Finding a homeomorphism between almost homeomorphic manifolds

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§ 1. Introduction

Throughout this paper we shall only be concerned with the piecewise linear category of polyhedra and piecewise linear maps. In this paper we investigate the following problem; Let W_1 and W_2 be two PL manifolds whose interiors and boundaries are PL homeomorphic each other. When are W_1 and W_2 PL homeomorphic?

We obtain the result that such homeomorphism problem is closely related to the h-cobordism near the boundary (see THEOREM 2).

∂M and $Int\ M$ stand for the boundary and the interior of the manifold M. $\cong$ means PL homeomorphic. $I=[0, 1]$ is a closed unit interval. $\# X$ means the order of a set X.

§ 2.

DEFINITION 1. Let W_i $(i=1, 2)$ be bounded manifolds. When $\partial W_1 \cong \partial W_2$ and $Int\ W_1 \cong Int\ W_2$, we say W_1 is *almost homeomorphic* to W_2. And we define $\mathcal{A}(W)$= set of PL homeomorphism classes of PL manifolds which are almost homeomorphic to W.

PROPOSITION 1. ([2. *Th.* 2, 4]) *Let* W_j^n $(j=1, 2)$ *be compact bounded n-manifolds* $(n \geqq 6)$. *Then* $Int\ W_1^n \cong Int\ W_2^n$ *if and only if* W_1 *and* W_2 *are boundary h-cobordant i.e. there are h-cobordisms* $(U^{(i)};\ \partial W_2^{(i)},\ M^{(i)})$ *such that*

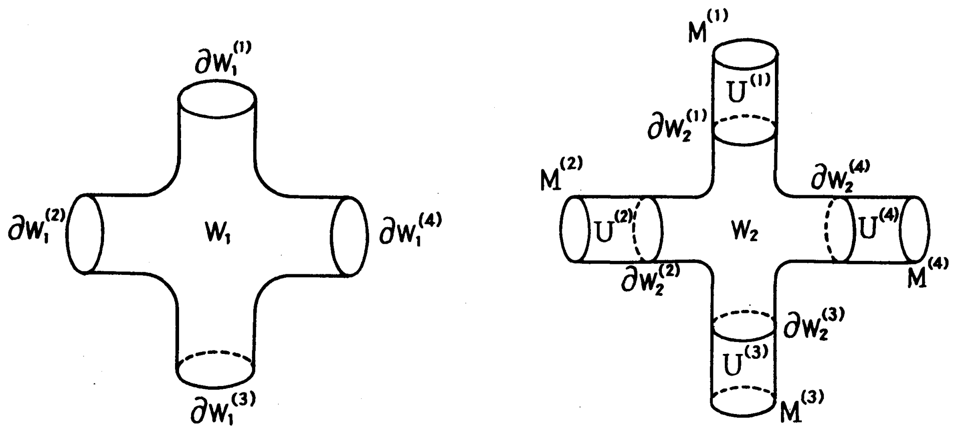

Fig. 1.

$$(W_1, \partial W_2) \cong (W_2 \underset{\partial W_2^{(1)}}{\cup} U^{(1)} \cup \cdots \underset{\partial W_2^{(p)}}{\cup} U^{(p)}, M^{(1)} \cup \cdots \cup M^{(p)})$$

where p is the number of the components of ∂W_i.

LEMMA 1. *Let W_i $(i=1, 2)$ be compact bounded n-manifolds such that $Int\ W_1 \cong Int\ W_2$ and g: $Int\ W_1 \to Int\ W_2$ be a given PL homeomorphism. Let c_1: $\partial W_1 \times I \to W_1$ be an PL embedding such that $c_1(x, 0)=x$ for $x \in \partial W_1$ i.e. c_1 is a boundary collar and let U_g be a region bounded by ∂W_2 and $gc_1(\partial W_1 \times \{1\})$ in W_2. If $U_g \cong \partial W_1 \times I$, there is a PL homeomorphism h: $W_1 \to W_2$ such that*

$$h \mid W_1 - c_1\big(\partial W_1 \times [0, 1]\big) = g \mid W_1 - c_1\big(\partial W_1 \times [0, 1]\big)$$

PROOF. Let ϕ: $\partial W_1 \times I \to U_g$ be a *PL* homeomorphism such that $\phi(\partial W_1 \times \{0\}) = \partial W_2$ and $\phi(\partial W_1 \times \{1\}) = gc_1(\partial W_1 \times \{1\})$. Then we may define a *PL* homeomorphism h: $W_1 \to W_2$ by

$$h(x) = \begin{cases} g(x) & \text{on } x \in W_1 - c_1(\partial W_1 \times [0, 1]) \\ \phi(u, \alpha) & \text{on } x \in c_1(\partial W_1 \times [0, 1]) \end{cases}$$

where u and α are decided as follows; since $x \in c_1(\partial W_1 \times [0, 1])$, it can be written $x = c_1(y, \alpha)$ $(0 \leqq \alpha \leqq 1)$ and since $\phi(\partial W_1 \times \{1\}) = gc_1(\partial W_1 \times \{1\})$, we can write $\phi^{-1}gc_1(y, 1) = (u, 1)$ $(u \in \partial W_1)$. Since $g = \phi$ on $c_1(\partial W_1 \times \{1\})$ by definition, h is well defined.

