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Classification of exotic circles of $PL_{+}(S^{1})$

Hiroyuki MINAKAWA

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Abstract. Let G be a subgroup of the group $\mathrm{Homeo}_{+}(S^{1})$ of orientation preserving homeomorphisms of the circle. An exotic circle of G is a subgroup of G which is topologically conjugate to $SO(2)$ but not conjugate to $SO(2)$ in G. The existence of an exotic circle shows us the fact that the subgroup G is far from being a Lie group. We previously proved that the group $PL_{+}(S^{1})$ of orientation preserving piecewise linear homeomorphisms of the circle has exotic circles. We give a more explicit construction of exotic circles of $PL_{+}(S^{1})$ and classify all exotic circles up to PL conjugacy.

Key words: topological circle, exotic circle, $PL_{+}(S^{1})$, topologically conjugate, PL conjugate, total derivative, half total derivative.

Introduction

Let G be a Lie group and M an oriented manifold of class C^{k} $(1\leq k\leq\infty)$. Let $\mathrm{Diff}_{+}^{k}(M)$ denote the group of all C^{k} diffeomorphisms of M. A topological action is a continuous map $\varphi:G\times M\rightarrow M$ such that

1) $\varphi_{e}(x)=x$,
2) $\varphi_{gh}(x)=\varphi_{g}(\varphi_{h}(x))$.

where e is the unit of G and $\varphi_{g}(x)=\varphi(g,x)$. D. Montgomery and L. Zippin proved the following theorem ([4]).

Theorem 0.1 *Let φ be a topological action. If every φ_{g} belongs to $\mathrm{Diff}_{+}^{k}(M)$ then φ is a map of class C^{k}.*

In the case where $G=M=S^{1}$, this theorem implies the following corollary.

Corollary 0.2 *If every $h\circ R_{x}\circ h^{-1}$ is contained in $\mathrm{Diff}_{+}^{k}(S^{1})$, then h belongs to $\mathrm{Diff}_{+}^{k}(S^{1})$. Here, $R_{x}:S^{1}\rightarrow S^{1}$ is the rotation of S^{1} by x, i.e., $R_{x}(y)=x+y$.*

Indeed, for $\varphi(x,y)=h\circ R_{x}\circ h^{-1}(y)$. $\varphi:S^{1}\times S^{1}\rightarrow S^{1}$ is a topological action with $\varphi_{x}\in \mathrm{Diff}_{+}^{k}(S^{1})$. Then φ is of class C^{k} by Theorem 0.1. Fix

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a point y_0 and define the C^k diffeomorphism ϕ of S^1 by $\phi(x) = \varphi(x, y_0)$. Then we can easily see $\phi^{-1} \circ \varphi_x \circ \phi = R_x$. So $\phi^{-1} \circ h = R_z$ for some $z \in S^1$. This implies h belongs to $\mathrm{Diff}^k_+(S^1)$.

Let $SO(2) = \{R_x \mid x \in S^1\}$ be the group of all rotations of S^1. Corollary 0.2 says that $\mathrm{Diff}^k_+(S^1)$ has no exotic circle in the following sense.

Let G be a subgroup of $\mathrm{Homeo}_+(S^1)$.

Definition 0.3 1) A subgroup $S \subset \mathrm{Homeo}_+(S^1)$ is called a *topological circle* if $S = h \circ SO(2) \circ h^{-1}$ for some $h \in \mathrm{Homeo}_+(S^1)$.

2) A topological circle $S \subset G$ is an *exotic circle* of G if h does not belong to G.

Contrary to this phenomenon, we proved that $PL_+(S^1)$ has exotic circles in [5]. This fact gives one of the reasons why the topological group $PL_+(S^1)$ is very far from being a Lie group.

In this paper, we give a more explicit construction of exotic circles of $PL_+(S^1)$ and a perfect classification of all exotic circles up to PL conjugacy.

