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Dual-bimodules and torsion theories

(In memory of Professor Masanori Kato)

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Abstract. Alin and Dickson have pointed out that the Goldie torsion theory is centrally splitting when the ring is QF. This is also true for a dual ring. More generally, it is shown that if R is semiperfect with $soc(_RR)^2 \cdot rad(R) = 0$, then there exists a 3-fold torsion theory with length 2, which reduces the Goldie torsion theory in case R is a dual ring. It is also shown that every left dual-bimodule $_RQ_S$ with R as above can be decomposed into a direct sum of two types of left dual-bimodules. In case $Q = R$, this means that R can be decomposed into a ring direct sum of a semisimple ring and a ring with essential left singular ideal. This is nothing but a result due to Alin and Dickson proved for QF rings.

Key words: dual-bimodule, 3-fold torsion theory, Goldie torsion theory, QF-ring.

A ring R is called a dual ring [7] if

$$A = \ell_R r_R(A) \quad \text{and} \quad B = r_R \ell_R(B)$$

for every left ideal A and every right ideal B of R. A QF-ring is nothing but a dual ring, when it is Artinian. Generalizing the notion of dual rings, we have defined dual-bimodules in [11]. Let R and S be rings and $_RQ_S$ an (R, S)-bimodule. We shall call Q a left dual-bimodule if

$$\ell_R r_Q(A) = A \quad \text{and} \quad r_Q \ell_R(Q') = Q'$$

for every left ideal A of R and every S-submodule Q' of Q.

In [1], Alin and Dickson have pointed out that the Goldie torsion theory is centrally splitting when the ring is QF. We shall show in Section 1 that this is also true for a dual ring. More generally, it is shown that if R is a semiperfect ring with $rad(R) \in T$ (for the definition see below), then there exists a 3-fold torsion theory with length 2 (Theorem 1.5), which reduces the Goldie torsion theory in case R is a dual ring.

In Section 2, we shall show that every left dual-bimodule $_RQ_S$ with R semiperfect and $rad(R) \in T$ can be decomposed into a direct sum of two

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types of left dual-bimodules (Theorem 2.4). In case R is a dual ring, this means that R can be decomposed into a ring direct sum of a semisimple ring and a ring with essential left singular ideal. This is nothing but a result due to Alin and Dickson [1] proved for QF rings.

Throughout this paper all rings considered are associative rings with identity and all modules are unitary. Let R be a ring and ${}_RM$ a left R-module. For a subset X of M and a subset Y of R, the right annihilator of Y in M and the left annihilator of X in R are denoted by $r_M(Y)$ and $\ell_R(X)$, respectively. The socle of M and the singular submodule of M are denoted by $soc(M)$ and $Z(M)$, respectively. We also denote the Jacobson radical of R by $rad(R)$. For notations, definitions and results we shall mainly follow [2] and [11].

1. For a semiperfect ring R and $I = soc({}_RR)^2$, let

$$C = \{{}_RM \mid IM = M\},\ T = \{{}_RM \mid IM = 0\} \quad \text{and}$$
$$F = \{{}_RM \mid r_M(I) = 0\}.$$

Then (C, T, F) is a 3-fold torsion theory for R-mod [8], since I is idempotent (cf. [4, p. 900]). Furthermore

(1) Each element of C is semisimple. Indeed, if $M \in C$, then $M = IM \leq soc({}_RR)M \leq soc(M)$. Hence, M is semisimple.

(2) C is hereditary. Indeed, if $M \in C$ and M' is a submodule of M, then there exists a submodule M'' of M such that $M = M' \oplus M''$. Hence $M = IM = IM' \oplus IM''$ and thus $IM' = M'$.

(3) Each element of F is nonsingular. Indeed, if $M \in F$ and $x \in Z(M)$, then $I \leq \mathrm{soc}({}_RR) \leq \ell_R(x)$. Hence $Ix = 0$ and $x \in r_M(I) = 0$. Thus we have $Z(M) = 0$.