DEFINITION 2. Let U_g be a region defined by LEMMA 1, then U_g is an h-cobordism by PROPOSITION 1. We define $\tau(g_\infty) = \tau(U_g, gc_1(\partial W_1 \times \{1\})) \in Wh(\pi_1(\partial W_1))$, Whitehead torsion near the boundary with respect to g.

PROPOSITION 2. *Let W_i $(i=1, 2)$ be compact bounded manifolds and f: $Int\ W_1 \to Int\ W_2$ be a PL homeomorphism with $\tau(f_\infty) \neq 0$, then there is no PL homeomorphism $\bar{f}$: $W_1 \to W_2$ such that*

$$\bar{f} \mid W_1 - c_1\big(\partial W_1 \times [0, 1]\big) = f \mid W_1 - c_1\big(\partial W_1 \times [0, 1]\big).$$

(See also [5. chap. IX]).

PROOF. If there is a *PL* homeomorphism $\bar{f}$ as above,

$$\tau(f_\infty) = \tau\big(U_f, fc_1(\partial W_1 \times \{1\})\big) = \tau\big(U_f, \bar{f}c_1(\partial W_1 \times \{1\})\big)$$
$$= \tau\big(\partial W_1 \times I, \partial W_1 \times \{1\}\big) = 0.$$

This is a contradiction.

THEOREM 1. *Let W^n $(n \geqq 6)$ be a connected compact bounded PL n-manifold with a connected boundary ∂W. If $Wh(\pi_1(\partial W)) = (0)$, $\# \mathscr{A}(W) = 1$.*

PROOF. For any $W_1, W_2 \in \mathscr{A}(W)$, we will show $W_1 \cong W_2$. Let f: *Int* $W_1 \to$ *Int* W_2 be a *PL* homeomorphism and U_f be a region bounded by ∂W_2 and $fc_1(\partial W_1 \times \{1\})$. Then U_f is an h-cobordism by PROPOSITION 1 and $\tau(f_\infty) = \tau(U_f, fc_1(\partial W_1 \times \{1\})) = (-1)^{n-1}\tau(U_f, \partial W_2) \in Wh(\pi_1(\partial W_2)) \cong Wh(\pi_1(\partial W)) = (0)$ ([4, p. 394]). Hence by s-cobordism Theorem [6], $U_f \cong \partial W_2 \times I \cong \partial W_1 \times I$ and so there is a *PL* homeomorphism $\bar{f}$: $W_1 \to W_2$ such that $\bar{f} | W_1 - c_1(\partial W_1 \times [0, 1]) = f | W_1 - c_1(\partial W_1 \times [0, 1])$ by LEMMA 1.

PROPOSITION 3. *Let W_i^n $(i=1, 2)$ be compact bounded manifolds $(n \geqq 6)$ with connected boundaries ∂W_i and Int $W_1 \cong$ Int W_2. If n is even and $\pi_1(\partial W_1)$ is finite abelian, $D_\alpha W_1 \cong DW_2$ where DW_2 is the double of W_2 and $D_\alpha W_1 = W_1 \underset{\alpha}{\cup} W_1$ by some identification homeomorphism α: $\partial W_1 \to \partial W_1$. Furthermore if α is isotopic to identity, $D_\alpha W_1 \cong DW_1$.*

PROOF. Let W^+ and W^- be the copy of W and $DW = W^+ \underset{id}{\cup} W^-$. Let $f_\pm$: *Int* $W_1^\pm \to$ *Int* $W_2^\pm$ be homeomorphisms and $U_{f\pm}$ be regions bounded by $\partial W_2^\pm$ and $f_\pm c_1^\pm(\partial W_1^\pm \times \{1\})$ i.e. $(U_{f+}; f_+c_1^+(\partial W_1^+ \times \{1\}), \partial W_2^+)$ and U_{f-}: $f_-c_1^-(\partial W_1^- \times \{1\}), \partial W_2^-)$. If $\tau = \tau(U_{f+}, f_+c_1^+(\partial W_1^+ \times \{1\}))$, $\tau(U_{f-}, \partial W_2^-) = (-1)^{n-1}\bar{\tau}$ [4. p. 394] and so $\tau(U_{f+} \underset{\partial W_2}{\cup} U_{f-}, f_+c_1^+(\partial W_1^+ \times \{1\})) = \tau + (-1)^{n-1}\bar{\tau} \in Wh(\pi_1(\partial W_1))$ [4. Th. 3. 2] where $(U_{f+} \cup U_{f-}; f_+c_1^+(\partial W_1^+ \times \{1\})), f_-c_1^-(\partial W_1^- \times \{1\}))$ is an h-cobordism obtained from $U_{f+} \cup U_{f-}$ by ∂W_2 identiffied. Since $\pi_1(\partial W)$ is finite abelian, $\tau = \bar{\tau}$ [4] and so $\tau(U_{f+} \cup U_{f-}, f_+c_1^+(\partial W_1^+ \times \{1\})) = 0$ if n is even. Hence $U_{f+} \underset{\partial W_2}{\cup} U_{f-} \cong \partial W_1 \times I$ by s-cobordism Theorem. Similarly if