1. Piecewise linear homeomorphisms of S^1

Let $\mathrm{Homeo}^\sim_+(S^1)$ be the group of all orientation preserving homeomorphisms of $\mathbf{R}$ which commutes with the translation T_1. Here $T_b(x) = x + b$ ($x, b \in \mathbf{R}$) is the translation by b. Every $F \in \mathrm{Homeo}^\sim_+(S^1)$ induces a homeomorphism $f : S^1 \to S^1$ ($S^1 = \mathbf{R}/\mathbf{Z}$). So we define

$$p : \mathrm{Homeo}^\sim_+(S^1) \to \mathrm{Homeo}_+(S^1)$$

by $p(F) = f$. Conversely for any $f \in \mathrm{Homeo}_+(S^1)$, there exists a $\tilde{f} \in \mathrm{Homeo}^\sim_+(S^1)$ such that $p(\tilde{f}) = f$. Such $\tilde{f}$ is called a *lift* of f. We can easily check that

$$p^{-1}(f) = \{T_n \circ \tilde{f} \mid n \in \mathbf{Z}\}.$$

Let $PL^\sim_+(S^1)$ be the group of $\mathrm{Homeo}^\sim_+(S^1)$ defined as follows. $F \in \mathrm{Homeo}^\sim_+(S^1)$ belongs to $PL^\sim_+(S^1)$ if F is piecewise linear and bending points of F have no accumulation points in $\mathbf{R}$. Then we define $PL_+(S^1) = p(PL^\sim_+(S^1))$.

Let $\pi : \mathbf{R} \to S^1 = \mathbf{R}/\mathbf{Z}$ denote the quotient map. A point $\tilde{x} \in \mathbf{R}$ with $\pi(\tilde{x}) = x$ is called a *lift* of x. We may use the notation $\pi(\tilde{x}) = [\tilde{x}]$.

An important construction of PL homeomorphisms of S^1 is given by

PL interval exchange maps. A pair of maps (f, g) is called an interval exchange map of $[a, a+1]$ if there exist $x, y \in (a, a+1)$ such that both $f : [a, x] \to [y, a+1]$ and $g : [x, a+1] \to [a, y]$ are homeomorphisms with $f(a) = y$, $g(x) = a$. Moreover if f and g are both piecewise linear or affine, (f, g) is respectively called a PL or an affine interval exchange map of $[a, a+1]$. We identify $[a, a+1]/a \sim a+1$ with S^1 by the inclusion map $[a, a+1] \to \mathbf{R}$. Then every interval exchange map (f, g) induces a homeomorphism F of $[a, a+1]/a \sim a+1$, so of S^1. We can easily check that if (f, g) is PL, then F is contained in $PL_+(S^1)$.

2. Examples

First we define intervals I_A $(A \in \mathbf{R}_+ - \{1\})$ by

$$I_A = \begin{cases} [1/(A-1), A/(A-1)] & \text{if} \quad A > 1, \\ [A/(A-1), 1/(A-1)] & \text{if} \quad 0 < A < 1. \end{cases}$$

Let $h_A : S^1 \to S^1$ $(A > 1)$ be the orientation preserving piecewise C^ω diffeomorphism whose lift $\tilde{h}_A$ is defined by $\tilde{h}_A \mid I_A = h \mid I_A$, $h \mid I_A : I_A \to [0, 1]$, $h(x) = \log((A-1)x)/\log A$. Let $\underline{h} : S^1 \to S^1$ be the orientation reversing homeomorphism defined by $\underline{h}(x) = -x$. Here the homeomorphism h_A is well-defined, because the length of the interval I_A is equal to 1 and $h \mid I_A : [1/(A-1), A/(A-1)] \to [0, 1]$ is an orientation preserving homeomorphism. Then we define, for any $A > 1$,

$$S_A = h_A^{-1} \circ SO(2) \circ h_A \quad S_{A^{-1}} = \underline{h} \circ S_A \circ \underline{h}.$$

We can easily check that S_A $(A > 1)$ is contained in $PL_+(S^1)$, since $h^{-1} \circ T_a \circ h = M_{A^a}$ holds for any $a \in R$. Here, $T_a(x) = x + a$ and $M_a(x) = ax$. Indeed, we can explicitly represent any element $h_A^{-1} \circ R_{[a]} \circ h_A$ $(0 < a < 1)$ by an affine interval exchange map $(r_{(A,a)}, l_{(A,a)})$ of I_A;

$$r_{(A,a)} = M_{A^a} \mid [1/(A-1), A^{1-a}/(A-1)]$$

and

$$l_{(A,a)} = M_{A^{a-1}} \mid [A^{1-a}/(A-1), A/(A-1)].$$

Since h_A is not contained in $PL_+(S^1)$, then every S_A is exotic.

