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On certain integral equations related to nonlinear wave equations

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§ 1. Introduction

This paper is concerned with the global in time existence of solutions for integral equations related to the Cauchy problem for nonlinear wave equations.

In order to describe integral equations we introduce some notations. For a function $\varphi(x, t)$ of $(x, t)\in \mathbf{R}^{n}\times \mathbf{R}$, we define, dividing into two cases of odd or even space dimensions,

$$
(1.1)\qquad \begin{aligned} M(\varphi|x, r;t)&=\int_{|\omega|=1}\varphi(x+r\omega, t)dS_{\omega},\ n=2m+1,\\ M(\varphi|x, r;t)&=\int_{|\xi|\leq 1}\frac{\varphi(x+r\xi, t)}{\sqrt{1-|\xi|^{2}}}d\xi,\ n=2m,\end{aligned}
$$

where dS_{ω} stands for the surface element of the unit sphere in $\mathbf{R}^{n}$. When $\varphi(x)$ is independent of t, we denote $M(\varphi|x, r;t)$ by $M(\varphi|x, r)$.

We consider the integral equations for scalar unknowns $u(x, t)$ of the form

$$
(1.2)\qquad u(x, t)=v(x, t)+L(F(u))(x, t),\ (x, t)\in \mathbf{R}^{n}\times[0, \infty),
$$

where

$$
(1.3)\qquad L(F(u))(x, t)=A_{n}\int_{0}^{t}(t-\tau)M(F(u)|x, t-\tau;\tau)d\tau.
$$

Moreover, v and F are given functions and A_{n} is a given positive constant. Note that L is a positive linear operator.

We now specify the constant A_{n} as follows;

$$
(1.4)\qquad A_{n}=\frac{1}{(n-2)\omega_{n}}(n=2m+1),\ A_{n}=\frac{2}{(n-1)\omega_{n+1}}(n=2m),
$$

where ω_{n} stands for the measure of the unit sphere in $\mathbf{R}^{n}$. Let $f(x)$ and $g(x)$ be given functions with compact support. And let $v=v_{0}(x, t)$ be a unique solution to the Cauchy problem for a linear wave equation

$$(1.5)\qquad \begin{aligned} &\partial_t^2 v_0(x, t)-\Delta v_0(x, t)=G(x, t), && (x, t)\in\mathbf{R}^n\times[0, \infty),\\ &v_0(x, 0)=f(x),\ \partial_t v_0(x, 0)=g(x), && x\in\mathbf{R}^n,\end{aligned}$$

where

$$(1.6)\qquad G(x, t)=2(m-1)A_n M(F(f)|x, t).$$

Then we will find in section 3 that a solution $u(x, t)$ to the integral equation (1.2) is a solution to the Cauchy problem for a nonlinear wave equation of the form

$$(1.7)\qquad \begin{aligned} &\partial_t^2 u(x, t)-\Delta u(x, t)=F(u)(x, t)-H(x, t), && (x, t)\in\mathbf{R}^n\times[0, \infty),\\ &u(x, 0)=f(x),\ \partial_t u(x, 0)=g(x), && x\in\mathbf{R}^n,\end{aligned}$$

where

$$(1.8)\qquad H(x, t)=2(m-1)A_n\int_0^t M(\partial_t(F(u))|x,\ t-\tau\,;\tau)d\tau.$$

The uniqueness of solutions to the Cauchy problem (1.7) follows from Appendix in [6]. Note that G and H vanish for $n=2$ or 3.

When $F(u)$ is of the form $A|u|^p$ $(A>0)$, F. John [5] has proved the global existence of solutions to (1.7) in three space dimensions provided $p>1+\sqrt{2}$ and initial data are small. R. T. Glassey [3] has also proved the same results in two space dimensions for $p>(3+\sqrt{17})/2$. Moreover, Y. Choquet-Bruhat [2] has studied the global existence in the Sobolev spaces for higher dimensions.

Let $p_0(n)$ be the positive root of

$$(1.9)\qquad (n-1)p^2-(n+1)p-2=0.$$

This quadratic equation appeared for the first time in W. A. Strauss [8]. Then it follows that $1<p_0(n)\leqq 2$ for $n\geqq 4$ and the equality holds only for $n=4$. In this paper we first establish the global existence of C^1-solutions to the integral equation (1.2), provided a suitable norm of v is small and the following hypothesis $(H)_1$ holds:

$(H)_1$ $F(s)$ is of class C^1 with Hölder exponent δ $(0<\delta<1)$ and $F(0)=F'(0)=0$. Hence there exists a positive constant A such that $|F^{(j)}(s)|\leqq A|s|^{p-j}(j=0, 1)$ for $p=1+\delta>p_0(n)$, $|s|\leqq 1$.

Note that, for $n=4$, a hypothesis $(H)_2$, stated in section 2, similar to $(H)_1$ holds and hence (1.2) has a global solution of class C^2. Next we establish the global existence of solutions to the nonlinear wave equations (1.7) provided some derivatives of f and g are small.

The plan of this paper is as follows. In section 2 we introduce the weighted L^∞-norms and state more precisely the above main results. Introducing the weights is related to the decay rates of solutions to (1.5) and to the condition $p>p_0(n)$. To illustrate our situations we assume that initial data f and g are supported in a ball $\{x\in\mathbf{R}^n : |x|\leqq k\}$. The solutions to (1.5) have the classical decay rate

$$t^{-(n-1)/2} \text{ as } t\to\infty.$$

This fact will be proved in section 3 by using the explicit representation of solutions and the methods in [7] and [9]. As is well known, Huygens' principle is valid for $n=2m+1$. In the case where $n=2m$, we will also find in section 3 that the solutions to (1.5) decay in the solid characteristic cone $\{(x, t)\in\mathbf{R}^n\times[0, \infty) : |x|<t-k\}$ as

$$(1.10)\qquad (t+|x|+2k)^{-(n-1)/2}(t-|x|+2k)^{-(n-3)/2}.$$

Note that for solutions to (1.5) with $G=0$ one can replace the power $-(n-3)/2$ by $-(n-1)/2$. The condition $p>p_0(n)$ guarantees the integrability of a function $s^{-pq(n,p)}$ over $[1, \infty)$, where

$$(1.11)\qquad q(n, p)=\frac{n-1}{2}p-\frac{n+1}{2}.$$

The existence of solutions to the integral equation (1.2) is proved in section 5 by using the basic estimates established in section 4 and the classical iteration method by Picard. The main tool to prove the basic estimates is the fundamental identity for the integral of a plane wave function (see [4] p. 8)

$$(1.12)\qquad \int_{|\omega|=1}g(y\cdot\omega)dS_\omega=\omega_{n-1}\int_{-1}^{1}(1-p^2)^{(n-3)/2}g(|y|p)dp,$$

where $g(s)$ is a function of the scalar variable s. Finally we point out that the observation in [3] p. 243 is also useful for general even space dimensions.

