * f(x, y) \right| \quad (6.12)$$

where $\tilde{O}_{k, j} = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m : a^k \leq |x| < a^{k+1}, a^j \leq |y| < a^{j+1}\}$ with $a = 2^{\log(\frac{1}{|I|})}$.

We have the following Fourier transform estimates of the measures $\tau_{\Gamma, \Lambda, \tilde{b}, k, j}$:

Proposition 17 Let $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $\gamma > 1$, $\tilde{b}$ be a q -blocklike function associated with an interval I on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ for some $q > 1$ and $\omega = \min\{2, \gamma, q\}$. Also, let $U^{(n, 1)}$ be a subspace of $\mathcal{V}_l^n$ and $U^{(m, 2)}$ be a subspace of $\mathcal{V}_s^m$ with $|x|^l \notin U^{(n, 1)}$ and $|y|^s \notin U^{(m, 2)}$. Suppose $F : \mathbf{R}^n \rightarrow \mathbf{R}$ and $G : \mathbf{R}^m \rightarrow \mathbf{R}$ are functions given by

$$F(x) = \sum_{\nu=0}^l P_\nu(x) + W_1(|x|) \quad \text{and} \quad G(y) = \sum_{d=0}^s Q_d(y) + W_2(|y|)$$

where $P_\nu(\cdot)$ and $Q_d(\cdot)$ are homogeneous polynomials of degree ν and d , respectively for $0 \leq \nu \leq l$, $0 \leq d \leq s$ and $W_1(\cdot)$ and $W_2(\cdot)$ are arbitrary functions. Let

$$\begin{aligned}\tilde{J}_{k,j} &= \tilde{J}_{k,j}(\tilde{b}) \\ &= \int_{a^k}^{a^{k+1}} \int_{a^j}^{a^{j+1}} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{m-1}} e^{i(F(tx)+G(ry))} \tilde{b}(x,y) h(t,r) d\sigma(y) d\sigma(x) \frac{dr}{r} \frac{dt}{t}\end{aligned}$$

for $k, j \in \mathbf{Z}$. If $|I| < e^{-2}$, $P_l(\cdot) \in U^{(n,1)}$ and $Q_s(\cdot) \in U^{(m,2)}$, then there exists a constant C such that

$$\begin{aligned}|\tilde{J}_{k,j}| &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{lk \log(\frac{1}{|I|})} \|P_l\| \right)^{\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{js \log(\frac{1}{|I|})} \|Q_s\| \right)^{\frac{1}{8s\omega' \log |I|}}\end{aligned}\quad (6.13)$$

holds for all $k, j \in \mathbf{Z}$, F and G . The constant C is independent of $\tilde{b}(\cdot, \cdot)$ and it may depend on the subspaces $U^{(n,1)}$ or $U^{(m,2)}$ if either l or s is even, but it is independent of these subspaces if l and s are odd.

Proof. By the definition of $\tilde{J}_{k,j}$ and Hölder's inequality we get

$$\begin{aligned}|\tilde{J}_{k,j}| &\leq \int_{\mathbf{S}^{m-1}} \left(\int_{a^j}^{a^{j+1}} \int_{a^k}^{a^{k+1}} |h(t,r)|^\gamma \frac{dt}{t} \frac{dr}{r} \right)^{\frac{1}{\gamma}} \\ &\quad \times \left(\int_{a^j}^{a^{j+1}} \int_{a^k}^{a^{k+1}} \left| \int_{\mathbf{S}^{n-1}} e^{iF(tx)} \tilde{b}(x,y) d\sigma(x) \right|^{\gamma'} \frac{dt}{t} \frac{dr}{r} \right)^{\frac{1}{\gamma'}} d\sigma(y).\end{aligned}$$

Since

$$\int_{a^j}^{a^{j+1}} \int_{a^k}^{a^{k+1}} |h(t)|^\gamma \frac{dt}{t} \frac{dr}{r} \leq \sum_{\eta=1}^{[\log(\frac{1}{|I|})]+1} \sum_{\nu=1}^{[\log(\frac{1}{|I|})]+1} \int_{a^j 2^{\eta-1}}^{a^j 2^\eta} \int_{a^k 2^{\nu-1}}^{a^k 2^\nu} |h(t,r)|^\gamma \frac{dt}{t} \frac{dr}{r}$$

we have

$$|\tilde{J}_{k,j}| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^{(1+\frac{1}{\gamma})} \int_{\mathbf{S}^{m-1}} S_k(y) d\sigma(y) \quad (6.14)$$

where $[x]$ denotes the greatest integer not exceeding x and

$$S_k(y) = \left(\int_{a^k}^{a^{k+1}} \left| \int_{\mathbf{S}^{n-1}} e^{iF(tx)} \tilde{b}(x, y) d\sigma(x) \right|^{\gamma'} \frac{dt}{t} \right)^{\frac{1}{\gamma'}}.$$

By rescaling we have

$$|S_k(y)|^{\gamma'} \leq \sum_{\nu=1}^{\lfloor \log(\frac{1}{|I|}) \rfloor + 1} \int_1^2 \left| \int_{\mathbf{S}^{n-1}} e^{iF(a^k 2^{\nu-1} tx)} \tilde{b}(x, y) d\sigma(x) \right|^{\gamma'} dt.$$

Therefore, by using Lemma 8 and Hölder's inequality we get

$$\begin{aligned} |S_k(y)|^{\gamma'} &\leq \sum_{\nu=1}^{\lfloor \log(\frac{1}{|I|}) \rfloor + 1} \left(\int_1^2 \left| \int_{\mathbf{S}^{n-1}} e^{iF(a^k 2^{\nu-1} tx)} \tilde{b}(x, y) d\sigma(x) \right|^{\omega'} dt \right)^{\frac{\gamma'}{\omega'}} \\ &\leq C \log \left(\frac{1}{|I|} \right) (a^{kl} \|P_l\|)^{-\frac{\gamma'}{4l\omega'}} \|\tilde{b}(\cdot, y)\|_{L^q(\mathbf{S}^{n-1})}^{\gamma'}. \end{aligned}$$

Hence, by (6.14) we get

$$|\tilde{J}_{k,j}| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 (a^{kl} \|P_l\|)^{-\frac{1}{4l\omega'}} \int_{\mathbf{S}^{m-1}} \|\tilde{b}(\cdot, y)\|_{L^q(\mathbf{S}^{n-1})} d\sigma(y).$$

Thus by Hölder's inequality and (4.11) we get

$$|\tilde{J}_{k,j}| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 |I|^{-\frac{1}{q'}} (a^{kl} \|P_l\|)^{\frac{-1}{4l\omega'}}.$$

By interpolating between this estimate and the trivial estimate

$$|\tilde{J}_{k,j}| \leq C \left(\log \left(\frac{1}{|I|} \right) \right)^2$$

we obtain

$$|\tilde{J}_{k,j}| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{\log(\frac{1}{|I|})kl} \|P_l\| \right)^{\frac{1}{4l\omega' \log |I|}}. \quad (6.15)$$

Similarly, we have

$$|\tilde{J}_{k,j}| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{\log(\frac{1}{|I|})js} \|Q_s\| \right)^{\frac{1}{4s\omega' \log |I|}}. \quad (6.16)$$

Combining the estimates (6.15) and (6.16) gives (6.13). $\square$

Theorem 18 Let $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $\gamma > 1$ and $\tilde{b}$ be defined as in (4.9) with $|I| < e^{-2}$. Then for every $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$, there exists a constant $C_p = C(p, n, m, N, M, \mathcal{P}, \mathcal{Q}) > 0$ such that

$$\|\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}}^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (6.17)$$

for $1 < p \leq \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. Furthermore, if $\mathcal{P}(-x) = -\mathcal{P}(x)$ and $\mathcal{Q}(-y) = -\mathcal{Q}(y)$, then the constant C_p depends only on $p, n, m, N, M, \deg(\mathcal{P})$ and $\deg(\mathcal{Q})$ and neither on the function $\tilde{b}$ nor on the coefficients of the polynomial components of the mappings $\mathcal{P}$ and $\mathcal{Q}$.

Proof. Without loss of generality we may assume that $\tilde{b} \geq 0$ and $h \geq 0$. We will prove (6.17) by double induction on $\deg(\mathcal{P})$ and $\deg(\mathcal{Q})$. The proof will be divided into three steps.

Step 1. The inequality (6.17) holds if $\deg(\mathcal{P}) = 0$ and $\deg(\mathcal{Q})$ is arbitrary.

Proof of Step 1. By definition of $\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}$ we have

$$\begin{aligned} \tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j} * f(x, y) &= \int_{a^k}^{a^{k+1}} \int_{\mathbf{S}^{n-1}} \int_{a^j}^{a^{j+1}} \int_{\mathbf{S}^{m-1}} f(x - \mathcal{P}(0), y - \mathcal{Q}(rv)) \tilde{b}(u, v) \\ &\quad \times h(t, r) d\sigma(v) \frac{dr}{r} d\sigma(u) \frac{dt}{t}. \end{aligned}$$

Then

$$\begin{aligned} &\left| \tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j} * f(x, y) \right| \\ &\leq C \int_{a^j}^{a^{j+1}} \int_{\mathbf{S}^{m-1}} |f(x - \mathcal{P}(0), y - \mathcal{Q}(rv))| \tilde{b}_0(v) \tilde{h}_0(r) d\sigma(v) \frac{dr}{r} \quad (6.18) \end{aligned}$$

where $\tilde{h}_0(r) = \int_{a^k}^{a^{k+1}} |h(t, r)| \frac{dt}{t}$ and $\tilde{b}_0(v) = \int_{\mathbf{S}^{n-1}} |\tilde{b}(u, v)| d\sigma(u)$. It is to easy to verify that $\tilde{h}_0 \in \Delta_\gamma(\mathbf{R}^+)$ with

$$\left(\int_{a^j}^{a^{j+1}} |\tilde{h}_0(r)|^\gamma \frac{dr}{r} \right)^{\frac{1}{\gamma}} \leq \left[\log \left(\frac{1}{|I|} \right) \right]^{(1+\frac{1}{\gamma})}$$

and $\tilde{b}_0$ is a function on $\mathbf{S}^{m-1}$ with

$$\|\tilde{b}_0\|_{L^q(\mathbf{S}^{m-1})} \leq 4|I|^{-\frac{1}{q'}} \quad \text{and} \quad \|\tilde{b}_0\|_{L^1(\mathbf{S}^{m-1})} \leq 4.$$

Then by following the proof of the L^p boundedness of the corresponding maximal function in the one-parameter setting in ([AqP], Theorem 8.1) we get

$$\|\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}}^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_{p,1} \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)}$$

for $1 < p \leq \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. $\square$

Step 2.

$$\|\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}}^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_{p,2} \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)}$$

holds for $1 < p \leq \infty$, $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$, $\deg(\mathcal{P})$ is arbitrary and $\deg(\mathcal{Q}) = 0$.

