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On the sharpness of Seeger-Sogge-Stein orders

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Abstract. We will extend the sharpness results on L^{p}- and $L^{p}-L^{q}$-continuity of Fourier integral operators for an arbitrary rank of the canonical projection. For the elliptic operators of small negative orders we will show that by a coordinate change they are equivalent to pseudo-differential operators.

Key words: Fourier integral operator, regularity, sharp estimates, pseudo-differential operator, Lagrangian manifold.

1. Introduction

Let X, Y be smooth paracompact n-dimensional manifolds. Let $d\sigma_{X}$ and $d\sigma_{Y}$ be the standard symplectic forms on $T^{*}X$ and $T^{*}Y$ and let Λ be a conic Lagrangian submanifold of $T^{*}X\backslash 0\times T^{*}Y\backslash 0$, equipped with the symplectic form $d\sigma_{X}-d\sigma_{Y}$. We will assume that Λ is a local graph of a symplectomorphism from $T^{*}Y\backslash 0$ to $T^{*}X\backslash 0$. Let $T\in I^{\mu}(X, Y;\Lambda)$ be a Fourier integral operator with the canonical relation Λ. The distributional kernel $K\in \mathcal{D}'(X\times Y)$ of T is a Lagrangian distribution of order μ whose wavefront set is contained in $\Lambda'=\{(x, \xi, y, \eta):(x, \xi, y, -\eta)\in\Lambda\}$. The global theory of such operators can be found in [1]. Let $\pi_{X\times Y}$ be the natural projection from $T^{*}X\backslash 0\times T^{*}Y\backslash 0$ to $X\times Y$. The deep result of Seeger, Sogge and Stein [5] states that for $1<p<\infty$ and $\mu\leq-(n-1)|1/p-1/2|$ the operators $T\in I^{\mu}(X, Y;\Lambda)$ are continuous from $L_{comp}^{p}(Y)$ to $L_{loc}^{p}(X)$. This result is sharp if T is elliptic and $d\pi_{X\times Y}|_{\Lambda}$ has full rank equal to $2n-1$ anywhere, which follows from the stationary phase method as in [3]. Somewhat different approaches to this are in [6] and [7]. If the rank of the canonical projection on Λ can be bounded from above by

$$\operatorname{rank} d\pi_{X\times Y}|_{\Lambda}\leq 2n-k \tag{1}$$

with some $1\leq k\leq n$, then under the so-called smooth factorization condition introduced in [5] the operators $T\in I_{\rho}^{\mu}(X, Y;\Lambda)$, $1/2\leq\rho\leq 1$, are continuous from $L_{comp}^{p}(Y)$ to $L_{loc}^{p}(X)$ for $1<p<\infty$ and $\mu\leq-(n-k\rho)|1/p-1/2|$.

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In [4] the factorization condition is shown to be satisfied in a number of important cases, if a phase function of the operator is analytic.

Using analysis of some convolution operators in [8], it was shown in [5] that there exist conormal operators with constant rank $d\pi_{X\times Y}|_\Lambda \equiv 2n-k$, for which the estimate of the critical order μ is sharp. We want to show that for $\rho = 1$ this order is sharp for an arbitrary elliptic operator whose canonical relation satisfies inequality (1). The basic idea to test the L^p-continuity of an operator will be to investigate its behavior on the functions obtained from a δ-distribution at some $y_0 \in Y$ after the application of elliptic pseudo-differential operators of sufficiently negative orders. The only singularities of such functions are at y_0, meanwhile the singularities of T applied to them happen only in the directions transversal to some $(n-k)$-dimensional subset Σ_{y_0} of X. Finally, this will be applied to the continuous Fourier integral operators of zero order.

It was pointed out in [7, p. 398], that in $\mathbb{R}^3$ the operator $T : f \mapsto \frac{\partial}{\partial x_j}(f * d\sigma)$ with $j = 1, 2$, or 3, and $d\sigma$ the usual measure on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, is essentially a Fourier integral operator of order 0, which is not continuous in $L^p(\mathbb{R}^3)$, $1 < p < \infty$. We will show that this is not a single example and derive a structural formula for the continuous elliptic Fourier integral operators of order 0 (Theorem 2) and then generalize it for small negative orders and $L^p \to L^q$ continuity (Theorem 3).

