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Cup products on the complete relative cohomologies of finite groups and group algebras

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Abstract. There is a theory of cup products on the complete relative cohomology of Frobenius extensions in [N1], especially, of ring extensions of group algebras of finite groups. In this paper, we construct a cup product on the complete relative cohomology of finite groups in view of a generalization of one on the Tate cohomology of finite groups. And we show that there is an isomorphism between the complete relative cohomology of finite groups and one of the group algebras, and that it preserves these cup products.

Key words: cup product, complete (co)homology, relative (co)homology, Tate cohomology.

Introduction

Let G be a finite group and K a subgroup of G. In [N1], the complete relative cohomology groups $H^r(\mathbb{Z}G, \mathbb{Z}K, A)$ and a cup product $\cup$: $H^r(\mathbb{Z}G, \mathbb{Z}K, A) \otimes H^s(\mathbb{Z}G, \mathbb{Z}K, B) \to H^{r+s}(\mathbb{Z}G, \mathbb{Z}K, A \otimes_{\mathbb{Z}G} B)$ are defined for any two-sided $\mathbb{Z}G$-modules A, B and any $r, s \in \mathbb{Z}$. On the other hand, in this paper, we define a cup product $\cup : H^r(G, K, M) \otimes H^s(G, K, N) \to H^{r+s}(G, K, M \otimes N)$ for any left G-modules M, N and any $r, s \in \mathbb{Z}$, in view of a generalization of cup product on the Tate cohomology groups of finite groups. And we will consider the relationship between these cup products.

In §1, we show that the complete relative cohomology group of finite groups has a unique cup product (Theorem 1.4), which is a generalization of the ordinary cup product on the Tate cohomology $\hat{H}^r(G, M)$. In §2, we first introduce a modified cup product $\cup_\rho : H^r(G, K, {}_\eta A) \otimes H^s(G, K, {}_\eta B) \to H^{r+s}(G, K, {}_\eta(A \otimes_{\mathbb{Z}G} B))$ for any two-sided $\mathbb{Z}G$-modules A, B and any $r, s \in \mathbb{Z}$, which is induced by the above cup product and a G-pairing $\rho : {}_\eta A \otimes {}_\eta B \to {}_\eta(A \otimes_{\mathbb{Z}G} B)$, where ${}_\eta A$ denotes the G-module A defined by the conjugation action of G. Next we show that there exists an isomorphism $\Phi^r : H^r(\mathbb{Z}G, \mathbb{Z}K, A) \xrightarrow{\sim} H^r(G, K, {}_\eta A)$ for any two-sided $\mathbb{Z}G$-module A. In Theorem 2.3, the main theorem in the paper, it is shown that the

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isomorphism Φ^r preserves cup products, that is, $\Phi^{r+s} \circ \cup = \cup_\rho \circ (\Phi^r \otimes \Phi^s)$. Accordingly, it follows that the ring $H^*(G, K, \mathbb{Z})$ is a direct summand of the ring $H^*(\mathbb{Z}G, \mathbb{Z}K, \mathbb{Z}G)$ (Corollary 2.5). In §3, we will give an alternate proof for the absolute complete cohomology case of Theorem 2.3. This is done by explicit (co)chain level formulas of cup products. The absolute case also induces an affirmative answer to a question by I. Hambleton to the second author: "Is it true that the complete cohomology ring $H^*(\mathbb{Z}G, \mathbb{Z}G)$ of the algebra $\mathbb{Z}G$ has an invertible element of non-zero degree if and only if so does the complete cohomology ring $\hat{H}^*(G, \mathbb{Z})$ of the finite group G?"

1. Cup product on the complete relative cohomology of finite groups

In this paper let G be a finite group and K a subgroup of G. Then in [H, §4], the complete relative cohomology group $H^r(G, K, M)$ is defined for any left G-module M and any $r \in \mathbb{Z}$. Let X be a complete $(\mathbb{Z}G, \mathbb{Z}K)$-resolution of $\mathbb{Z}$ in the sense of [N1, §1]. Then $H^r(G, K, M)$ is defined as the r-th cohomology group of the complex $\mathrm{Hom}_{\mathbb{Z}G}(X, M)$, and in the case $K = \{1\}$, where 1 is the unit element of G, $H^r(G, K, M)$ coincides with the Tate cohomology group $\hat{H}^r(G, M)$ exactly (cf. [B, Chapter 6, §4]). We put

$$M^G = \{m \in M \mid gm = m \text{ for any } g \in G\},$$
$$M^K = \{m \in M \mid km = m \text{ for any } k \in K\},$$
$$N_{G/K}(M^K) = \left\{\sum_{i=1}^{n} g_i m \mid m \in M^K\right\},$$

where $\{g_1, \dots, g_n\}$ are elements of G such that $G = \bigcup_{i=1}^{n} g_i K$ is a left coset decomposition. Then we have the following lemma.