$$U'_{f-} = \left(U'_{f-}:\ \partial W_2^-, f_-c_1^-\left(\partial W_1^- \times \left\{\frac{1}{2}\right\}\right)\right) \text{ and}$$

$$U''_{f-} = \left(U''_{f-}:\ f_-c_1^-\left(\partial W_1^- \times \left\{\frac{1}{2}\right\}\right), f_-c_1^-(\partial W_1^- \times \{1\})\right),$$

$$U_{f+} \cup U'_{f-} \cong \partial W_1 \times I \textit{ and } U''_{f-} \cong \partial W_1 \times I.$$

Let ϕ_1: $\partial W_1 \times I \to U_{f+} \cup U'_{f-}$ be a homeomorphism such that $\phi_1(\partial W_1 \times \{0\}) = f_+c_1(\partial W_1^+ \times \{1\})$

$$\phi_1\left(\partial W_1 \times \{1\}\right) = f_-c_1^-\left(\partial W_1^- \times \left\{\frac{1}{2}\right\}\right)$$

and let

$$\phi_2:\ \left(\partial W_1^+ \times [0, 1]\right) \cup \left(\partial W_1^- \times \left[0, \frac{1}{2}\right]\right) \to \partial W_1 \times I$$

be a homeomorphism defined by

$$\phi_2(y^+, 1-t) = \left(z^+(y^+), \frac{t}{2}\right), \ 0 \leqq t \leqq 1$$

$$\phi_2(y^-, t) = \left(z^+(y^+), \frac{1}{2} + t\right), \ 0 \leqq t \leqq \frac{1}{2}$$

where

$$y^+ \in \partial W_1^+, \ y^- \in \partial W_1^- \quad \text{and} \quad \phi_1^{-1} f_+ c_1^+ (y^+, 1) = \left(z^+(y^+), 0\right).$$

And we define a homeomorphism γ: $DW_1 \to DW_1$ by

$$\gamma \,|\, \left(W_1^+ - c_1^+(\partial W_1^+ \times [0, 1))\right) \cup \left(W_1^- - c_1^-(\partial W_1^- \times [0, 1))\right) = id \ .$$

$$\gamma c_1^+(y^+, t) = c_1^+\left(y^+, \frac{1}{2}(3t-1)\right) \quad \frac{1}{3} \leqq t \leqq 1 \, .$$

$$\gamma c_1^+(y^+, t) = c_1^-\left(y^-, \frac{1}{2}(1-3t)\right) \quad 0 \leqq t \leqq \frac{1}{3}$$

$$\gamma c_1^-(y^-, t) = c_1^-\left(y^-, \frac{1}{2}(1+t)\right) \quad 0 \leqq t \leqq 1 \, ,$$

and let β: $c_1^-\left(\partial W_1^- \times \left\{\frac{1}{2}\right\}\right) \to c_1^-\left(\partial W_1^- \times \left\{\frac{1}{2}\right\}\right)$ be a homeomorphism defined by

$$\beta c_1^- \left(y^-, \frac{1}{2}\right) = (f_-)^{-1} \phi_1 \phi_2 \left(y^-, \frac{1}{2}\right).$$

We define a homeomorphism α: $\partial W_1 \to \partial W_1$ by

$$\alpha = \gamma^{-1} \beta \left(\gamma | c_1^+ (\partial W_1 \times \{0\})\right).$$

Then there is a well-defined homeomorphism h: $D_\alpha W_1 \to DW_2$ defined by

$$h(x) = \begin{cases} f_+(x^+) & x^+ \in W_1^+ - c_1^+\left(\partial W_1^+ \times [0, 1)\right) \\ \phi_1 \phi_2 (c_1^-)^{-1} \gamma (x^+) & x^+ \in c_1^+\left(\partial W_1^+ \times \left[0, \frac{1}{3}\right]\right) \\ \phi_1 \phi_2 (c_1^+)^{-1} \gamma (x^+) & x^+ \in c_1^+\left(\partial W_1^+ \times \left[\frac{1}{3}, 1\right]\right) \\ f_- \gamma (x^-) & x^- \in W_1^- \end{cases}$$