2. Results

By the equivalence-of-phase-function theorem as in [1, Th. 2.3.4] and [5] it is sufficient to consider operators in $\mathbb{R}^n$ with kernel

$$K(x,y) = \int_{\mathbb{R}^n} e^{i[\langle x,\xi\rangle - \phi(y,\xi)]} b(x,y,\xi) d\xi, \tag{2}$$

with some symbol $b \in S^\mu$ vanishing for x, y outside a compact set and phase function satisfying

$$\det \phi''_{y\xi} \neq 0 \tag{3}$$

on the support of b, which is equivalent to Λ being a canonical graph. Locally Λ is the set of the form $\{(\nabla_\xi \phi, \xi, y, \nabla_y \phi)\}$. We begin with the following

Proposition 1 *Let $T \in I^\mu(X, Y; \Lambda)$ be elliptic. Assume that the canon-*

ical relation Λ *is a local graph and* $\operatorname{rank} d\pi_{X\times Y}|_\Lambda \equiv 2n-k$, $1 \le k \le n$. *Then* T *is not bounded as a linear operator* $L^p_{comp}(Y) \to L^p_{loc}(X)$, *if* $\mu > -(n-k)|1/p - 1/2|$, $1 < p < \infty$.

Proof. By the above reduction it is sufficient to restrict ourselves to the case of $\mathbb{R}^n$ and operators satisfying (2) and (3). Let $P_{-s} \in \Psi^{-s}(Y)$ be an elliptic pseudo-differential operator in Y and consider $f_s(y) = (P_{-s}\delta_{y_0})(y)$. Then by Schwartz kernel theorem $f_s(y) = \int K_{-s}(y,z)\delta_{y_0}(z)dz = K_{-s}(y,y_0)$, and in view of the kernel estimates for pseudo-differential operators in, for example, [7, p. 241, 245], we have $|K_{-s}(y,y_0)| \le C|y-y_0|^{-n+s}$ in some local coordinate system. It follows that $f_s \in L^p_{loc}$ if and only if $s > n(1-1/p)$. We assume here $1 < p \le 2$, for the rest would follow by considering the adjoint operators.

Let $\Sigma = \pi_{X\times Y}(\Lambda)$. Then in view of the assumption on the rank of $\pi_{X\times Y}$, $\Sigma \subset X \times Y$ is a smooth submanifold of codimension k. Let Σ be given by the set of equations $h_j(x,y) = 0$, $1 \le j \le k$, in a neighborhood of y_0, where $\nabla h_1, \ldots, \nabla h_k$ are linearly independent. Then Λ is the conormal bundle of Σ and the phase function of T may be given by

$$\psi(x,y,\lambda) = \sum_{j=1}^{k} \lambda_j h_j(x,y).$$

Let $T_s = T \circ P_{-s}$. Then $Tf_s(x) = T_s(\delta_{y_0})(x)$ and the canonical relations of T_s and T coincide, since a composition with a pseudo differential operator leaves it invariant. The operator T_s is of order $\mu - s$ and in local coordinates it can be expressed as

$$\begin{aligned} Tf_s(x) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^k} e^{i\sum\lambda_j h_j(x,y)} a(x,\bar\lambda)\delta_{y_0}(y)d\bar\lambda\right)dy \\ &= \int_{\mathbb{R}^k} e^{i\langle\bar\lambda,\bar h(x,y_0)\rangle} a(x,\bar\lambda)d\bar\lambda \\ &= (2\pi)^k \check a(x,\bar h(x,y_0)), \end{aligned} \tag{4}$$