Since the proof of this step is similar to the proof of Step 1, we shall omit the details.

Step 3. Assume that (6.17) holds for all $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$ with $\deg(\mathcal{P}) \leq l - 1$ and for arbitrary $\deg(\mathcal{Q})$ together with all $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$ with $\deg(\mathcal{P}) = l$ and $\deg(\mathcal{Q}) \leq s - 1$. Now suppose that $\deg(\mathcal{P}) = l$ and $\deg(\mathcal{Q}) = s$. Then $\mathcal{P}(x) = \Phi_1(x) + \mathcal{R}_1(x)$ and $\mathcal{Q}(y) = \Phi_2(y) + \mathcal{R}_2(y)$ for some nonzero $\Phi_1 \in \mathcal{F}_{n,N,l}$, $\Phi_2 \in \mathcal{F}_{m,M,s}$, $\mathcal{R}_1 \in \mathcal{F}(n, N)$ and $\mathcal{R}_2 \in \mathcal{F}(m, M)$ with $\deg(\mathcal{R}_1) \leq l - 1$ and $\deg(\mathcal{R}_2) \leq s - 1$. Then by the induction hypotheses we have for $|I| < e^{-2}$ the following:

$$\|\tau_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}}^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_{p,3} \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)}, \quad (6.19)$$

$$\|\tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}}^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_{p,4} \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (6.20)$$

and

$$\|\tau_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}}^*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_{p,5} \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (6.21)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$.

Let $N_l = \dim(\mathcal{V}_l^n)$, $M_s = \dim(\mathcal{V}_s^m)$, $L : \mathbf{R}^N \rightarrow \mathbf{R}^{N_l}$ and $H : \mathbf{R}^M \rightarrow \mathbf{R}^{M_s}$ be linear transformations such that $\|(\xi \cdot \Phi_1)(\cdot)\| = |L(\xi)|$ and $\|(\eta \cdot \Phi_2)(\cdot)\| = |H(\eta)|$ for $\xi \in \mathbf{R}^N$ and $\eta \in \mathbf{R}^M$. Then we have the following.

Lemma 19 *Let $\omega = \min\{2, \gamma, q\}$. Then*

$$\|\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}\|, \|\tau_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}\|, \|\tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j}\|, \|\tau_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}, k, j}\| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2; \quad (6.22)$$

$$\begin{aligned} |\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta)| &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{lk \log(\frac{1}{|I|})} |L(\xi)| \right)^{\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{js \log(\frac{1}{|I|})} |H(\eta)| \right)^{\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (6.23)$$

$$\begin{aligned} &|\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta)| \\ &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{lk \log(\frac{1}{|I|})} |L(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{js \log(\frac{1}{|I|})} |H(\eta)| \right)^{\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (6.24)$$

$$\begin{aligned} &|\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta)| \\ &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{lk \log(\frac{1}{|I|})} |L(\xi)| \right)^{\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{js \log(\frac{1}{|I|})} |H(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (6.25)$$

$$\begin{aligned}
& |\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta) + \hat{\tau}_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta)| \\
& \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{lk \log(\frac{1}{|I|})} |L(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}} \\
& \quad \times \left(2^{js \log(\frac{1}{|I|})} |H(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}}; \tag{6.26}
\end{aligned}$$

$$\begin{aligned}
& |\hat{\tau}_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta)| \\
& \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{lk \log(\frac{1}{|I|})} |L(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}}; \tag{6.27}
\end{aligned}$$

and

$$\begin{aligned}
& |\hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta)| \\
& \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{js \log(\frac{1}{|I|})} |H(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}}. \tag{6.28}
\end{aligned}$$

Proof of Lemma 19. The proof of (6.22) is trivial and the proof of (6.23) follows directly by Proposition 17.

Next,

$$\begin{aligned}
& |\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta)| \\
& \leq \int_{\mathbf{S}^{n-1}} \left(\int_1^a \int_1^a \left| \int_{\mathbf{S}^{m-1}} e^{-i\eta \cdot \mathcal{Q}(a^j r y)} \tilde{b}(x, y) d\sigma(y) \right| \right. \\
& \quad \left. \times \left| h(a^k t, a^j r) \right| \left| e^{-i\xi \cdot \mathcal{P}(a^k t x)} - e^{-i\xi \cdot \mathcal{R}_1(a^k t x)} \right| \frac{dt}{t} \frac{dr}{r} \right) d\sigma(x).
\end{aligned}$$

Then by Hölder's inequality we have

$$\begin{aligned}
& |\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta)| \\
& \leq C \left(2^{lk \log(\frac{1}{|I|})} |L(\xi)| \right) \left(\int_1^a \int_1^a \left| h(a^k t, a^j r) \right|^\gamma \frac{dt}{t} \frac{dr}{r} \right)^{\frac{1}{\gamma}} \\
& \quad \times \int_{\mathbf{S}^{n-1}} \left(\int_1^a \int_1^a \left| \int_{\mathbf{S}^{m-1}} e^{-i\eta \cdot \mathcal{Q}(a^j r y)} \tilde{b}(x, y) d\sigma(y) \right|^{\gamma'} \frac{dt}{t} \frac{dr}{r} \right)^{\frac{1}{\gamma'}} d\sigma(x).
\end{aligned}$$

Then by following the proof of Proposition 17 we obtain the bound

$$\begin{aligned} & |\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta)| \\ & \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 |I|^{-\frac{1}{q'}} \left(2^{js \log(\frac{1}{|I|})} |H(\eta)| \right)^{-\frac{1}{8s\omega'}} \left(2^{lk \log(\frac{1}{|I|})} |L(\xi)| \right). \end{aligned} \quad (6.29)$$

By (6.29) and interpolation with the trivial estimate

$$|\hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - \hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta)| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2$$

we get (6.24). Similarly, we can prove (6.25)–(6.28) (see also the proof of Lemma 13). This completes the proof of our lemma. $\square$

As in the proof of Theorem 11 we may assume that $N_l \leq N$, $M_s \leq M$, $L = \pi_{N_l}^N$ and $H = \pi_{M_s}^M$. Let $\Psi^1 \in \mathcal{S}(\mathbf{R}^{N_l})$, and $\Psi^2 \in \mathcal{S}(\mathbf{R}^{M_s})$ be two Schwartz functions such that $(\Psi^i)^\wedge(\zeta_i) = 1$ for $|\zeta_i| \leq \frac{1}{2}$ and $(\Psi^i)^\wedge(\zeta_i) = 0$ for $|\zeta_i| \geq 1$, $i = 1, 2$ and define

$$(\Psi_k^1)^\wedge(x) = (\Psi^1)^\wedge \left(2^{lk \log(\frac{1}{|I|})} x \right) \quad \text{and} \quad (\Psi_j^2)^\wedge(y) = (\Psi^2)^\wedge \left(2^{js \log(\frac{1}{|I|})} y \right).$$

Define the sequence of measures $\{\mathcal{U}_{k,j}\}$ by

$$\begin{aligned} \hat{\mathcal{U}}_{k,j}(\xi, \eta) &= \hat{\tau}_{\mathcal{P}, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) - (\Psi_k^1)^\wedge(\pi_{N_l}^N \xi) \hat{\tau}_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j}(\xi, \eta) \\ &\quad - (\Psi_j^2)^\wedge(\pi_{M_s}^M \eta) \hat{\tau}_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta) + (\Psi_k^1)^\wedge(\pi_{N_l}^N \xi) \\ &\quad \times (\Psi_j^2)^\wedge(\pi_{M_s}^M \eta) \hat{\tau}_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}, k, j}(\xi, \eta). \end{aligned} \quad (6.30)$$

Then by using an argument similar to the one employed in the proof of Theorem 12 we get

$$\begin{aligned} |\hat{\mathcal{U}}_{k,j}(\xi, \eta)| &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{lk \log(\frac{1}{|I|})} |\pi_{N_l}^N \xi| \right)^{\pm \frac{1}{16l\omega' \log |I|}} \\ &\quad \times \left(2^{js \log(\frac{1}{|I|})} |\pi_{M_s}^M \eta| \right)^{\pm \frac{1}{16s\omega' \log |I|}}. \end{aligned} \quad (6.31)$$

Let $g(f)(x, y) = \left(\sum_{k,j \in \mathbf{Z}} |\mathcal{U}_{k,j} * f(x, y)|^2 \right)^{\frac{1}{2}}$ and $\mathcal{U}^*(f) = \sup_{k,j \in \mathbf{Z}} |\mathcal{U}_{k,j}| * f$. Then by (6.30) we have

$$\begin{aligned}
& \tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}}^* f(x, y) \\
& \leq g(f)(x, y) + \sup_{k, j \in \mathbf{Z}} \left| \left((\Psi_k^1 \otimes \delta_{\mathbf{R}^{N-N_l}}) \otimes \delta_{\mathbf{R}^M} \right) * \tau_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j} * f(x, y) \right| \\
& \quad + \sup_{k, j \in \mathbf{Z}} \left| \left(\delta_{\mathbf{R}^N} \otimes (\Psi_j^2 \otimes \delta_{\mathbf{R}^{M-M_s}}) \right) * \tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j} * f(x, y) \right| \\
& \quad + \sup_{k, j \in \mathbf{Z}} \left| \left((\Psi_k^1 \otimes \delta_{\mathbf{R}^{N-N_l}}) \otimes (\Psi_j^2 \otimes \delta_{\mathbf{R}^{M-M_s}}) \right) * \tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j} * f(x, y) \right|.
\end{aligned} \tag{6.32}$$