§ 2. Statement of main results

In this section we assume $n\geqq 4$ and state main results on the global existence of solutions to the integral equation (1.2) and the nonlinear wave equation (1.7). For that we introduce the following norm for $u\in C^0(\mathbf{R}^n\times[0. \infty))$ with supp $u\subset\{(x, t) : |x|\leqq t+k\}$;

$$(2.1)\qquad \|u\|=\sup_{(x,t)\in\mathbf{R}^n\times[0,\infty)}\left[\left(\frac{t+r+2k}{k}\right)^{(n-1)/2}N\left(\frac{t-r+2k}{k}\right)|u(x, t)|\right]$$

where $r\equiv|x|$ and k is a fixed positive constant.

The function $N(s)$ of $s\in[1,\infty)$ in (2.1) is defined by dividing into three cases. For the odd dimensional case, $n=2m+1$, we set

$$(2.2)\qquad N(s)=s^{q(n,p)}\ \text{if}\ p>p_0(n),$$

where $p_0(n)$ and $q(n,p)$ are defined by (1,9) and (1.11), respectively. For the even dimensional case, we first set

$$(2.3)\qquad N(s)=\begin{cases} s^{q(n,p)} & \text{if } p_0(n)<p<\dfrac{2n}{n-1}, \\ \dfrac{s^{(n-1)/2}}{\log(1+s)} & \text{if } p=\dfrac{2n}{n-1}, \\ s^{(n-1)/2} & \text{if } p>\dfrac{2n}{n-1}. \end{cases}$$

When $n=2,3$, the above norms are essentially the same ones as in [3], [5]. However, in order to discuss the solution to the equation (1.7), we need another function $N(s)$ for the even dimensional case. For a fixed number $\bar{q}$ which satisfies

$$(2.4)\qquad \frac{1}{p_0(n)}\leqq\bar{q}<\frac{n-1}{2},$$

we next set

$$(2.5)\qquad N(s)=\begin{cases} s^{q(n,p)} & \text{if } p_0(n)<p<\dfrac{2}{n-1}(\bar{q}+\dfrac{n+1}{2}), \\ s^{\bar{q}} & \text{if } p\geqq\dfrac{2}{n-1}\left(\bar{q}+\dfrac{n+1}{2}\right). \end{cases}$$

We here give some remarks on the above norms and relations between p, $q(n,p)$ and $\bar{q}$. First of all, since

$$\frac{2}{n-1}\left(\bar{q}+\frac{n+1}{2}\right)<\frac{2n}{n-1}$$

and

$$(2.6)\qquad \begin{aligned} &\bar{q}\leqq q(n,p) && \text{if and only if } p\geqq\frac{2}{n-1}\left(\bar{q}+\frac{n+1}{2}\right), \\ &\frac{n-1}{2}<q(n,p) && \text{if and only if } p>\frac{2n}{n-1}, \end{aligned}$$

we know that the norm (2.1) with (2.5) is weaker than that with (2,3). Next, the factor $(t+r+2k)^{(n-1)/2}$ in (2.1) indicates the decay rate of a solution v_0 to (1.5) in its support and $N((t-r+2k)/k)$ is closely related to the decay rate of v_0 inside of the solid characteristic cone $\{(x, t)\in\mathbf{R}^n\times[0, \infty) : r<t-k\}$. Finally, since (1.9) and (1, 11) imply $q(n, p)>q(n, p_0(n))=1/p_0(n)$ for $p>p_0(n)$, we know that

(2.7) $\quad pq(n, p)>1,\ pq>1$ if $p>p_0(n),\ q\geqq 1/p_0(n)$.

For each $j=1, 2$ let X_j be a Banach space defined by

(2.8) $\quad X_j=\{u\in C^0(\mathbf{R}^n\times[0, \infty)) : \operatorname{supp} u\subset\{(x, t)\in\mathbf{R}^n\times[0, \infty) : r\leqq t+k\},\ \|D_x^\alpha u\|<\infty \text{ for } |\alpha|\leqq j\}$

equipped with a norm $\|u\|_{X_j}=\sum_{|\alpha|\leqq j}\|D_x^\alpha u\|$. Here we use the usual notations;

(2.9) $\quad D_x^\alpha u=D_1^{\alpha_1}\cdots D_n^{\alpha_n}u.\quad \alpha=(\alpha_1, \cdots, \alpha_n),\ \alpha_j\geqq 0,$
$D_j=\partial/\partial x_j$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$.

We also define a function space $C^{l+\delta}(\mathbf{R})$ consisting all functions of class C^l with Hörder exponent $\delta(0<\delta<1)$.

In order to show the global existence of C^2-solutions to (1.2), we require the following hypothesis $(H)_2$ instead of $(H)_1$ stated in the section 1.

$(H)_2$ $\quad F(s)\in C^{2+\delta}(\mathbf{R})$ and there exist positive constants p and A such that $p_0(n)<p<2+\delta$ and $|F^{(j)}(s)|\leqq A|s|^{p-j}$ for $|s|\leqq 1,\ 0\leqq j\leqq p$.

Note that a typical example $F(s)=s^2$ for $n\geqq 5$ satisfies the above hypothesis.

Now we state our theorems, the first of which will be proved in section 5.

THEOREM 1. *Assume the hypothesis $(H)_j$, where $j=1$ or 2. Then the integral equation (1.2) is uniquely and globally solvable in X_j, provied $v\in X_j$ and $\|v\|$ does not exceed a certain positive number which depends on A, k, n, p and $\bar{q}$.*

In the proof of the theorem the following a priori estimate proved in section 4 will play an essential role.

LEMMA 2.1. *Let L be the linear integral operator defined by (1.3). Assume that $u\in C^0(\mathbf{R}^n\times[0, \infty))$ with* supp $u\subset\{(x, t)\in\mathbf{R}^n\times[0, \infty) : |x|\leqq t+k\}$ *and $\|u\|<\infty$. Then there exists a positive constant C depending only on n, p and $\bar{q}$ such that*

(BE) $\quad \|L(|u|^p)\|\leqq Ck^2\|u\|^p$ *if* $p>p_0(n)$.

REMARK 2.1. When n is even, the basic estimate (BE) does not hold, if $N(s)=s^q$ and $q>(n-1)/2$. For details see Appendix at the end of this paper. Besides, if $n=3$ then (BE) coincides in essence with (50a) in John [5].

Next, for the global in time existence of solutions to the nonlinear wave equation (1.7), we will prove in section 3 the following.