Lemma 1.1 *Let G be a finite group, K a subgroup of G and M a left G-module. Then the isomorphism $H^0(G, K, M) \simeq M^G / N_{G/K}(M^K)$ holds as $\mathbb{Z}$-modules.*

Proof. Take the complete resolution in [O, §6] as a complete $(\mathbb{Z}G, \mathbb{Z}K)$-resolution of $\mathbb{Z}$ and denote it by X:

$$X : \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots,$$

where we set $X_1 = \mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z}$, $X_0 = X_{-1} = \mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z}$; $d_1(x_1 \otimes_{\mathbb{Z}K}$

$x_2 \otimes_{\mathbb{Z}K} 1) = x_1x_2 \otimes_{\mathbb{Z}K} 1 - x_1 \otimes_{\mathbb{Z}K} 1$, $d_0(x \otimes_{\mathbb{Z}K} 1) = \sum_{i=1}^{n} xg_i \otimes_{\mathbb{Z}K} 1$ for $x_1, x_2, x \in G$. We make the following identifications:

$$\varphi : \mathrm{Hom}_{\mathbb{Z}G}(X_1, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}K}(\mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z}, M)$$
$$\text{by } \varphi(f)(x \otimes_{\mathbb{Z}K} 1) = f(1 \otimes_{\mathbb{Z}K} x \otimes_{\mathbb{Z}K} 1);$$
$$\psi : \mathrm{Hom}_{\mathbb{Z}G}(X_0, M) \xrightarrow{\sim} M^K$$
$$\text{by } \psi(f) = f(1 \otimes_{\mathbb{Z}K} 1) \text{ and } \psi^{-1}(m)(x \otimes_{\mathbb{Z}K} 1) = xm;$$
$$\psi : \mathrm{Hom}_{\mathbb{Z}G}(X_{-1}, M) \xrightarrow{\sim} M^K \text{ by the same map } \psi \text{ as above.}$$

Then we have the following complex which gives the 0-th cohomology $H^0(G, K, M) \simeq \mathrm{Ker}\, d_1^{\#} / \mathrm{Im}\, d_0^{\#}$;

$$\mathrm{Hom}_{\mathbb{Z}K}(\mathbb{Z}G \otimes_{\mathbb{Z}K} \mathbb{Z}, M) \xleftarrow{d_1^{\#}} M^K \xleftarrow{d_0^{\#}} M^K$$

where $d_1^{\#}(m)(x \otimes_{\mathbb{Z}K} 1) = xm - m$ and $d_0^{\#}(m) = \sum_{i=1}^{n} g_i m$. Since $\mathrm{Ker}\, d_1^{\#} = (M^K)^G = M^G$ and $\mathrm{Im}\, d_0^{\#} = N_{G/K}(M^K)$, it follows that the isomorphism $H^0(G, K, M) \simeq M^G / N_{G/K}(M^K)$ holds. □

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a $(\mathbb{Z}G, \mathbb{Z}K)$-exact sequence of left G-modules in the sense of [H, §1]. Then we have the long exact sequence $\cdots \to H^r(G, K, M_1) \to H^r(G, K, M_2) \to H^r(G, K, M_3) \xrightarrow{\partial} H^{r+1}(G, K, M_1) \to \cdots$ as in [H, §4], where ∂ is the connecting homomorphism. Let M and N be left G-modules. By the diagonal action of G on $M \otimes N$, that is, $g(m \otimes n) = gm \otimes gn$ for $g \in G$ and $m \otimes n \in M \otimes N$, we regard $M \otimes N$ as a left G-module.

Lemma 1.2 *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a $(\mathbb{Z}G, \mathbb{Z}K)$-exact sequence of left G-modules and N a left G-module. Then short sequences of left G-modules $0 \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$ and $0 \to N \otimes M_1 \to N \otimes M_2 \to N \otimes M_3 \to 0$ are $(\mathbb{Z}G, \mathbb{Z}K)$-exact, and they induce the connecting homomorphisms $\partial : H^r(G, K, M_3 \otimes N) \to H^{r+1}(G, K, M_1 \otimes N)$ and $\partial : H^r(G, K, N \otimes M_3) \to H^{r+1}(G, K, N \otimes M_1)$ for any $r \in \mathbb{Z}$, respectively.*

Proof. The lemma is proved by the definition of relative exactness. □

Let M and N be any left G-modules, and let r and s be any integers. Assume that an element $\alpha \cup \beta \in H^{r+s}(G, K, M \otimes N)$ is defined for every $\alpha \in H^r(G, K, M)$ and $\beta \in H^s(G, K, N)$. If $\cup$ satisfies the following conditions (I)–(III), then we will call $\alpha \cup \beta$ a cup product of α and β.

(I) $\cup$ induces a $\mathbb{Z}$-homomorphism:

$$H^r(G,K,M) \otimes H^s(G,K,N) \longrightarrow H^{r+s}(G,K,M \otimes N).$$

(II_1) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a ($\mathbb{Z}G$, $\mathbb{Z}K$)-exact sequence of left G-modules. Then

$$\partial(\alpha \cup \beta) = \partial(\alpha) \cup \beta$$

holds for $\alpha \in H^r(G,K,M_3)$ and $\beta \in H^s(G,K,N)$, where ∂ denotes the connecting homomorphism.