By letting $\mathcal{M}_{\mathbf{R}^d}$ denote the classical Hardy-Littlewood maximal function on $\mathbf{R}^d$, we obtain

$$\begin{aligned}
& \sup_{k, j \in \mathbf{Z}} \left| \left((\Psi_k^1 \otimes \delta_{\mathbf{R}^{N-N_l}}) \otimes \delta_{\mathbf{R}^M} \right) * \tau_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}, k, j} * f(x, y) \right| \\
& \leq C((\mathcal{M}_{\mathbf{R}^{N_l}} \otimes \text{id}_{\mathbf{R}^{N-N_l}}) \otimes \text{id}_{\mathbf{R}^M})(\tau_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}}^* f(x, y)),
\end{aligned} \tag{6.33}$$

$$\begin{aligned}
& \sup_{k, j \in \mathbf{Z}} \left| \left(\delta_{\mathbf{R}^N} \otimes (\Psi_j^2 \otimes \delta_{\mathbf{R}^{M-M_s}}) \right) * \tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j} * f(x, y) \right| \\
& \leq C(\text{id}_{\mathbf{R}^N} \otimes (\mathcal{M}_{\mathbf{R}^{M_s}} \otimes \text{id}_{\mathbf{R}^{M-M_s}}))(\tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}}^* f(x, y))
\end{aligned} \tag{6.34}$$

and

$$\begin{aligned}
& \sup_{k, j \in \mathbf{Z}} \left| \left((\Psi_k^1 \otimes \delta_{\mathbf{R}^{N-N_l}}) \otimes (\Psi_j^2 \otimes \delta_{\mathbf{R}^{M-M_s}}) \right) * \tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}, k, j} * f(x, y) \right| \\
& \leq C((\mathcal{M}_{\mathbf{R}^{N_l}} \otimes \text{id}_{\mathbf{R}^{N-N_l}}) \otimes (\mathcal{M}_{\mathbf{R}^{M_s}} \otimes \text{id}_{\mathbf{R}^{M-M_s}}))(\tau_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}}^* f(x, y)).
\end{aligned} \tag{6.35}$$

Then by (6.30) and (6.32)–(6.35) we get

$$\begin{aligned}
\mathcal{U}^* f(x, y) & \leq g(f)(x, y) + 2C((\mathcal{M}_{\mathbf{R}^{N_l}} \otimes \text{id}_{\mathbf{R}^{N-N_l}}) \otimes \text{id}_{\mathbf{R}^M})(\tau_{\mathcal{R}_1, \mathcal{Q}, \tilde{b}}^* f(x, y)) \\
& \quad + 2C(\text{id}_{\mathbf{R}^M} \otimes (\mathcal{M}_{\mathbf{R}^{M_s}} \otimes \text{id}_{\mathbf{R}^{M-M_s}}))(\tau_{\mathcal{P}, \mathcal{R}_2, \tilde{b}}^* f(x, y)) \\
& \quad + 2C((\mathcal{M}_{\mathbf{R}^{N_l}} \otimes \text{id}_{\mathbf{R}^{N-N_l}}) \otimes (\mathcal{M}_{\mathbf{R}^{M_s}} \otimes \text{id}_{\mathbf{R}^{M-M_s}}))(\tau_{\mathcal{R}_1, \mathcal{R}_2, \tilde{b}}^* f(x, y)).
\end{aligned} \tag{6.36}$$

By (6.31) and Plancherel's theorem we obtain

$$\|g(f)\|_{L^2} \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^2}. \tag{6.37}$$

Therefore by the L^p boundedness of the Hardy-Littlewood maximal function, (6.19)–(6.21), (6.36)–(6.37) we get

$$\|\mathcal{U}^*(f)\|_{L^2} \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^2} \quad (6.38)$$

with C independent of $\tilde{b}$. At this point, by applying Lemma 14 (for $q = 2$) in conjunction with the fact $\|\mathcal{U}_{k,j}\| \leq C(\log(\frac{1}{|I|}))^2$ we get

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\mathcal{U}_{k,j} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq C_{p_0} \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \quad (6.39)$$

if $\frac{1}{4} = \left| \frac{1}{p_0} - \frac{1}{2} \right|$. By (6.30), (6.39) and Theorem 15 we get

$$\|g(f)\|_{L^p} \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p} \quad \text{for } \frac{4}{3} < p < 4 \quad (6.40)$$

which, when combined with the L^p boundedness of the Hardy-Littlewood maximal function, (6.19)–(6.21) and (6.36), implies that

$$\|\mathcal{U}^*(f)\|_{L^p} \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p} \quad \text{for } \frac{4}{3} < p < 4 \quad (6.41)$$

holds with a positive constant C_p independent of $\tilde{b}$. By repeating the arguments in (6.37) $\rightarrow$ (6.38) $\rightarrow$ (6.39) $\rightarrow$ (6.40) with $(p, q) = (2, 2)$ replaced by $(p, q) = (\frac{4}{3} + \delta, \frac{4}{3} + \delta)$ ($\delta \rightarrow 0^+$) we obtain that

$$\|g(f)\|_{L^p} \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p} \quad \text{for } \frac{8}{7} < p < 8. \quad (6.42)$$

By repeated use of such argument we eventually obtain that

$$\|g(f)\|_{L^p} \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p} \quad \text{for } 1 < p < \infty \quad (6.43)$$

which, when combined with (6.32)–(6.35), implies that

$$\|\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}}^*(f)\|_{L^p} \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p} \quad \text{for } 1 < p < \infty. \quad (6.44)$$

Since

$$\|\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}}^*(f)\|_{L^\infty} \leq C\|f\|_{L^\infty}$$

holds trivially, the proof of this step is now complete.

Finally, if $\mathcal{P}(-x) = -\mathcal{P}(x)$ and $\mathcal{Q}(-y) = -\mathcal{Q}(y)$, then the integers l and s at each step of our double induction are always odd. In Step 1, $\mathcal{P} \equiv 0$ and $\mathcal{Q}$ is a polynomial of odd degree. Thus the constant $C_{p,1}$ depends only on m, p, M, h , and $\deg(\mathcal{Q})$ by the proof of Theorem 7.1 in [AqP]. Similarly, the constant $C_{p,2}$ in Step 2 depends only n, p, N, h and $\deg(\mathcal{P})$. By induction and Proposition 17 the constants $C_{p,3}, C_{p,4}, C_{p,7}$ and the constant C in (6.19)–(6.28) depend only $n, m, p, N, M, h, \deg(\mathcal{P})$ and $\deg(\mathcal{Q})$. Therefore, the same holds for the constant C_p in (6.17). This concludes the proof of our theorem. $\square$

Theorem 20 *Let $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $\gamma > 1$ and $\tilde{b}$ be given by (4.9). Let $T_{\tilde{b}}$ be given as above and suppose that $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$. Then for $1 < p < \infty$, there exists a constant $C_p > 0$ such that*

$$\|T_{\tilde{b}}(f)\|_{L^p} \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p} \quad \text{if } |I| < e^{-2} \quad (6.45)$$

and

$$\|T_{\tilde{b}}(f)\|_{L^p} \leq C_p \|f\|_{L^p} \quad \text{if } |I| \geq e^{-2} \quad (6.46)$$

for every $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. If $\mathcal{P}(-x) = -\mathcal{P}(x)$ and $\mathcal{Q}(-y) = -\mathcal{Q}(y)$, then the constant C_p depends only on $p, n, m, N, M, \deg(\mathcal{P})$ and $\deg(\mathcal{Q})$ and not on the coefficients of the polynomial components of the mappings $\mathcal{P}$ and $\mathcal{Q}$.

Proof. Since $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$, there are integers $0 < n_1 < n_2 < \dots < n_{\tilde{N}} = \deg(\mathcal{P})$, $0 < m_1 < m_2 < \dots < m_{\tilde{M}} = \deg(\mathcal{Q})$, and nonzero $\mathcal{P}^l \in \mathcal{F}_{n, N, n_l}$ and $\mathcal{Q}^s \in \mathcal{F}_{m, M, m_s}$ for $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$ such that $\mathcal{P}(x) = \mathcal{P}(0) + \sum_{l=1}^{\tilde{N}} \mathcal{P}^l(x)$ and $\mathcal{Q}(y) = \mathcal{Q}(0) + \sum_{s=1}^{\tilde{M}} \mathcal{Q}^s(y)$. For $0 \leq l \leq \tilde{N}$ and $0 \leq s \leq \tilde{M}$, we define

$$\Gamma_0(x) = \mathcal{P}(0) \quad \text{and} \quad \Gamma_l(x) = \mathcal{P}(0) + \sum_{i=1}^l \mathcal{P}^i(x) \quad \text{for } 1 \leq l \leq \tilde{N}, \quad (6.47)$$

and

$$\Lambda_0(y) = \mathcal{Q}(0) \quad \text{and} \quad \Lambda_s(y) = \mathcal{Q}(0) + \sum_{u=1}^s \mathcal{Q}^u(y) \quad \text{for } 1 \leq s \leq \tilde{M}. \quad (6.48)$$

Let

$$\tau_{\tilde{b},k,j}^{(l,s)} = \tau_{\Gamma_l, \Lambda_s, \tilde{b},k,j} \quad \text{and} \quad \tau_{l,s,\tilde{b}}^*(f) = \tau_{\Gamma_l, \Lambda_s, \tilde{b}}^*(f) \quad (6.49)$$

for $0 \leq l \leq \tilde{N}$ and $0 \leq s \leq \tilde{M}$.

