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On the number of singularities of a generic surface with boundary in a 3-manifold

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Abstract. We consider a C^{∞} generic map $f: M \to N$ of a compact surface M with boundary into a 3-manifold N with boundary which is neat (i.e., $f^{-1}(\partial N)=\partial M$). The isolated singularities of the image $f(M)$ are triple points, cross caps and boundary double points. Under certain homological conditions, we give some formulae relating the numbers of these singularities. We also obtain some geometrical applications of these results.

Key words: singular surface, triple point, cross cap, boundary double point, selftranslation surface.

1. Introduction

It is well known that a C^{∞} generic map $f: M \to N$ of a closed surface M into a 3-manifold N is an immersion with normal crossings except at isolated points, at which f has cross caps (for example, see [9]). Thus the singular part of $f(M)$ is made by the curve of double points of f with intersections at the triple points and with end points at the cross caps. In particular, there are a finite number of triple points and cross caps. In [18], Szücs gives the following congruence relating the number of triple points to the number of cross caps.

Theorem 1.1 *Let $f: M \to \mathbf{R}^3$ be a C^{∞} generic map of a closed surface M into $\mathbf{R}^3$. Then*

$$T(f) \equiv \sum_{i=1}^{k} n(x_i, f) + \chi(M) \mod 2,$$

where $T(f)$ is the number of triple points of f, $\chi(M)$ is the Euler characteristic of M, $x_1, \ldots, x_k \in f(M)$ are the cross caps of f, and $n(x_i, f) (\in \{0,1\})$ is the index of the cross cap x_i conveniently defined.

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Note that in [18], the class modulo 2 of the sum $\sum_{i=1}^{k} n(x_i, f)$ is called the linking number of f. Moreover, the theorem is also true if we consider a connected 3-manifold N such that either $H_1(N) = 0$ or $f_*[M] = 0$ in $H_2(N)$, instead of $\mathbf{R}^3$, where $[M] \in H_2(M)$ is the fundamental class of M (the homologies are always considered with $\mathbf{Z}_2$-coefficients). Note also that the above theorem is a generalization of Banchoff's result [1].

The above theorem has been a topological background of some previous geometrical results, implicitly or explicitly, as follows. Banchoff, Gaffney and McCrory in [2] applied it to the dual surface of a generic space curve $\alpha : S^1 \to \mathbf{R}^3$, getting a congruence between the numbers of tritangent planes and the zero torsion points of the curve (see also Ozawa [15]). We can also apply the theorem to the dual surface of an immersed surface $f : M \to \mathbf{R}^3$ to prove a theorem by Ozawa [14], which gives a congruence between the numbers of tritangent planes and the godron points of the surface (see [13]). In fact, Theorem 1.1 can be used in some other new situations, like the tangent developable of a generic space curve so that we obtain a congruence between the numbers of "pyramids" (triple points of the tangent developable) and the zero torsion points of the curve (see [13]). Furthermore, we can also apply Theorem 1.1 to the discriminant set of a stable map into 3-manifolds, obtaining a modulo 2 formula concerning the numbers of triple points of the discriminant set and the swallowtails of the stable map (see [16]).

The purpose of this paper is to extend the result to the case where M and N have boundaries and the map f is neat (i.e., $f^{-1}(\partial N) = \partial M$). Generic maps in this class can allow a new kind of a singular point, namely a boundary double point. We can also define the index $n'(y, f)$ of such a point $y \in f(M)$, and the main result in this case is the following.

Theorem 1.2 *Let $f : M \to N$ be a C^∞ generic neat map of a compact surface M with boundary into a connected 3-manifold N with boundary. We assume one of the following:*

(1) *M is orientable and $H_1(N) = 0$;*
(2) *M is orientable and $f_*[M, \partial M] = 0$ in $H_2(N, \partial N)$, where $[M, \partial M] \in H_2(M, \partial M)$ is the fundamental class of M;*
(3) *∂N is orientable and $f_*[M, \partial M] = 0$ in $H_2(N, \partial N)$.*

Then

$$T(f) \equiv \sum_{i=1}^{k} n(x_i, f) + \sum_{j=1}^{m} n'(y_j, f) + \chi(M) + \beta_0(\partial M) \mod 2,$$

where $x_1, \ldots, x_k \in f(M)$ are the cross caps, $y_1, \ldots, y_m \in f(\partial M)$ are the boundary double points of f and β_0 denotes the number of connected components.

(The formula is obtained in cases (1) and (2) in Theorem 3.7 and in case (3) in Corollary 4.8.)

To prove Theorem 1.2, we begin with the case where M is oriented. In this particular case, we obtain a stronger result, since we get an integral formula instead of a modulo 2 congruence. We define a colouration in the set $D \subset f(M)$ of double points of f, so that D consists of red and blue edges. Then

$$2E(f) = 3T(f) + \sum_{i=1}^{k} n(x_i, f) + \sum_{j=1}^{m} n'(y_j, f),$$

where $E(f)$ is the number of red edges of D.

In the last part of this paper, we use Theorem 1.2 in order to obtain an interesting geometrical application: the selftranslation surface associated with a generic space curve $\alpha : S^1 \to \mathbf{R}^3$ is the image of a Möbius strip by a generic map. We prove that if the curve α is convex, this surface must have at least either a triple point or a cross cap, thus improving a result given in [12].

Remark. In this paper we consider only C^∞ generic maps. However, all the results are true also for topologically stable singular surfaces in the sense of [9]. In fact, any topologically stable singular surface $f : M \to N$ is topologically equivalent to a C^∞ generic map and all the results exposed here are preserved by topological equivalences. For instance, the tangent developable and the dual surface of a generic space curve are not generic; however, they give topologically stable singular surfaces.

Throughout the paper all maps and manifolds are of class C^∞ and the homology groups are always with $\mathbf{Z}_2$-coefficients. The symbol "$\cong$" denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects.

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2. Proof of Theorem 1.1 in the oriented case

In this section we prove the integral formula for the case where M is oriented and M and N have no boundary. We start by defining the index of a cross cap and follow the same process as in [2] for the definition of the index of a topological cross cap of the dual surface α^* of a generic space curve $\alpha : S^1 \to \mathbf{R}^3$. However, there is a problem: in their definition, they use the fact that the dual surface is the boundary of a region in $(\mathbf{P}^3)^*$, and this region is given in a natural way by the geometry of the curve α. We solve this with the following Two Colour Theorem.

Lemma 2.1 *Let $f : M \to N$ be a generic map of a closed surface M into a connected 3-manifold N such that either $H_1(N) = 0$ or $f_*[M] = 0$ in $H_2(N)$. Then*

$$N \setminus f(M) = R \cup B,$$

where R and B are disjoint nonempty open sets with common boundary $f(M)$ (we call R the red part of $N \setminus f(M)$, and B the blue part).

Proof. Let $x_0 \in N \setminus f(M)$ be a fixed point. Then for every $x \in N \setminus f(M)$ we take a piecewise regular curve γ connecting x_0 to x such that γ intersects $f(M)$ transversely in a finite number of regular simple points. We prove that the parity of this number, $\#(\gamma \cap f(M))$, does not depend on the chosen curve γ. In fact, let γ' be another curve as above. Then

$$\#(\gamma \cap f(M)) + \#(\gamma' \cap f(M)) \equiv [\gamma \cup \gamma'] \cdot [f(M)] \equiv 0 \mod 2,$$

where $[\gamma \cup \gamma'] \cdot [f(M)]$ is the modulo 2 intersection number of the homology class represented by the closed curve $\gamma \cup \gamma'$ and that represented by $f(M)$, provided that either $H_1(N) = 0$ or $f_*[M] = 0$ in $H_2(N)$. Thus we can define

$$\begin{aligned} R &= \{x \in N \setminus f(M) : \#(\gamma \cap f(M)) \text{ is even}\}, \\ B &= \{x \in N \setminus f(M) : \#(\gamma \cap f(M)) \text{ is odd}\}. \end{aligned}$$

With this definition, it is obvious that R and B are disjoint nonempty sets. Moreover, the local form of $f(M)$ implies that $\partial R = \partial B = f(M)$. $\square$

In Figure 1 we have the local distributions of the colours in $N \setminus f(M)$

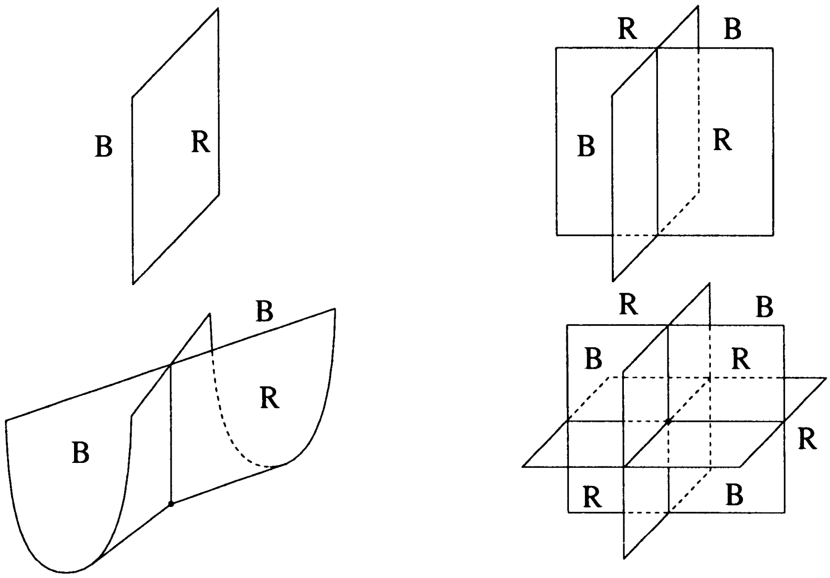

Fig. 1.