(II_2) Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a ($\mathbb{Z}G$, $\mathbb{Z}K$)-exact sequence of left G-modules. Then

$$\partial(\alpha \cup \beta) = (-1)^r \alpha \cup \partial(\beta)$$

holds for $\alpha \in H^r(G,K,M)$ and $\beta \in H^s(G,K,N_3)$, where ∂ denotes the connecting homomorphism.

(III) The diagram

$$\begin{array}{ccc} H^0(G,K,M) \otimes H^0(G,K,N) & \xrightarrow{\cup} & H^0(G,K,M \otimes N) \\ \downarrow & & \downarrow \\ M^G/N_{G/K}(M^K) \otimes N^G/N_{G/K}(N^K) & \longrightarrow & (M \otimes N)^G/N_{G/K}((M \otimes N)^K) \end{array}$$

commutes, in which the vertical homomorphisms are the isomorphisms given by Lemma 1.1 and the homomorphism in the bottom row is defined by

$$(m + N_{G/K}(M^K)) \otimes (n + N_{G/K}(N^K)) \mapsto m \otimes n + N_{G/K}((M \otimes N)^K).$$

Lemma 1.2 shows the existence of the connecting homomorphisms in the left hand sides of the equations in (II_1) and (II_2) above. It is easy to see that, in the case $K = \{1\}$, the cup product above coincides with the cup product in [B, Chapter 6, §5] since we can use dimension-shifting as in [B, Chapter 6, (5.4)].

Lemma 1.3 *Let C be a complex which is a $(\mathbb{Z}G, \mathbb{Z}K)$-exact sequence of left G-modules and M a $(\mathbb{Z}G, \mathbb{Z}K)$-projective G-module in the sense of* [H, §1]. *Then the complex $M \otimes C$ with diagonal G-action is contractible as a complex of G-modules.*

Proof. Since M is $(\mathbb{Z}G, \mathbb{Z}K)$-projective, M is a direct summand of the left G-module $\mathbb{Z}G \otimes_{\mathbb{Z}K} M$. Therefore it suffices to show that the complex $(\mathbb{Z}G \otimes_{\mathbb{Z}K} M) \otimes C$ (with diagonal G-action) is contractible as a complex of G-modules. We have the isomorphism $(\mathbb{Z}G \otimes_{\mathbb{Z}K} M) \otimes C \xrightarrow{\sim} \mathbb{Z}G \otimes_{\mathbb{Z}K} (M \otimes C)$ as complexes of G-modules, which is given by the map $(g \otimes_{\mathbb{Z}K} m) \otimes x \mapsto g \otimes_{\mathbb{Z}K} (m \otimes g^{-1}x)$ for any $g \in G$, $m \in M$ and $x \in C_\bullet$. Since $M \otimes C$ is contractible as a complex of K-modules, the proof is completed. □

Theorem 1.4 *There is a unique cup product.*

Proof. This theorem is proved in the same way as in [B, Chapter 6, Lemma 5.8] by adopting Lemma 1.3 instead of [B, Chapter 6, Lemma 5.11]. □

2. An isomorphism $\Phi^r : H^r(\mathbb{Z}G, \mathbb{Z}K, A) \to H^r(G, K, {}_\eta A)$ and cup products

We will denote the enveloping algebra $\mathbb{Z}G \otimes (\mathbb{Z}G)^\circ$ of $\mathbb{Z}G$ by P. Since we have the ring homomorphism $\eta : \mathbb{Z}G \to P$; $g \mapsto g \otimes (g^{-1})^\circ$ for $g \in G$, a left P-module A is regarded as a left G-module induced by η, which is denoted by ${}_\eta A$.

Let A and B be left P-modules. Then ${}_\eta A \otimes {}_\eta B$ is a left G-module with diagonal G-action and we have the G-pairing $\rho : {}_\eta A \otimes {}_\eta B \to {}_\eta(A \otimes_{\mathbb{Z}G} B)$ given by $\rho(m \otimes n) = m \otimes_{\mathbb{Z}G} n$. The map ρ induces the homomorphism

$$\rho^* : H^r(G, K, {}_\eta A \otimes {}_\eta B) \to H^r(G, K, {}_\eta(A \otimes_{\mathbb{Z}G} B))$$

for any $r \in \mathbb{Z}$, and we can define a modified cup product $\cup_\rho$ in terms of the cup product $\cup$ of §1 by the equation:

$$\alpha \cup_\rho \beta = \rho^*(\alpha \cup \beta)$$

for $\alpha \in H^r(G, K, {}_\eta A)$ and $\beta \in H^s(G, K, {}_\eta B)$. Let S be the image of the natural homomorphism from $\mathbb{Z}K \otimes (\mathbb{Z}K)^\circ$ to P. S is a subring of P. Then, for the modified cup product $\cup_\rho$, we have the following proposition.

