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Characterizations of multipliers in the distribution spaces with restricted growth

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Abstract. Let $\mathscr{K}'_M$ be the space of distributions on $\boldsymbol{R}^n$ which grow no faster than $e^{M(kx)}$ for some $k>0$ and an index function $M(x)$ and let K'_M be the Fourier transform of $\mathscr{K}'_M$. We establish the characterizations of the space $\mathscr{O}_M(\mathscr{K}'_M;\mathscr{K}'_M)$ of multipliers in $\mathscr{K}'_M$ and prove various types of the continuity from or into $\mathscr{O}_M(\mathscr{K}'_M;\mathscr{K}'_M)$. Also we define the space $\mathscr{O}_M(K'_M;K'_M)$ of multipliers in K'_M and find the relation between $\mathscr{O}_M(K'_M;K'_M)$ and the space $\mathscr{O}'_C(\mathscr{K}'_M;\mathscr{K}'_M)$ of convolution operators in $\mathscr{K}'_M$ by the Fourier transformation.

Let $\mathscr{K}'_1$ be the space of distributions of exponential growth. In $\mathscr{K}'_1$, M. Hasumi [4] and Z. Zielezny [8] established the characterizations of the space $\mathscr{O}_M(\mathscr{K}'_1;\mathscr{K}'_1)$ of multipliers and the space $\mathscr{O}'_C(\mathscr{K}'_1;\mathscr{K}'_1)$ of convolution operators in $\mathscr{K}'_1$. On the other hand, D. H. Pahk [5][6] introduced the space $\mathscr{K}'_M$, of distributions that grow no faster than $exp(M(kx))$ for some integer k and an index function $M(x)$, and the space $\mathscr{O}'_C(\mathscr{K}'_M;\mathscr{K}'_M)$ of convolution operators in $\mathscr{K}'_M$ which are extention of $\mathscr{K}'_1$ and $\mathscr{O}'_C(\mathscr{K}'_1;\mathscr{K}'_1)$, respectively. In $\mathscr{K}'_M$, he [5] and S. Abdullah [1][2] characterized the space $\mathscr{O}'_C(\mathscr{K}'_M;\mathscr{K}'_M)$ and the space $\mathscr{O}_M(\mathscr{K}'_M;\mathscr{K}'_M)$ of multipliers in $\mathscr{K}'_M$. In this note, we find the other characterizations of $\mathscr{O}_M(\mathscr{K}'_M;\mathscr{K}'_M)$ and prove the completeness of $\mathscr{O}_M(\mathscr{K}'_M;\mathscr{K}'_M)$ with the topology which we give, and various types of the continuity from or into $\mathscr{O}_M(\mathscr{K}'_M;\mathscr{K}'_M)$. Also we define the space $\mathscr{O}_M(K'_M;K'_M)$ of the multipliers in K'_M, the Fourier transform of $\mathscr{K}'_M$, and prove that the Fourier transformation from $\mathscr{O}'_C(\mathscr{K}'_M;\mathscr{K}'_M)$ into $\mathscr{O}_M(K'_M;K'_M)$ is continuous under given topology.

Before presenting our theorems we recall briefly the basic facts about the spaces $\mathscr{K}'_M$, $\mathscr{O}'_C(\mathscr{K}'_M;\mathscr{K}'_M)$ and K'_M. For further details we refer to [5].

The space $\mathscr{K}'_M$. Let $\mu(\xi)$ $(0\leq\xi\leq\infty)$ denote a continuous increasing function such that $\mu(0)=0$, $\mu(\infty)=\infty$. For $x\geq 0$, we define

$$M(x)=\int_0^x \mu(\xi)d\xi.$$

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The function $M(x)$ is an increasing, convex and continuous function with $M(0)=0$, $M(\infty)=\infty$. For $x<0$, we define $M(x)$ to be $M(-x)$ and for $x=(x_1, \cdots, x_n)\in \boldsymbol{R}^n$, $n\geq 2$, we define $M(x)$ to be $M(x_1)+\cdots+M(x_n)$.

Now we list some properties of $M(x)$ which will be used in the proof,

(i) $M(x)+M(y)\leq M(x+y)$ for all x, $y\geq 0$

(ii) $M(x+y)\leq M(2x)+M(2y)$ for all x, $y\geq 0$.