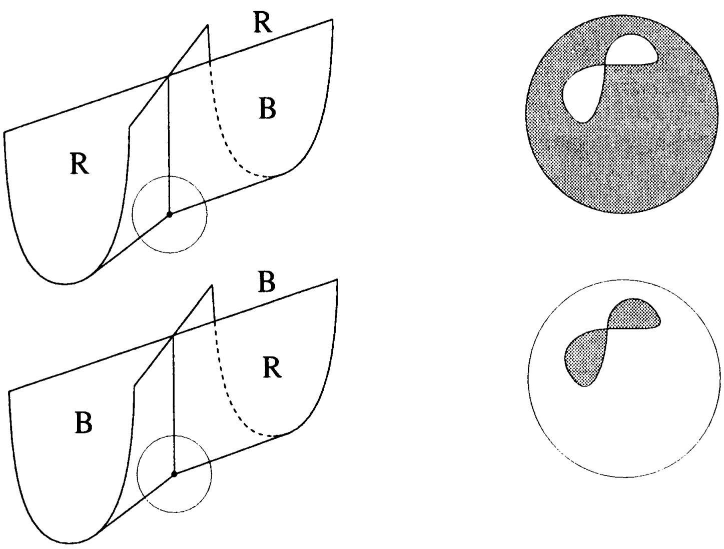

Fig. 2.

when we are in a neighbourhood of a regular point, a double point, a triple point or a cross cap. Note that in the case of the cross cap there would be another dual situation which is not equivalent to this: the "inside" components are red and the "outside" component is blue. Precisely, this fact will determine the index of a cross cap as follows.

Definition 2.2 Let $f : M \to N$ be a generic map of a closed surface M

into a connected 3-manifold N such that either $H_1(N) = 0$ or $f_*[M] = 0$ in $H_2(N)$. Then we define the index $n(x, f)$ of a cross cap $x \in f(M)$ by

$$n(x, f) = \chi(S_\varepsilon \cap \overline{B}),$$

where S_ε is a small 2-sphere in N centered at x and χ denotes the Euler characteristic.

We can see in Figure 2 that $n(x, f) = 1$ if and only if the inside components of $N \setminus f(M)$ in a neighbourhood of x are blue, and $n(x, f) = 0$ if and only if these components are red.

Note that the index of a cross cap depends on the colouration of $N \setminus f(M)$ as in Lemma 2.1. However, the parity of the sum $\sum_{i=1}^{k} n(x_i, f)$ over all cross caps x_i of f does not so depend, since the number of cross caps is always even.

Now we look at the graph structure of the singular part of $f(M)$. The edges of this graph are given by the double points of f and the vertices by the triple points and the cross caps. Moreover, the incidence rules are as follows: each triple point is incident to six edges and each cross cap to one. The following lemma asserts that when M is oriented, the colouration of $N \setminus f(M)$ induces a colouration in the edges of the graph.

Lemma 2.3 *Let $f : M \to N$ be as in Lemma* 2.1. *In addition we suppose that M is oriented. Then the colouration of $N \setminus f(M)$ induces a colouration in the set $D \subset f(M)$ of double points of f with the following properties*:

(1) *at each triple point of $f(M)$ there are three red edges and three blue edges of D*;

(2) *at a cross cap of f, the incident edge of D is red if the index is* 1 *or blue if the index is* 0.

Proof. We have that D and $f^{-1}(D)$ are smooth 1-manifolds, and that the map $f \,|\, f^{-1}(D) : f^{-1}(D) \to D$ is a double covering map. Given a point $y \in D$ such that $f^{-1}(y) = \{x_1, x_2\}$, we choose a nonzero tangent vector $v \in T_y D$. Then there exist tangent vectors $u_i \in T_{x_i} f^{-1}(D)$ such that $Tf(u_i) = v$ for $i = 1, 2$, where Tf denotes the differential of f. Each one of these vectors can be completed to an oriented basis $\{u_i, w_i\}$ of $T_{x_i} M$, and we consider the vectors $Tf(w_1)$, $Tf(w_2)$ in $T_y N$. On the other hand, let V be a small neighbourhood of y in N such that $V \setminus f(M)$ has exactly four connected components. Then we say that the point y is red or blue according to the colour of the quadrant determined by the tangent vectors $Tf(w_1)$ and $Tf(w_2)$. It is not difficult to see that this definition does not depend on the chosen vectors (see Figure 3).

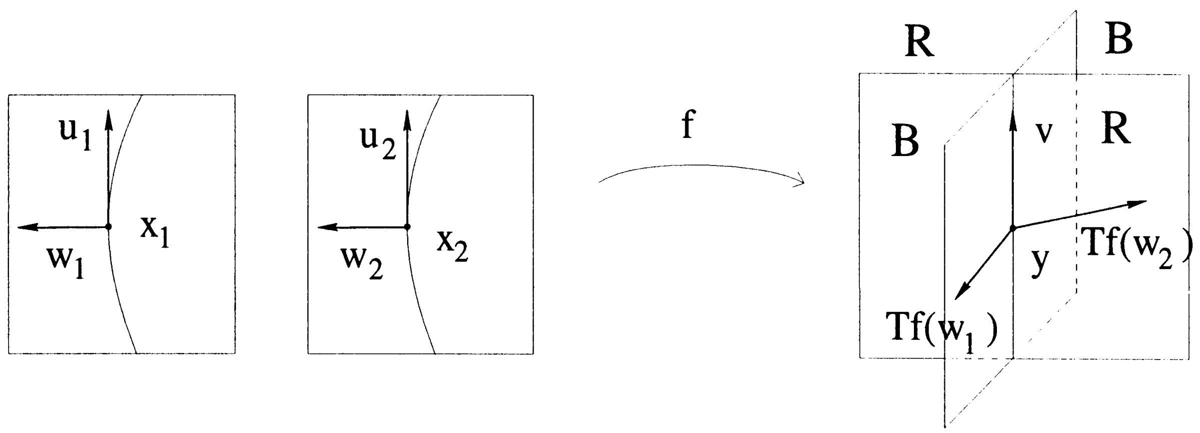

Fig. 3.

In particular, when the double point $y \in D$ is close enough to a cross cap, the component determined by the tangent vectors $Tf(w_1)$ and $Tf(w_2)$ is precisely the corresponding part of the outside component. Therefore, this point will be red when the index is 1, and blue when the index is 0 (see Figure 4). The local situation for a triple point is that we have six edges of D incident to this point, and three of these edges must be red and the other three must be blue (see Figure 4). □

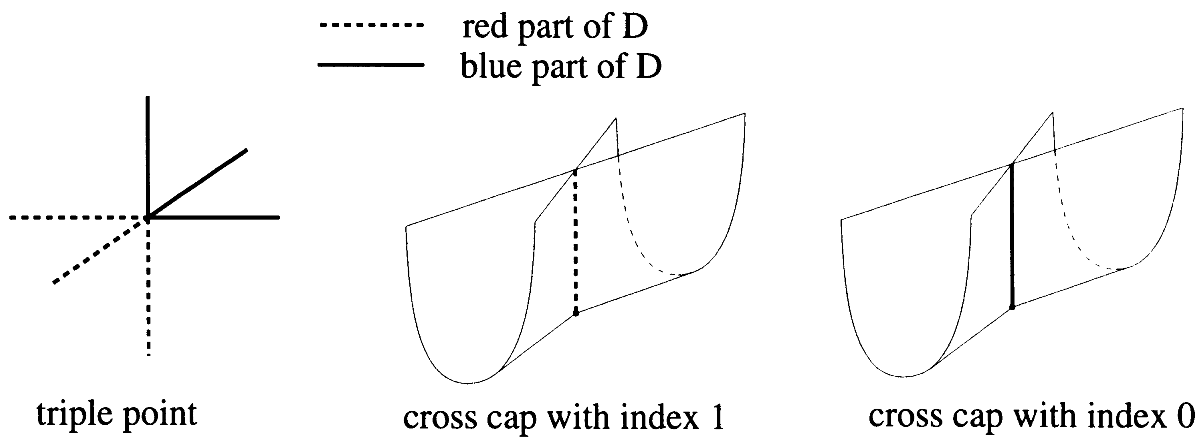

Fig. 4.