Proposition 2.1 *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a (P, S)-exact sequence of left P-modules, and let A and B be left P-modules. Then the following statements hold.*

(1) $0 \to {}_\eta A_1 \to {}_\eta A_2 \to {}_\eta A_3 \to 0$ *is $(\mathbb{Z}G, \mathbb{Z}K)$-exact.*

(2) *When* $0 \to {}_\eta(A_1 \otimes_{\mathbb{Z}G} B) \to {}_\eta(A_2 \otimes_{\mathbb{Z}G} B) \to {}_\eta(A_3 \otimes_{\mathbb{Z}G} B) \to 0$ *is*

$(\mathbb{Z}G, \mathbb{Z}K)$*-exact, the equation*

$$\partial(\alpha \cup_\rho \beta) = \partial(\alpha) \cup_\rho \beta$$

holds for $\alpha \in H^r(G, K, {}_\eta A_3)$ *and* $\beta \in H^s(G, K, {}_\eta B)$*, where* ∂ *denotes the connecting homomorphism.*

(3) *When* $0 \to {}_\eta(B \otimes_{\mathbb{Z}G} A_1) \to {}_\eta(B \otimes_{\mathbb{Z}G} A_2) \to {}_\eta(B \otimes_{\mathbb{Z}G} A_3) \to 0$ *is* $(\mathbb{Z}G, \mathbb{Z}K)$*-exact, the equation*

$$\partial(\alpha \cup_\rho \beta) = (-1)^r \alpha \cup_\rho \partial(\beta)$$

holds for $\alpha \in H^r(G, K, {}_\eta B)$ *and* $\beta \in H^s(G, K, {}_\eta A_3)$*, where* ∂ *denotes the connecting homomorphism.*

(4) *The diagram*

$$\begin{array}{ccc} H^0(G, K, {}_\eta A) \otimes H^0(G, K, {}_\eta B) & \xrightarrow{\cup_\rho} & H^0(G, K, {}_\eta(A \otimes_{\mathbb{Z}G} B)) \\ \downarrow & & \downarrow \\ \dfrac{({}_\eta A)^G}{N_{G/K}(({}_\eta A)^K)} \otimes \dfrac{({}_\eta B)^G}{N_{G/K}(({}_\eta B)^K)} & \longrightarrow & \dfrac{({}_\eta(A \otimes_{\mathbb{Z}G} B))^G}{N_{G/K}(({}_\eta(A \otimes_{\mathbb{Z}G} B))^K)} \end{array}$$

commutes, in which the vertical homomorphisms are the isomorphisms given by Lemma 1.1 *and the homomorphism in the bottom row is defined by*

$$\begin{aligned} &(m + N_{G/K}(({}_\eta A)^K)) \otimes (n + N_{G/K}(({}_\eta B)^K)) \\ &\qquad \mapsto m \otimes_{\mathbb{Z}G} n + N_{G/K}(({}_\eta(A \otimes_{\mathbb{Z}G} B))^K). \end{aligned}$$

Proof. The statement (1) follows by the definition of relative exactness. Therefore, by the condition (II_1) of the definition of the cup product, the diagram

$$\begin{array}{ccc} H^r(G, K, {}_\eta A_3) \otimes H^s(G, K, {}_\eta B) & \xrightarrow{\cup} & H^{r+s}(G, K, {}_\eta A_3 \otimes {}_\eta B) \\ {\scriptstyle \partial \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \partial} \\ H^{r+1}(G, K, {}_\eta A_1) \otimes H^s(G, K, {}_\eta B) & \xrightarrow[\cup]{} & H^{r+s+1}(G, K, {}_\eta A_1 \otimes {}_\eta B) \end{array}$$

commutes, and the diagram

$$\begin{array}{ccc} H^{r+s}(G, K, {}_\eta A_3 \otimes {}_\eta B) & \xrightarrow{\rho^*} & H^{r+s}(G, K, {}_\eta(A_3 \otimes_{\mathbb{Z}G} B)) \\ {\scriptstyle \partial} \downarrow & & \downarrow {\scriptstyle \partial} \\ H^{r+s+1}(G, K, {}_\eta A_1 \otimes {}_\eta B) & \xrightarrow[\rho^*]{} & H^{r+s+1}(G, K, {}_\eta(A_1 \otimes_{\mathbb{Z}G} B)) \end{array}$$

commutes. Hence the statement (2) holds. Using the condition (II_2) of the definition of the cup product, the statement (3) is shown by the same way as in the case (2). The diagram

$$\begin{array}{ccc} H^0(G,K,{}_\eta A\otimes {}_\eta B) & \xrightarrow{\rho^*} & H^0(G,K,{}_\eta(A\otimes_{\mathbb{Z}G} B)) \\ \downarrow & & \downarrow \\ ({}_\eta A\otimes {}_\eta B)^G/N_{G/K}(({}_\eta A\otimes {}_\eta B)^K) & \longrightarrow & ({}_\eta(A\otimes_{\mathbb{Z}G} B))^G/N_{G/K}(({}_\eta(A\otimes_{\mathbb{Z}G} B))^K) \end{array}$$

commutes for any left P-modules A and B, where the vertical homomorphisms are the isomorphisms shown in Lemma 1.1 and the homomorphism in the bottom row is defined by

$$a\otimes b + N_{G/K}(({}_\eta A\otimes {}_\eta B)^K) \mapsto a\otimes_{\mathbb{Z}G} b + N_{G/K}(({}_\eta(A\otimes_{\mathbb{Z}G} B))^K).$$

Therefore, by the condition (III) of the definition of the cup product, the statement (4) follows. □