THEOREM 2. *Assume that* $f\in C_0^{m+3}(\mathbf{R}^n)$, $g\in C_0^{m+2}(\mathbf{R}^n)$ *and supports of* f *and* g *are contained in* $\{x\in\mathbf{R}^n:|x|\leqq k\}$. *Furthermore, assume that* $F\in C^{m+1}(\mathbf{R})$ *and* F *satisfies the inequality in* $(H)_2$. *Let the norm* (2.1) *be given by* (2.5) *with* $\bar{q}=(n-3)/2$ *in even space dimensions. Then there exists a unique solution* $u\in X_2$ *to the Cauchy problem* (1.7) *provided* $|D_x^\alpha f|(|\alpha|\leqq m+1)$, $|D_x^\beta g|$ $(|\beta|\leqq m)$ *and* $|D_x^\gamma F(f)|(|\gamma|\leqq m-1)$ *are sufficiently small.*

§ 3. Proof of Theorem 2

Before proving Theorem 1 and Lemma 2.1, we shall give an example of v which satisfies the assumption of Theorem 1 for $j=2$. That is v_0, a unique solution to the Cauchy problem for a linear wave equation (1.5). Moreover, we will show in this section that the solution in X_2 to the integral equation (1.2) for $v=v_0$ is a solution to the Cauchy problem for the nonlinear wave equation (1.7). Since Theorem 2 readily follows from Theorem 1, Proposition 3.1 and 3.2 below, we shall concentrate to prove the propositions.

PROPOSITION 3.1. *Let* v_0 *be a unique solution to* (1.5) *with* $f\in C_0^{m+3}(\mathbf{R}^n)$, $g\in C_0^{m+2}(\mathbf{R}^n)$ *and*

$$\operatorname{supp} f,\ \operatorname{supp} g\subset\{x\in\mathbf{R}^n:|x|\leqq k\}.$$

Moreover, let $F\in C^{m+1}(\mathbf{R})$. *Then* v_0 *satisfies the assumptions on* v *for* $j=2$ *in Theorem* 1, *provided* $|D_x^\alpha f(x)|(|\alpha|\leqq m+1)$, $|D_x^\beta g(x)|(|\beta|\leqq m)$ *and* $|D_x^\gamma F(f)|(|\gamma|\leqq m-1)$ *are sufficiently small. Especially,* $v_0\in X_2$ *for the norm* (2.1) *given by* (2.5) *with* $\bar{q}=(n-3)/2$ *in the even space dimensions.*

PROPOSITION 3.2. *Let* v_0 *be a solution of class* C^2 *to* (1.5) *with* $F\in C^2(\mathbf{R})$. *Then a solution of class* C^2 *to* (1.2) *for* $v=v_0$ *is a unique solution to* (1.7).

Now, it is well known that there exists a unique solution v_0 to the Cauchy problem (1.5) by general theory for linear wave equation. So we here study becay rates of v_0. To this end, we shall derive an explicit expression for v_0. We first write v_0 in the form

(3. 1) $v_0=u_0+w$ in $\mathbf{R}^n\times[0,\infty)$.

Here u_0 is a unique solution to the Cauchy problem for the homogeneous linear wave equation

(3. 2) $$\begin{cases}\partial_t^2u_0-\Delta u_0=0 \text{ in } \mathbf{R}^n\times(0,\infty)\\ u_0(x,0)=f(x),\ \partial_tu_0(x,0)=g(x),\ x\in\mathbf{R}^n,\end{cases}$$

and w is a unique solution to the inhomogeneous problem

(3. 3) $$\begin{cases}\partial_t^2w-\Delta w=G \text{ in } \mathbf{R}^n\times(0,\infty)\\ w(x,0)=\partial_tw(x,0)=0,\ x\in\mathbf{R}^n,\end{cases}$$

where G is defined by (1. 6). As is known, u_0 is expressed in the form

(3. 4) $$u_0(x,t)=\sum_{i=0}^{m}f_it^i\partial_t^iM(f|x,t)+\sum_{i=0}^{m-1}g_it^{i+1}\partial_t^iM(g|x,t),$$

where f_i and g_i are constants depending only on n. For instance, see R. Courant and D. Hilbert [1], pp. 688-690, also see (3. 22) below. For w, we have

LEMMA 3. 1. *Let w be a unique solution to* (3. 3). *Then w is expressed in the form*

(3. 5) $$w(x,t)=\sum_{i=2}^{m}w_it^i\partial_t^{i-2}M(F(f)|x,t),$$

where w_i are positive constants depending only on n.

PROOF. We note that $M(F(f)|x,t)$ satisfies the Darboux equation

(3. 6) $$\partial_t^2M+2mt^{-1}\partial_tM=\Delta M.$$

For instance, see F. John [4], p. 97. Using this equation, we get

(3. 7) $$(\partial_t^2-\Delta)(t^i\partial_t^{i-2}M)=i(i-1)M_{i-2}+2iM_{i-1}-2m\sum_{j=0}^{i-2}(1-)^{i-2-j}\frac{(i-2)!}{j!}M_{j+1},$$

where $M_{i-2}=t^{i-2}\partial_t^{i-2}M$ for $i=2,\cdots,m$. Thus, by comparing the coefficients of M_{i-2} in each side of (3. 3), we see the following facts. Let $m=2$. Then we get $2w_2=2(2-1)A_n$, where A_n is defined by (1.4). Hence $w_2=A_n$.

Let $m=3$. Then we get

$$[3(3-1)+2\cdot3(3-2)]w_3-2w_2=0,$$
$$2w_2=2(3-1)A_n.$$

Hence $w_2=2A_n$, $w_3=3^{-1}A_n$.

Next, let $m\geqq 4$. It follows from (3.5) and (3.7) that

$$(\partial_t^2-\Delta)w=\sum_{i=2}^{m}w_i[i(i-1)M_{i-2}+2iM_{i-1}-2m\sum_{j=0}^{i-2}(-1)^{i-2-j}\frac{(i-2)!}{j!}M_{j+1}].$$

Using this equality, we regard (3.3) as an identity for M_{i-2} ($i=2, \cdots, m$). Then we get the following system of $(m-1)$-equations with respect to w_i.