Then by Proposition 17 and following an argument similar to the one used in the proof of (6.24) we get for $|I| < e^{-2}$ and for suitable linear transformations $L_l : \mathbf{R}^N \rightarrow \mathbf{R}^{\rho_l}$ and $H_s : \mathbf{R}^M \rightarrow \mathbf{R}^{\rho_s}$ the following:

$$\|\tau_{\tilde{b},k,j}^{(l,s)}\| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2; \quad (6.50)$$

$$\begin{aligned} |\hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta)| &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k \log(\frac{1}{|I|})} |L_l(\xi)| \right)^{\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{\log(\frac{1}{|I|}) j m_s} |H(\eta)| \right)^{\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (6.51)$$

$$\begin{aligned} &|\hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta)| \\ &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k \log(\frac{1}{|I|})} |L(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{\log(\frac{1}{|I|}) j m_s} |H(\eta)| \right)^{\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (6.52)$$

$$\begin{aligned} &|\hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta)| \\ &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k \log(\frac{1}{|I|})} |L(\xi)| \right)^{\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{\log(\frac{1}{|I|}) j m_s} |H(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (6.53)$$

$$\begin{aligned}
& \left| \hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) + \hat{\tau}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\
& \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k \log(\frac{1}{|I|})} |L(\xi)| \right)^{\frac{-1}{8l\omega' \log |I|}} \\
& \quad \times \left(2^{\log(\frac{1}{|I|}) j m_s} |H(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}}; \tag{6.54}
\end{aligned}$$

$$\begin{aligned}
& \left| \hat{\tau}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\
& \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k \log(\frac{1}{|I|})} |L(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}}; \tag{6.55}
\end{aligned}$$

and

$$\begin{aligned}
& \left| \hat{\tau}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\
& \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{\log(\frac{1}{|I|}) j m_s} |H(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}} \tag{6.56}
\end{aligned}$$

for all $k, j \in \mathbf{Z}$, $\xi \in \mathbf{R}^N$, $\eta \in \mathbf{R}^M$, $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$.

By Proposition 9 and following the proof of Lemma 13 we obtain, for $|I| \geq e^{-2}$, $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$, the following:

- (i) $\left\| \sigma_{\tilde{b},k,j}^{(l,s)} \right\| \leq C;$
- (ii) $\left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|)^{-\frac{1}{8n_l\omega'}} (2^{jm_s} |G_s(\eta)|)^{-\frac{1}{8m_s\omega'}};$
- (iii) $\left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|)(2^{jm_s} |G_s(\eta)|)^{-\frac{1}{4m_s\omega'}};$
- (iv) $\left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|)^{-\frac{1}{4n_l\omega'}} (2^{jm_s} |G_s(\eta)|);$
- (v) $\left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq C(2^{kn_l} |L_l(\xi)|);$
- (vi) $\left| \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \leq C(2^{jm_s} |G_s(\eta)|);$
- (vii) $\left| \hat{\sigma}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\sigma}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) + \hat{\sigma}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right|$
 $\leq C(2^{kn_l} |L_l(\xi)|)(2^{jm_s} |G_s(\eta)|)$

where C is independent of $k, j \in \mathbf{Z}$, $(\xi, \eta) \in \mathbf{R}^N \times \mathbf{R}^M$ and the blocklike function $\tilde{b}(\cdot, \cdot)$.

By Theorem 18 we have

$$\left\| \tau_{l,s,\tilde{b}}^*(f) \right\|_{L^p} \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \|f\|_{L^p} \quad (6.57)$$

for $|I| < e^{-2}$, $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$, $1 < p \leq \infty$ and $f \in L^p$.

By Proposition 9, (4.11) and following the ideas in the proof of Theorem 18 we get

$$\left\| \sigma_{\Gamma_l, \Lambda_s, \tilde{b}}^*(f) \right\|_{L^p} \leq C_p \|f\|_{L^p} \quad (6.58)$$

for $|I| \geq e^{-2}$, $1 \leq l \leq \tilde{N}$, $1 \leq s \leq \tilde{M}$, $1 < p \leq \infty$ and $f \in L^p$.

By (6.57)–(6.58) and Lemma 14 we get

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} \left| \tau_{k,j,\tilde{b}}^{(l,s)} * g_{k,j} \right|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_p \quad (6.59)$$

and

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} \left| \sigma_{\tilde{b},k,j}^{(l,s)} * g_k \right|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_p \quad (6.60)$$

for $1 < p < \infty$, $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$, and arbitrary functions $\{g_{k,j}\}_{k,j \in \mathbf{Z}}$ on $\mathbf{R}^N \times \mathbf{R}^M$. Then by (6.50)–(6.56), (6.59) and Theorem 16 we get (6.45). On the other hand, by the above estimates (i)–(vii), (6.60) and Theorem 12 we get (6.46). This completes the proof of our theorem. $\square$

Theorem 21 *Let $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ and $\Omega \in B_q^{0,1}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$ and $\gamma > 1$. Suppose that $(\mathcal{P}, \mathcal{Q}) \in \mathcal{F}(n, m, N, M)$. Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that*

$$\|T(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (6.61)$$

and

$$\|\mathcal{M}_{\mathcal{P}, \mathcal{Q}}(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (6.62)$$

for any $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ where

$$\begin{aligned} \mathcal{M}_{\mathcal{P}, \mathcal{Q}} f(x, y) = \sup_{R_1, R_2 > 0} \frac{1}{R_2^m R_1^n} \int_{|v| \leq R_2} \int_{|u| \leq R_1} |f(x - \mathcal{P}(u), y - \mathcal{Q}(v))| \\ \times |\Omega(u', v')| |h(|u|, |v|)| du dv. \end{aligned}$$

If $\mathcal{P}(-x) = -\mathcal{P}(x)$ and $\mathcal{Q}(-y) = -\mathcal{Q}(y)$, then the constant C_p depends only on $p, n, m, N, M, \deg(\mathcal{P})$ and $\deg(\mathcal{Q})$ and not on the coefficients of the polynomial components of the mappings $\mathcal{P}$ and $\mathcal{Q}$.

Proof. By assumption Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_{\mu} b_{\mu}$ where $c_{\mu} \in \mathbf{C}$, b_{μ} is a q -block with support on an interval I_{μ} on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$; $\|b_{\mu}\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq |I_{\mu}|^{-\frac{1}{q}}$ and $M_q^{0,1}(\{c_{\mu}\})$ satisfies (5.27).

By (5.27), (5.29)–(5.30), and (6.45)–(6.46) we get (6.61). On the other hand, by using Theorem 18 we have

$$\|\tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}_{\mu}}^*(f)\|_{L^p} \leq C_p \left[\log \left(\frac{1}{|I_{\mu}|} \right) \right]^2 \|f\|_{L^p} \quad (6.63)$$

for $|I_{\mu}| < e^{-2}$, $1 < p \leq \infty$ and $f \in L^p$.

By Proposition 9, (4.11) and following the proof of (6.17) we get

$$\|\sigma_{\mathcal{P}, \mathcal{Q}, \tilde{b}_{\mu}}^*(f)\|_{L^p} \leq C_p \|f\|_{L^p} \quad (6.64)$$

for $|I_{\mu}| \geq e^{-2}$, $1 < p \leq \infty$ and $f \in L^p$. By (5.28), it is easy to see that

$$\begin{aligned} \mathcal{M}_{\mathcal{P}, \mathcal{Q}} f(x, y) &\leq 4 \sum_{\mu=1}^{\infty} |c_{\mu}| \sigma_{\mathcal{P}, \mathcal{Q}, \tilde{b}_{\mu}}^*(|f|)(x, y) \\ &\leq 4 \sum_{\mu=1, |I_{\mu}| \geq e^{-2}}^{\infty} |c_{\mu}| \sigma_{\mathcal{P}, \mathcal{Q}, \tilde{b}_{\mu}}^*(|f|)(x, y) \\ &\quad + 16 \sum_{\mu=1, |I_{\mu}| < e^{-2}}^{\infty} |c_{\mu}| \tau_{\mathcal{P}, \mathcal{Q}, \tilde{b}_{\mu}}^*(|f|)(x, y). \end{aligned} \quad (6.65)$$

Hence, our inequality (6.62) follows from (6.64)–(6.66). This completes the proof of our theorem. $\square$

7. Maximal truncated singular integrals

A significant step towards proving the existence of singular integrals on product domains as the pointwise limit almost everywhere of the truncated integrals is contained in Theorem 22. This theorem is quite general and similar in spirit to Theorem E in [DR] and Lemma 6.3 in [FP]. Before we state our theorem we need a little more notation. Let $\mathcal{T}_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ be the subspace of $C_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ generated by functions of the form $\phi \otimes \psi$, where $\phi \in C_0^\infty(\mathbf{R}^n)$ and $\psi \in C_0^\infty(\mathbf{R}^m)$. For a given sequence of measures $\{\sigma_{k,j}\}_{k,j \in \mathbf{Z}}$, define $S_{k,j}$ and S_* by

$$S_{k,j}f(x, y) = \sum_{s=j}^{\infty} \sum_{l=k}^{\infty} \sigma_{l,s} * f(x, y)$$

and

$$S_*f(x, y) = \sup_{k,j \in \mathbf{Z}} |S_{k,j}f(x, y)|.$$

Theorem 22 *Let n, m, d_1, d_2 be positive integers. Let $\{\sigma_{k,j} : k, j \in \mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$ and let $L : \mathbf{R}^n \rightarrow \mathbf{R}^{d_1}$ and $Q : \mathbf{R}^m \rightarrow \mathbf{R}^{d_2}$ be linear transformations. Suppose that for all $k, j \in \mathbf{Z}$, $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$, for some $a \geq 2, b \geq 2, B > 1$ and $\alpha, \beta > 0$ we have*

- (i) $\|\sigma_{k,j}\| \leq CB^2$;
- (ii) $|\hat{\sigma}_{k,j}(\xi, \eta)| \leq CB^2(a^{kB} |L(\xi)|)^{\pm \frac{\alpha}{B}} (b^{jB} |Q(\eta)|)^{\pm \frac{\beta}{B}}$.

