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On a super class of p-hyponormal operators

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Abstract. Given an operator A on a Hilbert space $\mathcal{H}$, A is said to be p-hyponormal $(0 < p \leq 1)$ if $(AA^{\star})^{p} \leq (A^{\star}A)^{p}$. The class $H(p)$ of p-hyponormal operators has been studied in a number of papers in the recent past. Let $K(p)$ denote the class of operators A for which $((AA^{\star})^{p}x, x) \leq \|x\|^{2(1-p)} (A^{\star}Ax, x)^{p}$ for all $x \in \mathcal{H}$. Then $H(p) \subset K(p)$. In this note we study the spectral properties of operators in $K(p)$, and show that a number of the properties enjoyed by hyponormal operators carry over to $K(p)$. Our arguments often lead to an alternative, sometimes simpler, proof of the results for $H(p)$.

Key words: p-hyponormal operators, class $K(p)$, spectral properties.

1. Introduction

We consider operators (i.e., bounded linear transformations) on a complex Hilbert space $\mathcal{H}$. The operator A is said to be p-hyponormal, $0 < p \leq 1$, if $(AA^{\star})^{p} \leq (A^{\star}A)^{p}$. It is an easy consequence of the Löwner inequality that a p-hyponormal operator is q-hyponormal for all $0 < q \leq p$. In particular, a 1-hyponormal (or simply hyponormal) operator is p-hyponormal for all $0 < p < 1$, and in studying p-hyponormal operators for a general $0 < p < 1$ it is sufficient to consider $0 < p \leq \frac{1}{2}$. Semi-hyponormal (or, $\frac{1}{2}$-hyponormal) operators were introduced by Xia [20], and p-hyponormal operators for $0 < p < \frac{1}{2}$ were first studied by Aluthge [1]. Recently there have been a number of papers, especially by Muneo Cho et al. [2, 3, 4, 5, 6] and Masatoshi Fujii et al. [10, 11], on p-hyponormal operators, their spectral properties and their relationship to other classes of operators. Generally speaking p-hyponormal have properties very similar to hyponormal operators [1, 2, 3, 4, 5, 6, 10, 20, 21].

Let $H(p)$ denote the class of p-hyponormal operators, $0 < p \leq \frac{1}{2}$. Then $((AA^{\star})^{p}x, x) \leq ((A^{\star}A)^{p}x, x) \leq \|x\|^{2(1-p)}(A^{\star}Ax, x)^{p}$ for all $x \in \mathcal{H}$. Let $K(p)$, $0 < p \leq \frac{1}{2}$, denote the class of operators A for which $((AA^{\star})^{p}x, x) \leq \|x\|^{2(1-p)} (A^{\star}Ax, x)^{p}$. Then $H(p) \subset K(p)$ and the class $K(p)$ is monotone decreasing on p; also, operators $A \in K(p)$ are paranormal, i.e., if $A \in$

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$K(p)$, then $\|Ax\|^2 \leq \|A^2x\|$ for all unit vectors $x \in \mathcal{H}$ [11, Lemma 3 and Theorem 4]. In this note we study spectral properties of operators $A \in H(p)$ by studying $A \in K(p)$, and show that many a property of hyponormal operators is shared by operators in $K(p)$. On the way we give alternative (sometimes simpler) proofs of some of the results for the class $H(p)$.