Note that the colouration in the set D depends on the orientation of M in general. However, if M is connected, then it does not so depend.

Theorem 2.4 *Let $f : M \to N$ be a generic map of a closed oriented surface M into a connected 3-manifold N such that either $H_1(N) = 0$ or $f_*[M] = 0$ in $H_2(N)$. Then*

$$2E(f) = 3T(f) + \sum_{i=1}^{k} n(x_i, f),$$

where $T(f)$ is the number of triple points of f, $x_1, \ldots, x_k \in f(M)$ are the cross caps, and $E(f)$ is the number of red edges of the set D of double points of f. (Here we do not count the circle components of D.)

Proof. We just consider the graph structure of the red part of D. Each triple point is a vertex of order 3 and each cross cap with index 1 is a vertex of order 1. Then, by applying the classical theorem of graph theory, which asserts that the number of edges of a graph is equal to one half the sum of the orders of its vertices, we have the desired result. □

Since $\chi(M)$ is even for any closed orientable surface M, the following immediate consequence of Theorem 2.4 is the orientable version of Theorem 1.1.

Corollary 2.5 *Let $f : M \to N$ be as in Theorem 2.4. Then*

$$T(f) \equiv \sum_{i=1}^{k} n(x_i, f) \mod 2,$$

where $T(f)$ is the number of triple points of f and $x_1, \ldots, x_k \in f(M)$ are the cross caps.

Remark. In [18], Szücs proves Theorem 1.1, using the theorem of Banchoff [1] concerning the number of triple points of an immersed surface in $\mathbf{R}^3$ and using a surgery technique. Here we give another proof using a recent result of Carter and Saito [4], [5] concerning the normal Euler number of an embedded surface in $\mathbf{R}^4$.

First we review their result. We start with a closed embedded surface M in $\mathbf{R}^4$ and consider a generic projection into $\mathbf{R}^3$. Then the projected surface is a stable surface in $\mathbf{R}^3$, and we define the sign ($= \pm 1$) of each cross cap and each triple point, using the embedded surface M in $\mathbf{R}^4$. Furthermore, we use the Two Colour Theorem to define the colour (red or blue) of each cross cap. Here the cross cap is blue if and only if the index in our sense is 1. Define T to be the number of positive triple points minus the number of negative ones, B to be the number of positive blue cross caps minus the number of negative blue ones, and R to be the number of positive red cross caps minus the number of negative red ones. Then Carter and Saito have shown the following formula:

$$T + R + 2B = \frac{3}{2}e,$$

where e is the normal Euler number of the original embedded surface M in $\mathbf{R}^4$.

If we use their result, we can recover our modulo 2 formula (Theorem 1.1) concerning the numbers of triple points and cross caps as follows. First note that e is always even and that $e/2$ is congruent modulo 2 to the Euler characteristic of the surface by the Whitney congruence [19]. Consequently, since $B + R$ is always even, we see that $T + B$ is congruent modulo 2 to the Euler characteristic of the surface, using their formula. This is nothing but our modulo 2 formula, since T is congruent modulo 2 to the number of triple points and B is congruent modulo 2 to the sum of the indices of cross caps we defined.

Note that not every stable surface in $\mathbf{R}^3$ can be lifted to an embedded surface in $\mathbf{R}^4$ (see [7]). However, it is always lifted to an immersion with normal crossings into $\mathbf{R}^4$ (see, for example, [17]). Furthermore, we can generalize the formula of Carter and Saito *modulo* 2 to one for immersed surfaces in $\mathbf{R}^4$ as follows:

$$T + R + 2B \equiv \frac{3}{2}e + d \mod 2,$$

where d is the number of double points of the immersed surface in $\mathbf{R}^4$. Note that, in this case, $e/2 + d$ is congruent modulo 2 to the Euler characteristic of the surface (see [17]). Thus, we can prove our formula for an arbitrary stable surface in $\mathbf{R}^3$.

3. Extension of the results to the boundary case — the oriented case

In this section, we extend Theorems 1.1 and 2.4 to the case where M and N have boundaries. The first step toward such an extension is to define a reasonable class of maps of surfaces with boundary into 3-manifolds with boundary.

Definition 3.1 A smooth map $f : M \to N$ is said to be *neat* if $f^{-1}(\partial N) = \partial M$. A smooth neat map $f : M \to N$ is said to be *generic* if the following conditions are satisfied:

(1) $f \,|\, \mathrm{int}(M) : \mathrm{int}(M) \to \mathrm{int}(N)$ is generic in the sense of the preceding sections;

(2) f is an immersion at every boundary point;

(3) $f \,|\, \partial M : \partial M \to \partial N$ is a selftransverse immersion; i.e., it has only simple or transverse double points;

(4) if $x \in \partial M$ is a boundary point, then the planes $Tf(T_x M)$ and $T_{f(x)}\partial N$ are in general position in $T_{f(x)}N$.

Note that the conditions (2)–(4) imply the following:

(5) if x_1 and x_2 are distinct boundary points of M with $f(x_1) = f(x_2) = y$, then the planes $Tf(T_{x_1}M)$, $Tf(T_{x_2}M)$ and $T_y(\partial N)$ are in general position in $T_y N$.

If $f : M \to N$ is a generic neat map, then the local form of $f(M)$ in N is determined by the above conditions. We get a new kind of an isolated singular point: the boundary double point (see Figure 5).

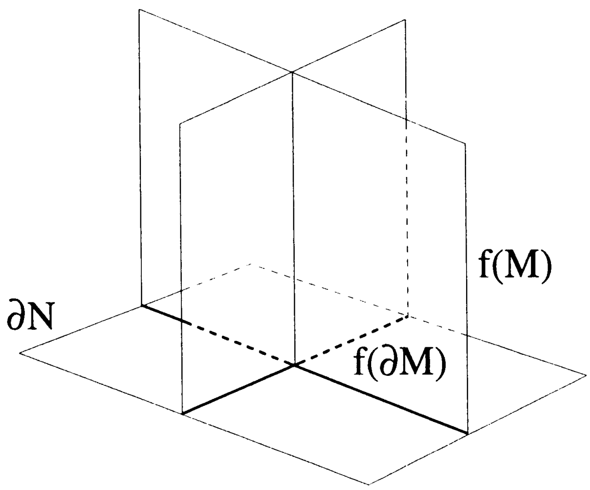

Fig. 5.

We denote by $N^\infty(M, N)$ the space of the smooth neat maps from M to N endowed with the Whitney C^∞ topology. We will prove that the set of the generic neat maps is open and dense in this space, provided that M

is compact.

Lemma 3.2 *If ∂M is compact, then every map $f \in N^\infty(M,N)$ can be C^∞ approximated by a map $g \in N^\infty(M,N)$ which is generic on some open collar neighbourhood of ∂M in M.*

Proof. Let W and Z be open collars of ∂M and ∂N in M and N respectively, and let $h : W \to \partial M \times [0,1)$ and $k : Z \to \partial N \times [0,1)$ be diffeomorphisms. Taking W smaller if necessary, we may assume that $k \circ f(W) \subset \partial N \times [0, 1-\delta]$ for some δ with $0 < \delta < 1$. Define $\alpha_t : \partial M \to \partial N$ ($t \in [0,1)$) and $\beta_x : [0,1) \to [0,1)$ ($x \in \partial M$) by

$$k \circ f \circ h^{-1}(x,t) = (\alpha_t(x), \beta_x(t)).$$

Note that $\beta_x^{-1}(0) = 0$ and that f is transverse to ∂N if and only if

$$\frac{d\beta_x}{dt}(0) \neq 0 \quad (\forall x \in \partial M).$$

Take a sufficiently small positive real number ε such that $0 < \varepsilon < \delta$. Furthermore, let $\xi : [0,1) \to [0,1]$ be a C^∞ function such that

$$\xi(t) = \begin{cases} 1 & (t \in [0,1/3]) \\ 0 & (t \in [2/3,1)). \end{cases}$$

Then the function

$$\beta_{x,\varepsilon}(t) = \beta_x(t) + \varepsilon t \xi(t)$$

C^∞ approximates β_x and satisfies $\beta_{x,\varepsilon}^{-1}(0) = 0$ and

$$\frac{d\beta_{x,\varepsilon}}{dt}(0) \neq 0 \quad (\forall x \in \partial M).$$

