Hokkaido Mathematical Journal Vol. 27 (1998) p. 393–407

# Life span and asymptotic behavior for a semilinear parabolic system with slowly decaying initial values

(Dedicated to Professor Rentaro Agemi on his 60th birthday)

Qing HUANG, Kiyoshi MOCHIZUKI and Kentaro MUKAI

(Received February 13, 1997; Revised April 22, 1997)

**Abstract.** We consider the semilinear parabolic system

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q,$$

where $x \in \mathbf{R}^N$ $(N \geq 1)$, $t > 0$ and $p,\ q \geq 1$. At $t = 0$, nonnegative, bounded and continuous initial values $(u_0(x), v_0(x))$ are prescribed. The main results are for the case when $(u_0, v_0)$ have polynomial decay near $x = \infty$. Assuming $u_0 \sim (\lambda|x|^{-a})^{1/(q+1)}$, $v_0 \sim (\lambda|x|^{-a})^{1/(p+1)}$ with $\lambda > 0$, $0 \leq a < N \min\{p+1, q+1\}$, we answer various questions of global existence and nonexistence, large time behavior or life span of the solutions in terms of simple conditions on $\lambda$, $a$, $p$, $q$ and the space dimension $N$.

*Key words*: blow-up, life span, global existence, asymptotic behavior, semilinear parabolic equation, slowly decaying initial value.

## 1. Introduction

We consider the initial value problem

$$\begin{cases} u_t = \Delta u + v^p, & x \in \mathbf{R}^N,\ t > 0, \\ v_t = \Delta v + u^q, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \\ v(x,0) = v_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1}$$

where $p,\ q \geq 1$, $pq > 1$, $N \geq 1$ and $(u_0(x), v_0(x))$ are nonnegative, bounded and continuous functions. The problem provides a simple example of a reaction-diffusion system. As a model of heat propagation in a two-component combustible mixture, $u$, $v$ represent the temperatures of the interacting components. It is assumed that thermal conductivity is constant and equal for both substance, and a volume energy release is given by some powers of $u$ and $v$.

It is well known that problem (1) has a unique, nonnegative and bounded

*1991 Mathematics Subject Classification* : 35K05, 35K15, 35K45 and 35K57.

solution at least locally in time. We define

$$T^* = T^*(u_0, v_0) = \sup\{T > 0;\ (u(t), v(t)) \text{ is bounded and solves (1) in } \mathbf{R}^N \times (0, T)\}.$$

$T^*$ is called the life span of solutions $(u(t), v(t))$. If $T^* = \infty$ the solutions are global. On the other hand, if $T^* < \infty$ one has

$$\limsup_{t\to T^*} \|u(t)\|_\infty = \infty \quad \text{or} \quad \limsup_{t\to T^*} \|v(t)\|_\infty = \infty \tag{2}$$

since otherwise solutions could be extended beyond $T^*$. When (2) holds we say that the solution blows up in finite time.

The blow-up and the global existence of solutions has been studied by Escobedo-Herrero [3], and the following results are proved there.

(I) Suppose that $1 < pq \le 1 + (2/N)\max\{p+1, q+1\}$. Then $T^* < \infty$ for every nontrivial solution $(u(t), v(t))$, and

$$\limsup_{t\to T^*} \|u(t)\|_\infty = \limsup_{t\to T^*} \|v(t)\|_\infty = \infty. \tag{3}$$

(II) Suppose that $pq > 1 + (2/N)\max\{p+1, q+1\}$. Let

$$u_0 \in L^\infty \cap L^{\alpha_1}, \quad v_0 \in L^\infty \cap L^{\alpha_2},$$

where $\alpha_1 = N(pq-1)/2(p+1)$, $\alpha_2 = N(pq-1)/2(q+1)$. If $\|u_0\|_{\alpha_1} + \|v_0\|_{\alpha_2}$ is sufficiently small, then $T^* = \infty$.

(III) Suppose that $pq > 1 + (2/N)\max\{p+1, q+1\}$. Let

$$u_0(x) \ge Ce^{-\alpha|x|^2}$$

for some $\alpha > 0$ and some $C > 0$ large enough. Then $T^* < \infty$ and (3) holds.

In this paper we shall study the behavior of solutions $(u(t), v(t))$ while the initial values $(u_0, v_0)$ have slow decay near $|x| = \infty$. For instance in case

$$u_0 \sim (\lambda|x|^{-a})^{1/(q+1)}, \quad v_0 \sim (\lambda|x|^{-a})^{1/(p+1)}$$

with $\lambda > 0$ and $0 \le a < N\min\{p+1, q+1\}$, we are interested in the question of global existence and nonexistence, large time behavior or life span of solutions in terms of $\lambda$ and $a$. These problems have been studied by Lee-Ni [8] and Gui-Wang [6] for the Cauchy problem of single equation $u_t = \Delta u + u^p$. Our results will partly extend theirs to the system of equations (1). Note that similar results can be obtained also for the Cauchy problem of quasilinear equation $u_t = \Delta u^m + u^p$ with $p > m > 1$ (see Mukai-Mochizuki-Huang [10]).