Lemma 2.2 *Let A be a left P-module. If A is either (P,S)-projective or (P,S)-injective in the sense of* [H, §1], *then ${}_\eta A$ is $(\mathbb{Z}G,\mathbb{Z}K)$-projective. Hence $H^r(G,K,{}_\eta A)=0$ holds for any $r\in\mathbb{Z}$.*

Proof. Let A be (P,S)-projective. Then A is a direct summand of the left P-module $P\otimes_S A$. $P\otimes_S A\simeq \mathbb{Z}G\otimes_{\mathbb{Z}K} A\otimes_{\mathbb{Z}K}\mathbb{Z}G$ holds as left P-modules and we have the isomorphism of left G-modules

$$\varphi : {}_\eta(\mathbb{Z}G\otimes_{\mathbb{Z}K} A\otimes_{\mathbb{Z}K}\mathbb{Z}G)\to \mathbb{Z}G\otimes_{\mathbb{Z}K} {}_\eta(A\otimes_{\mathbb{Z}K}\mathbb{Z}G)$$

given by $\varphi(\sigma\otimes_{\mathbb{Z}K} a\otimes_{\mathbb{Z}K}\tau)=\sigma\otimes_{\mathbb{Z}K}(a\otimes_{\mathbb{Z}K}\tau\sigma)$ for σ, $\tau\in G$ and $a\in A$. Therefore the left $\mathbb{Z}G$-module ${}_\eta A$ is a direct summand of $\mathbb{Z}G\otimes_{\mathbb{Z}K} {}_\eta(A\otimes_{\mathbb{Z}K}\mathbb{Z}G)$. This means that ${}_\eta A$ is $(\mathbb{Z}G,\mathbb{Z}K)$-projective. Note that the (P,S)-injectivity is equivalent to the (P,S)-projectivity by [O, Theorem 7]. □

Let A be a left P-module. As in [N1, §1], we have the complete relative cohomology group $H^r(\mathbb{Z}G,\mathbb{Z}K,A)$ from complete (P,S)-resolutions of $\mathbb{Z}G$. Then both $\{H^r(\mathbb{Z}G,\mathbb{Z}K,-)\}_{r\in\mathbb{Z}}$ and $\{H^r(G,K,{}_\eta(-))\}_{r\in\mathbb{Z}}$ are regarded as families of covariant functors from the category of P-modules to the category of $\mathbb{Z}$-modules with connecting homomorphisms $\partial : H^r(\mathbb{Z}G,\mathbb{Z}K,A_3)\to H^{r+1}(\mathbb{Z}G,\mathbb{Z}K,A_1)$ and $\partial : H^r(G,K,{}_\eta A_3)\to H^{r+1}(G,K,{}_\eta A_1)$ derived from a (P,S)-exact sequence $0\to A_1\to A_2\to A_3\to 0$, respectively. Since there exist the isomorphisms $H^0(\mathbb{Z}G,\mathbb{Z}K,A)\simeq({}_\eta A)^G/N_{G/K}(({}_\eta A)^K)\simeq$

$H^0(G, K, {}_\eta A)$ by [N1, Proposition 1.2] and Lemma 1.1, it follows that, by [N2, Proposition 4.1] and Lemma 2.2, the isomorphism $H^0(\mathbb{Z}G, \mathbb{Z}K, A) \xrightarrow{\sim} H^0(G, K, {}_\eta A)$ extends uniquely to the isomorphism

$$\Phi^r : H^r(\mathbb{Z}G, \mathbb{Z}K, A) \xrightarrow{\sim} H^r(G, K, {}_\eta A)$$

for any $r \in \mathbb{Z}$ such that $\Phi^{r+1} \circ \partial = \partial \circ \Phi^r$ holds for the connecting homomorphisms ∂'s. Thus we have the following theorem.

Theorem 2.3 *Let A and B be left P-modules. Then we have*

$$\Phi^{r+s}(\alpha \cup \beta) = \Phi^r(\alpha) \cup_\rho \Phi^s(\beta)$$

for any $\alpha \in H^r(\mathbb{Z}G, \mathbb{Z}K, A)$ and $\beta \in H^s(\mathbb{Z}G, \mathbb{Z}K, B)$, where $\cup$ denotes the cup product in [N1, §3].

Proof. By [N1, Definition 3.1 (iv)] and Proposition 2.1 (4), $\Phi^0(\alpha \cup \beta) = \Phi^0(\alpha) \cup_\rho \Phi^0(\beta)$ holds for any $\alpha \in H^0(\mathbb{Z}G, \mathbb{Z}K, A)$ and $\beta \in H^0(\mathbb{Z}G, \mathbb{Z}K, B)$. Therefore, by [N1, Definition 3.1 (ii) and (iii)] and Proposition 2.1 (2) and (3), this theorem is shown by induction for r and s. □

By the modified cup product $\cup_\rho$, the direct sum $H^*(G, K, {}_\eta\mathbb{Z}G) := \bigoplus_{r\in\mathbb{Z}} H^r(G, K, {}_\eta\mathbb{Z}G)$ is regarded as a ring, and the direct sum $H^*(\mathbb{Z}G, \mathbb{Z}K, \mathbb{Z}G) := \bigoplus_{r\in\mathbb{Z}} H^r(\mathbb{Z}G, \mathbb{Z}K, \mathbb{Z}G)$ is also regarded as a ring by the cup product. Therefore, by Theorem 2.3, we have the following corollary.