In the following we shall denote the spectrum, the point spectrum, the approximate point spectrum and the essential spectrum of the operator A by $\sigma(A), \sigma_o(A), \sigma_a(A)$ and $\sigma_e(A)$, respectively. We say that the complex number $\alpha, \alpha \in \mathcal{C}$, is in the joint point spectrum $\sigma_{jo}(A)$ (joint approximate point spectrum $\sigma_{ja}(A)$) of A if there exists a unit vector $x \in \mathcal{H}$ (respectively, a sequence of unit vectors $x_n \in \mathcal{H}$) such that $(A-\alpha)x = 0$ and $(A^\star - \bar{\alpha})x = 0$ (respectively, $(A - \alpha)x_n \to 0$ and $(A^\star - \bar{\alpha})x_n \to 0$ as $n \to \infty$). We shall denote the kernel and the closure of the range of A by $\ker A$ and $\overline{\operatorname{ran} A}$, and the restriction of A to an invariant subspace M will be denoted by $A|M$.The operator A will be said to be pure (= completely non-normal) if there exists no reducing subspace M of A such that $A|M$ is normal. Throughout the following A will have the (unique) polar decomposition $A = U|A|$, $|A| = (A^\star A)^{\frac{1}{2}}$, and the operator A_p will be defined by $A_p = U|A|^p$. We shall denote the boundary of a set S by ∂S. Any other notation will be defined as and when required.

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2. Results

If $A \in K(p)$, then A is paranormal [11, Theorem 4]. Since paranormal operators A are normaloid (i.e., $\|A\| = r(A)$, where $r(A)$ denotes the spectral radius of A) and the inverse, whenever it exists, of a paranormal operator is again paranormal [12], $r(A^{-1}) = \|A^{-1}\|$. Also $\sigma_e(A) = \sigma(A) - \sigma_{oo}(A)$, where $\sigma_{oo}(A)$ denotes the set of isolated eigen-values of A of finite multiplicity [9], and if $\sigma(A)$ is countable, then A is normal [17].

As mentioned in the introduction, $K(p)$ is monotone decreasing on $p(0 < p \leq \frac{1}{2})$. Thus, where need be, there is no loss of generality in assuming $p = 2^{-n}$ for some integer $n \geq 1$. Recall that the eigen-values of a p-hyponormal operator A are normal eigen-values, i.e., $\sigma_o(A) = \sigma_{jo}(A)$

[1, 6, 21]. That a similar result is true for $A \in K(p)$ is the content of the following theorem.

Theorem 1 (cf. [21, Theorem 2.3] and [6, Theorem 4]) *If $A \in K(p)$, then $\sigma_o(A) = \sigma_{jo}(A)$.*

Proof. It is immediate from the definition of $K(p)$ that if $0 \in \sigma_o(A)$, then 0 is a normal eigen-value of A. Let $\alpha = re^{i\theta}$, $r \neq 0$, be an eigen-value of A with a corresponding eigen-vector x. Then

$$\begin{aligned} \|Ax\| &= \|U|A|x\| = \||A|x\| = \||A|^{1-q}|A|^q x\| \quad \left(0 < q \leq \frac{1}{2}\right) \\ &\leq \||A|^{1+q}x\|^{1-q}\||A|^q x\|^q \\ &\qquad \text{(by the Hölder-McCarthy inequality [11])} \\ &= \||A|^q U^\star U|A|x\|^{1-q}\||A|^q x\|^q \\ &= r^{1-q}\||A|^q U^\star x\|^{1-q}\||A|^q x\|^q. \end{aligned}$$

Since $A \in K(q)$ for all $0 < q \leq p$, the definition of $K(p)$ implies

$$\begin{aligned} \||A|^q U^\star x\|^2 = (U|A|^{2q}U^\star x, x) &\leq \|x\|^{2(1-q)}\||A|x\|^{2q} \\ &= \|x\|^{2(1-q)}\|Ax\|^{2q}. \end{aligned}$$

Hence

$$\|Ax\| \leq r^{1-q}\|x\|^{(1-q)^2}\|Ax\|^{q(1-q)}\||A|^q x\|^q,$$

or,

$$\|x\|^{-(1-q)^2}\|Ax\|^{1-q(1-q)} \leq r^{1-q}\||A|^q x\|^q.$$

Since $Ax = re^{i\theta}x$, this implies

$$r^{q^2}\|x\|^q \leq \||A|^q x\|^q, \quad \text{or, } r^q\|x\| \leq \||A|^q x\|.$$