Throughout the rest of this paper we shall use the following notations. We set $C_b(\mathbf{R}^N)$ to be the space of all bounded continuous functions in $\mathbf{R}^N$ and, for $\alpha \geq 0$,

$$I^\alpha = \left\{\xi \in C_b(\mathbf{R}^N); \xi(x) \geq 0 \text{ and } \limsup_{|x|\to\infty} |x|^\alpha \xi(x) < \infty\right\},$$

$$I_\alpha = \left\{\xi \in C_b(\mathbf{R}^N); \xi(x) \geq 0 \text{ and } \liminf_{|x|\to\infty} |x|^\alpha \xi(x) > 0\right\}.$$

For two functions $f(r)$ and $g(r)$, we say that $f \sim g$ near $r = 0$ ($\infty$ respectively) if there exists two positive constants $C_1$, $C_2$ such that $C_1 f(r) \leq g(r) \leq C_2 f(r)$ near $r = 0$ ($\infty$ respectively). The letter $C$ denotes a positive generic constant which may vary from line to line. We shall use the notation $S(t)\xi$ to represent the solution of the heat equation with initial value $\xi(x)$:

$$[S(t)\xi](x) = (4\pi t)^{-N/2} \int_{\mathbf{R}^N} e^{-|x-y|^2/4t} \xi(y) dy.$$

Especially, we write $[S(t)\xi](x) = W(x,t;A,\alpha)$ when $\xi(x) = A|x|^{-\alpha}$ with $A > 0$ and $0 \leq \alpha < N$. $W$ has the explicit form

$$W(x,t;A,\alpha) = At^{-\alpha/2} h_\alpha(x/t^{1/2}),$$

where

$$h_\alpha(x) = (4\pi)^{-N/2} \int_{\mathbf{R}^N} e^{-|y|^2/4} |x-y|^{-\alpha} dy.$$

In the following we assume

$$q \geq p \geq 1 \quad \text{and} \quad pq > 1. \tag{4}$$

We put

$$(u_0(x), v_0(x)) = (\lambda^{1/(q+1)}\varphi(x), \lambda^{1/(p+1)}\psi(x))$$

in (1), where $\lambda > 0$, and write

$$T_\lambda^* = T^*(\lambda^{1/(q+1)}\varphi, \lambda^{1/(p+1)}\psi).$$

Moreover, we let

$$a^* = \frac{2(p+1)(q+1)}{pq-1}.$$

Then our results of this paper will be summarized in the following four theorems.

**Theorem 1** *Suppose $\psi(x) \in I_{a/(p+1)}$ for some $0 \le a < \min\{a^*, N(p+1)\}$. Then $T_\lambda^* < \infty$ for any $\lambda > 0$, and for given $\lambda_0 > 0$ there exists $C(\lambda_0) > 0$ such that*

$$T_\lambda^* \le C(\lambda_0)\lambda^{-2/(a^*-a)} \quad \text{for } \lambda < \lambda_0. \tag{5}$$

**Theorem 2** *Suppose that $\varphi \in I^{a/(q+1)}$ and $\psi \in I^{a/(p+1)} \cap I_{a/(p+1)}$ for some $0 \le a < \min\{a^*, N(p+1)\}$. Then we have*

$$T_\lambda^* \sim \lambda^{-2/(a^*-a)} \quad \text{near } \lambda = 0. \tag{6}$$

**Theorem 3** *Let $pq > 1 + (2/N)(q+1)$, or equivalently $a^* < N(p+1)$.*

(i) *Suppose that $\varphi \in I^{a/(q+1)}$, $\psi \in I^{a/(p+1)}$ for some $a^* < a < N(p+1)$. Then there exists $\lambda_1 > 0$ such that $T_\lambda^* = \infty$ for $\lambda < \lambda_1$, and*

$$\|u(t)\|_\infty \le Ct^{-a/2(q+1)}, \quad \|v(t)\|_\infty \le Ct^{-a/2(p+1)} \tag{7}$$