Next we will show $D_\alpha W_1 \cong DW_1$ if α is isotopic to identity. Since $\alpha \approx id.$, there is a level preserving homeomorphism H: $\partial W_1 \times I \to \partial W_1 \times I$ such that $H_0 = \alpha$ and $H_1 = id.$ Let c_1: $\partial W_1 \times I \to W_1$ be a collar (embedding) such that $c_1(y, 0) = y \in \partial W_1$ and c_1^α: $\partial W_1 \times I \to W_1$ be $c_1^\alpha(y, t) = c_1 H(y, t)$. Then we define a homeomorphism

$$F:\ DW_1 \to D_\alpha W_1$$

by

$$F(x^+) = x^+ \qquad x^+ \in W_1^+$$

$$F(x^-) = x^- \qquad x^- \in W_1^- - c_1^-\left(\partial W_1^- \times [0,1)\right)$$

$$F(x^-) = F\left(c_1^-(y^-, t)\right) = c_1^\alpha(y^-, t) \qquad x^- \in c_1^-\left(\partial W_1^- \times [0,1]\right).$$

Hence $DW_1 \cong D_\alpha W_1$.

DEFINITION 3. Let W^n $(n \geqq 6)$ be a compact bounded n-manifold. Then $I[W, \partial W]$ is the *inertia group* defined by [2] i.e.

$$I[W, \partial W] = \left\{\tau \in Wh(\pi_1(\partial W)) \mid (W, \partial W)\circ\tau = (W, \partial W)\right\}$$

where $(W, \partial W)\circ\tau = (W \cup U, \partial W')$ and $(U;\ \partial W, \partial W')$ is an h-cobordism with $\tau(U, \partial W) = \tau$. Similarly if M is a closed n-manifold $(n \geqq 5)$, $I[M] = \{\tau \in Wh(\pi_1(M)) \mid M\circ\tau = M\}$ where $M\circ\tau = M'$ *and* $(U; M, M')$ is an h-cobordism with $\tau(U, M) = \tau$.

Let $\langle W, \partial W\rangle$ be a set of manifolds $(W', \partial W)$ such that $(W, \partial W)\circ\tau = (W', \partial W)$, $\tau \in Wh(\pi_1(\partial W))$ and

$$\tilde{I}[W, \partial W] = \Big\{\tau \in Wh(\pi_1(\partial W)) \mid (W', \partial W)\circ\tau = (W', \partial W) \quad \text{for any } (W', \partial W) \in \langle (W, \partial W)\rangle\Big\}.$$

Then the following Lemma is obvious by definition.

LEMMA 2. $\#\tilde{I}[W, \partial W] \leqq \# I[W, \partial W] \leqq \# I[\partial W] \leqq \# Wh(\pi_1(\partial W))$. Using s-cobordism Theorem we obtain the following.

PROPOSITION 4. *Let W^n be a compact n-manifold $(n \geqq 6)$ with $\partial W = M_1 \cup M_2$ where M_i $(i=1, 2)$ are connected. If $(W; M_1, M_2)$ is an h-cobordism,* $\#\mathscr{A}(W) = \# I[M_1] = \# I[M_2]$.

PROOF. CASE 1. $M_1 \cong M_2$. We define a map α: $I[M_1] \to \mathscr{A}(W)$ by $\alpha(\tau) = \widetilde{W}$ where $\widetilde{W}$ is an h-cobordism from M_1 with $\tau(\widetilde{W}, M_1) = \tau$. Then $\widetilde{W}$ is uniquely determined by τ up to PL homeomorphism class [4. Th. 11. 3] and $\widetilde{W} \in \mathscr{A}(W)$ because $Int\ \widetilde{W} \cong M_1 \times R \cong Int\ W$ and $\partial\widetilde{W} = M_1 \cup M_1$ since $\tau \in I[M_1]$. And if $\tau_1 \neq \tau_2 \in I[M_1]$, $\widetilde{W}_1 \neq \widetilde{W}_2$ by [4. Th. 11. 3] where $\alpha(\tau_i) = \widetilde{W}_i$ $(i=1, 2)$. So α is injective, clearly for any $\widetilde{W} \in \mathscr{A}(W)$, $\tau \in I[M_1]$ where $\tau = \tau(\widetilde{W}, M_1)$. Hence α is onto.