where $\bar\lambda$ and $\bar h$ are the vectors with the components λ_j and h_j respectively, and $a \in S^{\mu-s+(n-k)/2}(\mathbb{R}^k)$ is a symbol of T_s after applying the stationary phase method and integrating away $(n-k)$-variables. Now, the inverse Fourier transform of a in the second variable is $(2\pi)^k\check a(x,\zeta) = \int_{\mathbb{R}^k} e^{i\langle\lambda,\zeta\rangle}a(x,\lambda)\hat\delta_0(\lambda)d\lambda = P_0\delta_0(\zeta) = K_0(\zeta,0)$ and this is equivalent to $|\zeta|^{-k-\operatorname{ord}(a)}$, where $P_0 \in \Psi^{\operatorname{ord}(a)}(\mathbb{R}^k)$ with symbol equal to $a(x,\lambda)$ and K_0

is a distributional kernel of P_0. In view of $\mathrm{dist}(x, \Sigma_{y_0}) \approx |\bar{h}(x, y_0)|$ with $\Sigma_{y_0} = \{x : (x, y_0) \in \Sigma\}$ and formulas above, we have $(2\pi)^k \breve{a}(x, \bar{h}(x, y_0)) \sim |\mathrm{dist}(x, \Sigma_{y_0})|^{-k-(\mu-s+(n-k)/2)}$, locally uniformly in x. Formula (4) implies that Tf_s is smooth along Σ_{y_0}, so $Tf_s \notin L^p_{loc}(\mathbb{R}^n)$ if and only if $p(k+\mu-s+(n-k)/2) \geq k$, or, equivalently, $s \leq \mu+k(1-1/p)+(n-k)/2$. Together with condition on $f_s \in L^p_{loc}$ this implies that T is not continuous in L^p-norms if such s exists, i.e. when $\mu > -(n-k)|1/p-1/2|$. This completes the proof. □

Assume now that the operator T is not conormal and that (1) is satisfied with $2n-k$ at some point. Then the set $\Lambda_0 = \{\lambda \in \Lambda : \mathrm{rank}\, d\pi_{X\times Y}|_\Lambda(\lambda) = 2n-k\}$ is nonempty and open in Λ. Applying the equivalence of the phase function and the same argument as in Proposition 1 at some $\lambda_0 = (x_0, \xi_0, y_0, \eta_0) \in \Lambda_0$, we get

Theorem 1 *Let $T \in I^\mu(X, Y; \Lambda)$ be elliptic. Assume that the canonical relation Λ is a local graph and that* $\mathrm{rank}\, d\pi_{X\times Y}|_\Lambda \leq 2n-k$, $1 \leq k \leq n$, *equal to $2n-k$ at some point. Then T is not bounded as a linear operator $L^p_{comp}(Y) \to L^p_{loc}(X)$, if $\mu > -(n-k)|1/p-1/2|$, $1 < p < \infty$.*

The application of the arguments of [5] to Theorem 1 yields that an operator T as in Theorem 1 is not bounded as a linear operator in Sobolev spaces $L^p_\alpha \to L^p_{\alpha-(n-k)|1/p-1/2|-\mu}$, $1 < p < \infty$.

It is well known ([2]) that pseudo-differential operators of zero order are continuous in L^p-spaces, $1 < p < \infty$. It turns out that all elliptic Fourier integral operators with this property can be obtain from pseudo-differential operators by a smooth coordinate change in one of the spaces X or Y. For a smooth map $\kappa : X \to Y$ the pullback by κ is a mapping $\kappa^* : C^\infty(Y) \to C^\infty(X)$ defined by $(\kappa^* f)(x) = f(\kappa(x))$. This pullback is a Fourier integral operator with the canonical relation corresponding to the phase function $\langle \kappa(x) - y, \eta\rangle$ and given by the graph of the induced transformation $\tilde{\kappa} : T^*X\backslash 0 \to T^*Y\backslash 0$ with $\tilde{\kappa}(x, \xi) = (\kappa(x), -({}^tD\kappa_x)^{-1}(\xi))$. See [1, 2.4] for more detailed discussion.