Suppose that for every $f \in \mathcal{T}_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ and $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$,

$$\begin{aligned} \text{the sums } & \sum_{s=j}^{\infty} \sum_{l=-\infty}^{\infty} \sigma_{l,s} * f(x, y), \quad \sum_{l=-\infty}^{\infty} \sum_{s=j}^{\infty} \sigma_{l,s} * f(x, y), \\ & \sum_{l=k}^{\infty} \sum_{s=-\infty}^{\infty} \sigma_{l,s} * f(x, y) \end{aligned}$$

and $\sum_{s=-\infty}^{\infty} \sum_{l=k}^{\infty} \sigma_{l,s} * f(x, y)$ are convergent with

$$\sum_{s=j}^{\infty} \sum_{l=-\infty}^{\infty} \sigma_{l,s} * f(x, y) = \sum_{l=-\infty}^{\infty} \sum_{s=j}^{\infty} \sigma_{l,s} * f(x, y)$$

and

$$\sum_{s=-\infty}^{\infty} \sum_{l=k}^{\infty} \sigma_{l,s} * f(x, y) = \sum_{l=k}^{\infty} \sum_{s=-\infty}^{\infty} \sigma_{l,s} * f(x, y).$$

Assume also that for $1 < p < \infty$ there exists a constant $\tilde{C}_p > 0$ such that

$$\|\sigma^*(f)\|_p \leq \tilde{C}_p B^2 \|f\|_p \quad (7.1)$$

$$\|S_1(f)\|_p \leq \tilde{C}_p B^2 \|f\|_p \quad (7.2)$$

$$\|S_2(f)\|_p \leq \tilde{C}_p B^2 \|f\|_p \quad (7.3)$$

hold for $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$, where $S_1(f) = \sup_{l \in \mathbf{Z}} |\sum_{s=-\infty}^{\infty} \sigma_{l,s} * f|$, $S_2(f) = \sup_{s \in \mathbf{Z}} |\sum_{l=-\infty}^{\infty} \sigma_{l,s} * f|$ and $\sigma^*(f) = \sup_{l,s \in \mathbf{Z}} |\sigma_{l,s} * f|$. Then for $1 < p < \infty$ there exists a constant $C_p > 0$ which is independent of L and Q such that

$$\|S_* f\|_p \leq C_p B^2 \|f\|_p$$

for $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

Proof. The proof given here is similar to an argument used in the one-parameter setting appearing in [FP]. Without loss of generality we may assume that $d_1 \leq n$, $d_2 \leq m$, $L = \pi_{d_1}^n$ and $Q = \pi_{d_2}^m$. Let

$$Sf(x, y) = \sum_{s=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sigma_{l,s} * f(x, y).$$

Then by condition (i), (7.1) and Lemma 14 we have for $1 < p < \infty$ there exists a constant $C_p > 0$

$$\left\| \left(\sum_{k,j \in \mathbf{Z}} |\sigma_{k,j} * g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p B^2 \left\| \left(\sum_{k,j \in \mathbf{Z}} |g_{k,j}|^2 \right)^{\frac{1}{2}} \right\|_p \quad (7.4)$$

for arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$. Then by (7.4), Theorem 15 and our assumptions we obtain

$$\|S(f)\|_p \leq C_p B^2 \|f\|_p \quad \text{for } 1 < p < \infty \quad \text{and} \quad f \in L^p(\mathbf{R}^n \times \mathbf{R}^m). \quad (7.5)$$

Now, choose and fix $\psi^{(i)} \in \mathcal{S}(\mathbf{R}^{d_i})$ such that $\psi^{(i)}(\xi_1, \dots, \xi_{d_i}) \equiv 1$ when $\xi_1^2 + \dots + \xi_{d_i}^2 < 1$ and $\psi^{(i)}(\xi_1, \dots, \xi_{d_i}) = 0$ for $\xi_1^2 + \dots + \xi_{d_i}^2 > 4$ for $i = 1, 2$. Define $\Psi^{(i)}$, $\Psi_k^{(1)}$ and $\Psi_j^{(2)}$ by $(\Psi^{(i)})^\wedge = \psi^{(i)}$, $\Psi_k^{(1)}(\xi) = \frac{1}{(a^k B)^{d_1}} \Psi^{(1)}(\frac{\xi}{a^k B})$ and $\Psi_j^{(2)}(\eta) = \frac{1}{(b^j B)^{d_2}} \Psi^{(2)}(\frac{\eta}{b^j B})$. Define the family of measures $\{\mathcal{U}_{k,j}^{(l,s)}: k, j \in \mathbf{Z}, l, s \geq 0\}$ as follows:

$$\begin{aligned} \mathcal{U}_{k,j}^{(l,s)} = & ((\delta_{\mathbf{R}^n} - A_k) \otimes (\delta_{\mathbf{R}^m} - B_j)) * \sigma_{k+l,s+j} \\ & - ((\delta_{\mathbf{R}^n} - A_k) \otimes B_j) * \sigma_{k+l,j-s-1} \\ & - (A_k \otimes (\delta_{\mathbf{R}^m} - B_j)) * \sigma_{k-l-1,s+j} \\ & + (A_k \otimes B_j) * \sigma_{k-l-1,j-s-1} \end{aligned} \quad (7.6)$$

where $A_k = \Psi_k^{(1)} \otimes \delta_{\mathbf{R}^{n-d_1}}$ and $B_j = \Psi_j^{(2)} \otimes \delta_{\mathbf{R}^{m-d_2}}$. Then for $f \in \mathcal{T}_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ we have

$$S_{k,j} f = S_{k,j}^1 f - S_{k,j}^2 f + S_{k,j}^3 f + S_{k,j}^4 f \quad (7.7)$$

where

$$\begin{aligned} S_{k,j}^1 f &= \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \mathcal{U}_{k,j}^{(l,s)} * f, \quad S_{k,j}^2 f = (A_k \otimes B_j) * S(f), \\ S_{k,j}^3 f &= (\delta_{\mathbf{R}^n} \otimes B_j) * \sum_{s=-\infty}^{\infty} \sum_{l=k}^{\infty} \sigma_{l,s} * f \quad \text{and} \\ S_{k,j}^4 f &= (A_k \otimes \delta_{\mathbf{R}^m}) * \sum_{s=j}^{\infty} \sum_{l=-\infty}^{\infty} \sigma_{l,s} * f. \end{aligned}$$

Now, by (7.5) we get immediately

$$\left\| \sup_{k,j \in \mathbf{Z}} |S_{k,j}^2 f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.8)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

Next we shall prove that

$$\left\| \sup_{k,j \in \mathbf{Z}} |S_{k,j}^3 f| \right\|_p \leq C_p B^2 \|f\|_p$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. Let $l \geq 0$, $k \in \mathbf{Z}$ and define the

measures $\{\nu_{k,l}\}$ as follows:

$$\nu_{k,l} = ((\delta_{\mathbf{R}^n} - A_k) \otimes \delta_{\mathbf{R}^m}) * \sum_{s=-\infty}^{\infty} \sigma_{l+k,s} - (A_k \otimes \delta_{\mathbf{R}^m}) * \sum_{s=-\infty}^{\infty} \sigma_{k-l-1,s}. \quad (7.9)$$

Then for $f \in \mathcal{T}_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ we have

$$T_{k,1}(f) = T_{k,2}(f) + T_{k,3}(f) \quad (7.10)$$

where

$$T_{k,1}(f) = \sum_{l=k}^{\infty} \sum_{s=-\infty}^{\infty} \sigma_{l,s} * f, \quad T_{k,2}(f) = (A_k \otimes \delta_{\mathbf{R}^m}) * S(f)$$

and

$$T_{k,3}(f) = \sum_{l=0}^{\infty} \nu_{k,l} * f.$$

We get directly from (7.5) that

$$\left\| \sup_{k \in \mathbf{Z}} |T_{k,2}f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.11)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

On the other hand,

$$\sup_{k \in \mathbf{Z}} |T_{k,3}f(x, y)| \leq \sum_{l=0}^{\infty} \sup_{k \in \mathbf{Z}} |\nu_{k,l} * f(x, y)|. \quad (7.12)$$

By the definition of the measures $\nu_{k,l}$ and (7.2) we obtain

$$\left\| \sup_{k \in \mathbf{Z}} |\nu_{k,l} * f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.13)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ with C_p independent of l .

For $l \geq 0$, let $G_l f(x, y) = (\sum_{k \in \mathbf{Z}} |\nu_{k,l} * f(x, y)|^2)^{\frac{1}{2}}$. Then by Plancherel's theorem we have

$$\begin{aligned} & \|G_l f\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m)}^2 \\ &= \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \sum_{k \in \mathbf{Z}} |\hat{\nu}_{k,l}(\xi, \eta)|^2 |\hat{f}(\xi, \eta)|^2 d\xi d\eta \end{aligned}$$

$$\begin{aligned}
&\leq 2 \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \sum_{k \in \mathbf{Z}} \left\{ \left| 1 - (\Psi_k^{(1)})^\wedge(\xi) \right|^2 \left| \sum_{s=-\infty}^{\infty} \hat{\sigma}_{k+l,s}(\xi, \eta) \right|^2 \right\} |\hat{f}(\xi, \eta)|^2 d\xi d\eta \\
&\quad + 2 \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \sum_{k \in \mathbf{Z}} \left\{ \left| (\Psi_k^{(1)})^\wedge(\xi) \right|^2 \left| \sum_{s=-\infty}^{\infty} \hat{\sigma}_{k-l-1,s}(\xi, \eta) \right|^2 \right\} |\hat{f}(\xi, \eta)|^2 d\xi d\eta \\
&= \mathbf{I}_1 + \mathbf{I}_2.
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{I}_1 &\leq CB^4 \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \left(\sum_{|a^{k+l} \pi_{d_1}^n \xi| \geq 1} |a^{(k+l)B} \pi_{d_1}^n \xi|^{-\frac{2\alpha}{B}} \right) \\
&\quad \times \left(\sum_{s \leq M} |b^{sB} \pi_{d_2}^m \eta|^{\frac{\beta}{B}} + \sum_{s \geq M} |b^{sB} \pi_{d_2}^m \eta|^{-\frac{\beta}{B}} \right)^2 |\hat{f}(\xi, \eta)|^2 d\xi d\eta
\end{aligned}$$

where M is chosen so that $|b^{BM} \pi_{d_2}^m \eta| \approx 1$. Then

$$\begin{aligned}
\mathbf{I}_1 &\leq CB^4 \sup_{\xi \neq 0} \left(\sum_{|a^{k+l} \pi_{d_1}^n \xi| \geq 1} |a^{k+l} \pi_{d_1}^n \xi|^{-\frac{2\alpha}{B}} \right) a^{-2l\alpha} \|f\|_2^2 \\
&\leq CB^4 a^{-2l\alpha} \|f\|_2^2.
\end{aligned}$$