*as $t \to \infty$.*

(ii) *Suppose that*

$$\lim_{|x|\to\infty} |x|^{a/(q+1)}\varphi(x) = A_1 > 0,$$

$$\lim_{|x|\to\infty} |x|^{a/(p+1)}\psi(x) = A_2 > 0$$

*for some $a^* < a < N(p+1)$. Then for $\lambda < \lambda_1$ we have*

$$t^{a/2(q+1)}|u(x,t) - W(x,t; A_1\lambda^{1/(q+1)}, a/(q+1))| \to 0,$$

$$t^{a/2(p+1)}|v(x,t) - W(x,t; A_2\lambda^{1/(p+1)}, a/(p+1))| \to 0 \tag{8}$$

*as $t \to \infty$ uniformly in $\mathbf{R}^N$.*

**Theorem 4** *Suppose that $\varphi$, $\psi \in C_b(\mathbf{R}^N)$ and $\varphi(0)\psi(0) > 0$. Then there* *exists* $\lambda_2 \geq 0$ *such that* $T_\lambda^* < \infty$ *for any* $\lambda > \lambda_2$, *and*

$$T_\lambda^* \sim \lambda^{-2/a^*} \quad as \ \ \lambda \to \infty. \tag{9}$$

Comparing Theorem 1 and (II) stated above (or Theorem 3 (i)), we see that the number $a^*$ gives another cutoff between the blow-up case and the global existence case. Theorem 3 (ii) is not treated in Lee-Ni [8]. The corresponding results for single equation have been obtained by Kamin-Peletier [7] in case of the heat equation with absorption. To show the theorems we shall frequently use a standard comparison principle. We refer Protter-Weinberger [11] and Bebernes-Eberly [1] on this principle. The condition $p \geq 1$ which guarantees the uniqueness of solutions to (1) is mainly required to verify this principle. In this paper we did not enter into the case $a \geq N(p+1)$. For single equation, this case is contained in [8], and some of their results can be extended also to our system. Finally, note that the critical exponent $a = a^*$ is expected to belong to the global existence case. In fact, if $N \geq 3$ and $pq > 1 + (2/(N-2)) \max\{p+1, q+1\}$, the functions

$$\Phi(x) = A|x|^{-a^*/(q+1)}, \quad \Psi(x) = B|x|^{-a^*/(p+1)}$$

become a stationary solution to (1) under suitably chosen positive constants $A$, $B$. We shall discuss these results elswhere.

The rest of the paper is organized as follows: Theorems 1 amd 2 are proved in the next §2, Theorems 3 and 4 are proved in §3 and §4, respectively. To show Theorem 2 we construct a super-solution to the system of equations (1). Its special form and estimate will also be used in §3 and §4.

## 2. Proof of Theorems 1 and 2

In order to obtain an estimate of $T_\lambda^*$ from above, the following lemma due to Escobedo-Herrero [3; Lemma 4.1] plays a key role in our proof.

**Lemma 1** *Assume that* $q \geq p \geq 1$ *and* $pq > 1$, *and let* $(u(t), v(t))$ *be the solution of* (1) *in some strip* $S_T = \mathbf{R}^N \times [0, T)$ *with* $0 < T \leq \infty$. *Assume also that* $u(t)$ *and* $v(t)$ *are bounded in* $S_T$. *Then there exists a constant* $C > 0$, *depending on* $p$, $q$ *but not on* $u_0$, $v_0$, *nor* $T$, *such that*

$$\lambda^{1/(p+1)} t^{(q+1)/(pq-1)} \|S(t)\psi\|_\infty \leq C \quad for\ any \ \ t \in [0, T). \tag{10}$$

*Proof of Theorem* 1. Since $\psi \in I_{a/(p+1)}$, we can choose a bounded continuous function $\tilde{\psi}(x)$ in $\mathbf{R}^N$ such that

$$\tilde{\psi}(x) = m|x|^{-a/(p+1)} \text{ for } |x| > R \text{ and } \tilde{\psi}(x) \le \psi(x) \text{ for } x \in \mathbf{R}^N,$$

where $m > 0$ is sufficiently small and $R > 0$ is sufficiently large. Let $t_0 = t_0(\lambda) > 0$ be a small number such that $t_0 < T_\lambda^*$. Then we have for $t > t_0$,

$$\begin{aligned} S(t)\psi \ge S(t)\tilde{\psi} &\ge (4\pi t)^{-N/2} \int_{|x-y|>R} e^{-|y|^2/4t} m|x-y|^{-a/(p+1)} dy \\ &\ge t^{-a/2(p+1)} (4\pi)^{-N/2} m \int_{|xt^{-1/2}-y|>Rt_0^{-1/2}} e^{-|y|^2/4} |x/t^{1/2} - y|^{-a/(p+1)} dy \\ &\equiv t^{-a/2(p+1)} k_{t_0}(x/t^{1/2}) > 0. \end{aligned}$$

Therefore,

$$\|S(t)\psi\|_\infty \ge t^{-a/2(p+1)} \|k_{t_0}\|_\infty.$$

Substituting this in (10), we see that the inequality

$$t^{(q+1)/(pq-1)} t^{-a/2(p+1)} \le C\lambda^{-1/(p+1)} \|k_{t_0}\|_\infty^{-1}$$

holds for any $t \in (t_0, T_\lambda^*)$.