CASE 2. $M_1 \not\cong M_2$. Let $\tau = \tau(W, M_1)$ fix. And we define a map α:

$I[M_1]\to\mathcal{A}(W)$ *by* $\alpha(\omega)=U\underset{M_1}{\cup}W$ where U is an h-cobordism from M_1 with $\tau(U, M_1)=\omega$. Then $\partial(U\underset{M_1}{\cup}W)=M_1\cup M_2$ and $Int\,(U\cup W)\cong M_1\times R\cong Int\,W$ by [1. vol. II]. So $U\underset{M_1}{\cup}W\in\mathcal{A}(W)$. *If* $\tau_1\neq\tau_2\in I[M_1]$, $U_1\underset{M_1}{\cup}W\neq U_2\underset{M_1}{\cup}W$ by [4. 11. 3]. Hence α is injective. Now we will show α surjective. For any $\widetilde{W}\in\mathcal{A}(W)$, since $\widetilde{W}-M_1\cong W-M_1\cong M_2\times[0,\infty)$. there is a PL homeomorphism f: $W-M_1\to\widetilde{W}-M_1$. Let U_f be a bounded region by M_1 and $fc_1(M_1\times\{1\})$ in $\widetilde{W}$ where c_1: $M_1\times I\to W$ is a boundary collar and let $\omega_f=\tau(U_f, M_1)$. Then $\omega_f\in I[M_1]$. Now ω_f does not depend on f because if g: $W-M_1\to\widetilde{W}-M_1$ is another PL homeomorphism,

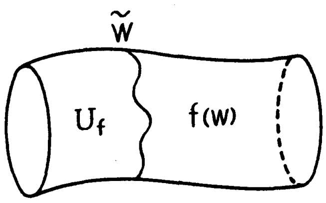

Fig. 2.

$$\begin{aligned}\tau(U_g, M_1)+\tau\big(g(W), g(M_1)\big)\\ &=\omega_g+\tau=\tau(\widetilde{W}, M_1)\\ &=\tau(U_f, M_1)+\tau\big(f(W, f(M_1)\big)=\omega_f+\tau\end{aligned}$$

and so $\omega_g=\omega_f$. Hence α is onto and $\#I[M_1]=\#\mathcal{A}(W)$. Similarly $\#I[M_2]=\#\mathcal{A}(W)$.

COROLLARY. *If W is the same as Proposition* 4, $\#\mathcal{A}(W)\geqq\#d^n(Wh(\pi_1(M_1)))$ *where* d_n: $Wh(\pi_1(M_1))\to Wh(\pi_1(M_1))$ *is an endomorphism defined by* $d_n(\tau)=\tau+(-1)^{n-1}\bar{\tau}$.

Proof follows by the fact that $d_n(Wh(\pi_1(M_1))$ is a subgroup of $I[M_1]$.

DEFINITION 4. Let W be a PL manifold and K be a subpolyhedron. We say K *homotopy spine* of W if the polyhedral pair (W, K) is an abstract h-neighborhood i.e. W, K satisfy the following conditions (see [2]):

(1) $K\subset Int\,W$

(2) for some regular neighborhood N of K in W, $(\overline{W-N}; \partial W, \partial N)$ is an h-cobordism.

DEFINITION 5. If $\tau\in I[W,\partial W]$, by the definition there exist an h-cobordism $(U; \partial W, \partial W)$ with $\tau(U,\partial W)=\tau$ and PL homeomorphism h: $W\underset{\partial W}{\cup}U\to W$. We define $I[W,\partial W; id]=\{\tau\in I[W,\partial W]\mid h|K$ *is homotopic to inclusion* i: $K\to W$ *for some homotopy spine K of* $W\}$.

LEMMA 3. *Let W^n be a compact bounded n-manifold* $(n\geqq 6)$ *with a connected boundary. If* i_*: $Wh(\pi_1(\partial W))\to Wh(\pi_1(W))$ *is a monomorphism,* $I[W,\partial W; id]=0$.

PROOF. Let $\tau\in I[W,\partial W; id]$, $(U; \partial W, \partial W)$ be an h-cobordism with

$\tau(U, \partial W)=\tau$ and $h: W\cup U\to W$ be a homeomorphism such that $h|K\simeq i$ for some homotopy spine K of W. Then by [4: 7.6, 7.7]

$$\tau\left(W, h(K)\right)=\tau(h)=\tau(i)=\tau(W, K)\in Wh\left(\pi_1(W)\right).$$

And since h is a homeomorphism,

$$\tau\left(W, h(K)\right)=i_*\tau(U, \partial W)+\tau(W, K).$$

Hence $i_*\tau(U, \partial W)=i_*\tau=0$ and $\tau=0$.