On the other hand, we can regard $\alpha_t : \partial M \to \partial N$ ($t \in [0,1)$) as a 1-parameter family of smooth maps of a closed 1-dimensional manifold into a surface. Then there exists a C^∞ approximation of the family, say $\{\alpha_{t,\varepsilon}\}_{t\in[0,1)}$, such that $\alpha_{0,\varepsilon}$ is a selftransverse immersion and that $\alpha_{t,\varepsilon} = \alpha_t$ for all $t \geq 2/3$. Note that, since ∂M is compact, $\alpha_{t,\varepsilon}$ is a selftransverse immersion not only for $t = 0$, but also for all t in the interval $[0,\gamma]$ for some γ with $0 < \gamma < 1$. Then the map

$$g' = k^{-1} \circ \varphi_\varepsilon \circ h : W \to Z$$

is a generic neat map on a sufficiently small neighbourhood of ∂M and C^∞

approximates $f\,|\,W$, where $\varphi_\varepsilon : \partial M \times [0,1) \to \partial N \times [0,1)$ is the smooth map defined by $\varphi_\varepsilon(x,t) = (\alpha_{t,\varepsilon}(x), \beta_{x,\varepsilon}(t))$. Furthermore, $g' = f$ outside of the closed collar neighbourhood $W' = h^{-1}(\partial M \times [0, 2/3])$. Then the map $g : M \to N$ defined by

$$g = \begin{cases} g' & \text{on } W \\ f & \text{on } M - W' \end{cases}$$

is a required map. This completes the proof. □

Proposition 3.3 *If M is compact, then the set of the generic neat maps is open and dense in $N^\infty(M,N)$.*

In the following, for smooth manifolds X, Y and positive integers s, k, we denote by $J^k_s(X,Y)$ the s-fold k-jet bundle of smooth maps of X into Y (for details, see [8], for example). Furthermore, we set $Y^s = \{(y_1,\dots,y_s) \in Y \times \cdots \times Y\}$, $\Delta^s_Y = \{(y,\dots,y) \in Y^s : y \in Y\}$, and we denote by $\pi_Y : J^k_s(X,Y) \to Y^s$ the natural projection and by $d : Y^s \to Y$ the projection to the first factor. For the proof of the above proposition, we need the following, which is proved in [11], §3.

Lemma 3.4 *Let $f : X \to Y$ be a smooth map between manifolds. Let W be a submanifold of $J^k_s(X,Y)$ such that $\pi_Y(W) \subset \Delta^s_Y$. Suppose that U is an open subset of Y and that $\mathcal{V}$ is an open neighbourhood of f in $C^\infty(X,Y)$. Then there exists a smooth map $g : X \to Y$ such that*

(1) $g \in \mathcal{V}$,
(2) $g = f$ *on* $f^{-1}(U) = g^{-1}(U)$,
(3) $j^k_s g$ *is transverse to* W *on* $W \cap \pi_Y^{-1}(d^{-1}(Y - \overline{U}))$.

Proof of Proposition 3.3. Let us begin by proving the density. Take an $f \in N^\infty(M,N)$. By Lemma 3.2 we can C^∞ approximate it by a map $g \in N^\infty(M,N)$ which is generic on an open collar W of ∂M in M. Since M is compact, there exists a closed collar neighbourhood $\tilde{Z}$ of ∂N in N such that $\tilde{W} = g^{-1}(\tilde{Z})$ $(\subset W)$ is a closed collar neighbourhood of ∂M in M and that $g\,|\,\tilde{W} : \tilde{W} \to \tilde{Z}$ is a generic neat map. Then, by Lemma 3.4, we see that there exists a C^∞ approximation g' of g such that $g = g'$ on $g^{-1}(Z) = (g')^{-1}(Z)$ and that g' is a generic neat map on $(g')^{-1}(N - (\tilde{Z} - Z))$, where $Z (\cong \partial N \times (0,1))$ is the topological interior of $\tilde{Z}$ in N. Since g' is a C^∞ approximation of g, we may assume that g' is also transverse to $\tilde{Z} - Z (\cong \partial N)$. Then it is easy to see that $g = g'$ on $(g')^{-1}(\tilde{Z}) = \overline{(g')^{-1}(Z)} =$

$\overline{g^{-1}(Z)} = g^{-1}(\tilde{Z})$. Thus g' is a generic neat map which C^∞ approximates f.

For the openness, first it is easy to see that the conditions (2)–(4) of Definition 3.1 are open conditions. In other words, for every generic neat map $f : M \to N$, every sufficiently close approximation satisfies the three conditions. Then one can choose a small closed collar neighbourhood $\tilde{Z}$ of ∂N in N such that every sufficiently close approximation f_ε of f satisfies the following.

(1) f_ε is transverse to $\partial \tilde{Z}$.

(2) $\tilde{W}_\varepsilon = f_\varepsilon^{-1}(\tilde{Z})$ is a closed collar neighbourhood of ∂M in M.

(3) Let $\tilde{h}_\varepsilon : \tilde{W}_\varepsilon \to \partial M \times [0,1]$ and $\tilde{k} : \tilde{Z} \to \partial N \times [0,1]$ be appropriate diffeomorphisms, where $\partial M \times \{0\}$ and $\partial N \times \{0\}$ correspond to ∂M and ∂N respectively. Define $\alpha_{t,\varepsilon} : \partial M \to \partial N$ $(t \in [0,1])$ and $\beta_{x,\varepsilon} : [0,1] \to [0,1]$ $(x \in \partial M)$ by $\tilde{k} \circ f_\varepsilon \circ \tilde{h}_\varepsilon^{-1}(x,t) = (\alpha_{t,\varepsilon}(x), \beta_{x,\varepsilon}(t))$. Then

$$\frac{d\beta_{x,\varepsilon}}{dt}(t) > 0 \quad ({}^\forall t \in [0,1]),$$

and the family $\{\alpha_{t,\varepsilon}\}_{t\in[0,1]}$ is a 1-parameter family of selftransverse immersions without bifurcations.

Note that, as a consequence of the above conditions (1)–(3), the following is also satisfied:

(4) $f_\varepsilon \,|\, \tilde{W}_\varepsilon : \tilde{W}_\varepsilon \to \tilde{Z}$ is a generic neat map.

Let $\{f_n\}$ be a sequence in $N^\infty(M,N)$ converging to f such that no f_n satisfies the condition (1) of Definition 3.1. For sufficiently large n, f_n satisfies the above conditions (1)–(4), if we replace f_ε with f_n. Since M is compact, we may also assume that the images of f_n are contained in a fixed compact neighbourhood V of $f(M)$ in N. Then there exists a point $y_n \in f_n(M)$ such that the stability condition (Definition 3.1 (1)) is broken at y_n for f_n. Since f_n satisfies the above conditions (1)–(4), we have $y_n \in V \cap \overline{(N - \tilde{Z})}$. Since $V \cap \overline{(N - \tilde{Z})}$ is compact, the sequence $\{y_n\}$ has a convergent subsequence. This implies that there exists a point in $f(M) \cap \overline{(N - \tilde{Z})}$ such that the stability condition is broken at y for f. This is a contradiction. Thus for sufficiently large n, f_n satisfies the condition (1) of Definition 3.1. Thus the set of the generic neat maps is open in $N^\infty(M,N)$. This completes the proof. □

Now we return to the initial purpose of this section, which is to extend Theorems 1.1 and 2.4 to the boundary case. For this purpose, we first need the Two Colour Theorem for generic neat maps.

Lemma 3.5 *Let $f : M \to N$ be a generic neat map of a compact surface M with boundary into a connected 3-manifold N with boundary such that either $H_1(N) = 0$ or $f_*[M, \partial M] = 0$ in $H_2(N, \partial N)$. Then*

$$N \setminus f(M) = R \cup B,$$

where R and B are disjoint nonempty open sets with common boundary $f(M)$. (Here the boundary of R (or B) is defined to be $\overline{R} \setminus R$ (resp. $\overline{B} \setminus B$).)

The above lemma can be proved by an argument similar to the proof of Lemma 2.1. Just use $[f(M), f(\partial M)] \in H_2(N, \partial N)$ instead of $[f(M)] \in H_2(N)$.

Using Lemma 3.5, we define the index $n(x, f)$ of a cross cap x of a generic neat map $f : M \to N$ in exactly the same way as in Definition 2.2.

We first consider the case where the surface M is oriented. As before, we look at the graph structure of the singular part of $f(M)$. The edges of this graph are given by the double points of f and the vertices by the triple points, the cross caps and the boundary double points. Moreover, the incidence rules are similar to the boundaryless case. We must note that each boundary double point is incident to one edge. Furthermore, under the homological assumption as in Lemma 3.5, using the same method as in the proof of Lemma 2.3, we can give a colouration to the set $D \subset f(M)$ of double points of f so that the conditions (1) and (2) of Lemma 2.3 are satisfied.

Definition 3.6 Let $f : M \to N$ be a generic neat map as in Lemma 3.5. When M is oriented, we can give a colouration to the set D of double points of f as remarked above. Then we define the index $n'(y, f)$ of a boundary double point $y \in f(\partial M)$ as $n'(y, f) = 1$ if the edge of D incident to y is red, and $n'(y, f) = 0$ if it is blue.