Remark. In [1], they studied about the following class of affine interval exchange maps A^*. For any $u, v \in (0,1)$, we define

$$f_{(u,v)} : [0,v] \to [u,1] \text{ by } f_{(u,v)}(x) = u + \frac{1-u}{v}x$$

and

$$g_{(u,v)} : [v,1] \to [0,u] \text{ by } g_{(u,v)}(x) = \frac{u}{1-v}(x-v).$$

Then A^* is the set of all interval exchange maps of form $(f_{(u,v)}, g_{(u,v)})$ $(u, v \in (0,1))$. We remark that

$$A^* = \bigcup_{a \in \mathbf{R}_+ - \{1\}} R^{-1}_{[1/(a-1)]} \circ S^*_a \circ R_{[1/(a-1)]},$$

where, $S^*_a = S_a - \{\mathrm{id}\}$ (see Lemma 4.9). They proved that every element of A^* has an absolutely continuous invariant probability measure in their paper. By the constructions of S_a, now we can get a very simple proof of this fact. Indeed, for any $f \in S_a$, an invariant measure μ for f is equal to

$$(h_a)^* m \quad \text{if } 1 < a, \text{ and}$$

$$(\underline{h} \circ h_a \circ \underline{h})^* m \quad \text{if } 0 < a < 1.$$

Here, m is the Lebesgue measure of S^1 with $m(S^1) = 1$.

3. Total derivative

Let f be any element of $PL_+(S^1)$. For any $x \in S^1$, we define the *right derivative* $d_R f(x)$ and the *left derivative* $d_L f(x)$ at x by

$$d_R f(x) = \lim_{\varepsilon \to 0, \varepsilon > 0} \frac{\tilde{f}(\tilde{x}+\varepsilon) - \tilde{f}(\tilde{x})}{\varepsilon},$$

and

$$d_L f(x) = \lim_{\varepsilon \to 0, \varepsilon > 0} \frac{\tilde{f}(\tilde{x}-\varepsilon) - \tilde{f}(\tilde{x})}{\varepsilon}.$$

This is well-defined, because each right-hand side does not depend on the choices of lifts $\tilde{f}$, $\tilde{x}$. We put

$$\Delta_x f = \log d_R f(x) - \log d_L f(x).$$

For given $f \in PL_+(S^1)$, a point x of S^1 is a *bending point* of f if $\Delta_x f \neq 0$. We denote all the bending points of f by $BP(f)$. It is trivially a finite set by the definition of $PL_+(S^1)$. Since $\Delta_x f = 0$ except for the points of $BP(f)$, we can define the *total derivative* $\Delta f(x)$ of f at x by

$$\Delta f(x) = \sum_{n \in \mathbf{Z}} \Delta_{f^n(x)} f = \sum_{y \in O_f(x)} \Delta_y f.$$

Here we denote the orbit of f through x by $O_f(x)$. That is, $O_f(x) = \{f^n(x) \mid n \in \mathbf{Z}\}$.

Lemma 3.1 *For any $f, g \in PL_+(S^1)$, and $x \in S^1$, the following formulas* (1), (2) *and* (3) *hold.*

(1) $\Delta_x(f \circ g) = \Delta_{g(x)} f + \Delta_x g,$

(2) $\Delta_{f(x)} f^{-1} = -\Delta_x f,$

(3) $\Delta_{g(x)}(g \circ f \circ g^{-1}) = \Delta_{f(x)} g + \Delta_x f - \Delta_x g.$

These are shown by the chain rules for d_R and d_L. The next lemma says that $\Delta f(x)$ is a PL conjugacy invariant.