Let $\mathscr{K}_M$ be the space of all C^∞-functions ϕ in $\boldsymbol{R}^n$ such that

$$\nu_k(\phi)=\sup_{\substack{x\in \boldsymbol{R}^n \\ |\alpha|\leq k}} e^{M(kx)}|D^\alpha \phi(x)|<\infty, \quad k=1, 2, 3, \cdots,$$

where $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ and $D_j=i^{-1}\left(\dfrac{\partial}{\partial x_j}\right)$. Provided with the topology defined by the seminorms ν_k, $\mathscr{K}_M$ is a Frechet space. The dual $\mathscr{K}'_M$ of $\mathscr{K}_M$ is the space of all continuous linear functionals on $\mathscr{K}_M$. Then a distribution u is in $\mathscr{K}'_M$ if and only if there exist $m\in \boldsymbol{N}^n$, $k\in \boldsymbol{N}$ and a bounded continuous function $f(x)$ on $\boldsymbol{R}^n$ such that

$$u=D^m(e^{M(kx)}f(x)).$$

$\mathscr{K}'_M$ is endowed with the topology of uniform convergence on all bounded sets in $\mathscr{K}_M$.

The space $\mathscr{O}'_C(\mathscr{K}'_M;\mathscr{K}'_M)$. If $u\in \mathscr{K}'_M$ and $\phi\in \mathscr{K}_M$, then the convolution $u*\phi$ is a C^∞-function defined by

$$u*\phi(x)=\langle u_y, \phi(x-y)\rangle$$

where $\langle u, \phi\rangle=u(\phi)$.

The space $\mathscr{O}'_C(\mathscr{K}'_M;\mathscr{K}'_M)$ of convolution operators in $\mathscr{K}'_M$ consists of distributions $S\in \mathscr{K}'_M$ satisfying one of the following equivalent conditions:

(i) the distribution $S_k=\gamma_k S$, $k=1, 2, \cdots$ are in the tempered distribution space; where $\gamma_k(x)=e^{M(kx)}$;

(ii) for every integer $k\geq 0$, there exists an integer $m\geq 0$ such that

$$S=\sum_{|\alpha|\leq m} D^\alpha f_\alpha$$

where f_α are continuous functions in $\boldsymbol{R}^n$ such that whose product with $e^{M(kx)}$ are bounded.

(iii) for every $\phi\in \mathscr{K}_M$, the convolution $S*\phi$ is in $\mathscr{K}_M$ and the map $\phi\rightarrow S*\phi$ from $\mathscr{K}_M$ into $\mathscr{K}_M$ is continuous.

The space K_M'. For $\phi\in\mathcal{K}_M$, the Fourier transform

$$\hat{\phi}(\xi)=\int_{\mathbf{R}^n}e^{-i<x,\xi>}\phi(x)dx$$

can be continued in $\mathbf{C}^n$ as an entire function of $\zeta=\xi+i\eta\in\mathbf{C}^n$ such that

$$\omega_k(\hat{\phi})=\sup_{\zeta\in\mathbf{C}^n}(1+|\zeta|)^k e^{-\Omega(\eta/k)}|\hat{\phi}(\zeta)|<\infty,\ k=1,\ 2,\cdots \tag{1}$$

where $\Omega(y)$ is the dual of $M(x)$ in the sense of Young. If K_M is the space of all entire functions with the property (1) and the topology in K_M is defined by the seminorms ω_k, then the Fourier transform is an isomorphism of $\mathcal{K}_M$ onto K_M. The dual K_M' of K_M is the space of the Fourier transformations of distributions in $\mathcal{K}'_M$. For $u\in\mathcal{K}'_M$, the Fourier transform $\hat{u}$ is defined by the Parseval formula. Also if $S\in\mathcal{O}'_C(\mathcal{K}'_M;\mathcal{K}'_M)$ and $\phi\in\mathcal{K}_M$, we have the formula

$$\widehat{S*\phi}=\hat{S}\hat{\phi}$$

where the product on the right-hand side is defined by $\langle\hat{S}\hat{u},\chi\rangle=\langle\hat{u},\hat{S}\chi\rangle$, $\chi\in\mathcal{K}_M$.