Also, by the Hölder-McCarthy inequality,

$$\begin{aligned} \||A|^q x\|^2 &= (|A|^{2q}x, x) \leq \|x\|^{2(1-q)}(|A|^2 x, x)^q \\ &= \|x\|^{2(1-q)}\||A|x\|^{2q} = \|x\|^{2(1-q)}\|Ax\|^{2q} \\ &= r^{2q}\|x\|^2. \end{aligned}$$

Thus

$$\||A|^q x\| = r^q\|x\|$$

for all $0 < q \leq p$. Choosing $q = p/2$ (so that $(|A|^p x, x) = r^p \|x\|^2$) and $q = p$ (so that $(|A|^{2p} x, x) = r^{2p} \|x\|^2$), we have

$$\begin{aligned} 0 \leq \||A|^p x - r^p x\|^2 &= (|A|^{2p} x, x) + r^{2p} \|x\|^2 \\ &\quad - r^p(|A|^p x, x) - r^p(x, |A|^p x) \leq 0, \end{aligned}$$

i.e.,

$$|A|^p x = r^p x, \tag{1}$$

Since there is no loss of generality in assuming $p = 2^{-n}$ for some integer $n \geq 1$, this implies

$$|A|x = rx$$

and (since $Ax = U|A|x = re^{i\theta}x$)

$$Ux = e^{i\theta}x. \tag{2}$$

Hence $\alpha \in \sigma_{jo}(A)$, and the proof is complete. □

Corollary 2 *$\alpha = re^{i\theta} \in \sigma_o(A)$, $A \in K(p)$, if and only if $r^p e^{i\theta} \in \sigma_{jo}(A_p)$.*

Proof. If $\alpha \in \sigma_o(A)$ and x is an eigen-vector corresponding to α, then (as seen in (1) and (2) above) $|A|^p x = r^p x$ and $Ux = e^{i\theta}x$. Hence $\alpha \in \sigma_{jo}(A_p)$. If, on the other hand, $\alpha \in \sigma_{jo}(A_p)$ and x is an eigen-vector corresponding to α, then $|A|^p x = r^p x$ and $Ux = e^{i\theta}x$ imply (as in the proof of the Theorem) that $\alpha \in \sigma_o(A)$. □

The operator $A_p (= U|A|^p)$ plays an important role in the study of p-hyponormal operators (see [1, 2, 3, 6]). The following theorem shows that there is a deep relationship between the reducing subspaces of $A \in K(p)$ and A_p.

Theorem 3 *An invariant subspace M of $A \in K(p)$ reduces A if an only if it reduces A_p.*

Proof. It will suffice to consider $p = 2^{-n}$ for some integer $n \geq 1$. Suppose M reduces A_p. Then

$$U|A|^p M \subset M, \ |A|^p U^\star M \subset M \text{ and } |A|^{2p} M \subset M$$

imply

$$|A| M \subset M \text{ and } |A|^p M \subset M.$$

Hence $U|A|M = AM \subset M$ and $|A|U^\star M = A^\star M \subset M$, i.e. M reduces A. The converse statement is similarly proved. □

We prove next $K(p)$ analogues of some well known results for hyponormal operators [15, 16]; $H(p)$ analogues of some of these results appear in [2], [5, Theorems 1, 2 and 3] and [6, Theorem 8].

Theorem 4 *Let $A \in K(p)$. Then :*

(i) *$\sigma_a(A) = \sigma_{ja}(A)$, and $\alpha = re^{i\theta} \in \sigma_a(A)$ if and only if there exists a sequence of unit vectors x_n such that $(|A| - r)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$ as $n \to \infty$.*

(ii) *$\alpha \in \sigma(A)$ and $\bar{\alpha} \notin \sigma_o(A^\star) \Rightarrow |\alpha| \in \sigma_e(|A|) \cap \sigma_e(|A^\star|)$.*

(iii) *$\alpha \in \partial\sigma(A) \Rightarrow |\alpha| \in \sigma(|A|) \cap \sigma(|A^\star|)$. If also A is pure and $(A - z) \in K(p)$ for all $z \in \mathcal{C}$, then $|\alpha| \in \sigma_e(|A|) \cap \sigma_e(|A^\star|)$.*

Proof. (i) Using the Berberian extension technique [21, p. 15] to extend A to an operator A^o on a Hilbert space H^o it is seen that $A^o \in K(p)$ with $\sigma_a(A) = \sigma_a(A^o) = \sigma_o(A^o)$. By Theorem 1, $\sigma_o(A^o) = \sigma_{jo}(A^o)$; this implies $\sigma_a(A) = \sigma_{ja}(A)$.