THEOREM 2. *If W^n is a compact bounded n-manifold ($n\geqq 6$) with a connected boundary, then $\#\mathscr{A}(W)=\#\langle(W, \partial W)\rangle\leqq\# I[\partial W]$. Furthermore if $ker(i_*: Wh(\pi_1(\partial W))\to Wh(\pi_1(W)))=0$ and $I[W, \partial W: id]=I[W, \partial W]$, $\#\mathscr{A}(W)=\# I[\partial W]$.*

PROOF. Let $W_1\in\mathscr{A}(W)$ such that $W_1\not\cong W$ and let f: *Int* $W_1\to$ *Int* W be a homeomorphism. Let U_f be a region bounded by ∂W and $fc_1(\partial W_1\times\{1\})$. Then $(U_f; \partial W, fc_1(\partial W_1\times\{1\}))$ is an h-cobordism by PROPOSITION 1 and $\tau(U_f, fc_1(\partial W_1\times\{1\}))\neq 0$ in $I[\partial W_1]=I[\partial W]$ for any f because $\partial W_1\cong\partial W$ and $W_1\not\cong W$. Similarly if $W_1, W_2\in\mathscr{A}(W)$ such that $W_1\not\cong W\not\cong W_2$, $W_1\not\cong W_2$, then $\tau(U_f, fc_1(\partial W_1\times\{1\}))\neq\tau(U_g, gc_2(\partial W_2\times\{1\}))$ where g: *Int* $W_2\to$ *Int* W is a homeomorphism. So $\#\mathscr{A}(W)\leqq\# I[\partial W]$.

Since *Int* $(W\cup_{\partial W} U)\cong$ *Int* W where U is an h-cobordism from ∂W, $W'\in\mathscr{A}(W)$ if $W'\in\langle(W, \partial W)\rangle$. So $\#\langle W, \partial W\rangle\leqq\#\mathscr{A}(W)$. And if $W_1, W_2\in\mathscr{A}(W)$, *Int* $W_1\cong$ *Int* W_2 so W_1 is boundary h-cobordant to W_2 by PROPOSITION 1 and $\partial W_1\cong\partial W_2$. Hence $W_1, W_2\in\langle W, \partial W\rangle$ and $\mathscr{A}(W)=\langle(W, \partial W)\rangle$. Since $\#\mathscr{A}(W)=\#(I[\partial W]/I[W, \partial W])$, by LEMMA 3

$$\#\mathscr{A}(W)=\# I[\partial W] \quad \text{if} \quad ker\ i_*=0$$

and $I[W, \partial W: id]=I[W, \partial W]$.

THEOREM 3. *Let M^n be a closed n-manifold ($n\geqq 5$) and let $G=\{\tau\in Wh(\pi_1(M\times S^1))\,|\,$if $A=(a_{ij})\in GL(p, Z\pi_1(M\times S^1))$ is a representative of τ, $a_{ij}\in Z\pi_1(M)\otimes\{1\}\}$. Then there is a homomorphism ϕ of $I[M]$ onto $G/(I[M\times S^1]\cap G)$ with $ker\,\phi\supset\{\omega\in I[M]|\omega+(-1)^n\bar{\omega}=0\}$.*

COROLLARY. *If M^n is a closed n-manifold ($n\geqq 5$ and odd) with $\pi_1(M)$ = finite abelian, $G\subset I[M\times S^1]$.*

PROOF. If M satisfies all conditions, for any $\tau\in Wh(\pi_1(M))$ $\tau+(-1)^n\bar{\tau}=0$ by [4]. So $\{\tau\in I[M]|\tau+(-1)^n\bar{\tau}=0\}=I[M]\subset ker\,\phi$. Hence $G/(I[M\times S^1]\cap G)=0$ by THEOREM 3.

PROOF OF THEOREM 3. Let U be an h-cobordism from M to itself.

Then $Int\ U \cong M\times R$ and so U is boundary h-cobordant to $M\times I$ by PROPOSITION 1 i.e. $(M\times I;\ M,\ M)\cong(V_1\underset{M}{\cup}U\underset{M}{\cup}V_2;\ M,\ M)$ for some h-cobordisms $(V_1;\ M,\ M)$, $(V_2;\ M,\ M)$. Let $(\widetilde{W};\ V_1\cup U\cup V_2,\ M\times I)$ be a trivial h-cobordism. Then $(W;U,\ M\times I)$ is an h-cobordism between U and $M\times I$ where

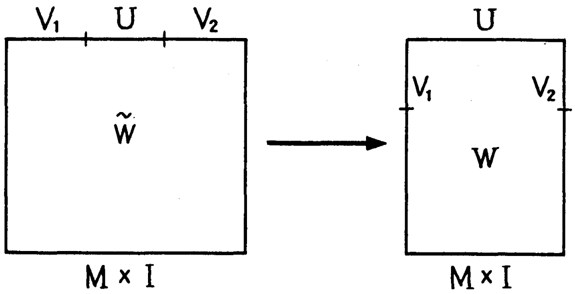

Fig. 3.