Similarly, we have

$$\mathbf{I}_2 \leq CB^4 a^{-2l\alpha} \|f\|_2^2.$$

Therefore,

$$\|G_l f\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m)}^2 \leq CB^4 a^{-2l\alpha} \|f\|_2^2. \quad (7.14)$$

By (7.13)–(7.14) and interpolation we obtain

$$\left\| \sup_{k \in \mathbf{Z}} |\nu_{k,l} * f| \right\|_p \leq C_p B^2 a^{-\alpha(p)l} \|f\|_p \quad (7.15)$$

for $1 < p < \infty$ and some $\alpha(p) > 0$. Hence by (7.12) and (7.15)

$$\left\| \sup_{k \in \mathbf{Z}} |T_{k,3} f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.16)$$

holds for $1 < p < \infty$ and $f \in \mathcal{T}_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$.

By (7.10)–(7.11), (7.16) and our assumptions we get

$$\left\| \sup_{k,j \in \mathbf{Z}} |S_{k,j}^3 f| \right\|_p \leq C_p B^2 \|f\|_p$$

for $1 < p < \infty$ and $f \in \mathcal{T}_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$. Since $\mathcal{T}_0^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ is dense in $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ we have

$$\left\| \sup_{k,j \in \mathbf{Z}} |S_{k,j}^3 f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.17)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. Similarly, we have

$$\left\| \sup_{k,j \in \mathbf{Z}} |S_{k,j}^4 f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.18)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

Finally, we shall prove that

$$\left\| \sup_{k,j \in \mathbf{Z}} |S_{k,j}^1 f| \right\|_p \leq C_p B^2 \|f\|_p$$

holds for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. The proof of this inequality is similar to the proof of (7.16). In fact, we have

$$\sup_{k,j \in \mathbf{Z}} |S_{k,j}^1 f| \leq \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \sup_{k,j \in \mathbf{Z}} |\mathcal{U}_{k,j}^{(l,s)} * f(x, y)|. \quad (7.19)$$

By the definition of $\mathcal{U}_{k,j}^{(l,s)}$ and (7.1) we have

$$\left\| \sup_{k,j \in \mathbf{Z}} |\mathcal{U}_{k,j}^{(l,s)} * f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.20)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ with C_p independent of l and s .

For $l, s \geq 0$, we let $F_{l,s}(f) = (\sum_{k,j \in \mathbf{Z}} |\mathcal{U}_{k,j}^{(l,s)} * f(x, y)|^2)^{\frac{1}{2}}$. Then by condition (ii) we get

$$\begin{aligned} \|F_{l,s}(f)\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m)}^2 &= \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \left(\sum_{k,j \in \mathbf{Z}} |\hat{\mathcal{U}}_{k,j}^{(l,s)}(\xi, \eta)|^2 |\hat{f}(\xi, \eta)|^2 \right) d\xi d\eta \\ &\leq 2(\mathbf{I}^{(1)}(f) + \mathbf{I}^{(2)}(f) + \mathbf{I}^{(3)}(f) + \mathbf{I}^{(4)}(f)) \end{aligned}$$

where

$$\mathbf{I}^{(1)}(f) = \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \sum_{k,j \in \mathbf{Z}} \left\{ |1 - (\Psi_k^{(1)})^\wedge(\xi)|^2 |1 - (\Psi_j^{(2)})^\wedge(\eta)|^2 |\hat{\sigma}_{k+l,s+j}(\xi, \eta)|^2 \right\} |\hat{f}(\xi, \eta)|^2 d\xi d\eta,$$

$$\mathbf{I}^{(2)}(f) = \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \sum_{k,j \in \mathbf{Z}} \left\{ |1 - (\Psi_k^{(1)})^\wedge(\xi)|^2 |(\Psi_j^{(2)})^\wedge(\eta)|^2 |\hat{\sigma}_{k+l,j-s-1}(\xi, \eta)|^2 \right\} |\hat{f}(\xi, \eta)|^2 d\xi d\eta,$$

$$\mathbf{I}^{(3)}(f) = \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \sum_{k,j \in \mathbf{Z}} \left\{ |(\Psi_k^{(1)})^\wedge(\xi)|^2 |1 - (\Psi_j^{(2)})^\wedge(\eta)|^2 |\hat{\sigma}_{k-l-1,s+j}(\xi, \eta)|^2 \right\} |\hat{f}(\xi, \eta)|^2 d\xi d\eta,$$

and

$$\mathbf{I}^{(4)}(f) = \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} \sum_{k,j \in \mathbf{Z}} \left\{ |(\Psi_k^{(1)})^\wedge(\xi)|^2 |(\Psi_j^{(2)})^\wedge(\eta)|^2 |\hat{\sigma}_{k-l-1,j-s-1}(\xi, \eta)|^2 \right\} |\hat{f}(\xi, \eta)|^2 d\xi d\eta.$$

By an argument similar to (7.14) we get easily

$$\|F_{l,s}(f)\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m)}^2 \leq CB^4 a^{-2l\alpha} b^{-2s\beta} \|f\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m)}^2. \quad (7.21)$$

By (7.20)–(7.21) and interpolation we get

$$\left\| \sup_{k,j \in \mathbf{Z}} \left| \mathcal{U}_{k,j}^{(l,s)} * f \right| \right\|_p \leq C_p B^2 a^{-l\alpha(p)} b^{-s\beta(p)} \|f\|_p \quad (7.22)$$

for $1 < p < \infty$, $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ and for some $\alpha(p), \beta(p) > 0$. By (7.19) and (7.22) we get

$$\left\| \sup_{k,j \in \mathbf{Z}} |S_{k,j}^1 f| \right\|_p \leq C_p B^2 \|f\|_p \quad (7.23)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. By (7.7)–(7.8), (7.17)–(7.18), and (7.23) we get our result. $\square$

Proof of Theorem B. We shall adopt the same notations as in Sections 5 and 6. By assumption Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_{\mu} b_{\mu}$ where $c_{\mu} \in \mathbf{C}$, b_{μ} is a q -block with support on an interval I_{μ} on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$; $\|b_{\mu}\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \leq |I_{\mu}|^{-\frac{1}{q}}$ and $M_q^{0,1}(\{c_{\mu}\})$ satisfies (5.27). For each $\mu \in \mathbf{N}$ let $\tilde{b}_{\mu}$ be the blocklike function corresponding to the block function b_{μ} . By (5.28) we have

$$T^*(f) \leq \sum_{\mu \in \mathbf{N}} |c_{\mu}| T_{\tilde{b}_{\mu}}^*(f) \quad (7.24)$$

where $T_{\tilde{b}_{\mu}}^*$ is the maximal truncated singular integral corresponding to the operator $T_{\tilde{b}_{\mu}}$. Thus, in order to prove the L^p boundedness of T^* , it suffices to establish appropriate bounds for $\|T_{\tilde{b}}^*\|_{p,p}$. In the remainder of the proof we shall let $\tilde{b}$ be any one of the $\tilde{b}_{\mu}$'s and $T^* = T_{\tilde{b}}^*$.

By Proposition 17 and following an argument similar to the one used in the proof of (6.24) we get for $|I| < e^{-2}$ and for suitable linear transformations $\tilde{L}_l : \mathbf{R}^N \rightarrow \mathbf{R}^{\rho_l}$ and $\tilde{H}_s : \mathbf{R}^M \rightarrow \mathbf{R}^{\varrho_s}$ the following:

$$\|\tau_{\tilde{b},k,j}^{(l,s)}\| \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2; \quad (7.25)$$

$$\begin{aligned} |\hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta)| &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k (\log \frac{1}{|I|})} |\tilde{L}_l(\xi)| \right)^{\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{(\log \frac{1}{|I|}) j m_s} |\tilde{H}(\eta)| \right)^{\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (7.26)$$

$$\begin{aligned} &|\hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta)| \\ &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k (\log \frac{1}{|I|})} |\tilde{L}(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}} \\ &\quad \times \left(2^{(\log \frac{1}{|I|}) j m_s} |\tilde{H}(\eta)| \right)^{\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (7.27)$$

$$\begin{aligned} &|\hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta)| \\ &\leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k (\log \frac{1}{|I|})} |\tilde{L}(\xi)| \right)^{\frac{1}{8l\omega' \log |I|}} \end{aligned}$$

$$\times \left(2^{(\log \frac{1}{|I|})jm_s} |\tilde{H}(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}}; \quad (7.28)$$

$$\begin{aligned} & \left| \hat{\tau}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) + \hat{\tau}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\ & \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k (\log \frac{1}{|I|})} |\tilde{L}(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}} \\ & \quad \times \left(2^{(\log \frac{1}{|I|})jm_s} |\tilde{H}(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}}; \end{aligned} \quad (7.29)$$

$$\begin{aligned} & \left| \hat{\tau}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\ & \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{n_l k (\log \frac{1}{|I|})} |\tilde{L}(\xi)| \right)^{-\frac{1}{8l\omega' \log |I|}}; \end{aligned} \quad (7.30)$$

and

$$\begin{aligned} & \left| \hat{\tau}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) - \hat{\tau}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \right| \\ & \leq C \left[\log \left(\frac{1}{|I|} \right) \right]^2 \left(2^{(\log \frac{1}{|I|})jm_s} |\tilde{H}(\eta)| \right)^{-\frac{1}{8s\omega' \log |I|}} \end{aligned} \quad (7.31)$$

for all $k, j \in \mathbf{Z}$, $\xi \in \mathbf{R}^N$, $\eta \in \mathbf{R}^M$, $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$. On the other hand, for $|I| \geq e^{-2}$ we will use the Fourier transform estimates of $\sigma_{\tilde{b},k,j}^{(l,s)}$ given in Lemma 13.