Since C is hereditary, by [8, Lemma 2.2] $C \subseteq F$ and so each element of C is semisimple projective. Conversely, let ${}_RM$ be semisimple projective. Then $soc({}_RR)M = soc(M) = M$. Hence, $IM = M$ and $M \in C$. Therefore, we have

$$C = \{{}_RM \mid M \text{ is semisimple projective}\}$$

(cf. [5]). Since ${}_R(R/I)$ has a projective cover, by [3, Theorems 3 and 8] it follows that $I = eR$ for some idempotent e in R.

Let $\bar{R} = R/rad(R)$ and let $\bar{1} = \sum_{i=1}^{n} \sum_{j=1}^{k(i)} \bar{e}_{ij}$ be a decomposition of the identity of $\bar{R}$ into orthogonal primitive idempotents according to a decomposition of $\bar{R}$ into simple left $\bar{R}$-modules, where $\bar{R}\bar{e}_{ij} \cong \bar{R}\bar{e}_{kl}$ if and only if $i = k$.

These idempotents can be lifted to a complete set of orthogonal primitive idempotents e_{ij} in R modulo $rad(R)$. Let $\bar{e}_i = \bar{e}_{i1}$ for each i. Then the set $\{\bar{R}\bar{e}_1, \ldots, \bar{R}\bar{e}_n\}$ forms a representative set of nonisomorphic simple left R-modules. We may assume that

$$\bar{R}\bar{e}_1, \ldots, \bar{R}\bar{e}_m \in C, \quad \bar{R}\bar{e}_{m+1}, \ldots, \bar{R}\bar{e}_n \in T.$$

Lemma 1.1 *With the notation as above, the following conditions are equivalent for any i and any j:*

(1) $\bar{R}\bar{e}_{ij} \in C$.
(2) $rad(R)e_{ij} = 0$.
(3) $e_{ij} \in soc({}_RR)$.
(4) $e_{ij}I \neq 0$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\bar{R}\bar{e}_{ij} \in C$. Then $rad(R)e_{ij}$ is a direct summand of Re_{ij} and Re_{ij} is indecomposable. Hence $rad(R)e_{ij} = 0$.

(2) $\Rightarrow$ (3). [2, Proposition 15.17]. (3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). Suppose that $e_{ij}I \neq 0$. If $e_{ij}I$ is a proper submodule of $e_{ij}R$, then $e_{ij}I \leq e_{ij}rad(R)$ and $e_{ij}I = e_{ij}I \cdot soc({}_RR) \leq e_{ij}rad(R) \cdot soc({}_RR) = 0$, a contradiction. Hence, $e_{ij}I = e_{ij}R$, $e_{ij} \in I$ and $rad(R)e_{ij} = 0$. Thus, $\bar{R}\bar{e}_{ij}$ is projective. □

Hence, we have:

Lemma 1.2 $I = e'R$ *with* $e' = \sum_{i=1}^{m} \sum_{j=1}^{k(i)} e_{ij}$.

Let $E = \{e_{ij}\}_{i,j}$ and define a relation $\sim$ on E by $e_{ij} \sim e_{kl}$ in case there exist s and t, $1 \leq s \leq n$, $1 \leq t \leq k(s)$ such that $e_{st}Re_{ij} \neq 0$ and $e_{st}Re_{kl} \neq 0$ ([2, p. 100]).

Lemma 1.3 *Let $1 \leq i \leq m$ and suppose that $rad(R) \in T$. Then $e_{i1} \sim e_{kl}$ for some k and l if and only if $i = k$.*

Proof. Suppose that $e_{i1} \sim e_{kl}$ for some k and l. Then by definition $e_{st}Re_{i1} \neq 0$ and $e_{st}Re_{kl} \neq 0$ for some s and t. By [2, Exercise 27.9] Re_{i1} contains submodules K and L such that $K \leq L$ and $L/K \cong \bar{R}\bar{e}_{st}$. Since $1 \leq i \leq m$, $\bar{R}\bar{e}_{i1} = \bar{R}\bar{e}_i \in C$. Hence $rad(R)e_{i1} = 0$ by Lemma 1.1 and Re_{i1} is simple. Therefore $L = Re_{i1}$ and $K = 0$ and thus $\bar{R}\bar{e}_i \cong Re_{i1} = L/K \cong \bar{R}\bar{e}_s$. This means that $i = s$.