$\partial W\cong\partial\widetilde{W}$. Let $\omega_1=\tau(V_1,M)$, $\omega_2=\tau(V_2,M)$ and let $(V_1';\ M,\ M)$, $(V_2';\ M,\ M)$ be h-cobordisms with $\tau(V_1',M)=-\omega_1$, $\tau(V_2',M)=-\omega_2$. Then $V_1\cup V_1'$ and $V_2\cup V_2'$ are both trivial h-cobordism and so there are trivial h-cobordisms W_1, W_2 between $V_1\cup V_1'$, $V_2\cup V_2'$ and $M\times I$. Therefore $V_1'\cup U\cup V_2'$ is h-cobordant to $M\times I$ by the h-cobordism $W_1\cup W\cup W_2$ such that

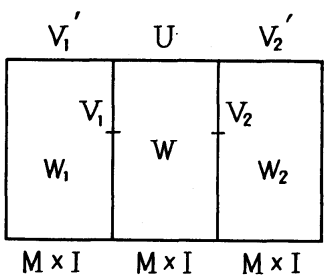

Fig. 4.

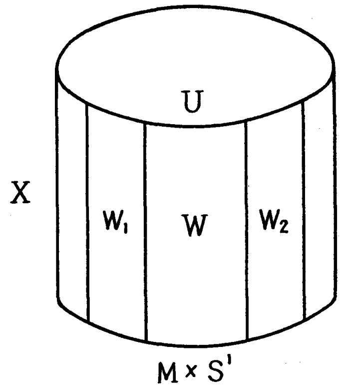

Fig. 5.

$\partial(V_1'\cup U\cup V_2')\cong M\times S^0$ is trivially h-cobordant to $M\times S^0$ by the same $W_1\cup W\cup W_2$. Let $X=(M\times I\times J)\cup(W_1\cup W\cup W_2)$ by identifying $M\times(0)\times(t)\sim M\times(t)\subset$ "*free*" part of ∂W_1, $M\times\{1\}\times\{t\}\sim M\times\{t\}\subset$ "*free*" part of ∂W_2. Then X is an h-cobordism from $M\times S^1$. So there is a map

$$\tilde{\phi}:\ I[M]\to Wh\big(\pi_1(M\times S^1)\big)$$

defined by

$$\tilde{\phi}(\tau)=\tau(X,\ M\times S^1)\quad\text{where}\quad\tau=\tau(U,\ M)\,.$$

Let $(\widetilde{U};\ M,\ M)$ be an h-cobordism with $\tau(\widetilde{U};\ M)=\tilde{\tau}\neq\tau\in I[M]$. Then

$U\not\cong\widetilde{U}$. And let $\widetilde{V}_1'$, $\widetilde{V}_2'$ be the corresponding h-cobordism for $\widetilde{U}$ as above. If $(\widetilde{V}_1'\cup\widetilde{U}\cup\widetilde{V}_2')\cup M\times I\times\{1\}\neq(V_1'\cup U\cup V_2')\cup(M\times I\times\{1\})$, $\tau(\widetilde{X}, M\times S^1)\neq\tau(X, M\times S^1)$. But if $(\widetilde{V}_1'\cup U\cup\widetilde{V}_2')\cup(M\times I\times\{1\})\cong(V_1'\cup U\cup V_2')\cup(M\times I\times\{1\})$, I don't know whether $\tau(\widetilde{X}, M\times S^1)=\tau(X, M\times S^1)$ or not. So we define a map ϕ: $I[M]\rightarrow Wh(\pi_1(M\times S^1))/I[M\times S^1]$ by $\phi(\tau)=[\tau(X, M\times S^1)]$. Then ϕ is well defined. And since $M\times I\times J\cup N(M\times 3I, W_1\cup W\cup W_2)\subset X$ is homeomorphic to $M\times S^1\times J$ where $M\times 3I=(M\times I)\cup(M\times I)\cup(M\times I)\cup\partial(W_1\cup W\cup W_2)$ and $N(M\times 3I, W_1\cup W\cup W_2)$ is a regular neighborhood of $M\times 3I$ in $W_1\cup W\cup W_2$, we may consider that X is constructed from $M\times S^1\times J$ by attaching handles on $M\times I\times\{1\}\subset M\times S^1\times\{1\}$. So if $A=(a_{ij})\in GL(p, Z\pi_1(M\times S^1))$ is a representation of $\tau(X, M\times S^1)$, $a_{ij}\in Z\pi_1(M)\otimes\{1\}$. Hence $Im\ \phi\subset G/(I[M\times S^1]\cap G)$. Now we will show ϕ surjective. Let $[\omega]$ be an element of $G/(I[M\times S^1]\cap G)$ such that $\omega=\tau(X, M\times S^1)$. Since a representative $A=(a_{ij})$ of ω in $GL(p, Z\pi_1(M\times S^1))$ has a form $a_{ij}\in Z\pi_1(M)\otimes\{1\}$, we may assume

$$X=(M\times S^1\times J)\cup\{handles\}$$
$$=(M\times I_1\times J)\cup(M\times I_2\times J)\cup\{handles\}$$