Without loss of generality, we may assume $\Gamma_l(0) = 0$ and $\Lambda_s(0) = 0$ for all l and s with $0 \leq l \leq \tilde{N}$ and $0 \leq s \leq \tilde{M}$. By using standard arguments it is easy to verify the following: For $f \in \mathcal{T}_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$ and $(x, y) \in \mathbf{R}^N \times \mathbf{R}^M$ we have

$$\begin{aligned} & \int_{\omega^j \leq |v| < \omega^{j+1}} \lim_{\varepsilon \rightarrow 0} \int_{|u| > \varepsilon} f(x - \Gamma_l(u), y - \Lambda_s(v)) \frac{\tilde{b}(u', v')}{|u|^n |v|^m} h_1(|u|) h_2(|v|) du dv \\ & = \lim_{\varepsilon \rightarrow 0} \int_{\omega^j \leq |v| < \omega^{j+1}} \int_{|u| > \varepsilon} f(x - \Gamma_l(u), y - \Lambda_s(v)) \frac{\tilde{b}(u', v')}{|u|^n |v|^m} h_1(|u|) h_2(|v|) du dv \end{aligned} \quad (7.32)$$

and

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{|v| > \varepsilon} \int_{\omega^k \leq |u| < \omega^{k+1}} f(x - \Gamma_l(u), y - \Lambda_s(v)) \frac{\tilde{b}(u', v')}{|u|^n |v|^m} h_1(|u|) h_2(|v|) du dv \\
&= \int_{\omega^k \leq |u| < \omega^{k+1}} \lim_{\varepsilon \rightarrow 0} \int_{|v| > \varepsilon} f(x - \Gamma_l(u), y - \Lambda_s(v)) \frac{\tilde{b}(u', v')}{|u|^n |v|^m} h_1(|u|) h_2(|v|) du dv
\end{aligned} \tag{7.33}$$

where

$$\omega = 2^{\log \delta} \quad \text{with} \quad \delta = \frac{1}{|I|} \quad \text{if} \quad |I| < e^{-2} \quad \text{and} \quad \delta = e \quad \text{if} \quad |I| \geq e^{-2}. \tag{7.34}$$

In addition, for $f \in \mathcal{T}_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$ and $(x, y) \in \mathbf{R}^N \times \mathbf{R}^M$ the sums

$$\begin{aligned}
& \sum_{d=j}^{\infty} \sum_{i=-\infty}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y), \quad \sum_{i=-\infty}^{\infty} \sum_{d=j}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y), \\
& \sum_{d=-\infty}^{\infty} \sum_{i=k}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y), \quad \text{and} \quad \sum_{i=k}^{\infty} \sum_{d=-\infty}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y)
\end{aligned} \tag{7.35}$$

converge with

$$\sum_{d=j}^{\infty} \sum_{i=-\infty}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y) = \sum_{i=-\infty}^{\infty} \sum_{d=j}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y) \tag{7.36}$$

and

$$\sum_{d=-\infty}^{\infty} \sum_{i=k}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y) = \sum_{i=k}^{\infty} \sum_{d=-\infty}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(l, s)} * f(x, y). \tag{7.37}$$

where

$$\Upsilon_{\tilde{b}, k, j}^{(l, s)} = \tau_{\tilde{b}, k, j}^{(l, s)} \quad \text{if} \quad |I| < e^{-2} \quad \text{and} \quad \Upsilon_{\tilde{b}, k, j}^{(l, s)} = \sigma_{\tilde{b}, k, j}^{(l, s)} \quad \text{if} \quad |I| \geq e^{-2}. \tag{7.38}$$

Since $h(t, s) = h_1(t)h_2(s) \in L^\infty(\mathbf{R}^+ \times \mathbf{R}^+)$ we have by using Theorem 5, (4.12), and Theorem 18

$$\left\| \Upsilon_{\tilde{b}}^{*(l, s)}(f) \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \tag{7.39}$$

for $1 < p \leq \infty$, $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$, $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$ where

$$\Upsilon_{\tilde{b}}^{*(l,s)} f(x, y) = \sup_{k,j \in \mathbf{Z}} \left| \left| \Upsilon_{\tilde{b},k,j}^{(l,s)} \right| * f(x, y) \right|.$$

We shall now verify the L^p boundedness of the maximal operators

$$\sup_{k \in \mathbf{Z}} \left| \sum_{d=-\infty}^{\infty} \Upsilon_{\tilde{b},k,d}^{(l,s)} * f \right| \quad \text{and} \quad \sup_{j \in \mathbf{Z}} \left| \sum_{i=-\infty}^{\infty} \Upsilon_{\tilde{b},i,j}^{(l,s)} * f \right|.$$

For $f \in \mathcal{T}_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$, we have by (7.32)

$$\begin{aligned} \sup_{j \in \mathbf{Z}} \left| \sum_{i=-\infty}^{\infty} \Upsilon_{\tilde{b},i,j}^{(l,s)} * f(x, y) \right| &= \\ \sup_{j \in \mathbf{Z}} \left| \int_{\omega^j \leq |v| < \omega^{j+1}} \text{p.v.} \int_{\mathbf{R}^n} f(x - \Gamma_l(u), y - \Lambda_s(v)) \frac{\tilde{b}(u', v')}{|u|^n |v|^m} h_1(|u|) h_2(|v|) du dv \right| \\ &\leq \int_{\mathbf{S}^{m-1}} \sup_{j \in \mathbf{Z}} \left| \int_{\omega^j}^{\omega^{j+1}} T_{\Gamma_l, v} f(\cdot, y - \Lambda_s(vt))(x) h_2(t) \frac{dt}{t} \right| d\sigma(v) \end{aligned}$$

where

$$T_{\Gamma_l, v} g(x) = \text{p.v.} \int_{\mathbf{R}^n} g(x - \Gamma_l(u)) \frac{\tilde{b}(u', v')}{|u|^n} h_1(|u|) du.$$

Hence

$$\sup_{j \in \mathbf{Z}} \left| \sum_{i=-\infty}^{\infty} \Upsilon_{\tilde{b},i,j}^{(l,s)} * f(x, y) \right| \leq C \int_{\mathbf{S}^{m-1}} \mathcal{M}_{\mathcal{B}_{v,s}}(T_{\Gamma_l, v} f(x, \cdot))(y) d\sigma(v) \quad (7.40)$$

where

$$\mathcal{M}_{\mathcal{B}_{v,s}} g(y) = \sup_{j \in \mathbf{Z}} \left(\int_{\omega^j}^{\omega^{j+1}} |g(y - \mathcal{B}_{v,s}(t))| \frac{dt}{t} \right) \quad \text{with} \quad \mathcal{B}_{v,s}(t) = \Lambda_s(tv).$$

By following the proof of Theorem 1.1 in [AqP] we get that for $1 < p <$

∞ there exist $\theta_p \in (0, 1]$ and $C_p > 0$ such that

$$\begin{aligned} & \|T_{\Gamma_l, v}(g_1)\|_{L^p(\mathbf{R}^N)} \\ & \leq C_p \log(\delta) \|\tilde{b}(\cdot, v')\|_{L^1(S^{n-1})}^{(1-\frac{\theta_p}{\log(\delta)})} \|\tilde{b}(\cdot, v')\|_{L^q(S^{n-1})}^{\frac{\theta_p}{\log(\delta)}} \|g_1\|_{L^p(\mathbf{R}^N)} \end{aligned} \quad (7.41)$$

holds for all $g_1 \in L^p(\mathbf{R}^N)$. Also, by using Lemma 2 one can easily see that

$$\|\mathcal{M}_{\mathcal{B}_{v,s}}(g_2)\|_{L^p(\mathbf{R}^M)} \leq C_p \log(\delta) \|g_2\|_{L^p(\mathbf{R}^M)} \quad (7.42)$$

holds for $1 < p \leq \infty$, and $g_2 \in L^p(\mathbf{R}^M)$ with a positive constant C_p independent of v and $\tilde{b}$. By (7.40)–(7.42), (4.11)–(4.12), Hölder's inequality and the fact that $\mathcal{T}_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$ is dense in $L^p(\mathbf{R}^N \times \mathbf{R}^M)$ we get

$$\left\| \sup_{j \in \mathbf{Z}} \left| \sum_{i=-\infty}^{\infty} \Upsilon_{\tilde{b}, i, j}^{(l, s)} * f \right| \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.43)$$

for $1 < p < \infty$, and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ with $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$.

By (7.33) and an argument similar to the one used the proof of (7.43) we have

$$\left\| \sup_{k \in \mathbf{Z}} \left| \sum_{d=-\infty}^{\infty} \Upsilon_{\tilde{b}, k, d}^{(l, s)} * f \right| \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.44)$$

for $1 < p < \infty$, and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ with $1 \leq l \leq \tilde{N}$ and $1 \leq s \leq \tilde{M}$.