$(3.8)_{m-2}$ $[m(m-1)+2m(m-2)]w_m+[2(m-1)-2m]w_{m-1}=0,$

$(3.8)_{m-3}$ $[-2m(m-2)(m-3)]w_m$
$+[(m-1)(m-2)-2m(m-3)]w_{m-1}$
$+[2(m-2)-2m]w_{m-2}=0,$

$(3.8)_{m-4}$ $[2m(m-2)(m-3)(m-4)]w_m+[-2m(m-3)(m-4)]w_{m-1}$
$+[(m-2)(m-3)+2m(m-4)]w_{m-2}$
$+[2(m-3)-2m]w_{n-3}=0,$

$\cdots\cdots,$

$(3.8)_2$ $[-2m(-1)^{m-3}(m-2)!]w_m+[-2m(-1)^{m-4}(m-3)!]w_{m-1}$
$+\cdots+(-12m)w_5+(4\cdot 3+4m)w_4+(6-2m)w_3=0,$

$(3.8)_1$ $[-2m(-1)^{m-2}(m-2)!]w_m+[-2m(-1)^{m-3}(m-3)!]w_{m-1}$
$+\cdots+(-4m)w_4+(3\cdot 2+2m)w_3+(4-2m)w_2=0,$

$(3.8)_0$ $2w_2=2(m-1)A_n.$

Note that the left hand side of $(3.8)_{i-2}$ coincides with the coefficient of M_{i-2} in $(\partial_t^2-\Delta)w$ for each $i=2, \cdots, m$. Moreover, in this system, we first add $(m-3)$ times $(3.8)_{m-2}$ to $(3.8)_{m-3}$, next $(m-4)$ times $(3.8)_{m-3}$ to $(3.8)_{m-4}$, and so on. Finally we add one times $(3.8)_2$ to $(3.8)_1$. Thus we obtain the following system of $(m-1)$-equations with respect to $w_2, \cdots, w_m$;

$$\begin{aligned}
&[m(m-1)+2m(m-2)]w_m=2w_{m-1},\\
&m(m-1)(m-3)w_m\\
&+[(m-1)(m-2)+(2m-2)(m-3)]w_{m-1}=4w_{m-2},\\
&(m-1)(m-2)(m-4)w_{m-1}\\
&+[(m-2)(m-3)+(2m-4)(m-4)]w_{m-2}=6w_{m-3},\\
&\cdots\cdots\cdots\\
&5\cdot 4\cdot 2w_5+[4\cdot 3+8\cdot 2]w_4=(2m-6)w_3,\\
&4\cdot 3\cdot 1w_4+[3\cdot 2+6\cdot 1]w_3=(2m-4)w_2,\\
&w_2=(m-1)A_n.
\end{aligned}$$

Since all the coefficients of w_i are positive, we see from the first $(m-2)$-equations that there are positive constants $c_2, \cdots, c_m$ such that $w_i = c_i w_2$ for $i=2, \cdots, m$. Hence we conclude that there exist positive constants w_i $(i=2, \cdots, m)$ which satisfy the equation (3. 3) when w is expressed by (3. 5), as required.

Now, we have already obtained the explicit expression of v_0 given by (3. 1)-(3. 5). Thus we are in a position to estimate v_0. The following estimates are in essence due to S. Klainerman [7] or W. von Wahl [9].

LEMMA 3. 2. *Let f, g and F be as in Proposition* 3. 1. *Then*

$$|u_0(x, t)| \leqq C_1(|f|_{m+1} + |g|_m)(t+1)^{-(n-1)/2} \tag{3.9}$$

and

$$|w(x, t)| \leqq C_2 |F(f)|_{m-1}(t+1)^{-(n-1)/2} \tag{3.10}$$

for $|x| \leqq t+k$ and $t \geqq 0$, where C_1, C_2 are positive constants depending only on n, k, and

$$|\varphi|_i = \sum_{|\alpha| \leqq i} \sup_{x \in \mathbf{R}^n} |D_x^\alpha \varphi(x)|.$$

PROOF. Suppose $t \leqq 2$. Then it immediately follows from (1. 1), (3. 4) and (3. 5) that

$$|u_0(x, t)| \leqq C(|f|_m + |g|_{m-1})$$

and

$$|w(x, t)| \leqq C'|F(f)|_{m-2},$$

which imply (3, 9) and (3. 10).

In what follows, suppose $t \geqq 2$. Then one can derive (3.9) from the results in [7], replacing the L^1-norms by L^∞-norms.

We shall here review briefly Klainerman [7], pp. 52-55. First of all, the following inequality will pkay a key role; Let i, j be nonnegative integers with $j \leqq n-1$ and let $\varphi \in C_0^{i+j+1}(\mathbf{R}^n)$. Then

$$\begin{aligned} \rho^{n-1-j} \int_{|\omega|=1} \left| \left(\frac{d}{d\rho} \right)^i \varphi(x+\rho\omega) \right| dS_\omega \\ \leqq \frac{1}{j!} \|D^{i+j+1}\varphi\|_{L^1} \text{ for } \rho > 0. \end{aligned} \tag{3.11}$$

First, suppose $n=2m+1$ and $\varphi \in C_0^m(\mathbf{R}^n)$. Then it follows from (1. 1) that

$$t^m t^{i+1}\partial_t^i M(\varphi|x,t)=t^{m+i+1}\int_{|\omega|=1}\left(\frac{d}{dt}\right)^i\varphi(x+t\omega)dS_\omega.$$

By virtue of (3.11) with $j=m-1-i$, we therefore get

$$|t^{i+1}\partial_t^i M(\varphi|x,t)|\leqq\frac{1}{(m-1-i)!}\|D^m\varphi\|_{L^1}\cdot t^{-m}$$

for $i=0,\cdots,m-1$. Moreover, if $\varphi\in C_0^{m+1}(\mathbf{R}^n)$, then (3.11) with $i=m$, $j=0$ yields

$$|t^m\partial_t^m M(\varphi|x,t)|\leqq\|D^{m+1}\varphi\|_{L^1}\cdot t^{-m}.$$

Hence, by (3.4), we obtain (3.9), because $\|D^i\varphi\|_{L^1}\leqq C|\varphi|_i$. Similarly, (3.10) follows from (3.5) and (3.11) with $j=m-2-i$ for $i=0,\cdots,m-2$.