This proves that $T_\lambda^* < \infty$ for any $\lambda > 0$. Inequality (5) also follows from this since we can choose $t_0 = t_0(\lambda_0)$ for any $0 < \lambda \le \lambda_0$. □

In order to obtain an estimate of $T_\lambda^*$ from below, we shall construct a suitable supersolution of (1).

Let $(x(t), y(t))$ be a solution of the ordinary differential equation

$$\begin{cases} x' = f(t)y^p, & t > 0, \\ y' = f(t)x^q, & t > 0, \\ x(0) = x_0 > 0, \quad y(0) = y_0 > 0, \end{cases} \tag{11}$$

where $pq > 1$ and $f(t) > 0$ is a bounded continuous function of $t \ge 0$.

**Lemma 2** *Assume that*

$$(q+1)^{-1} x_0^{q+1} \le (p+1)^{-1} y_0^{p+1}. \tag{12}$$

*Then we have*

$$y(t) \le \left\{ y_0^{-(pq-1)/(q+1)} - \frac{pq-1}{q+1}\left(\frac{q+1}{p+1}\right)^{q/(q+1)} \int_0^t f(s)ds \right\}^{-(q+1)/(pq-1)} \tag{13}$$

*Proof.* From equation (11) it follows that $x^q dx = y^p dy$. Integrate both sides from 0 to $t$. Then by virtue of (12)

$$x(t) \le f(t)\left(\frac{q+1}{p+1}\right)^{1/(q+1)} y(t)^{(p+1)/(q+1)}.$$

Substitute this in the second eqution of (11). Then we have

$$y^{-q(p+1)/(q+1)} y' \le \left(\frac{q+1}{p+1}\right)^{q/(q+1)} f(t).$$

Integrating this again from 0 to $t$, we obtain (13). □

We put

$$\begin{aligned} W_1(x,t) &= W(x, t+t_1; M_1\lambda^{1/(q+1)}, a/(q+1)), \\ W_2(x,t) &= W(x, t+t_1; M_2\lambda^{1/(p+1)}, a/(p+1)), \end{aligned} \tag{14}$$

where $M_1$, $M_2$ and $t_1$ are positive constants. Note that for $\varphi \in I^{a/(q+1)}$, $\psi \in I^{a/(p+1)}$ with $0 \le a < N(p+1)$, we can choose $M_1$, $M_2$ large enough to satisfy

$$W_1(x,0) \ge \lambda^{1/(q+1)}\varphi(x), \quad W_2(x,0) \ge \lambda^{1/(p+1)}\psi(x). \tag{15}$$

**Lemma 3** (i) *$W_j(x,t) > 0$ $(j = 1,2)$ and $|x|^{a/(q+1)}W_1(x,t)$, $|x|^{a/(p+1)}W_2(x,t)$ are bounded in $\mathbf{R}^N \times [0,\infty)$.*

(ii) *There exists a constant $C > 0$ such that for any $t \ge 0$,*

$$\|W_1(\cdot,t)\|_\infty \le C(t+t_1)^{-a/2(q+1)},$$
$$\|W_2(\cdot,t)\|_\infty \le C(t+t_1)^{-a/2(p+1)}.$$

(iii) *There exists a constant $C_1 > 0$ such that for any $t \ge 0$,*

$$\|W_2(\cdot,t)^p/W_1(\cdot,t)\|_\infty \le C_1\lambda^{2/a^*}(t+t_1)^{-a/a^*},$$

$$\|W_1(\cdot,t)^q/W_2(\cdot,t)\|_\infty \le C_1\lambda^{2/a^*}(t+t_1)^{-a/a^*}.$$

*Proof.* For any $0 \leq \alpha < N$ and $x \in \mathbf{R}^N$, we have (cf., Rapnikov-Eidelman [12]) $h_\alpha(x) > 0$, and

$$\lim_{|x|\to\infty} |x|^\alpha h_\alpha(x) = 1.$$

Since $W(x, t+t_0; A, \alpha) = A(t+t_0)^{-\alpha/2} h_\alpha(x(t+t_0)^{-1/2})$, these properties of $h_\alpha(x)$ prove the assertions of the lemma. □

Now, let $(\alpha(t), \beta(t))$ be the solution of

$$\begin{cases} \alpha' = \|W_2(\cdot,t)^p/W_1(\cdot,t)\|_\infty \beta^p, & t > 0 \\ \beta' = \|W_1(\cdot,t)^q/W_1(\cdot,t)\|_\infty \alpha^q, & t > 0 \\ \alpha(0) = \beta(0) = 1, \end{cases} \tag{16}$$

and let us define $(\bar{u}(x,t), \bar{v}(x,t))$ as follows:

$$\bar{u}(x,t) = \alpha(t) W_1(x,t), \quad \bar{v}(x,t) = \beta(t) W_2(x,t). \tag{17}$$