Similarly, Re_{kl} contains submodules K' and L' such that $K' \leq L'$ and $L'/K' \cong \bar{R}\bar{e}_{st}$. If $m+1 \leq k \leq n$, $\bar{R}\bar{e}_{kl} \cong \bar{R}\bar{e}_k \in T$. Moreover, as $rad(R) \in T$, $I \cdot rad(R) = 0$ and hence $rad(R)e_{kl} \in T$. Thus $Re_{kl} \in T$ since T is closed under extensions. This implies that $\bar{R}\bar{e}_{st} \cong L'/K' \in T$. However, as $i = s$, $\bar{R}\bar{e}_{st} \cong \bar{R}\bar{e}_s \in C$, a contradiction. Therefore $1 \leq k \leq m$. It follows that Re_{kl} is simple and $\bar{R}\bar{e}_{st} \cong L'/K' = Re_{kl}$ as is stated above. Hence $\bar{R}\bar{e}_s \cong Re_{kl} \cong \bar{R}\bar{e}_k$ and thus $s = k$.

Conversely, suppose that $i = k$. Then, since $\bar{R}\bar{e}_{i1} \cong \bar{R}\bar{e}_{il}$, $Re_{i1} \cong Re_{il}$ by [2, Proposition 17.18]. Now $e_{i1}Re_{i1} \neq 0$ and $e_{i1}Re_{i1} \cong e_{i1}Re_{il}$. Hence we have $e_{i1} \sim e_{il}$. □

From this lemma it follows that for each i, $1 \leq i \leq m$, $e_{i1} \approx e_{kl}$ ([2, p. 100]) for some k and l if and only if $i = k$. Hence $u_i = \sum_{j=1}^{k(i)} e_{ij}$ is the block idempotent and is cental ([2, Theorem 7.9]). Thus, $e' = \sum_{i=1}^{m} u_i$ is also central. Moreover, $e = e'$ as the following lemma shows.

Lemma 1.4 *With the notation as above, $e' = e$.*

Proof. Since $eR = e'R$, $e = e'u'$ and $e' = eu$ for some u, u' in R. Then $e'e = e$ and $ee' = e'$. Hence, as e' is central, $e = e'$. □

By [8, Theorem 2.7], we have:

Theorem 1.5 *Let R be a semiperfect ring with $rad(R) \in T$. Then (C, T, F) is a 3-fold torsion theory with length 2. R has a ring decomposition $R = eR \oplus (1-e)R$ with the central idempotent $e = \sum_{i=1}^{m} \sum_{j=1}^{k(i)} e_{ij}$.*

Note that a semiperfect ring cannot always satisfy the condition that $rad(R) \in T$. For example, let k be a field and R the ring of all 2×2 upper triangular matrices over k. Then $soc({}_RR)^2 \cdot rad(R) \neq 0$ and hence $rad(R) \notin T$.

To apply Theorem 1.5 to the case of dual rings, we need the following:

Lemma 1.6 *Let R' and S' be arbitrary rings and ${}_{R'}Q_{S'}$ a left dual-bimodule. Then the following conditions are equivalent:*

(1) *$soc({}_{R'}R')$ is essential in ${}_{R'}R'$.*
(2) *Every nonzero left ideal of R' contains a minimal left ideal.*
(3) *Every proper submodule of $Q_{S'}$ is contained in a maximal sub-*

module.

(4) $rad(Q_{S'})$ *is small in* $Q_{S'}$.

Proof. (1) $\Rightarrow$ (2) follows from [2, Corollary 9.10] and (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) follow from [2, Proposition 9.18] and [11, Lemma 1.5 and Proposition 1.6], respectively. □

Note that, in the above lemma, (3) is not equivalent to (4) in general (see [2, Exercise 9.4]).