Now we study the space of multipliers in $\mathcal{K}'_M$. Let $S\in\mathcal{K}'_M$, we find for conditions on the C^∞-function $\alpha(x)$ in order that $\alpha S\in\mathcal{K}'_M$. If $\alpha(x)=exp(exp|x|^2)$, $\alpha S\notin\mathcal{K}'_M$ for $M(x)=|x|^2/2$, the reason being that $\alpha(x)$ grows very fast at infinity. Hence we consider the following definition:

DEFINITION 1 [1]. We denote by $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ of all $\phi\in C^\infty(\mathbf{R}^n)$ such that for every $\alpha\in\mathbf{N}^n$, there exist $k_0\in\mathbf{N}$ and $C_0>0$ such that

$$|D^\alpha\phi(x)|\le C_0e^{M(k_0x)},\ x\in\mathbf{R}^n.$$

We present the characterizations of the element of $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$.

THEOREM 2. *Let $\phi\in C^\infty(\mathbf{R}^n)$. The following statements are equivalent:*

(a) $\phi\in\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$.

(b) *for every $f\in\mathcal{K}_M$, $\phi\cdot f\in\mathcal{K}_M$.*

(c) *for every $\alpha\in\mathbf{N}^n$ and $f\in\mathcal{K}_M$, $D^\alpha\phi\cdot f$ is bounded in $\mathbf{R}^n$.*

PROOF. (a)$\Longleftrightarrow$(b). This equivalence follows from the Theorem 6 of [1].

(b)$\Longrightarrow$(c). Since D^α maps $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ into itself, the implication follows immediately.

(c)$\Longrightarrow$(a). First, we remark that condition (a) is implied by the following condition: there exist $k_0 \in \boldsymbol{N}$ and $L>0$ so that for any $\alpha \in \boldsymbol{N}^n$,

$$|D^\alpha \phi(x)| \leq e^{M(k_0 x)} \text{ for } |x| \geq L \tag{2}$$

Indeed, if (2) holds, condition (a) is satisfied for

$$C_0 = \max\{\sup_{\substack{x \in \boldsymbol{R}^n \\ |x| \leq L}} |D^\alpha \phi(x)|, 1\}.$$

The proof will be by contradiction. Suppose that (2) does not hold. Hence by induction, we can find a sequence $\{x_j\} \in \boldsymbol{R}^n$ with $|x_{j+1}| \geq |x_j| + 2$ such that, for some $\alpha_0 \in \boldsymbol{N}^n$

$$|D^{\alpha_0} \phi(x_j)| > e^{M(2jx_j)}.$$

Let $\gamma \in C_c^\infty(\boldsymbol{R}^n)$ be such that $0 \leq \gamma \leq 1$, $\operatorname{supp}\gamma \subset \overline{B(0,1)}$, the ball centered 0 with radius 1, and $\gamma(0)=1$. Define

$$\psi(x) = \sum_{j=1}^\infty \frac{\gamma(x-x_j)}{e^{M(jx_j)}}.$$

The sum is well-defined, since the supports of the functions $\gamma(x-x_j)$ are disjoint. Then $\psi(x) \to 0$ in $\mathcal{K}_M$ as $|x| \to \infty$, i.e., $\psi \in \mathcal{K}_M$. Indeed, let $t=(t_1, t_2, \cdots, t_n)$, $t_j=(t_{j1}, t_{j2}, \cdots, t_{jn})$ in $\boldsymbol{R}^n$ with $|t_i|-1 \leq |t_{ji}| \leq |t_i|+1$ for all $i=1,2,\cdots,n$ and $\alpha \in \boldsymbol{N}^n$, $k \in \boldsymbol{N}$. By the properties of $M(x)$ and the definition of $M(x)$ for $x<0$ and $x \in \boldsymbol{R}^n$, it follows that

$$\begin{aligned}
&\sup_t e^{M(kt)} D^\alpha \psi(t) \\
&= \sup_t \frac{e^{M(kt)} D^\alpha \gamma(t-t_j)}{e^{M(jt_j)}} \\
&\leq \sup_t e^{M(2k(t-t_j)) + M(2kt_j) - M(jt_j)} D^\alpha \gamma(t) \\
&\leq C_{\alpha,\gamma} \sup_t e^{nM(2k) - M((j-2k)t_j)} \to 0 \text{ as } |t| \to \infty
\end{aligned}$$

where $C_{\alpha,\gamma} = \sup_t D^\alpha \gamma(t) < \infty$ and $t-t_j = (t_1-t_{j1}, t_2-t_{j2}, \cdots, t_n-t_{jn})$.