**Lemma 4** (i) *$(\alpha(t), \beta(t))$ is a subsolution of (11) with $f(t) = C_1 \lambda^{2/a^*}(t+t_1)^{-a/a^*}$ and $x_0 = y_0 = 1$.*

(ii) *Suppose that $\varphi \in I^{a/(q+1)}$ and $\psi \in I^{a/(p+1)}$. Then $(\bar{u}(t), \bar{v}(t))$ gives a supersolution of (1).*

*Proof.* (i) is obvious from Lemma 3 (iii). We have

$$\begin{aligned} \bar{u}_t &= \alpha'(t) W_1(x,t) + \alpha(t) W_{1t}(x,t) \\ &= \|W_2^p/W_1\|_\infty \beta^p W_1 + \alpha \Delta W_1 \geq \Delta \bar{u} + \bar{v}^p. \end{aligned}$$

Similarly, we have $\bar{v}_t \geq \Delta \bar{v} + \bar{u}^q$. These inequalities and (15) show the assertion (ii). □

*Proof of Theorem* 2. It follows from Lemma 4 (ii) and a standard comparison argument that

$$u(x,t) \leq \bar{u}(x,t) \quad \text{and} \quad v(x,t) \leq \bar{v}(x,t). \tag{18}$$

Then we see from (17) that $T_\lambda^*$ is not less than the life span of $(\alpha(t), \beta(t))$.

By means of Lemma 4 (i) and a comparison principle, we see from Lemma 2 that

$$\beta(t) \le \left\{1 - \frac{pq-1}{q+1}\left(\frac{q+1}{p+1}\right)^{q/(q+1)} C_1\lambda^{2/a^*} \times \int_0^t (s+t_1)^{-a/a^*} ds\right\}^{-(q+1)/(pq-1)}. \qquad (19)$$

Remember that we have assumed $0 \le a < a^*$. Then (19) implies that $\beta(t)$ remains finite at least for $t$ satisfying

$$\frac{C_1(pq-1)a^*}{(q+1)(a^*-a)}\left(\frac{q+1}{p+1}\right)^{q/(q+1)} \lambda^{2/a^*} t^{(a^*-a)/a^*} \le 1.$$

Integrating the first equation of (16) shows that $\alpha(t)$ is finite in the same interval. Thus, we obtain

$$T_\lambda^* > \frac{1}{2} C_2 \lambda^{-2/(a^*-a)} \quad \text{for any} \quad \lambda > 0.$$

Combining this and (5) of Theorem 1, we conclude the assertion of Theorem 2. □

## 3. Proof of Theorem 3

In this section we restrict ourselves to the case $a^* < N(p+1)$, and assume that $a^* < a < N(p+1)$.

We see from (19) that $\beta(t)$ is global and bounded in $t \ge 0$ if $\lambda < \lambda_1$, where $\lambda_1 > 0$ is given by

$$\frac{C_1(pq-1)a^*}{(q+1)(a-a^*)}\left(\frac{q+1}{p+1}\right)^{q/(q+1)} \lambda_1^{2/a^*} t_1^{-(a-a^*)/a^*} = 1.$$

Moreover, noting $a > a^*$, we see that the right side of the first equation of (16) is integrable in $t \in (0, \infty)$. This implies that $\alpha(t)$ is also global and bounded in $t > 0$. Then we have (7) from (17) and Lemma 3 (ii). Theorem 3 (i) is thus complete.

To show Theorem 3 (ii) we put

$$u_k(x,t) = k^{a/(q+1)} u(kx, k^2 t), \quad v_k(x,t) = k^{a/(p+1)} u(kx, k^2 t)$$

for $k > 0$. Then $(u_k(t), v_k(t))$ solves

$$\begin{cases} u_{kt} = \Delta u_k + k^{-2a/a^*+2} v_k^p, \\ v_{kt} = \Delta v_k + k^{-2a/a^*+2} u_k^q, \\ u_k(x,0) = k^{a/(q+1)} \lambda^{1/(q+1)} \varphi(kx), \\ v_k(x,0) = k^{a/(p+1)} \lambda^{1/(p+1)} \psi(kx). \end{cases} \tag{20}$$

It follows from (7) that

$$\|u_k(t)\|_\infty \leq k^{a/(q+1)} C(k^2 t)^{-a/2(q+1)} = Ct^{-a/2(q+1)}.$$

Thus, $\{u_k(x,t)\}$ is uniformly bounded in $\mathbf{R}^N \times [\delta, \infty)$ for any $\delta$. As is easily seen, the uniform boundedness implies the equicontinuity of $\{u_k(x,t)\}$ in any bounded set of $\mathbf{R}^N \times [\delta, \infty)$. Then using the Ascoli-Arzela theorem and a diagonal sequence method in $\delta$, we see that for any sequence $\{k_j\} \to \infty$, there exists a subsequence $\{k_j'\}$ and a function $w_1(x,t) \in C(\mathbf{R}^N \times (0,\infty))$ such that

$$u_{k_j'}(x,t) \to w_1(x,t) \quad \text{as} \quad k_j' \to \infty$$

locally uniformly in $\mathbf{R}^N \times (0, \infty)$.