Remark. In the above situation, *reversing the orientation of a component of M changes the indices of even number of boundary double points.* This fact is proved as follows. Let M_1 be a component of M. It is enough to show that $f(\partial M_1)$ meets with $c = f(\partial M \setminus \partial M_1)$ at an even number of points. Suppose that $f(\partial M_1)$ intersects with c at an odd number of points. Then the modulo 2 intersection number $f_*[M_1, \partial M_1] \cdot [c]$ in N does not vanish, which implies that $[c] \in H_1(N)$ is not zero. This is a contradiction, since c

is the boundary of $f(M \setminus M_1)$.

Note that the index as in Definition 3.6 depends both on the colouration of $N \setminus f(M)$ as in Lemma 3.5 and on the orientation of M. However, from the above fact together with the fact that $k+m$ is always even, it follows that the parity of the sum $\sum_{i=1}^{k} n(x_i, f) + \sum_{j=1}^{m} n'(y_j, f)$ over all cross caps and all boundary double points does not depend on the colouration of $N \setminus f(M)$ nor on the orientation of M.

By the same graph theoretical argument as in the proof of Theorem 2.4, we obtain the following.

Theorem 3.7 *Let $f : M \to N$ be a generic neat map of a compact oriented surface M with boundary into a connected 3-manifold N with boundary such that $H_1(N) = 0$ or $f_*[M, \partial M] = 0$ in $H_2(N, \partial N)$. Then*

$$2E(f) = 3T(f) + \sum_{i=1}^{k} n(x_i, f) + \sum_{j=1}^{m} n'(y_j, f),$$

where $T(f)$ is the number of triple points of f, $E(f)$ is the number of red edges of the set D of double points of f, $x_1, \ldots, x_k \in f(M)$ are the cross caps and $y_1, \ldots, y_m \in f(\partial M)$ are the boundary double points of f. (Here we do not count the circle components of D.)

We have the following immediate consequence of Theorem 3.7, which is Theorem 1.2 in the case where M is orientable.

Corollary 3.8 *Let $f : M \to N$ be as above. Then*

$$T(f) \equiv \sum_{i=1}^{k} n(x_i, f) + \sum_{j=1}^{m} n'(y_j, f) \mod 2,$$

where $T(f)$ is the number of triple points of f, $x_1, \ldots, x_k \in f(M)$ are the cross caps of f and $y_1, \ldots, y_m \in f(\partial M)$ are the boundary double points of f.

4. Extension of the results to the boundary case — the general case

For the general case, we must modify the arguments of the previous section. First we prove the following refinement of Lemma 3.5.

Lemma 4.1 *Let $f : M \to N$ be a generic neat map of a compact surface M with boundary into a connected 3-manifold N with boundary. If $f_*[M, \partial M] = 0$ in $H_2(N, \partial N)$, then there exists a two-colour decomposition $N \setminus f(M) = R \cup B$ as in Lemma 3.5 such that $\bar{B}$ is compact.*

Proof. Let us assume that in a two-colour decomposition $N \setminus f(M) = R \cup B$, neither $\bar{R}$ nor $\bar{B}$ is compact. Then we see that there exists an embedded proper arc in N which intersects $f(M)$ transversely at an odd number of points. Let $\gamma \in H_1^c(N)$ be the class represented by this proper arc, where H_*^c denotes the homology with closed support (or the homology of infinite chains). Then the modulo 2 intersection number $\gamma \cdot f_*[M, \partial M]$ in N does not vanish. This contradicts our assumption. Thus, choosing the colours appropriately, we may assume that $\bar{B}$ is compact. This completes the proof. □

Remark. A two-colour decomposition as above does not always exist in general even if $H_1(N) = 0$. For example, consider the embedding $D^2 = D^2 \times \{0\} \hookrightarrow D^2 \times \mathbf{R}$.

Theorem 4.2 *Let $f : M \to N$ be a generic neat map of a compact surface M with boundary into a connected 3-manifold N with boundary. Suppose that $f_*[M, \partial M] = 0$ in $H_2(N, \partial N)$. We fix a two-colour decomposition $N \setminus f(M) = R \cup B$ as in Lemma 3.5 such that $\bar{B}$ is compact (see Lemma 4.1). Then*

$$T(f) \equiv \sum_{i=1}^{k} n(x_i, f) + \chi(M) + \chi(B \cap \partial N) \mod 2,$$

where $x_1, \ldots, x_k \in f(M)$ are the cross caps of f.

Proof. We prove the theorem by using surgeries on f. Let $x \in f(M)$ be a triple point. Then we can eliminate the triple point by performing the surgery operation as in Figure 6.

Note that this does not affect the hypotheses on homology and on B. Note also that two cross caps with distinct indices are created. Under this surgery operation, $T(f)$ decreases by 1, $\chi(M)$ decreases by 2, $\sum_{i=1}^{k} n(x_i, f)$ increases by 1, and $\chi(B \cap \partial N)$ is invariant. Thus the changes modulo 2 on the two sides of the required congruence are the same. Thus we may assume that f has no triple points.

Let γ be a double point arc of f connecting two cross caps. Then we

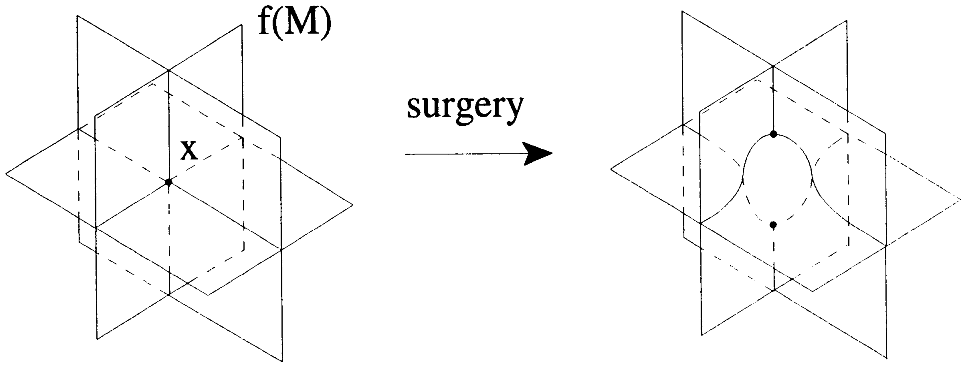

Fig. 6.

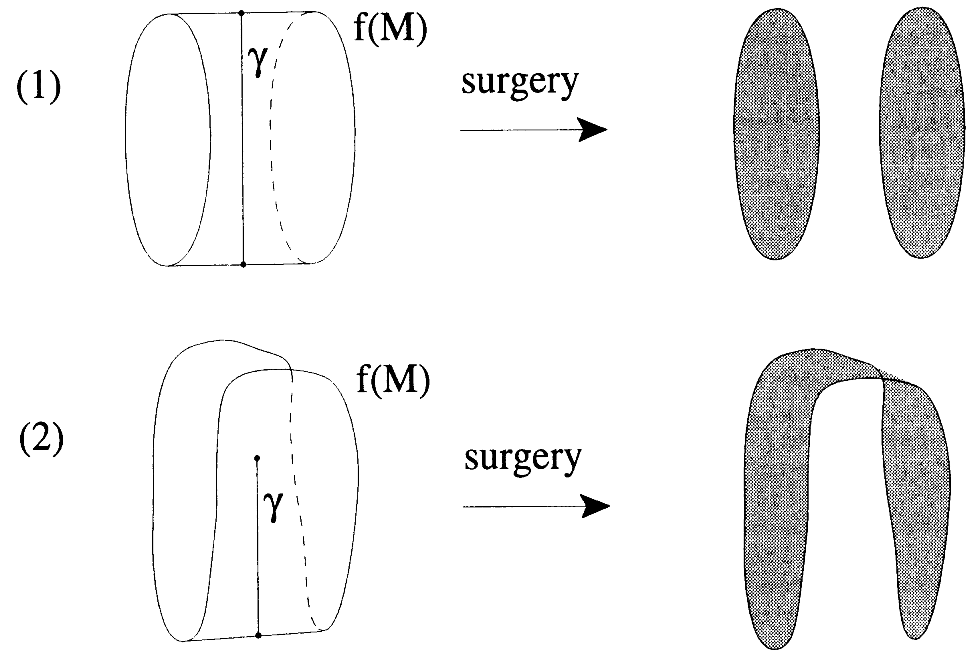

Fig. 7.

perform the surgery operation as in Figure 7. Note that this does not affect the hypotheses on homology and on B. Furthermore, no triple point is created. Under the surgery operation as in Figure 7 (1), $\chi(M)$ increases by 2, $\sum_{i=1}^{k} n(x_i, f)$ is invariant or decreases by 2, since the two cross caps have the same index, and $\chi(B \cap \partial N)$ is invariant. Under the surgery operation as in Figure 7 (2), $\chi(M)$ increases by 1, $\sum_{i=1}^{k} n(x_i, f)$ decreases by 1, since the two cross caps have distinct indices, and $\chi(B \cap \partial N)$ is invariant. Thus we may assume that f has no such double point arcs nor triple points.