Next, suppose $n=2m$ and $\varphi\in C_0^{i+1}(\mathbf{R}^n)$. Then we see from (1.1) that, for each $i=0,\cdots,m$,

$$\begin{aligned}t^{i+1}\partial_t^i M(\varphi|x,t)&=t^{2-n}\int_0^t\frac{\rho^{n-1+i}}{\sqrt{t^2-\rho^2}}d\rho\int_{|\omega|=1}\left(\frac{d}{d\rho}\right)^i\varphi(x+\rho\omega)dS_\omega\\&\equiv I_1+I_2,\end{aligned}$$

where I_1 stands for the integral over $t-1\leqq\rho\leqq t$ and I_2 the one over $0\leqq\rho\leqq t-1$. Since

$$\frac{\rho^i}{\sqrt{t^2-\rho^2}}\leqq\frac{t^i}{\sqrt{2t-1}}\quad\text{for } 0\leqq\rho\leqq t-1,$$

we have

$$|I_2|\leqq t^{(3/2)-n+i}\|D^i\varphi\|_{L^1}.$$

Moreover, (3.11) with $j=0$ yields

$$|I_1|\leqq 2\sqrt{2}\|D^{i+1}\varphi\|_{L^1}\cdot t^{(3/2)-n+i}$$

because $\rho^{n-1+i}\leqq 2t^{i-1}\rho\rho^{n-1}$ for $t-1\leqq\rho\leqq t$, $t\geqq 2$, and

$$\int_{t-1}^t\frac{\rho}{\sqrt{t^2-\rho^2}}d\rho\leqq\sqrt{2t}.$$

Hence we obtain

$$|t^{i+1}\partial_t^i M(\varphi|x,t)|\leqq 2\sqrt{2}(\|D^i\varphi\|_{L^1}+\|D^{i+1}\varphi\|_{L^1})\cdot t^{(3/2)-n+i}$$

for $i=0,\cdots,m$. Therefore we get (3.9) for $n=2m$ by this inequality and (3.4). Analogously, we obtain (3.10) by (3.5). The proof is complete.

In order to prove Proposition 3.1, we also employ the following two lemmas. For convenience, we set

(3.12) $\quad M_0(\varphi|x, t)=t^{2m-1}M(\varphi|x, t).$

LEMMA 3.3. *Let*

(3.13) $\quad r<t-2k$ *and* $t>0$, *where* $r=|x|$.

Suppose that $\varphi\in C_0^0(\mathbf{R}^{2m})$ *and supp* $\varphi\subset\{x\in\mathbf{R}^{2m}: r\leqq k\}$. *Then*

(3.14) $$\left|\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^j M_0(\varphi|x, t)\right|\leqq c_j\|\varphi\|_{L^1}(t+r+2k)^{-(1/2)-j}(t-r+2k)^{-(1/2)-j}$$

for each nonnegative integer j, where

$$c_j=(2j-1)!!2^{(1/2)+j}\cdot 4^{(1/2)+j}$$

with $(-1)!!=1$. *Besides,*

(3.15) $$\left|\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^j M_0(\varphi|x, t)\right|$$
$$\leqq(2j+1)k^{-1}c_j\|\varphi\|_{L^1}(t+r+2k)^{-(1/2)-j}(t-r+2k)^{-(1/2)-j}.$$

PROOF. The definitions of M_0 and M; (3.12) and (1.1) imply

(3.16) $$M_0(\varphi|x, t)=\int_{|x-y|\leqq t}\frac{\varphi(y)}{\sqrt{t^2-|x-y|^2}}dy$$
$$=\int_{|y|\leqq k}\frac{\varphi(y)}{\sqrt{t^2-|x-y|^2}}dy.$$

The second equality follows from (3.13), because

$$t-|x-y|\geqq t-r-k>k \text{ for } y\in\text{supp }\varphi.$$

In view of (3.13) and (3.16), we have

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^j M_0(\varphi|x, t)$$
$$=(-1)^j(2j-1)!!\int_{|y|\leqq k}\varphi(y)(t^2-|x-y|^2)^{-(1/2)-j}dy.$$

Moreover, (3.13) implies that

$$2(t+|x-y|)\geqq 2t>t+r+2k \text{ and}$$
$$4(t-|x-y|)\geqq 4(t-r-k)>t-r+2k \text{ for } |y|\leqq k.$$

Therefore we obtain (3.14). Note that (3.15) follows from

$$\frac{t}{(t+|x-y|)(t-|x-y|)}\leqq\frac{1}{t-r-k}<\frac{1}{k} \text{ for } |y|\leqq k.$$

LEMMA 3.4. *Let f, g and F be as in Proposition* 3.1, *and let w be a unique solution to the Cauchy problem* (3.3). *Then there is a constant C_1 depening only on n such that*

(3.17) $$|w(x, t)| \leqq C_1 \|F(f)\|_{L^1}(t+r+2k)^{-(n-1)/2}(t-r+2k)^{-(n-3)/2}$$

for $n=2m$ whenever (3.13) *holds.*

PROOF. First, we shall express w in terms of $(t^{-1}\partial/\partial t)^j M_0(F(f)|x, t)$, more precisely, show that

(3.18) $$\left(\frac{\partial}{\partial t}\right)^l M(F(f)|x, t) = t^{1-2m-l} \sum_{j=0}^{l} \alpha_j t^{2j} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^j M_0(F(f)|x, t)$$

with some constants α_j, depending only on m and l, such that $\alpha_l=1$. Set $s=t^2$, so that

$$\frac{\partial}{\partial t} = 2\sqrt{s}\frac{\partial}{\partial s}, \quad \frac{\partial}{\partial s} = \frac{1}{2t}\frac{\partial}{\partial t}.$$

Then it follows from (3.12) and (3.16) that

$$M(F(f)|x, t) = s^{-m+(1/2)} \int_{|y| \leqq k} \frac{F(f(y))}{\sqrt{s-|x-y|^2}} dy.$$

Hence we obtain (3.18) easily.

From (3.5) and (3.18) we have

$$\begin{aligned} w(x, t) &= \sum_{i=2}^{m} w_i t^i t^{1-2m-(i-2)} \sum_{j=0}^{i-2} \alpha_j t^{2j} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^j M_0(F(f)|x, t) \\ &= \sum_{i=2}^{m} w_i t^{3-2m} t^{2(i-2)} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{i-2} M_0(F(f)|x, t) \\ &\quad + \sum_{i=3}^{m} w_i t^{3-2m} \sum_{j=0}^{i-3} \alpha_j t^{2j} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^j M_0(F(f)|x, t) \end{aligned}$$

because $\alpha_{i-2}=1$. Hence one can write

(3.19) $$w = w' + w'',$$

where

$$w'(x, t) = w_m \frac{1}{t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-2} M_0(F(f)|x, t)$$

and

$$w''(x, t) = \sum_{j=0}^{m-3} \beta_j t^{3-2m+2j} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^j M_0(F(f)|x, t)$$

with some constants $\beta_0, \cdots, \beta_{m-3}$ depending only on w_i and α_j.