We shall show

$$w_1(x,t) = W(x,t; A_1 \lambda^{1/(q+1)}, a/(q+1)). \tag{21}$$

It follows from the first equation of (20) that

$$\int_{\mathbf{R}^N} u_k(x,t)\zeta(x,t)dx - \int_{\mathbf{R}^N} u_k(x,0)\zeta(x,0)dx$$
$$= \int_0^t \int_{\mathbf{R}^N} \left\{ u_k \zeta_t + u_k \Delta\zeta + k^{-2a/a^*+2} v_k^p \zeta \right\} dxdt \tag{22}$$

for any $t > 0$ and nonnegative $\zeta(x,t) \in C_0^\infty(\mathbf{R}^N \times [0,\infty))$. By assumption on the initial values

$$\int_{\mathbf{R}^N} u_k(x,0)\zeta(x,0)dx = \int_{\mathbf{R}^N} k^{a/(q+1)} \lambda^{1/(q+1)} \varphi(kx)\zeta(x,0)dx$$
$$\to A_1 \lambda^{1/(q+1)} \int_{\mathbf{R}^N} |x|^{-a/(q+1)} \zeta(x,0)dx \quad \text{as} \quad k = k_j' \to \infty.$$

On the other hand,

$$\int_0^t \int_{\mathbf{R}^N} k^{-2a/a^*+2} v_k^p \zeta dxdt$$
$$= \int_0^{k^2 t} \int_{\mathbf{R}^N} k^{a/(q+1)} v(kx,\tau)^p \zeta(x, k^{-2}\tau) dxd\tau.$$

Since $\alpha(t)$ is bounded, it follows from Lemma 3 (i) that

$$\begin{aligned} k^{a/(q+1)} v(kx,\tau)^p &\le C k^{a/(q+1)} W_2(kx,\tau)^p \\ &\le C\left\{(k|x|)^{a/(p+1)} W_2(kx,\tau)\right\}^{(p+1)/(q+1)} \\ &\qquad \times |x|^{-a/(q+1)} \left(W_2(kx,\tau)\right)^{(pq-1)/(q+1)} \\ &\le C M_2^p \lambda^{p/(p+1)} |x|^{-a/(q+1)} (\tau+t_1)^{-a/a^*} \\ &\qquad \times \left(h_{a/(p+1)}(kx/(\tau+t_1)^{1/2})\right)^{(pq-1)/(q+1)}. \end{aligned}$$

Note here that $a/(q+1) < N$, $a/a^* > 1$ and

$$h_{a/(p+1)}(kx/(\tau+t_1)^{1/2}) \to 0 \quad \text{as} \quad k \to \infty$$

for $x \neq 0$. Then we can apply the Lebesgue dominated convergence theorem to obtain

$$\int_0^{k^2 t} \int_{\mathbf{R}^N} k^{a/(q+1)} v(kx,\tau)^p \zeta(x, k^{-2}\tau) dxd\tau \to 0 \quad \text{as} \quad k = k_j' \to \infty.$$

Thus, letting $k = k_j' \to \infty$ in (22), we obtain

$$\int_{\mathbf{R}^N} w_1(x,t)\zeta(x,t)dx - \int_{\mathbf{R}^N} A_1 \lambda^{1/(q+1)} |x|^{-a/(q+1)} \zeta(x,0) dx$$
$$= \int_0^t \int_{\mathbf{R}^N} \{w_1 \zeta_t + w_1 \Delta\zeta\} dxdt.$$

The uniqueness of solutions of

$$u_t = \Delta u, \quad u(x,0) = A_1 \lambda^{1/(q+1)} |x|^{-a/(q+1)},$$

then implies (21).

We have thus proved that

$$u_k(x,t) \to W(x,t; A_1\lambda^{1/(q+1)}, a/(q+1)) \quad \text{as} \quad k \to \infty \tag{23}$$

uniformly in compact sets of $\mathbf{R}^N \times (0,\infty)$.