Now suppose that $\alpha = re^{i\theta} \in \sigma_a(A)$, $r > 0$. Then there exists a sequence of unit vectors $\{x_n\} \in \mathcal{H}$ such that $(A - \alpha)x_n \to 0$ and $(A^\star - \bar{\alpha})x_n \to 0$ as $n \to \infty$. Let $u = [x_n]$ denote the equivalence class of $\{x_n\}$ in H^o. Then u is a unit vector such that $A^o u = \alpha u$ and $A^{o\star} u = \bar{\alpha} u$. Thus $|A^o|^2 u = |\alpha|^2 u = r^2 u$; hence $P(|A^o|^2)u = P(r^2)u$ for every polynomial $P(z)$ with $P(0) = 0$. In particular, $|A^o|u = ru$. This, since $U^o|A^o|u = re^{i\theta}u$, implies $Uu = e^{i\theta}u$. Consequently, $(|A| - r)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$ as $n \to \infty$. Since the reverse implication is obviously true, the proof is complete.

(ii) If $\bar{\alpha} \notin \sigma_o(A^\star)$, then (since $\sigma_o(A) = \sigma_{jo}(A)$ for $A \in K(p)$)$\alpha \notin \sigma_o(A)$. This, since $\sigma_e(A) = \sigma(A) - \sigma_{oo}(A)$ [9], implies $\alpha \in \sigma_e(A)$ and (so) there exists a sequence of unit vectors x_n, x_n converges to 0 weakly, such that $(A - \alpha)x_n \to 0$ as $n \to \infty$. Let $\alpha = re^{i\theta}$. As in the proof of (i) above, there exists a sequence of unit vectors x_n converging weakly to 0 such that

$$(|A| - r)x_n \to 0 \text{ and } (U - e^{i\theta})x_n \to 0 \text{ as } n \to \infty.$$

This implies $r = |\alpha| \in \sigma_e(|A|) \cap \sigma_e(|A^\star|)$. (Recall that $|A^\star| = U|A|U^\star$.)

(iii) If $\alpha \in \partial\sigma(A) \subset \sigma_a(A)$, then there exists a sequence of unit vectors x_n such that $(A - \alpha)x_n \to 0$ as $n \to \infty$. Let $\alpha = re^{i\theta}$. Since $\sigma_a(A) =$

$\sigma_{ja}(A)$, $(|A| - r)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$ as $n \to \infty$. This implies $r = |\alpha| \in \sigma(|A|) \cap \sigma(|A^\star|)$. Suppose now that $(A - z) \in K(p)$ for all $z \in \mathcal{C}$. Then $(A - z)$ is paranormal [11, Theorem 4], and so, since paranormal operators are normaloid, normaloid (for all $z \in \mathcal{C}$). Consequently A is convexoid [18; pp. 539, 542], and hence satisfies growth condition (G_1) (i.e., $\|(A - z)^{-1}\| \leq \frac{1}{d(z, \mathrm{conv}\sigma(A))}$, where $d(z, \mathrm{conv}\sigma(A))$ denotes the distance of z from the convex hull of $\sigma(A)$ [18; p. 606]). Recall that if $\alpha \in \partial\sigma(A)$, then either $\alpha \in \sigma_e(A)$ (indeed, α is in the intersection of the left and the right essential spectra of A) or α is an isolated point of $\sigma(A)$ [14]. Since isolated points of the spectrum of an operator satisfying growth condition (G_1) are eigen-values of the operator [18] and since $\sigma_o(A) = \sigma_{jo}(A)$, it follows that if A is pure then $\alpha \in \sigma_e(A)$. This, as in part (ii) implies $|\alpha| \in \sigma_e(|A|) \cap \sigma_e(|A^\star|)$.