Now, we shall derive (3. 17). It follows from Lemma 3. 3 that

$$|w'(x, t)| \leqq w_m c_{m-2} \|F(f)\|_{L^1} \cdot t^{-1}(t+r+2k)^{(3/2)-m}(t-r+2k)^{(3/2)-m}.$$

Since $2t > t+r+2k$ by (3. 13), we have

$$|w'(x, t)| \leqq 2w_m c_{m-2} \|F(f)\|_{L^1}(t+r+2k)^{(1/2)-m}(t-r+2k)^{(3/2)-m}.$$

Besides,

$$\begin{aligned}|w''(x, t)| &\leqq \sum_{j=0}^{m-3} |\beta_j| t^{3-2m+2j} c_j \|F(f)\|_{L^1} \times \\ &\quad \times (t+r+2k)^{-(1/2)-j}(t-r+2k)^{-(1/2)-j} \\ &\leqq (t+r+2k)^{(1/2)-m}(t-r+2k)^{(3/2)-m} \|F(f)\|_{L^1} \times \\ &\quad \times \sum_{j=0}^{m-3} 2^{2m-3-2j} |\beta_j| c_j \left(\frac{t-r+2k}{t+r+2k}\right)^{m-2-j},\end{aligned}$$

hence we have

$$\begin{aligned}|w''(x, t)| \leqq \tilde{C}_0 \|F(f)\|_{L^1}(t+r+2k)^{(1/2)-m} \times \\ \times (t-r+2k)^{(3/2)-m}\left(\frac{t-r+2k}{t+r+2k}\right).\end{aligned}$$

Therefore by (3. 19) we obtain (3. 17).

PROOF OF PROPOSITION 3. 1. It follows from (3. 4), (3. 5) and (1. 1) that, for each $t \geqq 0$,

$$\operatorname*{supp}_{x \in \mathbf{R}^n} u_0, \ \operatorname*{supp}_{x \in \mathbf{R}^n} w \subset \{x \in \mathbf{R}^n : r \leqq t+k\},$$

in particular, if n is odd, then

$$\operatorname*{supp}_{x \in \mathbf{R}^n} u_0, \ \operatorname*{supp}_{x \in \mathbf{R}^n} w \subset \{x \in \mathbf{R}^n : t-k \leqq r \leqq t+k\},$$

namely, the strong Huygens' principle holds.

We shall first examine $\|v_0\|$. It follows from (3. 1), (3. 9) and (3. 10) that

$$(3.20) \qquad |v_0(x, t)| \leqq (C_1 + C_2)(|f|_{m+1} + |g|_m + |F(f)|_{m-1})(t+1)^{-(n-1)/2}$$

for $r \leqq t+k$ and $t \geqq 0$. Hence, if n is odd, by virtue of (2. 1), (2. 2) and the strong Huygens' principle,

$$(3.21) \qquad \|v_0\| \leqq C(|f|_{m+1} + |g|_m + |F(f)|_{m-1}),$$

where the constant C depends only on n, p and k, because $t-r+2k \leqq 3k$

for $r \geqq t-k$. Thus if $|f|_{m+1}$, $|g|_m$ and $|F(f)|_{m-1}$ are small then so is $\|v_0\|$.

Now let n be even and suppose (3.13) holds. Then we adopt another expression for u_0;

$$u_0(x, t)=\partial_t R(f|x, t)+R(g|x, t), \tag{3.22}$$

where

$$R(\varphi|x, t)=\frac{1}{(2\pi)^m}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}[t^{2m-1}M(\varphi|x, t)].$$

See for instance Courant and Hilbert [1], p. 682. Applying (3.14) and (3.15) with (3.12) to (3.22), we have

$$\begin{aligned}|u_0(x, t)| \leqq c_{m-1}(\|g\|_{L^1}+(2m-1)k^{-1}\|f\|_{L^1})\times \\ \times(t+r+2k)^{(1/2)-m}(t-r+2k)^{(1/2)-m}.\end{aligned}$$

Therefore it follows from (3.18) and (3.1) that

$$|v_0(x, t)| \leqq C(|f|_0+|g|_0+|F(f)|_0)(t+r+2k)^{-(n-3)/2}\ (t-r+2k)^{-(n-3)/2}$$

for $r<t-2k$. Thus, according to (3.20), (2.1) and (2.5) with $\bar{q}=(n-3)/2$, we obtain

$$\|v_0\| \leqq C'(|f|_{m+1}+|g|_m+|F(f)|_{m-1}), \tag{3.23}$$

where the constant C' depends only on n and k, because $t-r+2k \leqq 4k$ for $r \geqq t-2k$. Thus if $\|f\|_{m+1}$, $\|g\|_m$ and $\|F(f)\|_{m-1}$ are small then so is $\|v_0\|$.

Finally we examine $\|D_x^\alpha v_0\|$ for $|\alpha| \leqq 2$. It follows from (1.1) that D_x^α and M commute. Hence the procedure we derived (3.21) and (3.23) yields

$$\|D_x^\alpha v_0\| \leqq (C+C')(|D_x^\alpha f|_{m+1}+|D_x^\alpha g|_m+|D_x^\alpha F(f)|_{m-1}).$$

Thus $v_0 \in X_2$ provided $f \in C_0^{m+3}(\mathbf{R}^n)$, $g \in C_0^{m+2}(\mathbf{R}^n)$ and $F \in C^{m+1}(\mathbf{R})$. The proof is complete.

Proof of Proposition 3.2. Since the usiqueness of solutions to (1.7) follows from Appendix of John [6], it suffices to show that a C^2-solution $u(x, t)$ to (1.2) with $v=v_0$ is a solution to the Cauchy problem (1.7).

From (1.3) we see that u has the same Cauchy data as v_0 and that

$$\begin{aligned}\partial_t^2 L(F(u))(x, t)=A_n M(F(u)|x, 0\,; t) \\ +A_n\int_0^t[2\partial_t M(F(u)|x, t-\tau\,; \tau)+(t-\tau)\partial_t^2 M(F(u)|x, t-\tau\,; \tau)]d\tau.\end{aligned}$$

Moreover, (1.4) yields

$$A_nM(F(u)|x, 0; t)=(2m-1)^{-1}F(u)(x, t).$$

Noting that $M(F(u)|x, t-\tau; \tau)$ satisfies the Darboux equation, like (3. 6),

$$\partial_t^2 M+2m(t-\tau)^{-1}\partial_t M=\Delta M,$$

we thus get

$$\begin{aligned}&(\partial_t^2-\Delta)L(F(u))(x, t)\\&=\frac{1}{2m-1}F(u)(x, t)-(2m-2)A_n\int_0^t \partial_t M(F(u)|x, t-\tau; \tau)d\tau.\end{aligned}$$

Since

$$\begin{aligned}\partial_t M(F(u)|x, t-\tau; \tau)=&M(\partial_t(F(u))|x, t-\tau; \tau)\\&-\partial_\tau M(F(u)|x, t-\tau; \tau)\end{aligned}$$

and

$$\begin{aligned}\int_0^t \partial_\tau M(F(u)|x, t-\tau; \tau)d\tau&=M(F(u)|x, 0; t)-M(F(u)|x, t; 0)\\&=[(2m-1)A_n]^{-1}F(u)(x, t)-M(F(f)|x, t),\end{aligned}$$

we obtain

$$\begin{aligned}&(\partial_t^2-\Delta)u(x, t)\\&=(\partial_t^2-\Delta)[v_0(x, t)+L(F(u))(x, t)]\\&=G(x, t)+\frac{1}{2m-1}F(u)(x, t)\\&-(2m-2)A_n[\frac{H(x, t)}{(2m-2)A_n}-\frac{F(u)(x, t)}{(2m-1)A_n}+M(F(f)|x, t)]\\&=G(x, t)+F(u)(x, t)-H(x, t)-(2m-2)A_nM(F(f)|x, t).\end{aligned}$$

After all, (1. 6) yields (1. 7). The proof is completed.