Note again that $(\alpha(t), \beta(t))$ is bounded. Then it follows from (14) and (17) that

$$u_k(x,t) \le Ck^{a/(q+1)} W(kx, k^2t + t_1; M_1\lambda^{1/(q+1)}, a/(q+1)).$$

Let $t = 1$ in this inequality. Then by use of the selfsimilarity of $W(x,t; A, \alpha)$:

$$W(x,t; A, \alpha) = k^\alpha W(kx, k^2t; A, \alpha),$$

we have

$$u_k(x,1) \le CW(x, 1 + k^{-2}t_1; M_1\lambda^{1/(q+1)}, a/(q+1)).$$

This inequality implies with Lemma 3 (i) that for any $\epsilon > 0$ there exists an $R > 0$ independent of $k > (2t_1)^{-2}$ such that $\{u_k(x,1)\}$ are uniformly less than $\epsilon$ in $|x| > R$. Therefore, we have from (23) that

$$u_k(x,1) - W(x, 1; A_1\lambda^{1/(q+1)}, a/(q+1)) \to 0 \quad \text{as} \quad k \to \infty$$

uniformly in $\mathbf{R}^N$.

We let $y = kx$ and $s = k^2$ in this relation. Then noting again the selfsimilarity of $W(x,t; A, \alpha)$, we conclude that

$$s^{a/2(q+1)}|u(y,s) - W(y, s; A_1\lambda^{1/(q+1)}, a/(q+1))| \to 0 \quad \text{as} \quad s \to \infty$$

uniformly in $\mathbf{R}^N$.

Relation (8) is now proved for $u(x,t)$. The same argument can be applied also to $v(x,t)$, and Theorem 3 (ii) is complete.

## 4. Proof of Theorem 4

In this section we consider the case where $\lambda$ goes to $\infty$.

In order to obtain an estimate of $T^*_\lambda$ from below, we choose $f(t) \equiv 1$ and $x_0 = \lambda^{1/(q+1)}\|\varphi_0\|_\infty$, $y_0 = \lambda^{1/(p+1)}\|\psi_0\|_\infty$ in (11). Then the solution $(x(t), y(t))$ gives a supersolution of (1), and we have from Lemma 2

$$y(t) \le \left\{ y_0^{-(pq-1)/(q+1)} - \frac{pq-1}{q+1}\left(\frac{q+1}{p+1}\right)^{q/(q+1)} t \right\}^{-(q+1)/(pq-1)}$$

if $(q+1)^{-1}x_0^{q+1} \le (p+1)^{-1}y_0^{p+1}$. Similarly, we can have

$$x(t) \le \left\{ x_0^{-(pq-1)/(p+1)} - \frac{pq-1}{p+1}\left(\frac{p+1}{q+1}\right)^{p/(p+1)} t \right\}^{-(p+1)/(pq-1)}$$

if $(q+1)^{-1}x_0^{q+1} \geq (p+1)^{-1}y_0^{p+1}$. From these inequalities we conclude

$$T_\lambda^* \geq C\left[\max\left\{(q+1)^{-1}\|\varphi\|_\infty^{q+1}, (p+1)^{-1}\|\psi\|_\infty^{p+1}\right\}\right]^{-2/a^*}\lambda^{-2/a^*} \tag{24}$$

for any $\lambda > 0$, where $C = a^*(p+1)^p(q+1)^q/2$.

To obtain an estimate of $T_\lambda^*$ from above, we put

$$u_\lambda(x,t) = \lambda^{-1/(q+1)}u(\lambda^{-1/a^*}x, \lambda^{-2/a^*}t),$$

$$v_\lambda(x,t) = \lambda^{-1/(p+1)}v(\lambda^{-1/a^*}x, \lambda^{-2/a^*}t).$$

Then $(u_\lambda(t), v_\lambda(t))$ solves

$$\begin{cases} u_{\lambda t} = \Delta u_\lambda + v_\lambda^p, \\ v_{\lambda t} = \Delta v_\lambda + u_\lambda^q, \\ u_\lambda(x,0) = \varphi_\lambda(x) = \varphi(\lambda^{-1/a^*}x), \\ v_\lambda(x,0) = \psi_\lambda(x) = \psi(\lambda^{-1/a^*}x). \end{cases} \tag{25}$$

Let $\tilde{T}_\lambda^*$ be the life span of $(u_\lambda(t), v_\lambda(t))$. Then obviously

$$T_\lambda^* = \lambda^{-2/a^*}\tilde{T}_\lambda^*. \tag{26}$$

We define

$$F(t) = \int_{\mathbf{R}^N} u_\lambda(x,t)\rho_\epsilon(x)dx, \quad G(t) = \int_{\mathbf{R}^N} v_\lambda(x,t)\rho_\epsilon(x)dx,$$

where $\rho_\epsilon(x) = (\pi^{-1}\epsilon)^{N/2}e^{-\epsilon|x|^2}$ (cf. e.g., Mochizuki-Suzuki [9]). Then by the Jensen inequality, the following inequalities hold for $t > 0$.

$$F'(t) \geq -2N\epsilon F(t) + G^p(t), \quad G'(t) \geq -2N\epsilon G(t) + F^q(t).$$