Lemma 3.2 *For any $f, g \in PL_+(S^1)$ and $x \in S^1$, we have*

$$\Delta(g \circ f \circ g^{-1})(g(x)) = \Delta f(x).$$

Proof. We use the following notations.

$y = g(x)$,

$x_n = f^n(x)$, and

$y_n = g(x_n) = (g \circ f \circ g^{-1})^n(g(x))$.

Case 1: Suppose that $\sharp O_f(x) = n$ for some $n \in N$. That is, $O_f(x) = \{x_1, x_2, \ldots, x_n = x\}$. Then we have

$$\begin{aligned} \Delta(g \circ f \circ g^{-1})(g(x)) &= \sum_{i=1}^{n} \Delta_{y_i}(g \circ f \circ g^{-1}) \\ &= \sum_{i=1}^{n} (\Delta_{x_{i+1}} g + \Delta_{x_i} f - \Delta_{x_i} g) \\ &= \sum_{i=1}^{n} \Delta_{x_i} f \\ &= \Delta f(x). \end{aligned}$$

Case 2: Suppose that $\sharp O_f(x) = \infty$. Since both $BP(f)$ and $BP(g)$ are finite sets, there exists an integer M such that $\Delta_{x_n} f = \Delta_{x_n} g = 0$ for any

$|n| \geq M$. This implies that $\Delta_{y_n}(g \circ f \circ g^{-1}) = 0$ for any $|n| \geq M+1$. So we have

$$\begin{aligned} \Delta(g \circ f \circ g^{-1})(g(x)) &= \sum_{i=-M}^{M} \Delta_{y_i}(g \circ f \circ g^{-1}) \\ &= \sum_{i=-M}^{M} (\Delta_{x_{i+1}} g + \Delta_{x_i} f - \Delta_{x_i} g) \\ &= \Delta_{x_{M+1}} g + \sum_{i=-M}^{M} \Delta_{x_i} f - \Delta_{x_{-M}} g \\ &= \Delta f(x) \end{aligned}$$

□

Lemma 3.3 *Let f, g be elements of $PL_+(S^1)$ such that $f \circ g = g \circ f$. If $\langle f, g \rangle$ is isomorphic to $\mathbf{Z} + \mathbf{Z}$ and acts on an orbit $\langle f, g \rangle(x)$ freely, then we have that $\Delta(f^m \circ g^n)(x) = 0$ for any $n, m \in \mathbf{Z}$.*

Proof. If $m = n = 0$, then it is trivial. Suppose that $m \neq 0$ or $n \neq 0$. Take integers p, q such that $mp + nq = 0$ and $(p, q) \neq (0, 0)$. Then $\langle f^m \circ g^n, f^p \circ g^q \rangle$ is also isomorphic to $\mathbf{Z} + \mathbf{Z}$ and acts on $\langle f^m \circ g^n, f^p \circ g^q \rangle(x)$ freely. So it suffices to prove that $\Delta f(x) = 0$. Since the action of $\langle f, g \rangle$ on its orbit $\langle f, g \rangle(x)$ is free, then $\langle f, g \rangle(x)$ is divided into infinitely many orbits of f. That is,

$$\langle f, g \rangle(x) = \bigcup_{n \in \mathbf{Z}} O_f(g^n(x)) \quad \text{(disjoint union)}$$

So there exists an integer n such that $O_f(g^n(x)) \cap BP(f) = \emptyset$, because $BP(f)$ is a finite set. By Lemma 3.2, we have that

$$\Delta f(x) = \Delta(g^n \circ f \circ g^{-n})(g^n(x)) = \Delta f(g^n(x)) = 0.$$

□

The following corollary is very important to characterize the elements of a topological circle of $PL_+(S^1)$. Before stating it, we recall the notion of the rotation number. The rotation number $\rho : \mathrm{Homeo}_+(S^1) \to S^1$ is a well-known semi-conjugacy invariant which has the following properties ([1], [7], [8]);

1) $\rho(R_a) = a$ for any $a \in S^1$.