But we have

$$|\psi(x_j) D^{\alpha_0} \phi(x_j)| > e^{M(jx_j)} \quad \text{for } j=1,2,\cdots,$$

which contradicts condition (c). Q. E. D.

We define on $\mathcal{O}_M(\mathcal{K}'_M; \mathcal{K}'_M)$ the topology by the family of seminorms

$$\delta_{\phi,k}(f) = \sup_{\substack{x \in \boldsymbol{R}^n \\ |\alpha| \leq k}} e^{M(kx)} |\phi(x) D^\alpha f(x)|,$$

where $\phi\in\mathcal{K}_M$ and $k\in\boldsymbol{N}$. In [1], S. Abdullah introduced the family of seminorms $\rho_{\phi,k}$ on $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$, that are equivalent to $\delta_{\phi,k}$, defined by

$$\rho_{\phi,k}(f)=\sup_{\substack{x\in\boldsymbol{R}^n\\|\alpha|\leq k}} e^{M(kx)}|D^\alpha(f\cdot\phi)|,\ k\in\boldsymbol{N} \text{ and } \phi\in\mathcal{K}_M.$$

But he did not mention about the properties of the space $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ itself. In this direction we have

THEOREM 3. *The space $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ with the topology defined by the family of seminorms $\delta_{\phi,k}$, $k\in\boldsymbol{N}$ and $\phi\in\mathcal{K}_M$, is complete.*

PROOF. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ and let $\alpha\in\boldsymbol{N}^n$. Then $\{f_n\}$ is obviously a Cauchy sequence in C^∞. By the completeness of C^∞, there exist $f\in C^\infty$ such that $f_n\to f$ in C^∞, hence $D^\alpha f_n\to D^\alpha f$ in $\boldsymbol{R}^n$. Since $f_n\in\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$, there exist $k_0\in\boldsymbol{N}$ which depends on α and n and $C_0>0$ such that

$$|D^\alpha f_n(x)|\leq C_0e^{M(k_0x)} \text{ for all } n.$$

Hence, for such $k_0\in\boldsymbol{N}$ and $C_1>0$,

$$|D^\alpha f(x)|\leq C_1e^{M(k_0x)},\ C_1=C_0+1,$$

i.e., $f\in\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$. Now we claim that $f_n\to f$ in $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$. Since $\{f_n\}$ is a Cauchy sequence in $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$, for all $k\in\boldsymbol{N}$ and $\phi\in\mathcal{K}_M$, there is a constant $C_{\phi,k}$ which is dependent on ϕ and k such that

$$\sup_{\substack{x\in\boldsymbol{R}^n\\|\alpha|\leq k}} e^{M(kx)}|\phi(x)D^\alpha f_n(x)|\leq C_{\phi,k},\ \forall n\in\boldsymbol{N}. \tag{3}$$

But this inequality implies that, given $\varepsilon>0$, there is a constant $M>0$ such that

$$\sup_{\substack{x\in\boldsymbol{R}^n\\|\alpha|\leq k}} e^{M(kx)}|\phi(x)D^\alpha f_n(x)|\leq\varepsilon,\ \forall n\in\boldsymbol{N} \text{ and } x \text{ with } |x|>M. \tag{4}$$

Indeed, suppose that (4) is false. For some $k'\in\boldsymbol{N}$, $\alpha_0\in\boldsymbol{N}^n$ with $|\alpha_0|\leq k'$ and $\phi_0\in\mathcal{K}_M$, there exist $\varepsilon_0>0$ and a sequence $\{x_j\}$ with $|x_j|\to\infty$ as $j\to\infty$ such that

$$e^{M(k'x_j)}|\phi_0(x_j)D^{\alpha_0}f_{n_0}(x_j)|\geq\varepsilon_0,$$

where n_0 is some integer. Then

$$e^{M((k'+1)x_j)}|\phi_0(x_j)D^{\alpha_0}f_{n_0}(x_j)|\geq\varepsilon_0e^{M(x_j)}\to\infty \text{ as } j\to\infty$$

which is contradict to (3).