Consider a double point circle of f in $\operatorname{int} N$. We can eliminate it by using an argument similar to that used in [10], Lemma 2.3.1 (see also [10], proof of Theorem 1.3.2). In this case, all the quantities in the required congruence are invariant. Thus we may assume that f has no double point circles.

Let γ be a double point arc connecting a cross cap and a boundary double point. Then we perform the surgery operation as in Figure 8. Note that this does not affect the hypotheses on homology and on B. Under the surgery operation as in Figure 8 (1), $\chi(M)$ increases by 1, $\sum_{i=1}^{k} n(x_i, f)$ decreases by 1, and $\chi(B \cap \partial N)$ is invariant. Under the surgery operation as in Figure 8 (2), $\chi(M)$ increases by 1, $\sum_{i=1}^{k} n(x_i, f)$ is invariant, and $\chi(B \cap \partial N)$ decreases by 1. Thus we may assume that f has no such double point arcs.

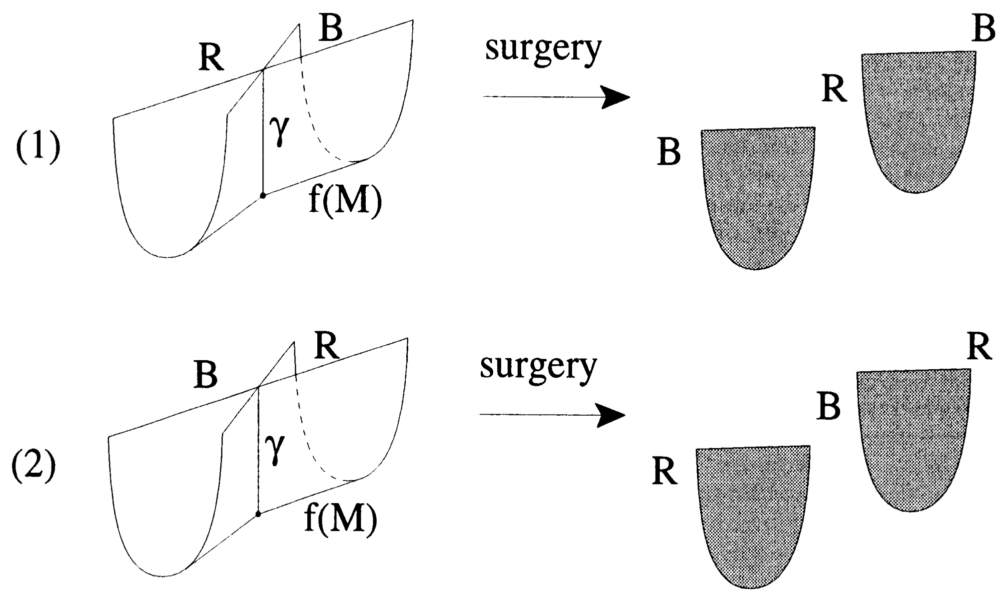

Fig. 8.

Let γ be a double point arc connecting two boundary double points of f. Then we perform the surgery operation as in Figure 9. Then all the quantities in question are invariant and we may assume that f has no such double point arcs.

Thus we may assume that f is an embedding from the beginning. In this case, $W = \bar{B}$ is a compact 3-manifold with $\partial W = f(M) \cup (B \cap \partial N)$.

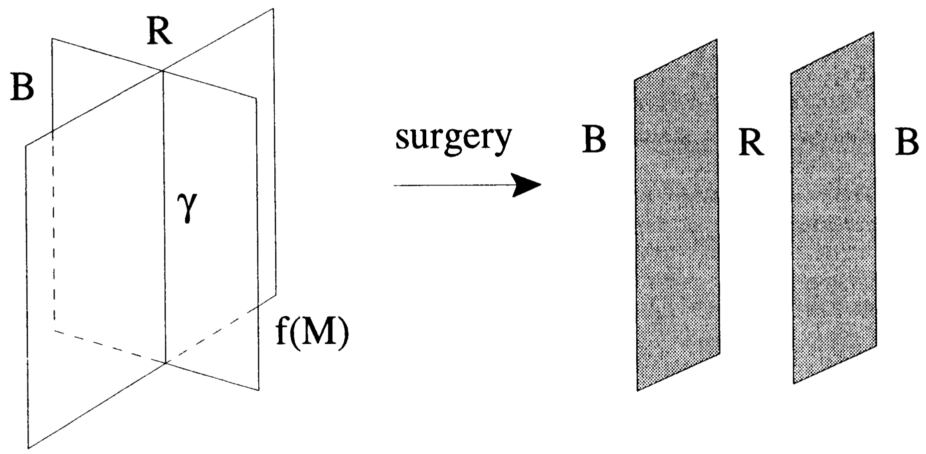

Fig. 9.

Hence we have

$$0 \equiv \chi(\partial W) \equiv \chi(f(M)) + \chi(B \cap \partial N) \quad \text{mod } 2$$

and hence

$$\chi(M) \equiv \chi(B \cap \partial N) \quad \text{mod } 2.$$

Note that in this case $T(f) = 0$ and $\sum_{i=1}^{k} n(x_i, f) = 0$. Thus the required congruence holds in this case. This completes the proof. □

Remark. Here we give an alternative proof of Theorem 4.2, which relies on Theorem 1.1. The union of $f(M)$ with $\bar{B} \cap \partial N$ is topologically equivalent to the image of a closed surface $\widetilde{M}$ under a generic smooth map $\tilde{f}$ into int N having additional cross caps corresponding to the boundary double points of f. Note that all of these additional cross caps have index 1 and that $\tilde{f}_*[\widetilde{M}] = 0$ in $H_2(N)$. Then, applying Theorem 1.1, we have

$$T(f) \equiv \sum_{i=1}^{k} n(x_i, f) + \partial d(f) + \chi(\widetilde{M}) \quad \text{mod } 2,$$

where $\partial d(f)$ is the number of boundary double points of f. (Recall that Theorem 1.1 holds for maps into a connected 3-manifold N instead of $\mathbf{R}^3$ under our homological condition, as noted in the introduction.) Furthermore, it is not difficult to see that

$$\chi(\widetilde{M}) = \chi(M) - \partial d(f) + \chi(B \cap \partial N).$$

Hence we have the conclusion.

We note that this proof is inspired by the proof of Theorem 2 of [6].

Obviously, the above proof relies on Theorem 1.1. However, the proof which we have given just before this remark is independent of Theorem 1.1. Thus our proof also gives an alternative proof of Theorem 1.1 (and also Theorem 2 of [18]), since Theorem 1.1 is an easy corollary of Theorem 4.2.

Remark. In Theorem 4.2, $\sum_{i=1}^{k} n(x_i, f) + \chi(B \cap \partial N)$ modulo 2 is invariant under the change of the colouration, provided that N is compact. Note also that

$$\chi(B \cap \partial N) \equiv \chi(\bar{R} \cap \partial N) \mod 2,$$

provided that N is compact.

As a corollary to Theorem 4.2, we have the following, which has originally been pointed out in [6] (see also [3]).

Corollary 4.3 *Let $f : M \to D^3$ be a generic neat immersion, where M is a compact connected surface with one boundary component. Then*

$$\begin{aligned} T(f) &\equiv \chi(M) + \beta_0(B \cap \partial D^3) \mod 2 \\ &\equiv \chi(M) + \beta_0(R \cap \partial D^3) \mod 2, \end{aligned}$$

where $D^3 \setminus f(M) = R \cup B$ is a two-colour decomposition as in Lemma 3.5 and β_0 denotes the number of connected components.

Proof. By Theorem 4.2, we have

$$T(f) \equiv \chi(M) + \chi(B \cap \partial D^3) \mod 2.$$

Since $f(\partial M)$ is connected, we see easily that each connected component of $\partial D^3 \setminus f(\partial M)$ is homeomorphic to the open 2-disk. Hence we have $\chi(B \cap \partial D^3) = \beta_0(B \cap \partial D^3)$. This completes the proof. □

In view of Theorem 4.2, we need to clarify the quantity $\chi(B \cap \partial N)$. For this purpose, we first define an index of a double point of an immersion with normal crossings of a 1-dimensional manifold into a surface as follows.

Let F be an *orientable* surface and $\alpha : \Gamma \to F$ an immersion with normal crossings, where $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s$ and each Γ_i is diffeomorphic to S^1. We suppose that $\alpha_*[\Gamma] = 0$ in $H_1(F)$, where $[\Gamma] \in H_1(\Gamma)$ is the fundamental class of Γ. Then by an argument similar to that in the proof of Lemma 2.1, we see that $F \setminus \alpha(\Gamma) = R \cup B$, where R and B are disjoint nonempty open sets with common boundary $\alpha(\Gamma)$. In the following, we fix such a colouration.