We note here that if the operator A in Theorem 4(iii) is in $K(p) \cap H(p)$, then A_p being hyponormal satisfies the property that $(A_p - z)$ is hyponormal for all $z \in \mathcal{C}$. Hyponormal operators satisfy growth condition (G_1). Since $\alpha = re^{i\theta} \in \partial\sigma(A)$ implies $r^p e^{i\theta} \in \partial\sigma(A_p)$, $r^p \in \sigma_e(|A_p|) \cap \sigma_e(|A_p^\star|) = \sigma_e(|A|^p) \cap \sigma_e(|A^\star|^p)$. Hence $r \in \sigma_e(|A|) \cap \sigma_e(|A^\star|)$. (See also [5, Theorem 2].)

Let $KU(p)$ denote the class of $A \in K(p)$ for which U in the polar decomposition $A = U|A|$ is unitary. □

Theorem 5 *If $A \in KU(p)$ is pure, then neither* $\min\sigma(|A|)$ *nor* $\max\sigma(|A|)$ *is in* $\sigma_{oo}(|A|)$.

Proof. Suppose $\alpha = \min\sigma(|A|) \in \sigma_{oo}(|A|)$. Let $M_\alpha = \{x \in \mathcal{H} : |A|x = \alpha x\}$. Since U is unitary, $\sigma(|A|^{2p}) = \sigma(U|A|^{2p}U^\star)$ and $\alpha^{2p} = \min\sigma(U|A|^{2p}U^\star)$. Letting $x \in M_\alpha$ the definition of $K(p)$ implies

$$((AA^\star)^p x, x) = (U|A|^{2p}U^\star x, x) \leq \|x\|^{2(1-p)} \, (A^\star Ax, x)^p$$

Also, since $\alpha^{2p} = \min\sigma(U|A|^{2p}U^\star)$,

$$\alpha^{2p}\|x\|^2 \leq (U|A|^{2p}U^\star x, x).$$

Hence

$$\alpha^{2p}\|x\|^2 = (U|A|^{2p}U^\star x, x), \text{ or, } U|A|^{2p}U^\star x = \alpha^{2p}x.$$

Thus

$$U|A|U^{\star}x = \alpha x,\ |A|U^{\star}x = \alpha U^{\star}x \text{ and } U^{\star}M_{\alpha} \subset M_{\alpha}.$$

The subspace M_{α} being finite dimensional there exist non-trivial $y \in M_{\alpha}$ and $(0 \neq)\beta \in \mathcal{C}$ such that $U^{\star}y = \beta y$, and then $U|A|y = \alpha U y = \frac{\alpha}{\beta} y$. Hence $\sigma_o(A) \neq \phi$. Since $\sigma_o(A) = \sigma_{jo}(A)$ and A is pure, we have a contradiction. Consequently, $\alpha \notin \sigma_{oo}(|A|)$. □

Now let $a = \max \sigma(|A|) \in \sigma_{oo}(|A|)$, and let $M_a = \{x \in \mathcal{H} : |A|x = ax\}$. Then

$$a^{2p} = \max \sigma(|A|^{2p}),\ (|A|^{2p}x, x) \leq a^{2p}\|x\|^2\ (x \in M_a)$$

and

$$\sigma(|A|^2) = \sigma(U^{\star}|A|^2 U).$$

The definition of $K(p)$ implies

$$(|A|^{2p}x, x) \leq \|x\|^{2(1-p)}\ (U^{\star}|A|^2 U x, x)^p$$

for all $x \in \mathcal{H}$. Hence, for $x \in M_a$,

$$a^{2p}\|x\|^2 = (|A|^{2p}x, x) \leq \|x\|^{2(1-p)}(U^{\star}|A|^2 U x, x)^p \leq a^{2p}\|x\|^2.$$

Thus

$$U^{\star}|A|^2 U x = a^2 x.$$

Following an argument similar to that above this implies $\sigma_o(A) = \sigma_{jo}(A) \neq \phi - a$ contradiction. Hence $a \notin \sigma_{oo}(|A|)$.