Since $f_n \to f$ in C^∞, it follows that f satisfies inequality (3) . Hence by (4), it follows that for x with $|x|>M$, we have

$$\sup_{\substack{x\in \boldsymbol{R}^n\\ |\alpha|\leq k}} e^{M(kx)}|\phi(x)D^\alpha f(x)|\leq \varepsilon. \tag{5}$$

On the other hand, since $f_n \to f$ in C^∞, $D^\alpha f_n$ converges uniformly to $D^\alpha f$ on the compact set $\{x\in \boldsymbol{R}^n : |x|\leq M\}$. This implies that, given $\varepsilon>0$, there exist $n_0 \in \boldsymbol{N}$ such that for $n\geq n_0$,

$$\sup_{\substack{x\in \boldsymbol{R}^n\\ |\alpha|\leq k}} e^{M(kx)}|\phi(x)D^\alpha (f_n-f)|<\varepsilon \text{ for all } |x|\leq M. \tag{6}$$

Our last three inequalities (4), (5) and (6) implies that, given $\varepsilon>0$, there exist $n_0\in \boldsymbol{N}$ such that for $n\geq n_0$,

$$\sup_{\substack{x\in \boldsymbol{R}^n\\ |\alpha|\leq k}} e^{M(kx)}|\phi(x)D^\alpha (f_n-f)|<\varepsilon. \quad \text{Q. E. D.}$$

THEOREM 4. *We have the inclusions*

$$\mathscr{K}_M \hookrightarrow \mathscr{O}_M(\mathscr{K}'_M ; \mathscr{K}'_M) \hookrightarrow \mathscr{K}'_M$$

with continuous imbeddings. Moreover, $\mathscr{K}_M$ is dense in $\mathscr{O}_M(\mathscr{K}'_M ; \mathscr{K}'_M)$.

PROOF. Let $\psi \in \mathscr{O}_M(\mathscr{K}'_M ; \mathscr{K}'_M)$ and ϕ, $f\in \mathscr{K}_M$. It is clear that $\mathscr{K}_M \subset \mathscr{O}_M(\mathscr{K}'_M ; \mathscr{K}'_M)$. Since, for k_0 in the Definition 1 and $k>k_0$,

$$|\langle \psi, \phi\rangle| \leq C_\psi \nu_k(\phi),$$

where $C_\psi = \int_{\boldsymbol{R}^n} e^{-M(kx)}\psi(x)dx \leq C_0 \int_{\boldsymbol{R}^n} e^{-M((k-k_0)x)}dx<\infty$, ψ defines an element of $\mathscr{K}'_M$. Hence $\mathscr{O}_M(\mathscr{K}'_M ; \mathscr{K}'_M)\subset \mathscr{K}'_M$. The two inequalities

$$\delta_{\phi,k}(f)\leq C_\phi \nu_k(f),$$

where $C_\phi = \sup_{x\in \boldsymbol{R}^n} |\phi(x)|<\infty$, and

$$|\langle \psi, \phi\rangle| \leq C' \delta_{\phi,k}(\psi),$$

where $C' = \int_{\boldsymbol{R}^n} e^{-M(kx)}dx<\infty$ imply that the imbeddings are continuous. To prove the second assertion, we shall use the idea in the proof of Theorem 3. For $\psi \in \mathscr{O}_M(\mathscr{K}'_M ; \mathscr{K}'_M)$, since ψ is also in C^∞ and C_c^∞ is a dense subspace of C^∞ (Theorem 4. 2 [3]), there is a sequence $\{\psi_n\}$ in C_c^∞ such that $\psi_n \to \psi$ in C^∞. Since for all $n\in \boldsymbol{N}$, $\psi_n \in C_c^\infty$, for all $k\in \boldsymbol{N}$ and $\phi \in \mathscr{K}_M$, there is a constant $C_{\phi,k}$ which is dependent on ϕ, n and k such that

$$\sup_{\substack{x\in \boldsymbol{R}^n \\ |\alpha|\leq k}} e^{M(kx)}|\phi(x)D^\alpha\psi_n(x)|\leq C_{\phi,k}.$$

Then the inequality (4) in the proof of Theorem 3 holds for ψ_n instead of f_n by the same reason in the proof. Now by $\psi_n\to\psi$ in C^∞, the inequality (5) and (6) in the proof of Theorem 3 for ψ_n instead of f_n do hold. Hence $\psi_n\to\psi$ in $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$. But $\psi_n\in C_C^\infty$ is also in $\mathcal{K}_M$, $\mathcal{K}_M$ is dense in $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$. Q. E. D.