2) $\rho(f \circ g) = \rho(f) + \rho(g)$ if $f \circ g = g \circ f$.

3) If $\rho(f) = a$, then $R_a^{-1} \circ f$ has a fixed point.

4) Suppose that $\rho(f)$ is irrational, that is, $\rho(f) \notin \mathbf{Q}/\mathbf{Z}$. If $\rho(f) = \rho(g)$, then $f^{-1} \circ g$ has a fixed point.

5) If f^n has no fixed points for any $n \in \mathbf{Z} - \{0\}$, then $\rho(f)$ is irrational.

Corollary 3.4 *Let S be a topological circle of $PL_+(S^1)$. Then $\Delta f(x) = 0$ for any $f \in S$ and any $x \in S^1$.*

Proof. If f has a finite orbit, then f must be finite order.

$$\begin{aligned}\Delta f(x) &= \sum_{i=0}^{n-1} \Delta_{x_i} f \\ &= \Delta_x f^n = 0\end{aligned}$$

Here, the integer n is the order of f and $x_i = f^i(x)$.

Next, if f has no finite orbit, then f has an irrational rotation number $\rho(f)$. Take any $g \in S$ which has no finite orbit and with $\rho(g) \neq \rho(f)$. Then $\langle f, g \rangle$ is isomorphic to $\mathbf{Z} + \mathbf{Z}$ and acts on its orbit $\langle f, g \rangle(x)$ freely for any $x \in S^1$. So we have $\Delta f(x) = 0$ for any $x \in S^1$ by Lemma 3.3. □

Remark. It is well known that every element $f \in PL_+(S^1)$ of finite order is PL conjugate to $R_{\rho(f)}$.

4. Half total derivative

Definition 4.1 Let f be an element of $PL_+(S^1)$. A point $x \in S^1$ is called a *center* of f, if $\sharp O_f(x) = \infty$ and there exist a non-negative integer m and a positive integer n such that both $f^m(x)$ and $f^{-n}(x)$ are bending points. A symbol $C(f)$ denotes the set of all centers of f.

We can easily see that every $f \in PL_+(S^1)$ has at most finite number of centers. We prepare another terminology.

Definition 4.2 An element $f \in PL_+(S^1)$ is *good* if it satisfies the following two conditions.

(1) f has no finite orbit.

(2) $\Delta f(x) = 0$ for any $x \in S^1$.

Moreover we define $\Delta^\omega f(x) = \sum_{i \geq 0} \Delta_{x_i} f$, where $x_i = f^i(x)$.

Lemma 4.3 *Let f be a good element of $PL_+(S^1)$. Then $\Delta^\omega f(x) = 0$ for any $x \notin C(f)$.*

Proof. Since $x \notin C(f)$, there are two possibilities 1) $f^m(x) \notin BP(f)$ for any $m \geq 0$, or 2) $f^{-n}(x) \notin BP(f)$ for any $n \geq 1$. So 1) implies that $\Delta^\omega f(x) = 0$, since $\Delta_{x_i} f = 0$ for any integer $i \geq 0$. Next 2) implies that $\Delta^\omega f(x) = \Delta f(x)$, since $\Delta_{x_{-i}} f = 0$ for any positive integer i. This right-hand side is equal to zero, because f is good. □

Definition 4.4 Let $f \in PL_+(S^1)$ be a good element. We define the *half total derivative* $\Sigma^\omega f$ of f by

$$\Sigma^\omega f = \sum_{x \in S^1} \Delta^\omega f(x).$$

In the right-hand side, $\Delta^\omega f(x)$ vanishes outside of $C(f)$ by Lemma 3.4. So this is well-defined.

Lemma 4.5 *Let f, g be elements of $PL_+(S^1)$. Suppose f is good. Then we have that*

$$\Delta^\omega (g \circ f \circ g^{-1})(g(x)) = \Delta^\omega f(x) - \Delta_x g.$$

The proof of this lemma is almost same as that of the case 2 of Lemma 3.2. So we omit the proof.