Next we shall prove that

$$\|T_*(f)\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.45)$$

holds for $1 < p < \infty$, and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ where

$$T_* f(x, y) = \sup_{k, j \in \mathbf{Z}} \left| \sum_{d=j}^{\infty} \sum_{i=k}^{\infty} \Upsilon_{\tilde{b}, i, d}^{(\tilde{N}, \tilde{M})} * f(x, y) \right|.$$

We may assume without loss of generality that $\rho_l \leq N$, $\varrho_s \leq M$, $L_l = \pi_{\rho_l}^n$ and $G_s = \pi_{\varrho_s}^m$ for $1 \leq l \leq \tilde{N}$, and $1 \leq s \leq \tilde{M}$. Exactly as in the proof of Theorem 12, choose and fix a Schwartz function $\phi \in \mathcal{S}(\mathbf{R})$ such that $(\hat{\phi})(t) \equiv 1$ if $|t| \leq \frac{1}{2}$ and $(\hat{\phi})(t) \equiv 0$ if $|t| \geq 1$. For each $1 \leq l \leq \tilde{N}$ and

$1 \leq s \leq \tilde{M}$ define $\varphi_k^{(l)}$ on $\mathbf{R}^{\rho_l}$ and $\psi_j^{(s)}$ on $\mathbf{R}^{\varrho_s}$ by

$$(\varphi_k^{(l)})^\wedge(x) = (\phi)^\wedge(|\omega^{kn_l} x|^2), \quad \text{and} \quad (\psi_j^{(s)})^\wedge(y) = (\phi)^\wedge(|\omega^{jm_s} y|^2).$$

Define the family of measures $\{\lambda_{\tilde{b},k,j}^{(l,s)}\}$ by

$$\begin{aligned} \hat{\lambda}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) &= \hat{\Upsilon}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) \prod_{l < i \leq \tilde{N}} (\varphi_k^{(i)})^\wedge(\pi_{\rho_i}^N \xi) \prod_{s < d \leq \tilde{M}} (\psi_j^{(d)})^\wedge(\pi_{\varrho_d}^M \eta) \\ &\quad - \hat{\Upsilon}_{\tilde{b},k,j}^{(l-1,s)}(\xi, \eta) \prod_{l-1 < i \leq \tilde{N}} (\varphi_k^{(i)})^\wedge(\pi_{\rho_i}^N \xi) \prod_{s < d \leq \tilde{M}} (\psi_j^{(d)})^\wedge(\pi_{\varrho_d}^M \eta) \\ &\quad - \hat{\Upsilon}_{\tilde{b},k,j}^{(l,s-1)}(\xi, \eta) \prod_{l < i \leq \tilde{N}} (\varphi_k^{(i)})^\wedge(\pi_{\rho_i}^N \xi) \prod_{s-1 < d \leq \tilde{M}} (\psi_j^{(d)})^\wedge(\pi_{\varrho_d}^M \eta) \\ &\quad + \hat{\Upsilon}_{\tilde{b},k,j}^{(l-1,s-1)}(\xi, \eta) \prod_{l-1 < i \leq \tilde{N}} (\varphi_k^{(i)})^\wedge(\pi_{\rho_i}^N \xi) \\ &\quad \times \prod_{s-1 < d \leq \tilde{M}} (\psi_j^{(d)})^\wedge(\pi_{\varrho_d}^M \eta). \end{aligned} \quad (7.46)$$

By (7.25)–(7.31), Lemma 13 and the argument used in the proof of (5.8) we obtain

$$\begin{aligned} & \left| \hat{\lambda}_{\tilde{b},k,j}^{(l,s)}(\xi, \eta) \right| \\ & \leq C [\log(\delta)]^2 (\omega^{kn_l} |\pi_{\rho_l}^N \xi|)^{\pm \frac{1}{16n_l \omega^l \log(\delta)}} (\omega^{jm_s} |\pi_{\varrho_s}^M \eta|)^{\pm \frac{1}{16m_s \omega^s \log(\delta)}}. \end{aligned} \quad (7.47)$$

By (7.32)–(7.34) and standard arguments it is easy to see that for $f \in \mathcal{T}_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$ and $(x, y) \in \mathbf{R}^N \times \mathbf{R}^M$ the sums

$$\begin{aligned} & \sum_{d=j}^{\infty} \sum_{i=-\infty}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y), \quad \sum_{i=-\infty}^{\infty} \sum_{d=j}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y), \\ & \sum_{d=-\infty}^{\infty} \sum_{i=k}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y), \end{aligned}$$

and

$$\sum_{i=k}^{\infty} \sum_{d=-\infty}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y) \quad (7.48)$$

converge with

$$\sum_{d=j}^{\infty} \sum_{i=-\infty}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y) = \sum_{i=-\infty}^{\infty} \sum_{d=j}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y) \quad (7.49)$$

and

$$\sum_{d=-\infty}^{\infty} \sum_{i=k}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y) = \sum_{i=k}^{\infty} \sum_{d=-\infty}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f(x, y). \quad (7.50)$$

Also, by (7.39), (7.43)–(7.44), (7.46) and the ideas used in their proofs we have

$$\left\| \lambda_{\tilde{b}}^{*(l,s)}(f) \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)}, \quad (7.51)$$

$$\left\| \sup_{j \in \mathbf{Z}} \left| \sum_{i=-\infty}^{\infty} \lambda_{\tilde{b},i,j}^{(l,s)} * f \right| \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.52)$$

and

$$\left\| \sup_{k \in \mathbf{Z}} \left| \sum_{d=-\infty}^{\infty} \lambda_{\tilde{b},k,d}^{(l,s)} * f \right| \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.53)$$

for $1 < p < \infty$, and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$ where

$$\lambda_{\tilde{b}}^{*(l,s)} f(x, y) = \sup_{k,j \in \mathbf{Z}} \left| \lambda_{\tilde{b},k,j}^{(l,s)} * f(x, y) \right|.$$

Therefore, by (7.47)–(7.53) and Theorem 22 we have

$$\left\| \sup_{k,j \in \mathbf{Z}} \left| \sum_{d=j}^{\infty} \sum_{i=k}^{\infty} \lambda_{\tilde{b},i,d}^{(l,s)} * f \right| \right\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.54)$$

for $1 < p < \infty$, and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. By (7.46) and the cancellation

property of $\tilde{b}$ it is easy to see that

$$\Upsilon_{\tilde{b},k,j}^{(\tilde{N},\tilde{M})} = \sum_{s=1}^{\tilde{M}} \sum_{l=1}^{\tilde{N}} \lambda_{\tilde{b},k,j}^{(l,s)}. \quad (7.55)$$

Then by (7.54)–(7.55) we get (7.45).

Now, for any $\varepsilon_1, \varepsilon_2 > 0$, there exist integers $k, j \in \mathbf{Z}$ such that $\omega^k \leq \varepsilon_1 < \omega^{k+1}$ and $\omega^j \leq \varepsilon_2 < \omega^{j+1}$. Hence we have for $f \in \mathcal{T}_0^\infty(\mathbf{R}^N \times \mathbf{R}^M)$

$$T^*f(x, y) \leq T_*f(x, y) + T_1f(x, y) + T_2f(x, y) + \Upsilon_{\tilde{b}}^{*(\tilde{N},\tilde{M})}f(x, y) \quad (7.56)$$

where

$$\begin{aligned} T_1f(x, y) &= \|h_2\|_\infty \sup_{k,j \in \mathbf{Z}} \int_{\omega^j \leq |v| < \omega^{j+1}} \frac{1}{|v|^m} \\ &\quad \times \left| \int_{|u| > \omega^k} f(x - \Gamma_{\tilde{N}}(u), y - \Lambda_{\tilde{M}}(v)) \frac{\tilde{b}(u', v')}{|u|^n} h_1(|u|) du \right| dv \end{aligned}$$

and

$$\begin{aligned} T_2f(x, y) &= \|h_1\|_\infty \sup_{k,j \in \mathbf{Z}} \int_{\omega^k \leq |u| < \omega^{k+1}} \frac{1}{|u|^n} \\ &\quad \times \left| \int_{|v| > \omega^j} f(x - \Gamma_{\tilde{N}}(u), y - \Lambda_{\tilde{M}}(v)) \frac{\tilde{b}(u', v')}{|v|^m} h_2(|v|) dv \right| du. \end{aligned}$$

For T_1 we have

$$T_1f(x, y) \leq C \int_{\mathbf{S}^{m-1}} \mathcal{M}_{\mathcal{J}_v} \left(S_{\Gamma_{\tilde{N}},v}^* f(x, \cdot) \right) (y) d\sigma(v) \quad (7.57)$$

where

$$S_{\Gamma_{\tilde{N}},v}^* g_1(x) = \sup_{k \in \mathbf{Z}} \left| \int_{|u| > \omega^k} g_1(x - \Gamma_{\tilde{N}}(u)) \frac{\tilde{b}(u', v')}{|u|^n} h_1(|u|) du \right|$$

and

$$\mathcal{M}_{\mathcal{J}_v} g_2(y) = \sup_{j \in \mathbf{Z}} \int_{\omega^j}^{\omega^{j+1}} |g_2(y - \mathcal{J}_v(t))| \frac{dt}{t} \quad \text{with } \mathcal{J}_v(t) = \Lambda_{\tilde{M}}(tv).$$

By following the proof of Theorem 1.2 in [AqP] we get that for $1 < p < \infty$ there exist $\theta_p \in (0, 1]$ and $C_p > 0$ such that

$$\begin{aligned} & \|S_{\Gamma_{\tilde{N}}, v}^* g_1\|_{L^p(\mathbf{R}^N)} \\ & \leq C_p \log(\delta) \|\tilde{b}(\cdot, v')\|_{L^1(S^{n-1})}^{(1-\frac{\theta_p}{\log(\delta)})} \|\tilde{b}(\cdot, v')\|_{L^q(S^{n-1})}^{\frac{\theta_p}{\log(\delta)}} \|g_1\|_{L^p(\mathbf{R}^N)} \end{aligned} \quad (7.58)$$

for all $g_1 \in L^p(\mathbf{R}^N)$. By using Lemma 2 and (7.57)–(7.58) we have

$$\|T_1 f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.59)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. Similarly,

$$\|T_2 f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.60)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. Therefore, by (7.39), (7.45) and (7.59)–(7.60) we have

$$\|T^* f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \leq C_p [\log(\delta)]^2 \|f\|_{L^p(\mathbf{R}^N \times \mathbf{R}^M)} \quad (7.61)$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^N \times \mathbf{R}^M)$. This completes the proof of Theorem B. $\square$

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