For $\phi\in \mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ and $S\in\mathcal{K}'_M$, we define the product ϕS by

$$\langle\phi S, f\rangle=\langle S, \phi f\rangle \text{ for } f\in\mathcal{K}_M. \tag{7}$$

In [1], S. Abdullah proved that if $\phi\in \mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$, $f\in\mathcal{K}_M$ and $S\in\mathcal{K}'_M$, the mapping $f\to\phi f$ from $\mathcal{K}_M$ into $\mathcal{K}_M$ and the mapping $S\to\phi S$ from $\mathcal{K}'_M$ into $\mathcal{K}'_M$ are continuous. We can also prove the following sequential continuity.

THEOREM 5. *If $\phi\in \mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$, $f\in\mathcal{K}_M$ and $S\in\mathcal{K}'_M$, then the mapping $\phi\to\phi f$ from $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ into $\mathcal{K}_M$ and the mapping $\phi\to\phi S$ from $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ into $\mathcal{K}'_M$ are sequentially continuous.*

PROOF. Let $\phi\in \mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$. By the Leibniz formula,

$$\sup_{\substack{x\in \boldsymbol{R}^n \\ |\alpha|\leq k}} e^{M(kx)}|D^\alpha(f\cdot\phi)|\leq C_{\alpha',\alpha''} \sup_{\substack{x\in \boldsymbol{R}^n \\ |\alpha'|,|\alpha''|\leq k}} e^{M(kx)}|D^{\alpha'}f\cdot D^{\alpha''}\phi|.$$

Since $D^{\alpha'}f=f_{\alpha'}\in\mathcal{K}_M$, $\nu_k(f\cdot\phi)\leq C_{\alpha',\alpha''}\delta_{f_{\alpha'},k}(\phi)$ for a constant $C_{\alpha',\alpha''}$. Hence if $\phi_j\to 0$ in $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$, then $\phi_j\to 0$ in $\mathcal{K}_M$, i. e. the first mapping is sequentially continuous. The second part of the theorem follows from (7) and the first part of the theorem. Q. E. D.

REMARK. We will not mention the further topological properties of $\mathcal{O}_M(\mathcal{K}'_M;\mathcal{K}'_M)$ in this paper. The report of our progress in this direction will be publish soon.

Finally, let K_M and K'_M be the Fourier transformation of $\mathcal{K}_M$ and $\mathcal{K}'_M$, respectively. Now we define the space $\mathcal{O}_M(K'_M;K'_M)$ and find the relation between $\mathcal{O}'_C(\mathcal{K}'_M;\mathcal{K}'_M)$ and $\mathcal{O}_M(K'_M;K'_M)$ by the Fourier transformation.

DEFINITION 6. We denote by $\mathcal{O}_M(K'_M;K'_M)$ of all $\psi\in C^\infty$ extendable over $\boldsymbol{C}^n$ as entire functions such that there exist $k_1\in\boldsymbol{N}$ and $C_1>0$ such that

$$|\psi(\zeta)|\leq C_1(1+|\zeta|)^{k_1}e^{-\Omega(\eta/k_1)},$$

where $\zeta=\xi+i\eta\in \boldsymbol{C}^n$ and $\Omega(y)$ is the dual of $M(x)$ in the sense of Young.

We define on $\mathcal{O}_M(K'_M; K'_M)$ the topology by the family of seminorms

$$\sigma_{\hat{\phi},k}(f)=\sup_{\zeta\in \boldsymbol{C}^n}(1+|\zeta|)^k e^{-\Omega(\eta/k)}|\hat{\phi}\cdot f|$$

where $\phi\in \mathscr{K}_M$, i.e., $\hat{\phi}\in K_M$ and $k\in \boldsymbol{N}$.

By the Paley-Wiener-Schwartz Theorem for $\mathcal{O}'_C(\mathscr{K}'_M; \mathscr{K}'_M)$ (Theorem 4.1 (b) in [5]), the element in $\mathcal{O}_M(K'_M; K'_M)$ is a Fourier transformation of the element of $\mathcal{O}'_C(\mathscr{K}'_M; \mathscr{K}'_M)$. And, the space $\mathcal{O}_M(K'_M; K'_M)$ is the multipliers in K'_M by the following;

THEOREM 7. *Let $\psi\in \mathcal{O}_M(K'_M; K'_M)$. Then for all $f\in \mathscr{K}_M$, $\hat{f}\cdot\psi\in K_M$, and the map $\hat{f}\rightarrow \hat{f}\psi$ from K_M into itself is continuous.*