Corollary 4.6 *Let $f \in PL_+(S^1)$ be a good element. Then we have that*

$$\Sigma^\omega (g \circ f \circ g^{-1}) = \Sigma^\omega f$$

for any $g \in PL_+(S^1)$.

Proof. There exists a finite subset F of S^1 such that $\Delta^\omega (g \circ f \circ g^{-1})(x) = \Delta^\omega f = \Delta_x g = 0$ for any $x \notin F$. Then we have that

$$\begin{aligned}
\Sigma^\omega (g \circ f \circ g^{-1}) &= \sum_{x \in F} \Delta^\omega (g \circ f \circ g^{-1})(x) \\
&= \sum_{x \in F} (\Delta^\omega f(x) - \Delta_x g) \\
&= \sum_{x \in F} \Delta^\omega f(x) - \sum_{x \in F} \Delta_x g \\
&= \Sigma^\omega f
\end{aligned}$$

The last equality is due to the fact that $\sum_{x \in BP(g)} \Delta_x g = 0$ holds for any $g \in PL_+(S^1)$. □

Lemma 4.7 *Let $f \in PL_+(S^1)$ be a good element. For any $x_0 \in S^1 - C(f)$, there exists a unique element $h \in PL_+(S^1)$ such that*

1) $h(x_0) = x_0$,
2) $\Delta_x h = \Delta^\omega f(x)$ *if* $x \neq x_0$.

Proof. If $C(f)$ is an empty set, then h must be equal to the identity. Suppose that $C(f) = \{x_1, \dots, x_n\}$ for some positive integer n. We can assume that there exist a set of lifts $\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n$ such that $\tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_n < \tilde{x}_0 + 1$. For any real number λ, we define a step function $H_\lambda : [\tilde{x}_0, \tilde{x}_0 + 1) \to \mathbf{R}$ by

$$H_\lambda(\tilde{y}) = \lambda_i \quad \text{if} \quad \tilde{y} \in [\tilde{x}_i, \tilde{x}_{i+1}).$$

Here, $\tilde{x}_{n+1} = \tilde{x}_0 + 1$, $\lambda_0 = \lambda$ and $\lambda_i = \lambda + \sum_{j=1}^{i} \Delta^\omega f(x_i)$ $(n \geq i \geq 1)$. Then we put that

$$h_\lambda(\tilde{y}) = \int_{\tilde{x}_0}^{\tilde{y}} e^{H_\lambda(t)} dt + \tilde{x}_0.$$

We can easily see that h_λ is piecewise linear and monotone increasing. Furthermore $h_\lambda(\tilde{y}) > h_\mu(\tilde{y})$ for any $\tilde{y} \in [\tilde{x}_0, \tilde{x}_0 + 1)$ if $\lambda > \mu$, because $\lambda_i > \mu_i$ $(i = 0, 1, \dots, n)$ if $\lambda > \mu$. So it follows that the map $\phi : \mathbf{R} \to (\tilde{x}_0, \infty)$, $\phi(\lambda) = h_\lambda(\tilde{x}_0 + 1)$ is an orientation preserving homeomorphism. Then there exists a unique real number λ such that $h_\lambda(\tilde{x}_0 + 1) = \tilde{x}_0 + 1$. This h_λ induces the element $h \in PL_+(S^1)$ which is required. □

Lemma 4.8 *Let f, h, x_0 be as in Lemma 4.7. Then we have*

1) $BP(h \circ f \circ h^{-1}) \subset \{h(x_0), h \circ f^{-1}(x_0)\}$
2) $\Delta_{h(x_0)}(h \circ f \circ h^{-1}) = \Sigma^\omega f$.