Corollary 2.4 *The ring $H^*(\mathbb{Z}G, \mathbb{Z}K, \mathbb{Z}G)$ is isomorphic to the ring $H^*(G, K, {}_\eta\mathbb{Z}G)$.*

$\mathbb{Z}$ is regarded as a left $\mathbb{Z}G$-module with the trivial G-action, and $H^*(G, K, \mathbb{Z}) := \bigoplus_{r\in\mathbb{Z}} H^r(G, K, \mathbb{Z})$ is regarded as a ring by the cup product in §1. The augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ induces the homomorphism

$$\varepsilon^* : H^r(G, K, {}_\eta\mathbb{Z}G) \to H^r(G, K, \mathbb{Z})$$

for any $r \in \mathbb{Z}$. On the other hand, the inclusion homomorphism $\iota : \mathbb{Z} \to {}_\eta\mathbb{Z}G$ induces the homomorphism

$$\iota^* : H^r(G, K, \mathbb{Z}) \to H^r(G, K, {}_\eta\mathbb{Z}G)$$

for any $r \in \mathbb{Z}$. Since $\varepsilon^* \circ \iota^*$ is the identity homomorphism, it follows that the ring $H^*(G, K, \mathbb{Z})$ is a direct summand of the ring $H^*(G, K, {}_\eta\mathbb{Z}G)$. Therefore, by Corollary 2.4, we have the following corollary.

Corollary 2.5 *The ring $H^*(G,K,\mathbb{Z})$ is a direct summand of the ring $H^*(\mathbb{Z}G,\mathbb{Z}K,\mathbb{Z}G)$.*

3. Alternate proof for the absolute case of Theorem 2.3

We can verify that the isomorphism $\Phi^r : H^r(\mathbb{Z}G,A) \xrightarrow{\sim} \hat{H}^r(G,{}_\eta A)$ for every $r \in \mathbb{Z}$ preserves the cup products by means of its formula on the (co)chain level, which is the absolute complete cohomology case of Theorem 2.3.

To be precise, let $X_{\mathbb{Z}G}$ be the standard complex of $\mathbb{Z}G$ and $X_G = X_{\mathbb{Z}G}\otimes_{\mathbb{Z}G}\mathbb{Z}$ the standard G-complex. $\sigma_0[\sigma_1,\ldots,\sigma_n]\sigma_{n+1}$ denotes the element $\sigma_0\otimes\sigma_1\otimes\cdots\otimes\sigma_n\otimes\sigma_{n+1}$ of the n-th component $(X_{\mathbb{Z}G})_n$ for $\sigma_i\in G$, and $\sigma_0[\sigma_1\mid\cdots\mid\sigma_n]$ denotes the element $\sigma_0[\sigma_1,\ldots,\sigma_n]\otimes_{\mathbb{Z}G}1$ of the n-th component $(X_G)_n$ for $\sigma_i\in G$. P denotes the enveloping algebra $\mathbb{Z}G\otimes(\mathbb{Z}G)^\circ$ of $\mathbb{Z}G$. Let A be a left P-module and we regard A as a two-sided $\mathbb{Z}G$-module. Then we have the following isomorphism Φ^r of the (co)chains which give the previous isomorphism of the complete cohomologies:

$$\text{if } r(=:p)\geqslant 0,\quad \Phi^p : \mathrm{Hom}_P((X_{\mathbb{Z}G})_p,A)\to \mathrm{Hom}_{\mathbb{Z}G}((X_G)_p,{}_\eta A);$$
$$\Phi^p(f)(\sigma_0[\sigma_1\mid\cdots\mid\sigma_p])=f(\sigma_0[\sigma_1,\ldots,\sigma_p](\sigma_0\cdots\sigma_p)^{-1}),$$
$$\text{and if } r(=:-q)\leqslant -1,\quad \Phi^{-q} : (X_{\mathbb{Z}G})_{q-1}\otimes_P A\to {}_\eta A\otimes_{\mathbb{Z}G}(X_G)_{q-1};$$
$$\Phi^{-q}(\sigma_0[\sigma_1,\ldots,\sigma_{q-1}]\sigma_q\otimes_P a)=(\sigma_0\cdots\sigma_q)a\otimes_{\mathbb{Z}G}\sigma_0[\sigma_1\mid\cdots\mid\sigma_{q-1}],$$

where ${}_\eta A$ is regarded as a right G-module by $a*g = g^{-1}\circ a(=\eta(g^{-1})a = g^{-1}ag)$ for $a\in{}_\eta A$ and $g\in G$.