PROOF. For any k and for C_1, k_1, in the Definition [6] and some l,

$$\begin{aligned}\omega_k(\psi\cdot \hat{f})&=\sup_{\zeta\in \boldsymbol{C}^n}(1+|\zeta|)^k e^{-\Omega(\eta/k)}|\psi\cdot \hat{f}|\\ &\le C_1\sup_{\zeta\in \boldsymbol{C}^n}(1+|\zeta|)^{k+k_1}e^{-\Omega(\eta/k)-\Omega(\eta/k_1)}|\hat{f}|\\ &\le C_1\sup_{\zeta\in \boldsymbol{C}^n}(1+|\zeta|)^{2(k+k_1)}e^{-\Omega(\eta/2(k+k_1))}|\hat{f}|\\ &=C_1\omega_{2(k+k_1)}(\hat{f})\le C_1\nu_l(f),\end{aligned}$$

where we use the properties of $\Omega(x)$ and the Paley-Wiener-Schwartz Theorem for $\hat{f}$ (Theorem 4.1 [a] in [5]) and the continuity of the Fourier transform from $\mathscr{K}_M$ onto K_M. This completes the proof of the theorem.

THEOREM 8. *The Fourier transformation from $\mathcal{O}'_C(\mathscr{K}'_M; \mathscr{K}'_M)$ into $\mathcal{O}_M(K'_M; K'_M)$ is continuous.*

PROOF. Let $\mathscr{S}\in \mathcal{O}'_C(\mathscr{K}'_M; \mathscr{K}'_M)$ and $\phi\in \mathscr{K}_M$. We recall that if $\mathscr{S}_j\rightarrow \mathscr{S}$ in $\mathcal{O}'_C(\mathscr{K}'_M; \mathscr{K}'_M)$ then $\mathscr{S}_j*\phi\rightarrow \mathscr{S}*\phi$ in $\mathscr{K}_M$ by the equivalent condition (iii) of $\mathcal{O}'_C(\mathscr{K}'_M; \mathscr{K}'_M)$.

For any $k\in \boldsymbol{N}$ and $\alpha\in \boldsymbol{N}^n$ with $|\alpha|\le k$, by Lemma 1.3 in [5].

$$\begin{aligned}|\zeta^\alpha \hat{\mathscr{S}}\cdot\hat{\phi}|&=|\zeta^\alpha(\widehat{\mathscr{S}*\phi})|=\left|\int_{\boldsymbol{R}^n}e^{-i<x,\zeta>}D^\alpha(\mathscr{S}*\phi)(x)dx\right|\\ &\le\int_{\boldsymbol{R}^n}e^{<x,\eta>}e^{-M((k+1)x)+M((k+1)x)}|D^\alpha(\mathscr{S}*\phi)(x)|dx\\ &\le\sup_{\substack{x\in \boldsymbol{R}^n\\ |\alpha|\le k+1}}\{e^{M((k+1)x)}|D^\alpha(\mathscr{S}*\phi)(x)|\}\sup_{x\in \boldsymbol{R}^n}\{e^{<x,\eta>-M(kx)}\}\\ &\qquad\times\int_{\boldsymbol{R}^n}e^{-M(x)}dx\\ &\le C''e^{\Omega(\eta/k)}\nu_{k+1}(\mathscr{S}*\phi),\end{aligned}$$

where C'' does not dependent on ϕ.

Hence

$$\sigma_{\hat{\phi},k}(\hat{\mathscr{S}})=\sup_{\zeta\in C^n}(1+|\zeta|)^k e^{-\Omega(\eta/k)}|\hat{\mathscr{S}}\cdot\hat{\phi}|$$
$$\leq C''_k \nu_{k+1}(\mathscr{S}*\phi).$$

Thus if $\mathscr{S}_j\to\mathscr{S}$ in $\mathscr{O}'_c(\mathscr{K}'_M;\mathscr{K}'_M)$, $\hat{\mathscr{S}}_j\to\hat{\mathscr{S}}$ in $\mathscr{O}_M(K'_M;K'_M)$. Since $\mathscr{O}'_c(\mathscr{K}'_M;\mathscr{K}'_M)$ is bornological [2] and the Fourier transformation is linear, the result holds. Q. E. D.

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