Proof. We have that $\Delta_{h(x)}(h \circ f \circ h^{-1}) = \Delta_{f(x)} h + \Delta_x f - \Delta_x h$ by Lemma 3.1. If $\{x, f(x)\}$ does not contain x_0, then $\Delta_{f(x)} h = \Delta^\omega f(f(x))$ and $\Delta_x h = \Delta^\omega f(x)$ by Lemma 4.7 2). So we have

$$\begin{aligned} \Delta_{h(x)}(h \circ f \circ h^{-1}) &= \Delta^\omega f(f(x)) + \Delta_x f - \Delta^\omega f(x) \\ &= \Delta^\omega f(x) - \Delta^\omega f(x) \\ &= 0. \end{aligned}$$

This shows 1).

In order to prove 2), we calculate $\Delta_{h(x_0)}(h \circ f \circ h^{-1})$. Since x_0 is not contained in $C(f)$, $\Delta^\omega f(x_0) = 0$. Moreover $\Delta_{f(x_0)}h = \Delta^\omega f(f(x_0))$, because $f(x_0) \neq x_0$ and $f^2(x_0) \neq x_0$ by the goodness of f. This implies that

$$\begin{aligned}\Delta_{h(x_0)}(h \circ f \circ h^{-1}) &= \Delta_{f(x_0)}h + \Delta_{x_0} f \Delta_{x_0} h \\ &= \Delta^\omega f(x_0) - \Delta_{x_0} h \\ &= -\Delta_{x_0} h.\end{aligned}$$

Since $BP(h)$ is contained in $C(f) \cup \{x_0\} = \{x_0, x_1, \ldots, x_n\}$, $\sum_{i=0}^n \Delta_{x_i} h = \sum_{x \in S^1} \Delta_x h = 0$. So we have

$$\begin{aligned}-\Delta_{x_0} h &= \sum_{i=1}^n \Delta_{x_i} h \\ &= \sum_{i=1}^n \Delta^\omega f(x_i) \\ &= \sum_{x \in S^1} \Delta^\omega f(x) \\ &= \Sigma^\omega f.\end{aligned}$$

□

Lemma 4.9 *Let f, g be elements of $PL_+(S^1)$. Suppose there exists a point $z \in S^1$ such that*

1) $BP(f) = \{z, f^{-1}(z)\}$, $f^{-1}(z) \neq z$,
2) $BP(g) = \{z, g^{-1}(z)\}$, $g^{-1}(z) \neq z$,
3) $\Delta_z f = \Delta_z g$.

If $f \circ g^{-1}$ has a fixed point, then $f = g$.

Proof. By the hypothesis 3), we have that

$$\Delta_{g(z)}(f \circ g^{-1}) = \Delta_z f + \Delta_{g(z)} g^{-1} = \Delta_z f - \Delta_z g = 0.$$

Since $BP(f \circ g^{-1}) \subset g(BP(f)) \cup BP(g^{-1}) = \{g(z), g \circ f^{-1}(z), z\}$, it follows that $BP(f \circ g^{-1}) \subset \{z, g \circ f^{-1}(z)\}$. If $g \circ f^{-1}(z) = z$, then $f \circ g^{-1}$ can not have any bending points. So it has to be an element of $SO(2)$ with fixed point. That is, $f \circ g^{-1} = id_{S^1}$. In order to complete the proof of this lemma, it suffices to show that $g \circ f^{-1}(z) = z$. Suppose not, then we can see that $f^{-1}(z) \notin BP(g)$. So we have $BP(f \circ g^{-1}) = \{z, g \circ f^{-1}(z)\}$. Since $f \circ g^{-1}(g \circ f^{-1}(z)) = z$, $f \circ g^{-1}$ can have no fixed point. This is contradiction. □

We use the notation $S_1 = SO(2)$ from now. The following theorem is the goal of this paper.

Theorem 4.10 *Let S be a topological circle of $PL_+(S^1)$. Then the number $\Sigma^\omega f$ does not depend on the choice of $f \in S$ with irrational rotation numbers. Furthermore S is PL conjugate to $S_{A(S)}$ (see Section 2), where $\log A(S) = \Sigma^\omega f$ ($\rho(f)$: irrational).*

Proof. Take any element $f \in S$ with an irrational rotation number α and fix it. By Corollary 3.4, f is a good element. Fix a point $z \in S^1 - C(f)$. There exists $h \in PL_+(S^1)$ such that

$$BP(h \circ f \circ h^{-1}) \subset \{h(z), h \circ f^{-1}(z)\}$$

and

$$\Delta_{h(z)}(h \circ f \circ h^{-1}) = \Sigma^\omega f$$

by Lemma 4.8.