§ 4. Proof of Lemma 2. 1

Throughout this section we assume $n\geqq 3$ and adopt the following form $N(s)=s^q$ with appropriate q for the function N in (2. 1), unless n is even and $p=2n/(n-1)$. For odd n we set $q=q(n, p)$. When n is even, we take a number q such that

$$1/p_0(n)\leqq q\leqq \min\{q(n, p), (n-1)/2\}$$

in each case. We also suppose that $r\leqq t+k$ and $u(x, t)$ is such a function as stated in Lemma 2. 1. Then it follows from (1. 3) and (1. 1) that $L(|u|^p)$ also has the same properties as u except for $\|L(|u|^p)\|<\infty$.

Besides, we denote various constants depending only on n, p and q by C or C'.

Let n be odd. Then it follows from (1.3), (1.1), (2.1) and (2.2) that

$$|L(|u|^p)(x, t)| \leqq A_n \|u\|^p k^{((n-1)p/2)+pq} J(x, t), \tag{4.1}$$

where

$$J(x, t) = \int_0^t (t-\tau)d\tau \int_{|\omega|=1} b(|x+(t-\tau)\omega|, \tau)dS_\omega \tag{4.2}$$

and

$$b(\lambda, \tau) = (\tau+\lambda+2k)^{-(n-1)p/2}(\tau-\lambda+2k)^{-pq} \tag{4.3}$$

$$\text{for } 0 \leqq \lambda \leqq \tau+k, \text{ and } b(\lambda, \tau)=0 \text{ for } \lambda > \tau+k \text{ or } \lambda < 0.$$

When n is even, we also obtain similarly (4.1) with

$$J(x, t) = \int_0^t (t-\tau)d\tau \int_{|\xi|\leqq 1} \frac{b(|x+(t-\tau)\xi|, \tau)}{\sqrt{1-|\xi|^2}} d\xi \tag{4.4}$$

unless $p = 2n/(n-1)$.

We shall here employ the following fundamental identity for spherical means.

LEMMA 4.1. *Let* $b(\lambda, \tau)$ *be the function defined by* (4.3). *Let* $\tau \geqq 0$ *and* $\rho > 0$. *Then*

$$\begin{aligned} &\int_{|\omega|=1} b(|x+\rho\omega|, \tau)dS_\omega \\ &= 2^{3-n}\omega_{n-1}(r\rho)^{2-n} \int_{|\rho-r|}^{\rho+r} \lambda b(\lambda, \tau)h(\lambda, \rho, r)d\lambda, \end{aligned} \tag{4.5}$$

where

$$h(\lambda, \rho, r) = \{\lambda^2-(\rho-r)^2\}^{(n-3)/2}\{(\rho+r)^2-\lambda^2\}^{(n-3)/2}. \tag{4.6}$$

PROOF. Since $|x+\rho\omega| \geqq |\rho-r|$, if $|\rho-r| \geqq \tau+k$ then it follows from (4.3) that each side of (4.5) vanishes. Hence we suppose

$$|\rho-r| < \tau+k$$

and set

$$\begin{aligned} &g(s) = b(\sqrt{r^2+\rho^2+s}, \tau) \text{ for } s_1 \leqq s \leqq s_2, \\ &g(s) = 0 \text{ for } s < s_1 \text{ or } s > s_2, \\ &\text{where } s_1 = -2r\rho \text{ and } s_2 = (\tau+k)^2-(r+\rho)^2. \end{aligned} \tag{4.7}$$

We here want to apply (1.12). Although it is assumed in [4] that $g(s)$ is a continuous function, we find that (1.12) is still valid for the function defined by (4.7) provided $n\geqq 3$. For, the $g(s)$ can be approximated by a decreasing sequence of uniformly bounded continuous functions supported in the interval $[s_1-1, s_2+1]$.

We are now in a position to prove (4.5). Since

$$|x+\rho\omega|=\sqrt{r^2+\rho^2+2\rho x\cdot\omega},$$

it follows from (4.7) that

$$b(|x+\rho\omega|, \tau)=g(2\rho x\cdot\omega).$$

Therefore, setting $y=2\rho x$ in (1.12), we have

$$\int_{|\omega|=1} b(|x+\rho\omega|, \tau)dS_\omega$$
$$=\omega_{n-1}\int_{-1}^{1}(1-\eta^2)^{(n-3)/2}b(\sqrt{r^2+\rho^2+2\rho r\eta},\ \tau)d\eta.$$

Moreover we introduce a variable of integration λ instead of η by

$$\lambda=\sqrt{r^2+\rho^2+2\rho r\eta},$$

as in [4], p. 80. Then, since

$$\frac{d\lambda}{d\eta}=\frac{r\rho}{\lambda}$$

and

$$1-\eta^2=\frac{(\lambda^2-(\rho-r)^2)((\rho+r)^2-\lambda^2)}{4r^2\rho^2},$$

we obtain (4.5). The proof is complete.

For $h(\lambda, \rho, r)$ we will use only the following three estimates.