Definition 4.4 Fix an orientation of Γ. For a double point $y \in \alpha(\Gamma)$ of α, we define the index i_y of y by

$$i_y = \begin{cases} 0 & \text{if } y \text{ is as in Figure 10 (1),} \\ 1 & \text{if } y \text{ is as in Figure 10 (2).} \end{cases}$$

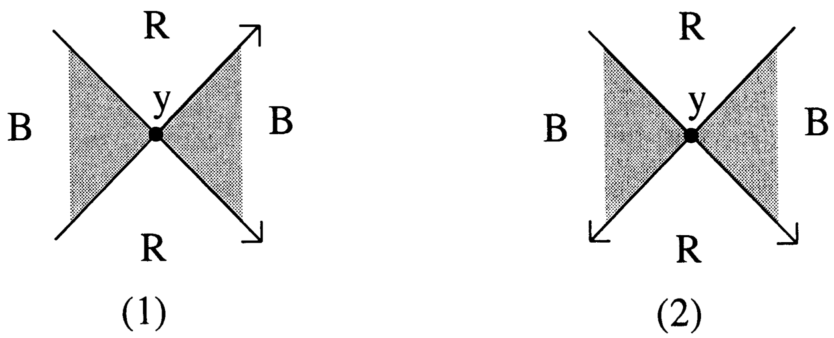

Fig. 10.

Lemma 4.5 *The parity of the sum $\sum_y i_y$ over all double points y of α does not depend on the orientation of Γ.*

Proof. In the following, the symbol "$\cdot$" denotes the intersection number modulo 2 in F. First note that the selfintersection number modulo 2 $(\alpha_*[\Gamma_1]) \cdot (\alpha_*[\Gamma_1])$ in F vanishes, since F is orientable. Thus we have

$$\begin{aligned}(\alpha_*[\Gamma_1]) \cdot (\alpha_*[\Gamma_2] + \cdots + \alpha_*[\Gamma_s]) \\ &= (\alpha_*[\Gamma_1]) \cdot (\alpha_*[\Gamma]) - (\alpha_*[\Gamma_1]) \cdot (\alpha_*[\Gamma_1]) \\ &= 0,\end{aligned}$$

since $\alpha_*[\Gamma] = 0$ in $H_1(F)$ by our assumption. Thus the number of intersection points of $\alpha(\Gamma_1)$ with $\alpha(\Gamma_2 \cup \cdots \cup \Gamma_s)$ is even.

Now let $y \in \alpha(\Gamma)$ be a double point of α. If y is a double point of $\alpha(\Gamma_1)$, then the index i_y does not change when we reverse the orientation of Γ_1. If y is an intersection point of $\alpha(\Gamma_1)$ with $\alpha(\Gamma_2 \cup \cdots \cup \Gamma_s)$, the index i_y changes when we reverse the orientation of Γ_1. However, the number of such intersection points is always even by the argument in the previous paragraph so that the sum $\sum_y i_y$ modulo 2 over all double points y of $\alpha(\Gamma)$ does not change when we reverse the orientation of Γ_1. Similar argument can be applied to $\Gamma_2, \ldots, \Gamma_s$. This completes the proof. □

Remark. If we drop the hypothesis that F be orientable, the above lemma does not hold. Consider two simple closed curves in the projective plane $\mathbf{R}P^2$ intersecting transversely with each other at exactly one point.

Lemma 4.6 *Suppose that $\bar{B}$ is compact. Then*

$$\chi(B) \equiv \sum_{y \in \Delta(\alpha)} i_y + \beta_0(\Gamma) \mod 2,$$

where $\Delta(\alpha)$ is the set of double points of α.

Proof. We prove the lemma by using surgeries on α. The following proof can be regarded as a 1-dimensional version of the proof of Theorem 4.2. We proceed by the induction on the number of elements of $\Delta(\alpha)$, which we denote by n. When $n = 0$, we see that $\chi(B) + \beta_0(\Gamma)$ is even, since $\bar{B}$ is a compact orientable surface with $\beta_0(\partial \bar{B}) = \beta_0(\Gamma)$ and $\chi(B) = \chi(\bar{B})$. Thus the required congruence holds. When $n > 0$, take a point $y \in \Delta(\alpha)$ and consider the surgery operation on α as in Figure 11, which decreases n by 1. Note that this does not affect our homological hypothesis on $\alpha_*[\Gamma]$ nor

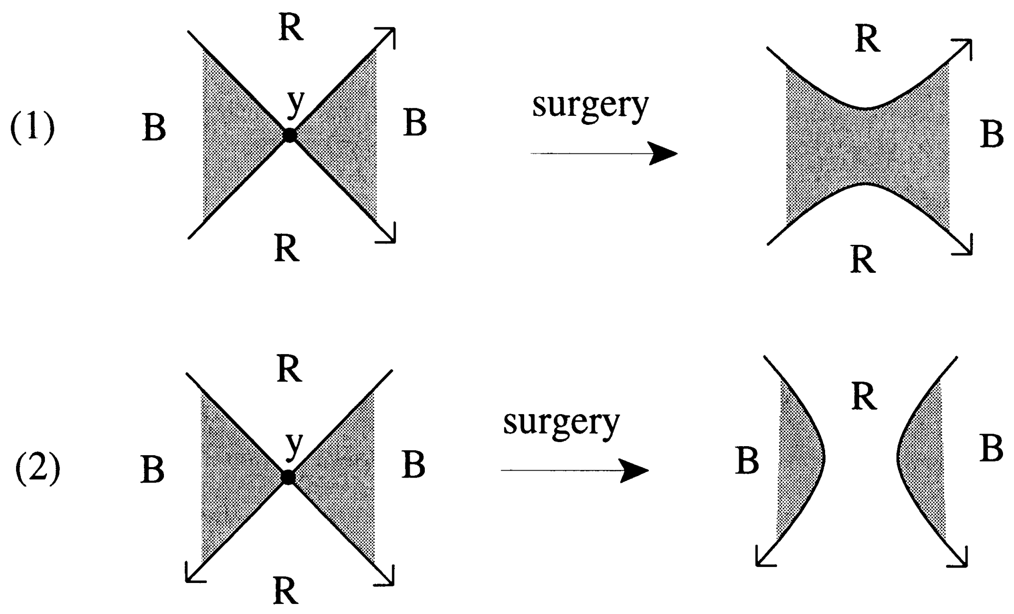

Fig. 11.

the compactness of $\bar{B}$. In the first case as in Figure 11 (1), $\chi(B)$ decreases by 1, $\beta_0(\Gamma)$ changes by ± 1, and $\sum_{y \in \Delta(\alpha)} i_y$ does not change. In the second case as in Figure 11 (2), $\chi(B)$ does not change, $\beta_0(\Gamma)$ changes by ± 1, and $\sum_{y \in \Delta(\alpha)} i_y$ decreases by 1. In both cases, $\chi(B) + \beta_0(\Gamma) + \sum_{y \in \Delta(\alpha)} i_y$ modulo 2 does not change. Thus by the induction hypothesis, this must be even. This completes the proof. $\square$

Remark. Let M be a compact oriented surface and $f : M \to N$ a generic neat map, where N is a connected 3-manifold with ∂N orientable. Suppose that $H_1(N) = 0$ or $f_*[M, \partial M] = 0$ in $H_2(N, \partial N)$. Then it is easy to show that for every boundary double point y of f, the index $n'(y, f)$ defined in Definition 3.6 coincides with the index i_y with respect to the immersion with normal crossings $f \mid \partial M : \partial M \to \partial N$ as defined in Definition 4.4, where ∂M is oriented as the boundary of M.

Definition 4.7 Consider the situation as in Theorem 4.2. We suppose that ∂N is orientable. For a boundary double point y of f, define the index $n'(y, f)$ to be the index i_y with respect to the immersion with normal crossings $f \mid \partial M : \partial M \to \partial N$. By the above remark, this definition is compatible with that given in Definition 3.6 when M is oriented.

As a corollary to the above observations, we have the following immediately.

Corollary 4.8 *Under the hypothesis of Theorem* 4.2, *if ∂N is orientable, then*

$$T(f) \equiv \sum_{i=1}^{k} n(x_i, f) + \sum_{j=1}^{m} n'(y_j, f) + \chi(M) + \beta_0(\partial M) \mod 2,$$

where $y_1, \ldots, y_m \in f(\partial M)$ are the boundary double points of f.

Theorem 3.7 together with Corollary 4.8 implies Theorem 1.2 in the introduction.

Remark. Consider the embedding $f : F \to F \times \{0\} \hookrightarrow F \times [-1, 1]$, where F is the Möbius strip. Then the congruence in Theorem 4.2 holds, but not the congruence in Corollary 4.8. This is because $\partial(F \times [-1, 1])$ is not orientable. This example also shows that in Theorem 1.2, the orientability hypothesis on either M or ∂N is essential.