Theorem 2 *Let $T \in I^0(X, Y; \Lambda)$ be elliptic and assume Λ to be a local graph, $1 < p < \infty$, $p \neq 2$. Then T is continuous from $L^p_{comp}(Y)$ to $L^p_{loc}(X)$ if and only if there exist $P \in \Psi^0(X), Q \in \Psi^0(Y)$, such that $T = P \circ \kappa^*_- = \kappa^*_+ \circ Q$, where κ^*_- and κ^*_+ are the pullbacks by smooth coordinate changes $X \to Y$.*

Proof. The operators κ_-^* and κ_+^* are clearly L^p continuous, and this together with the continuity of pseudo-differential operators of order 0 imply the continuity of T. Conversely, let k be a minimal codimension of $\Sigma = \pi_{X\times Y}(\Lambda)$ in $X \times Y$, i.e. $2n - k = \max_{\lambda\in\Lambda} \operatorname{rank} d\pi_{X\times Y}|_\Lambda(\lambda)$. Then Theorem 1 together with our assumption of the continuity of T imply $k = n$. This means that $\operatorname{rank} d\pi_{X\times Y}|_\Lambda \equiv n$ and Σ is a smooth n-dimensional submanifold of $X \times Y$. The rank of $d\pi_X|_\Sigma$ of the projection $\pi_X : X \times Y \to X$ is equal to n in view of the assumption on Λ to be a local graph. The surjectivity of $d\pi_X|_\Sigma$ together with $\dim \Sigma = n$ imply that $\pi_X|_\Sigma$ is a diffeomorphism, and locally $\Sigma = \{(x, \sigma(x))\}$, σ a diffeomorphism. The pullback operator $\kappa_+^* = \sigma^*$ has the canonical relation equal to the conormal bundle of Σ, which is Λ, implying that the operator Q in $T = \kappa_+^* \circ Q$ is pseudo-differential. The same argument applies for Y space to yield the second part of the Theorem. □

Finally we would like to make some remarks about $L^p(Y) \to L^q(X)$-continuity. Under the factorization assumptions of [5], the interpolation between $L^p \to L^p$ and $H^1 \to L^2$ for operators of order $-n/2$ ([7, Ch. 3,5.21]) yields that for $1 < p \le q \le 2$ and $2 \le p \le q < \infty$ the operators $T \in I^\mu(X, Y; \Lambda)$ are continuous from $L^p(Y)$ to $L^q(X)$ for $\mu \le -n/p + k/q + (n - k)/2$. Note that for $k = 1$ we get the orders of [7, Ch. 9,6.15]. The technique of the proof of Proposition 1 can be applied to show that an elliptic operator $T \in I^\mu(X, Y; \Lambda)$ with maximal rank equal to $2n - k$ at some point is not continuous from $L^p(Y)$ to $L^q(X)$ if $\mu > (n - k)/2 - n/p + k/q$, which shows that the orders above are sharp. A straightforward generalization of Theorem 2 yields

Theorem 3 *Let $T \in I^\mu(X, Y; \Lambda)$ be elliptic and assume Λ to be a local graph, $1 < p \le q < 2$. Assume that $-n(1/p - 1/q) \ge \mu > -(1/q - 1/2) - n(1/p - 1/q)$. Then T is continuous from $L^p_{comp}(Y)$ to $L^q_{loc}(X)$ if and only if there exist $P \in \Psi^\mu(X)$, $Q \in \Psi^\mu(Y)$, such that $T = P \circ \kappa_-^* = \kappa_+^* \circ Q$, where κ_-^* and κ_+^* are the pullbacks by smooth coordinate changes $X \to Y$.*

The converse statement follows from $L^p \to L^q$-continuity of pseudo-differential operators of order $-n(1/p - 1/q)$, which can be obtained from [7, Ch. 9,6.15] by Hardy-Littlewood argument or by interpolation between $H^1 \to L^2$ and $L^p \to L^p$ for zero order operators. Note that the argument of Proposition 1 with $k = n$ implies that this order is also sharp. By duality the same conclusion holds for $2 < p \leq q < \infty$. Finally we would like to note that because the graphs of the transformations κ_+^* and κ_-^* in Theorems 2 and 3 are the same, it follows that κ_+ and κ_- are equal.

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