so that all handles do not attach one of $M\times I\times J_s'$, say $M\times I_1\times J$ [4]. Let $U=\partial X-(M\times S^1\times\{0\}\cup M\times I_1\times\{1\})$. Then U is an h-cobordism between M. So there is an element $\tau(U, M)\in I[M]$ such that $\phi(\tau(U, M))=[\omega]$. Now let ω be an element of $I[M]$ such that $\omega+(-1)^n\overline{\omega}=0$. Then $U\cup\overline{U}\cong M\times I$ where $\tau(U, M)=\omega$ and $\tau(\overline{U}, M)=(-1)^n\overline{\omega}$. So $((U\cup\overline{U})\times J)\underset{M\times\partial I\times J}{\cup}(M\times I\times J)\cong M\times S^1\times J$. Hence $\phi(\omega)=0$ and so $ker\ \phi\supset\{\omega\in I[M]|\omega+(-1)^n\overline{\omega}=0\}$.

Now, suppose R and R' are rings which are also algebras over the commutative ring A, and let C be a free *R-complex* with a preferred basis, and C' a free R' complex with a preferred basis. Then $C\otimes_A C'$ is a free $R\otimes_A R'$ compelx with a preferred basis.

We obtain the following proposition by [4. § 3].

PROPOSITION 5. *Let C be a free R-complex with a preferred basis, and C' a free R'-complex with a preferred basis. Then if $H_*(C)$ and $H_*(C')$ are both free, so is $H_*(C\otimes_A C')$ and $\tau(C\otimes_A C')=\tau(C\otimes B)+\tau(C\otimes B')+\tau(\mathscr{H})$ where B and B' are R'-complexes such that*

$$B:\ 0\rightarrow C_n'\rightarrow C_{n-1}'\rightarrow\cdots\rightarrow C_{p+1}'\rightarrow 0$$
$$B':\ 0\rightarrow C_p'\rightarrow C_{p-1}'\rightarrow\cdots\rightarrow C_0'\rightarrow 0$$

for any p when

$$0 \to C'_n \to C'_{n-1} \to \cdots \to C'_p \to C'_{p-1} \to \cdots \to C'_0 \to 0$$

is a chain complex C' and where $\mathscr{H}$ is an $R \otimes_A R'$ complex $H_m(C \otimes B) \to H_m(C \otimes C') \to H_m(C \otimes B') \to H_{m-1}(C \otimes B) \to \cdots \to H_0(C \otimes B') \to 0$ induced by the short exact sequence

$$0 \to C \otimes B \to C \otimes C' \to C \otimes B' \to 0 .$$

COROLLARY. *If M is a manifold such that $H_*(\widetilde{M})$ is free $Z\pi_1(M)$-module where $\widetilde{M}$ is a universal covering space, $\tau(M \times S^1)(\overset{\text{def}}{=} \tau(c(\widetilde{M \times S^1}))) \in Wh(\pi_1(M \times S^1))$ is equal to $\tau(c(\widetilde{M})) + \tau(\mathscr{H})$, where $\mathscr{H}$ is a $Z\pi_1(M) \otimes_Z Z\pi_1(S^1)$ complex*

$$H_m\Big(c(\widetilde{M}) \otimes c_1(\widetilde{S}^1)\Big) \to H_m\Big(c(\widetilde{M}) \otimes c(\widetilde{S}^1)\Big) \to H_m\Big(c(\widetilde{M}) \otimes c_0(\widetilde{S}^1)\Big)$$
$$\to \cdots \to H_0\Big(c(\widetilde{M}) \otimes c_0(\widetilde{S}^1)\Big) \to 0 .$$

PROOF. By PROPOSITION 5

$$\begin{aligned}
&\tau\Big(c(\widetilde{M}) \otimes c(\widetilde{S}^1)\Big) \\
&\quad = \tau\Big(c(\widetilde{M}) \otimes c_1(S^1)\Big) + \tau\Big(c(\widetilde{M}) \otimes c_0(\widetilde{S}^1)\Big) + \tau(\mathscr{H}) \\
&\quad = \chi\Big(c_1(\widetilde{S}^1)\Big)\tau\Big(c(\widetilde{M})\Big) + \chi\Big(c_0(\widetilde{S}^1)\Big)\tau\Big(c(\widetilde{M})\Big) + \tau(\mathscr{H}) \\
&\quad = \chi\Big(c(\widetilde{S}^1)\Big)\tau\Big(c(\widetilde{M})\Big) + \tau(\mathscr{H})
\end{aligned}$$

where $\chi(c(\widetilde{S}^1))$ is the euler characteristic as $Z\pi_1(S^1)$-complex. Since $\widetilde{S}^1 = \widetilde{R}^1$, $\chi(c(\widetilde{S}^1)) = 1$. Hence $\tau(c(\widetilde{M}) \otimes c(\widetilde{S}^1)) = \tau(c(\widetilde{M})) + \tau(\mathscr{H})$. And since $\widetilde{M \times S} = \widetilde{M} \times \widetilde{S}^1$, we obtain the result.

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