Izumiya and Marar [9] have shown that for a map as in Theorem 4.2, we have

$$\chi(f(M)) = \chi(M) + T(f) + \frac{C(f) - \partial d(f)}{2},$$

where $C(f)$ is the number of cross caps and $\partial d(f)$ is the number of boundary double points of f. This formula together with our Corollary 4.8 implies the following.

Proposition 4.9 *Under the hypothesis of Theorem* 4.2, *if* ∂N *is orientable, then*

$$\chi(f(M)) \equiv \frac{(B_c(f) - R_c(f)) - (B_b(f) - R_b(f))}{2} + \beta_0(\partial M) \mod 2,$$

where

$$\begin{aligned} B_c(f) &= \sharp\{x_i : n(x_i, f) = 0\} \\ &\qquad \text{(number of blue cross cap points)}, \\ R_c(f) &= \sharp\{x_i : n(x_i, f) = 1\} \\ &\qquad \text{(number of red cross cap points)}, \\ B_b(f) &= \sharp\{y_j : n'(y_j, f) = 0\} \\ &\qquad \text{(number of blue boundary double points)}, \\ R_b(f) &= \sharp\{y_j : n'(y_j, f) = 1\} \\ &\qquad \text{(number of red boundary double points)}, \end{aligned}$$

$x_1, \ldots, x_k \in f(M)$ *are the cross caps of* f, *and* $y_1, \ldots, y_m \in f(\partial M)$ *are the boundary double points of* f.

Note that the above proposition generalizes [10], Theorem 1.3.2.

5. A geometrical application

In this last section we apply Theorem 1.2 in a particular case: the selftranslation surface of a generic space curve $\alpha : S^1 \to \mathbf{R}^3$. This surface has been studied in [12] and it is defined as the image of the map $\mathcal{T} : S^1 \times S^1 \to \mathbf{R}^3$ given by $\mathcal{T}(s_1, s_2) = (1/2)(\alpha(s_1) + \alpha(s_2))$. Because of the symmetry of $\mathcal{T}$, it induces another map $\tilde{\mathcal{T}} : S^1 \times S^1/\sim \to \mathbf{R}^3$ whose domain, $S^1 \times S^1/\sim$, is homeomorphic to the Möbius strip. The boundary corresponds to the diagonal of $S^1 \times S^1$ and its image is the curve $\alpha(S^1)$.

In [12] it is proved that when the curve α is generic, the map $\widetilde{\mathcal{T}}$ is stable. Moreover, it is obtained as a geometrical application of this result that when α is convex, the selftranslation surface cannot be embedded and hence it must have a double point. This double point is the center of a parallelogram inscribed in the curve.

This result can be improved by using our theorem. A curve $\alpha : S^1 \to \mathbf{R}^3$ is *convex*, by definition, if $\alpha(S^1)$ lies in the boundary of its convex hull, which is homeomorphic to the closed 3-disk. If α is generic, then the interior of the selftranslation surface is contained in the interior of the convex hull. In this way, the map $\widetilde{\mathcal{T}}$ is a generic neat map in the sense of the previous sections with no boundary double points.

Corollary 5.1 *Let $\alpha : S^1 \to \mathbf{R}^3$ be a generic convex curve. Then*

$$T(\widetilde{\mathcal{T}}) \equiv \sum_{i=1}^{k} n(s_i, t_i, \widetilde{\mathcal{T}}) + 1 \mod 2,$$

where $T(\widetilde{\mathcal{T}})$ is the number of triple points and $(s_1, t_1), \ldots, (s_k, t_k) \in S^1 \times S^1/\sim$ are the cross caps of the selftranslation surface $\widetilde{\mathcal{T}}$ associated with α.

Note that a cross cap of the selftranslation surface is the middle point of a pair of points of the curve with parallel tangents, and a triple point is the center of a (nonregular) octahedron inscribed in the curve.

Corollary 5.2 *Let $\alpha : S^1 \to \mathbf{R}^3$ be a generic convex curve. Then there exists either a pair of points of the curve with parallel tangents, or an octahedron inscribed in the curve.*

Remark. We can also prove Corollary 5.2 directly, using a result of [1]. Indeed, the map $\widetilde{\mathcal{T}}$ extends to a generic map of $\mathbf{R}P^2$ (seen as a Möbius strip with a 2-disk attached) to $\mathbf{R}^3$, the extension f being an embedding on the 2-disk. If f is an immersion, then it is known to have at least a triple point by [1]; if it is not an immersion, then obviously it has a cross cap.

We can generalize the above corollary to a more general class of space curves as follows. Let $\alpha : S^1 \to \mathbf{R}^3$ be an immersion with a finite number of selfintersection points such that at each intersection point, there exist exactly two branches of α and that they are not tangent to each other. We call such α a *normal* space curve. For a normal space curve α, we can also define $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ as above. Furthermore, when α is generic, $\widetilde{\mathcal{T}}$ is a generic neat map except in a neighbourhood of the double points of α. Then we have the following.

Proposition 5.3 *Let $\alpha : S^1 \to \mathbf{R}^3$ be a generic convex normal space curve. Let $\partial C(\alpha) \setminus \alpha(S^1) = R \cup B$ be a two-colour decomposition as in the paragraph just before Definition* 4.4, *where $C(\alpha)$ is the convex hull of α. If $\beta_0(R)$ or $\beta_0(B)$ is odd, then there exists either a pair of points of the curve with parallel tangents, or an octahedron inscribed in the curve.*

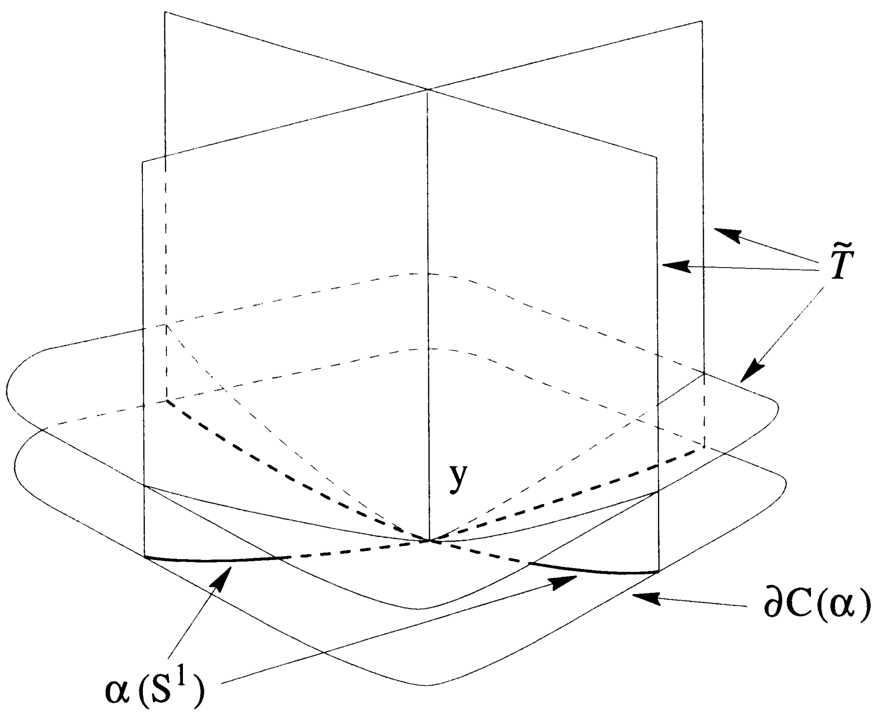

Fig. 12.

Proof. Let $y \in \alpha(S^1)$ be a double point of α. It is not difficult to see that the image of $\widetilde{T}$ in a neighbourhood of y looks like as in Figure 12. Then, modifying $\widetilde{T}$ slightly in a neighbourhood of the double points of α, we obtain a generic neat map $f : M \to C(\alpha)$ of the Möbius strip M with the following properties:

(1) $f(\partial M) = \alpha(S^1)$,

(2) the cross caps of f are in one to one correspondence with the cross caps of $\widetilde{T}$,

(3) the triple points of f are in one to one correspondence with the union of the triple points of $\widetilde{T}$ and the double points of α.

Now suppose that $\widetilde{\mathcal{T}}$ has neither cross caps nor triple points. In this case, f is a generic neat immersion. Then by Corollary 4.3 we see that the number of double points of α is congruent modulo 2 to $\beta_0(R)$ and $\beta_0(B)$. On the other hand, since f is a generic neat immersion, the number of boundary double points is even. Thus $\beta_0(R)$ and $\beta_0(B)$ are even. This contradicts our assumption. Thus $\widetilde{\mathcal{T}}$ has either a cross cap or a triple point. This completes the proof. □

Remark. Note that Proposition 5.3 is a generalization of Corollary 5.2. Furthermore, in Proposition 5.3, the condition that $\beta_0(R)$ or $\beta_0(B)$ be odd is equivalent to that either the number of double points of α with index 0 or that with index 1 (in the sense of Definition 4.4) be odd.

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