Then it is shown that the following diagram commutes:

$$\begin{array}{ccc} H^r(\mathbb{Z}G,A)\otimes H^s(\mathbb{Z}G,B) & \xrightarrow{\Phi^r\otimes\Phi^s} & \hat{H}^r(G,{}_\eta A)\otimes\hat{H}^s(G,{}_\eta B) \\ \cup\downarrow & & \downarrow\cup_\rho \\ H^{r+s}(\mathbb{Z}G,A\otimes_{\mathbb{Z}G}B) & \xrightarrow[\Phi^{r+s}]{} & \hat{H}^{r+s}(G,{}_\eta(A\otimes_{\mathbb{Z}G}B)). \end{array}$$

In the above, the cup product $\cup$ is given in [S, Section 2.4] and the product $\cup_\rho$ is induced by the cup product $\cup_G : \hat{H}^r(G,M)\otimes\hat{H}^s(G,N)\to\hat{H}^{r+s}(G,M\otimes N)$ and the G-pairing $\rho : {}_\eta A\otimes{}_\eta B\to{}_\eta(A\otimes_{\mathbb{Z}G}B)$ defined in the previous section.

The formulas in [K, page 69] are slightly modified into the following formulas for $\cup_G$, that is, the cases (ii), (iv) and (vi) are altered by means of the anti-commutativity of the cup product.

(i) the case $r = p$ and $s = q$ where $p \geqslant 0$ and $q \geqslant 0$

$\cup_G : \mathrm{Hom}_{\mathbb{Z}G}((X_G)_p, M) \otimes \mathrm{Hom}_{\mathbb{Z}G}((X_G)_q, N) \to \mathrm{Hom}_{\mathbb{Z}G}((X_G)_{p+q}, M \otimes N);$

$(f \cup_G g)([\sigma_1 \mid \cdots \mid \sigma_{p+q}]) = f([\sigma_1 \mid \cdots \mid \sigma_p]) \otimes \sigma_1 \cdots \sigma_p g([\sigma_{p+1} \mid \cdots \mid \sigma_{p+q}])$

(ii) the case $r = -p$ and $s = -q$ where $p \geqslant 1$ and $q \geqslant 1$

$\cup_G : (M \otimes_{\mathbb{Z}G} (X_G)_{p-1}) \otimes (N \otimes_{\mathbb{Z}G} (X_G)_{q-1}) \to (M \otimes N) \otimes_{\mathbb{Z}G} (X_G)_{p+q-1};$

$$\begin{aligned}&(m \otimes_{\mathbb{Z}G} [\sigma_1 \mid \cdots \mid \sigma_{p-1}]) \cup_G (n \otimes_{\mathbb{Z}G} [\tau_1 \mid \cdots \mid \tau_{q-1}])\\ &\quad = (-1)^{pq} \sum_{\sigma \in G} (\tau_1 \cdots \tau_{q-1} \sigma m \otimes n) \otimes_{\mathbb{Z}G} [\tau_1 \mid \cdots \mid \tau_{q-1} \mid \sigma \mid \sigma_1 \mid \cdots \mid \sigma_{p-1}]\end{aligned}$$

(iii) the case $r = p$ and $s = -q$ where $p \geqslant q \geqslant 1$

$\cup_G : \mathrm{Hom}_{\mathbb{Z}G}((X_G)_p, M) \otimes (N \otimes_{\mathbb{Z}G} (X_G)_{q-1}) \to \mathrm{Hom}_{\mathbb{Z}G}((X_G)_{p-q}, M \otimes N);$

$$\begin{aligned}&(f \cup_G (n \otimes_{\mathbb{Z}G} [\sigma_1 \mid \cdots \mid \sigma_{q-1}]))\,([\tau_1 \mid \cdots \mid \tau_{p-q}])\\ &\quad = (-1)^{(q-1)q/2} \sum_{\sigma \in G} f([\tau_1 \mid \cdots \mid \tau_{p-q} \mid \sigma \mid \sigma_1 \mid \cdots \mid \sigma_{q-1}]) \otimes \tau_1 \cdots \tau_{p-q} \sigma n\end{aligned}$$

(iv) the case $r = p$ and $s = -q$ where $q > p \geqslant 0$

$\cup_G : \mathrm{Hom}_{\mathbb{Z}G}((X_G)_p, M) \otimes (N \otimes_{\mathbb{Z}G} (X_G)_{q-1}) \to (M \otimes N) \otimes_{\mathbb{Z}G} (X_G)_{q-p-1};$

$$\begin{aligned}&f \cup_G (n \otimes_{\mathbb{Z}G} [\sigma_1 \mid \cdots \mid \sigma_{q-1}])\\ &\quad = (-1)^{pq+p(p+1)/2} (\sigma_1 \cdots \sigma_{q-p-1} f([\sigma_{q-p} \mid \cdots \mid \sigma_{q-1}]) \otimes n)\\ &\qquad \otimes_{\mathbb{Z}G} [\sigma_1 \mid \cdots \mid \sigma_{q-p-1}]\end{aligned}$$

(v) the case $r = -p$ and $s = q$ where $q \geqslant p \geqslant 1$

$\cup_G : (M \otimes_{\mathbb{Z}G} (X_G)_{q-1}) \otimes \mathrm{Hom}_{\mathbb{Z}G}((X_G)_p, N) \to \mathrm{Hom}_{\mathbb{Z}G}((X_G)_{q-p}, M \otimes N);$

$$\begin{aligned}&((m \otimes_{\mathbb{Z}G} [\sigma_1 \mid \cdots \mid \sigma_{p-1}]) \cup_G f)\,([\tau_1 \mid \cdots \mid \tau_{q-p}])\\ &\quad = (-1)^{(p-1)p/2} \sum_{\sigma \in G} (\sigma_1 \cdots \sigma_{p-1} \sigma)^{-1} m\\ &\qquad \otimes (\sigma_1 \cdots \sigma_{p-1} \sigma)^{-1} f([\sigma_1 \mid \cdots \mid \sigma_{p-1} \mid \sigma \mid \tau_1 \mid \cdots \mid \tau_{q-p}])\end{aligned}$$