Given a pure hyponormal contraction A (on a separable Hilbert space $\mathcal{H}$) with Hilbert-Schmidt class defect operator $D_A = (1 - A^{\star}A)^{\frac{1}{2}}$, Takahashi and Uchiyama [19] showed that A belongs to the class C_{10} of contractions. That the same is true for p-hyponormal contractions has been proved in [8, Theorems 1 and 2]. The following theorem extends this result to contractions $A \in K(p)$. We assume in the following that $\mathcal{H}$ is separable.

Recall that a contraction A is said to be of the class

$C_{\cdot 0}(C_{0\cdot})$ if $\|A^{\star n}x\| \to 0$

(resp., $\|A^n x\| \to 0$) as $n \to \infty$ for all $x \in \mathcal{H}$;

$C_{\cdot 1}(C_{1\cdot})$ if $\inf_n \|A^{\star n}x\|$

$$(\text{resp. } \inf_n \|A^n x\|) > 0 \text{ for all non-trivial } x \in \mathcal{H};$$
$$C_{\alpha\beta}, \alpha, \beta = 0, 1, \text{ if } A \in C_{\alpha\cdot} \cap C_{\cdot\beta};$$
$$C_0 \text{ if there exists an inner function } u \text{ such that } u(A) = 0$$

[13]. The contraction A is said to be c.n.u. (= completely non-unitary) if there exists no non-trivial reducing subspace M of A such that $A|M$ is unitary. If the c.n.u. contraction $A \in C_{00}$ has Hilbert-Schmidt class defect operator, then $A \in C_0$ [19]. Every $C_{\cdot 0}$ has contraction A with Hilbert-Schmidt class defect operator has a triangulation

$$A = \begin{vmatrix} A_o & \star \\ 0 & A_3 \end{vmatrix} \text{ of type } \begin{vmatrix} C_0 & \star \\ 0 & C_{10} \end{vmatrix}, \tag{3}$$

and every C_0 contraction A_o has a triangulation

$$\begin{vmatrix} A_1 & \star \\ 0 & A_2 \end{vmatrix},$$

where $\sigma(A_1)$ consists of a countable number of characteristic values in the open unit disc $\mathcal{D}$ and $\sigma(A_2) \subset \partial\mathcal{D}$ [13].

Theorem 6 *If the contraction $A \in K(p)$ is pure and has Hilbert-Schmidt class defect operator, then $A \in C_{10}$.*

Proof. The hypothesis $A \in K(p)$ implies A is paranormal. Since paranormal contractions have $C_{\cdot 0}$ c.n.u. part [7, Theorem 4], $A \in C_{\cdot 0}$ and has a triangulation (3). The restriction of a paranormal operator to an invariant subspace being paranormal, A_1 is paranormal, and so, since $\sigma(A_1)$ is countable, normal [17] with $\sigma(A_1) = \sigma_o(A_1)$. By Theorem 1, $\sigma_o(A) = \sigma_{jo}(A)$; hence, since A is pure, A_1 acts on the trivial space and

$$A = \begin{vmatrix} A_2 & \star \\ 0 & A_3 \end{vmatrix}.$$

As stated above, $\sigma(A_2) \subset \partial\mathcal{D}$; hence A_2 is an invertible paranormal operator, with A_2^{-1} also paranormal [12, 18]. Paranormal operators are normaloid [12]; hence $r(A_2) = r(A_2^{-1}) = 1$, which implies A_2 is unitary. Since A is pure, and so c.n.u., A_2 acts on the trivial space. This implies $A = A_3 \in C_{10}$. □

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