Alin and Dickson [1, Example 2] have pointed out that the Goldie torsion theory is centrally splitting (cf. [8, p. 562]) when the ring is QF. However, by Theorem 1.5, this is true for dual rings. In fact, if R is a dual ring, then R is semiperfect by [7, Theorem 3.9] and $soc({}_RR)$ is essential in ${}_RR$ by Lemma 1.6. Hence by [9, Theorem 6] or [12, Theorem 3.1] I is the smallest element of the left Gabriel topology of the Goldie torsion functor G. By [9, Lemma 3], $G(M) = r_M(I)$ for each R-module ${}_RM$. Therefore, the torsion class of G is $\{{}_RM \mid IM = 0\}$ and coincides with T.

Moreover, it follows from [11, Proposition 1.6] that $soc({}_RR)\cdot rad(R) = 0$ and hence $rad(R) \in T$. Thus, we have:

Corollary 1.7 *If R is a dual ring, then the Goldie torsion theory is centrally splitting.*

2. In this section, let ${}_RQ_S$ be a left dual-bimodule and R a semiperfect ring with $rad(R) \in T$. Then by [11, Proposition 1.8] there are only finitely many nonisomorphic simple submodules of Q_S. We may take one of these as uS with $u \in Q$. Then Ru is simple and is in either C or T. As is easily seen, we have:

Lemma 2.1 *With the notation as above, the following conditions are equivalent:*

(1) $Ru \in C$.

(2) $\ell_R(u)$ *is a direct summand of* ${}_RR$.

(3) $\ell_R(u)$ *is not essential in* ${}_RR$.

(4) uS *is not small in* Q_S.

(5) $u \notin rad(Q_S)$.

Proposition 2.2 *Let ${}_RQ_S$ be a left dual-bimodule and R a semiperfect ring with $rad(R) \in T$. Consider the following conditions:*

(1) ${}_RQ \in C$.
(2) Q_S *is semisimple.*
(3) *No nonzero submodule of* Q_S *is small.*
(4) *Every simple left* R*-module is projective.*
(5) *Every minimal left ideal of* R *is projective.*
(6) $soc({}_RR)$ *is projective.*
(7) $soc({}_RR)^2 = soc({}_RR)$.

Then each of (1) *to* (4) *is equivalent. Each* (i) *from* (4) *to* (6) *implies* ($i+1$) *and if* $soc({}_RR)$ *is essential in* ${}_RR$, *then* (7) *implies* (1).

Proof. (1) ⇒ (2) follows from [11, Proposition 1.12] and (2) ⇒ (3) is trivial.

(3) ⇒ (4). Every simple left R-module is isomorphic to Ru with some u in Q ([11, p. 95]). Then uS is simple and is not small in Q_S. Hence by Lemma 2.1 Ru is projective.

(4) ⇒ (1). If every simple left R-module is projective, then R is semisimple and so is ${}_RQ$ by [11, Proposition 1.12].

(4) ⇒ (5) ⇒ (6) are trivial.

(6) ⇒ (7). As $soc({}_RR) \in C$, $soc({}_RR)^2 \cdot soc({}_RR) = soc({}_RR)$ and hence $soc({}_RR)^2 = soc({}_RR)$.

(7) ⇒ (1). Suppose that $soc({}_RR)$ is essential in ${}_RR$. Then since $soc({}_RR) = I = Re$ and $rad(R) \in T$, $soc({}_RR) \cap rad(R) = I \cdot rad(R) = 0$, from which it follows that $rad(R) = 0$. Hence R is semisimple and so is ${}_RQ$. □

The equivalence of (6) and (7) was also shown by [12, Theorem 3.6]. We shall call a left dual-bimodule ${}_RQ_S$ with $soc({}_RR)^2 = soc({}_RR)$ a left dual-bimodule of the first type. Every semisimple ring is a left dual-bimodule of the first type. If R is a dual ring, it is of the first type if and only if it is a semisimple ring.