(vi) the case $r = -p$ and $s = q$ where $p > q \geqslant 0$

$\cup_G : (M \otimes_{\mathbb{Z}G} (X_G)_{q-1}) \otimes \mathrm{Hom}_{\mathbb{Z}G}((X_G)_p, N) \to (M \otimes N) \otimes_{\mathbb{Z}G} (X_G)_{p-q-1};$

$$\begin{aligned}&(m \otimes_{\mathbb{Z}G} [\sigma_1 \mid \cdots \mid \sigma_{p-1}]) \cup_G f\\ &\quad = (-1)^{pq+q(q+1)/2} ((\sigma_1 \cdots \sigma_q)^{-1} m \otimes (\sigma_1 \cdots \sigma_q)^{-1} f([\sigma_1 \mid \cdots \mid \sigma_q]))\\ &\qquad \otimes_{\mathbb{Z}G} [\sigma_{q+1} \mid \cdots \mid \sigma_{p-1}]\end{aligned}$$

In the following, we only show the commutativity of the diagram in the case (ii). First we have

$$\begin{aligned}
&\cup_{\mathbb{Z}G} \circ (\Phi^{-p} \otimes \Phi^{-q}) (([\sigma_1, \ldots, \sigma_{p-1}] \otimes_P a) \otimes ([\tau_1, \ldots, \tau_{q-1}] \otimes_P b)) \\
&\quad = \cup_{\mathbb{Z}G}((\sigma_1 \cdots \sigma_{p-1} a \otimes_{\mathbb{Z}G} [\sigma_1 \mid \cdots \mid \sigma_{p-1}]) \\
&\qquad\qquad \otimes (\tau_1 \cdots \tau_{q-1} b \otimes_{\mathbb{Z}G} [\tau_1 \mid \cdots \mid \tau_{q-1}])) \\
&\quad = (-1)^{pq} \sum_{\sigma \in G} (\eta(\tau_1 \cdots \tau_{q-1}\sigma)\sigma_1 \cdots \sigma_{p-1} a \otimes_{\mathbb{Z}G} \tau_1 \cdots \tau_{q-1} b) \\
&\qquad\qquad \otimes_{\mathbb{Z}G} [\tau_1 \mid \cdots \mid \tau_{q-1} \mid \sigma \mid \sigma_1 \mid \cdots \mid \sigma_{p-1}] \\
&\quad = (-1)^{pq} \sum_{\sigma \in G} (\tau_1 \cdots \tau_{q-1}\sigma\sigma_1 \cdots \sigma_{p-1} a \sigma^{-1} \otimes_{\mathbb{Z}G} b) \\
&\qquad\qquad \otimes_{\mathbb{Z}G} [\tau_1 \mid \cdots \mid \tau_{q-1} \mid \sigma \mid \sigma_1 \mid \cdots \mid \sigma_{p-1}].
\end{aligned}$$

On the other hand, by noticing that $(\sigma)_{\sigma\in G}$ and $(\sigma^{-1})_{\sigma\in G}$ are $\mathbb{Z}$-dual bases of $\mathbb{Z}G$ (cf. [S, §1.1]), we have

$$\begin{aligned}
&\Phi^{-p-q} \circ \cup (([\sigma_1, \ldots, \sigma_{p-1}] \otimes_P a) \otimes ([\tau_1, \ldots, \tau_{q-1}] \otimes_P b)) \\
&\quad = \Phi^{-p-q}\Big((-1)^{pq} \sum_{\sigma \in G} [\tau_1, \ldots, \tau_{q-1}, \sigma, \sigma_1, \ldots, \sigma_{p-1}] \otimes_P (a\sigma^{-1} \otimes_{\mathbb{Z}G} b)\Big) \\
&\quad = (-1)^{pq} \sum_{\sigma \in G} (\tau_1 \cdots \tau_{q-1}\sigma\sigma_1 \cdots \sigma_{p-1} a \sigma^{-1} \otimes_{\mathbb{Z}G} b) \\
&\qquad\qquad \otimes_{\mathbb{Z}G} [\tau_1 \mid \cdots \mid \tau_{q-1} \mid \sigma \mid \sigma_1 \mid \cdots \mid \sigma_{p-1}].
\end{aligned}$$

Hence the commutativity was shown. The other cases are proved similarly.

Remark. By means of the ring homomorphisms induced by ε^* and ι^* in §2, we see that $H^*(\mathbb{Z}G, \mathbb{Z}G)$ has an invertible element of non-zero degree if and only if so does $\hat{H}^*(G, \mathbb{Z})$. Of course, the existence of an invertible element of non-zero degree in the complete cohomology ring means that the cohomology is periodic.

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