Case 1: Suppose that $\Sigma^\omega f = 0$. $h \circ f \circ h^{-1}$ has no bending point, that is, $h \circ f \circ h^{-1} = R_\alpha$. So we have

$$h \circ S \circ h^{-1} = h \circ \overline{\langle f \rangle} \circ h^{-1} = \overline{\langle h \circ f \circ h^{-1} \rangle} = SO(2).$$

Here, the overline means taking a closure with respect to C^0-topology in $\mathrm{Homeo}_+(S^1)$.

Case 2: Suppose that $\Sigma^\omega f > 0$. Put $u = [1/(A(S) - 1)] \in S^1$, $\beta = u - h(z)$ and $f_1 = R_\beta \circ h \circ f \circ (R_\beta \circ h)^{-1}$. Then we have

$$BP(f_1) = \{u, f_1^{-1}(u)\}$$

and

$$\Delta_u f_1 = \Sigma^\omega f = \log A(S).$$

By the construction of $S_{A(S)}$, each element $g \in S_{A(S)}$ has same properties

$$BP(g) = \{u, g^{-1}(u)\}$$

and

$$\Delta_u g = \log A(S).$$

If $g \in S_{A(S)}$ has the rotation number α, then g has to be equal to f_1 by Lemma 4.9, that is, $f_1 \in S_{A(S)}$. So we have

$$\begin{aligned} R_\beta \circ h \circ S \circ (R_\beta \circ h)^{-1} &= \overline{\langle R_\beta \circ h \circ f \circ (R_\beta \circ h)^{-1} \rangle} \\ &= \overline{\langle f_1 \rangle} = S_{A(S)}. \end{aligned}$$

We can easily check that $\Sigma^\omega g = \log A(S)$ for any $g \in S_{A(S)}$ with irrational rotation numbers. Since $\Sigma^\omega f$ is PL conjugacy invariant, this value does not depend on the choices of $f \in S$ with irrational rotation numbers.

Case 3: Suppose that $\Sigma^\omega f < 0$. The proof is reduced to the case above by Proposition 4.11 stated below. This completes the proof of this theorem. □

Let $\underline{h} : S^1 \to S^1$ be the orientation reversing homeomorphism defined by $\underline{h}(x) = -x$. We note that $\underline{h}^2 = \mathrm{id}$.

Proposition 4.11 *If S is a topological circle, then $\underline{S} = \underline{h} \circ S \circ \underline{h}^{-1}$ is also a topological circle. Moreover, if S is contained in $PL_+(S^1)$, then $\Sigma^\omega(\underline{h} \circ f \circ \underline{h}^{-1}) = -\Sigma^\omega f$ for any $f \in S$ with an irrational rotation number.*

Proof. There exists an orientation preserving homeomorphism $h : S^1 \to S^1$ such that $S = h \circ SO(2) \circ h^{-1}$. Since $\underline{h} \circ SO(2) \circ \underline{h}^{-1} = SO(2)$,

$$\underline{h} \circ S \circ \underline{h}^{-1} = \underline{h} \circ h \circ \underline{h} \circ SO(2) \circ (\underline{h} \circ h \circ \underline{h})^{-1}.$$

This means that $\underline{h} \circ S \circ \underline{h}^{-1}$ is a topological circle. We can easily check that the last statement in this lemma by the following formulas

$$\begin{aligned} d_R(\underline{h} \circ f \circ \underline{h}^{-1})(x) &= d_L f(\underline{h}(x)), \\ d_L(\underline{h} \circ f \circ \underline{h}^{-1})(x) &= d_R f(\underline{h}(x)) \end{aligned}$$

for any $f \in PL_+(S^1)$. □

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Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan
E-mail: minakawa@math.sci.hokudai.ac.jp