Note that

$$\lim_{\lambda\to\infty} F(0) = \varphi(0) > 0, \quad \lim_{\lambda\to\infty} G(0) = \psi(0) > 0.$$

Then there exist $\epsilon_0 > 0$ and $\lambda_0 > 0$ such that when $0 < \epsilon < \epsilon_0$ and $\lambda > \lambda_0$,

$$-2N\epsilon F(0) + G^p(0) > 0, \quad -2N\epsilon G(0) + F^q(0) > 0.$$

Thus, if we put

$$\Omega = \{(x,y);\ -2N\epsilon x + y^p > 0,\ -2N\epsilon y + x^q > 0\},$$

then $\Omega$ becomes an invariant set for $(F(t), G(t))$.

As in Galaktionov-Kurdymov-Samarskii [4], [5] (cf., also Caristi-Mitidier [2]), we let $V(t) = F(t)G(t)$, and differentiate it in $t$. Then by use of the Hölder inequality, we obtain

$$V'(t) \geq -4N\epsilon V(t) + C(p,q)V(t)^{(p+1)(q+1)/(p+q+2)} \tag{27}$$

where

$$C(p,q) = \left(\frac{p+q+2}{q+1}\right)^{(p+1)/(p+q+2)} \left(\frac{p+q+2}{p+1}\right)^{(q+1)/(p+q+2)}.$$

Since

$$\lim_{\lambda\to\infty} V(0) = \varphi(0)\psi(0) > 0$$

for $0 < \epsilon < \epsilon_0$, we have from (27)

$$\limsup_{\lambda\to\infty} \tilde{T}_\lambda^* \leq \int_{\varphi(0)\psi(0)}^{\infty} \left\{C(p,q)\xi^{(p+1)(q+1)/(p+q+2)} - 4N\epsilon\xi\right\}^{-1} d\xi.$$

Thus, $V(t)$ blows up in a finite time. Moreover, letting $\epsilon \to 0$, we have

$$\limsup_{\lambda\to\infty} \tilde{T}_\lambda^* \leq C\{\varphi(0)\psi(0)\}^{-(pq-1)/(p+q+2)}.$$

From this and (26) it follows that

$$\limsup_{\lambda\to\infty} \lambda^{2/a^*} T_\lambda^* \leq C\{\varphi(0)\psi(0)\}^{-(pq-1)/(p+q+2)}. \tag{28}$$

Combinig (24) and (28), we conclude Theorem 4.

## References


[1] Bebernes J. and Eberly D., *Mathematicl problems from combustion theory.* Springer-Verlag, New York, 1989.

[2] Caristi G. and Mitidieri E., *Blow-up estimate of positive solutions of a parabolic system.* J. Diff. Eqns **113** (1994), 265–271.

[3] Escobedo M. and Herrero M.A., *Boundedness and blow up for a semilinear reaction-diffusion system.* J. Diff. Eqns. **89** (1991), 176–202.

[4] Galaktionov V.A., Kurdyumov S.P. and Samarskii A.A., *A parabolic system of quasilinear equations I.* Differential Equations **19** (1983), 2133–2143.

[5] Galaktionov V.A., Kurdyumov S.P. and Samarskii A.A., *A parabolic system of quasilinear equations II.* Differential Equations **21** (1985), 1544–1559.

[6] Gui C. and Wang X., *Life span of solutions of the Cauchy problem for a semilinear heat equation.* J. Diff. Eqns **115** (1995), 166–172.

[7] Kamin S. and Peletier L.A., *Large time behaviour of solutions of the heat equation with absorption.* Ann. Scu. Norm. Sup. Pisa **XII** (1985), 393–408.
[8] Lee T. and Ni W., *Global existence, largetime behavior and life span on solutions of a semilinear parabolic Cauchy problem.* Trans. Amer. Math. Soc. **333** (1992), 365–378.
[9] Mochizuki K. and Suzuki R., *Critical exponent and critical blow-up for quasilinear parabolic equations.* Israel J. Math. **98** (1997), 141–156.
[10] Mukai K., Mochizuki K. and Huang Q., *Large time behavior and life span for a quasilinear parabolic equation with slow decay initial values.* Nonlinear Anal. TMA, to appear.
[11] Protter M.H. and Weinberger H.F., *Maximum principles in differential equations.* Springer-Verlag, New York, 1984.
[12] Repnikov V.D. and Eidelman S.D., *A new proof of the theorem of the stbilization of the solutions of the Cauchy problem for the heat equation.* Math. Sb. **73** (1967), 155–159.

Qing Huang
Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo 192-0397, Japan

Kiyoshi Mochizuki
Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo 192-0397, Japan
E-mail: mochizuk@math.metro-u.ac.jp

Kentaro Mukai
Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo 192-0397, Japan