Proposition 2.3 *Let* ${}_RQ_S$ *be a left dual-bimodule and* R *a semiperfect ring with* $rad(R) \in T$. *Then the following conditions are equivalent*:

(1) ${}_RQ \in T$.
(2) $soc(Q_S)$ *is small in* Q_S.
(3) *Every simple submodule of* Q_S *is small in* Q_S.
(4) *No simple left* R*-module is projective.*
(5) *Every minimal left ideal of* R *is nilpotent.*

(6) $soc({}_RR)^2 = 0$.
(7) ${}_RR \in T$.

Proof. (1) $\Rightarrow$ (2). Let $soc(Q_S) = \bigoplus_{i=1}^n u_iS$ with each u_iS simple ([11, Proposition 1.8]). Then each Ru_i is simple and is in T. Hence by Lemma 2.1 each u_iS is small in Q_S. Thus $soc(Q_S)$ is also small in Q_S.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Every simple left R-module is isomorphic to Ru for some $u \in Q$. If Ru is projective, then uS is simple and is not small in Q_S by Lemma 2.1, a contradiction.

(4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (6). Since every minimal left ideal is contained in $rad(R)$, it follows that $soc({}_RR)$ is in T, which shows that $soc({}_RR)^2 = 0$.

(6) $\Rightarrow$ (7) $\Rightarrow$ (1) are trivial. □

The equivalence of (4) and (7) was pointed out in [1, Example 2] for a ring with minimum condition on left ideals. We shall call a left dual-bimodule ${}_RQ_S$ with $soc({}_RR)^2 = 0$ a left dual-bimodule of the second type. For example, let ${}_RQ_R$ be the dual-bimodule of [11, Example 4.1], $Q' = p^{-2}R/R$ and $\bar{R} = R/Rp^2$. Then, ${}_{\bar{R}}Q'_R$ is a left dual-bimodule ([11, Example 4.2]) of the second type. If R is a dual ring, it is of the second type if and only if it is a ring with essential left singular ideal.

Theorem 2.4 *Every left dual-bimodule ${}_RQ_S$ for which R is a semiperfect ring with $rad(R) \in T$ can be decomposed into a direct sum of the left dual-bimodule ${}_{Re}eQ_S$ of the first type and the left dual-bimodule ${}_{R(1-e)}(1-e)Q_S$ of the second type.*

Proof. By [11, Proposition 1.15] ${}_{Re}eQ_S$ is a left dual-bimodule and, as $soc({}_{Re}Re) = soc({}_RRe) = soc({}_RR)e$, $soc({}_{Re}Re)^2 = soc({}_RR)^2e = Re$. Hence $soc({}_{Re}Re)^2 = soc({}_{Re}Re)$ and thus ${}_{Re}eQ_S$ is of the first type. Similarly, ${}_{R(1-e)}(1-e)Q_S$ is a left dual-bimodule with $soc({}_{R(1-e)}R(1-e))^2 = soc({}_RR)^2(1-e) = 0$ and hence it is of the second type. □

Alin and Dickson [1, Example 2] have pointed out that any QF-ring R has G-global dimension zero, i.e. $R = R' \oplus R''$ (ring direct sum) where R' is a ring with essential left singular ideal and R'' is semisimple with minimum condition. However, as an application of Theorem 2.4, we can generalize this result to the case of dual ring.

Corollary 2.5 *If R is a dual ring, then R has G-global dimension zero, i.e. $R = Re \oplus R(1-e)$ (ring direct sum) where Re is semisimple and $R(1-e)$ is a ring with essential left singular ideal.*

It is to be noted that, as is seen from the following example due to K. Koike, a semiperfect ring R with $rad(R) \in T$ need not be a dual ring in general, even if its $soc({}_RR)$ is essential in ${}_RR$. Let k be a field and $R = k \times k^2$, the trivial extension of k by the (k,k)-bimodule k^2. Then R is a semiperfect ring with $rad(R) = 0 \times k^2$. Hence by [2, Proposition 15.17] $soc({}_RR) = 0 \times k^2$. As is easily seen, $soc({}_RR)$ is essential in ${}_RR$ and $rad(R) \in T$. However, R is not a dual ring.

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