LEMMA 4.2. *Let $h(\lambda, \rho, r)$ be the function defined by* (4.6). *Suppose that $|\rho-r|\leqq\lambda\leqq\rho+r$ and $\rho\geqq 0$. Then*

(4.8) $|\lambda-r|\leqq\rho\leqq\lambda+r,$

(4.9) $h(\lambda, \rho, r)\leqq 4^{n-3}r^{n-3}\lambda^{n-3},$

(4.10) $h(\lambda, \rho, r)\leqq 2^{n-3}\rho^{n-3}r^{(n-3)/2}\lambda^{(n-3)/2}$

and

(4.11) $h(\lambda, \rho, r)\leqq 8^{n-3}\rho^{n-3}r^{n-3}.$

PROOF. By assumption we get $\rho-r\leqq\lambda\leqq\rho+r$ and $r-\rho\leqq\lambda$, which imply (4.8). Moreover (4.6) can be rewritten as

$$h(\lambda, \rho, r)=\{\rho^2-(\lambda-r)^2\}^{(n-3)/2}\{(\lambda+r)^2-\rho^2\}^{(n-3)/2}.$$

Since

$$\rho^2-(\lambda-r)^2\leqq(\lambda+r)^2-(\lambda-r)^2=4\lambda r$$

and

$$(\lambda+r)^2-\rho^2\leqq(\lambda+r)^2-(\lambda-r)^2,$$

we have (4.9). In addition,

$$\rho^2-(\lambda-r)^2\leqq\rho^2$$

yields (4.10). Moreover $\lambda\leqq\rho+r$ implies that $\lambda\leqq 2\rho$ or $\lambda\leqq 2r$. If $\lambda\leqq 2\rho$ then (4.11) follows from (4.9). If $\lambda\leqq 2r$ then (4.10) gives

$$h(\lambda, \rho, r)\leqq(2\sqrt{2})^{n-3}\rho^{n-3}r^{n-3},$$

which implies (4.11). The proof is complete.

Now, suppose n is odd. By virtue of (4.2) and Lemma 4.1 we then get

$$J(x, t)=2^{3-n}\omega_{n-1}I(r, t),$$

where

$$I(r, t)=r^{2-n}\int_0^t(t-\tau)^{3-n}d\tau\int_{|t-\tau-r|}^{t-\tau+r}\lambda b(\lambda, \tau)h(\lambda, t-\tau, r)d\lambda, \tag{4.12}$$

so that (4.1) can be written as

$$|L(|u|^p)(x, t)|\leqq A_n 2^{3-n}\omega_{n-1}\|u\|^p k^{((n-1)p/2)+pq}I(r, t). \tag{4.13}$$

Next suppose n is even. Then, changing variables $(t-\tau)\xi=y-x$ and switching to polar coordinates $y-x=\rho\omega$, $|\omega|=1$, we have similarly from (4.4)

$$J(x, t)=2^{3-n}\omega_{n-1}I(r, t),$$

where

$$I(r, t)=r^{2-n}\int_0^t(t-\tau)^{2-n}d\tau\times \tag{4.14}$$
$$\times\int_0^{t-r}\frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}}d\rho\int_{|\rho-r|}^{\rho+r}\lambda b(\lambda, \tau)h(\lambda, \rho, r)d\lambda$$

with $b(\lambda, \tau)$ and $h(\lambda, \rho, r)$ defined by (4.3) and (4.6) respectively. Moreover (4.13) is still valid unless the function $N(s)$ in (2.1) is given by (2.3) with $p=2n/(n-1)$.

Furthermore, inverting the order of the (ρ, λ)-integral in (4.14), we find from (4.8) that

$$I(r, t)=I_1(r, t)+I_2(r, t),$$

where

$$(4.15)\qquad I_1(r, t)=r^{2-n}\int_0^t(t-\tau)^{2-n}d\tau\int_{|t-r-\tau|}^{t+r-\tau}\lambda b(\lambda, \tau)d\lambda \times\int_{|\lambda-r|}^{t-\tau}\frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2-\rho^2}}d\rho$$

and

$$(4.16)\qquad I_2(r, t)=r^{2-n}\int_0^{(t-r)_+}(t-\tau)^{2-n}d\tau\int_0^{t-r-\tau}\lambda b(\lambda, \tau)d\lambda \times\int_{|\lambda-r|}^{\lambda+r}\frac{\rho h(\lambda, \rho, r)}{\sqrt{(t-\tau)^2-\rho^2}}d\rho$$

with $(t-r)_+=\max\{t-r, 0\}$. Therefore (4.1) can be written as

$$(4.17)\qquad |L(|u|^p)(x, t)|\leqq A_n2^{3-n}\|u\|^pk^{((n-1)p/2)+pq}(I_1(r, t)+I_2(r, t)).$$

Consequently we have only to estimate the quantities given by (4.12), (4.15) and (4.16).

From now on we often use for convenience the following notations

$$(4.18)\qquad \alpha=\tau+\lambda,\ \beta=\tau-\lambda,$$

so that (4.3) can be written as

$$(4.19)\qquad b(\lambda, \tau)=\begin{cases}(\alpha+2k)^{-(n-1)p/2}(\beta+2k)^{-pq} & \text{for } \alpha\geqq\beta\geqq-k,\\ 0 & \text{for } \beta<-k \text{ or } \alpha<\beta.\end{cases}$$

First consider the case where n is odd.

LEMMA 4.3. *Let $I(r, t)$ be given by (4.12) with $q=q(n, p)$, where $q(n, p)$ is the number befined by (1.11). Suppose that $n\geqq3$ is odd and $p>p_0(n)$. Then there is a constant C_1, depending only on n and p, such that*

$$(4.20)\qquad I(r, t)\leqq C_1k^{1-pq(n, p)}(t+r+2k)^{-(n-1)/2}(t-r+2k)^{-q(n, p)}$$

for $r\leqq t+k$.

PROOF. First suppose $4r \leqq t+r+2k$, namely,

(4.21) $$t+r+2k \leqq 2(t-r+2k).$$

By (4.11) we have

(4.22) $$I(r, t) \leqq 8^{n-3} \frac{1}{r} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda b(\lambda, \tau) d\lambda.$$

Moreover, changing variables by (4.18), we get, by (4.19),

(4.23) $$I(r, t) \leqq 8^{n-3} \frac{1}{r} \int_{|t-r|}^{t+r} (\alpha+2k)^{1-((n-1)p/2)} d\alpha \int_{-k}^{\infty} (\beta+2k)^{-pq(n,p)} d\beta.$$

Besides, the α-integral is dominated by

$$(|t-r|+2k)^{1-((n-1)p/2)} \int_{t-r}^{t+r} d\alpha,$$

since (1.11) can be written as

(4.24) $$1-\frac{(n-1)p}{2} = -\frac{n-1}{2} - q(n, p).$$

Therefore in virtue of (2.7), (4.21) and (4.23) we obtain (4.20).

Next suppose

(4.25) $$4r \geqq t+r+2k.$$

Then by (4.10) we get

(4.26) $$\begin{aligned} I(r, t) &\leqq 2^{n-3} r^{(1-n)/2} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda^{(n-1)/2} b(\lambda, \tau) d\lambda \\ &\leqq 2^{2n-4} (t+r+2k)^{(1-n)/2} \int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau} \lambda^{(n-1)/2} b(\lambda, \tau) d\lambda. \end{aligned}$$

Moreover, similarly to (4.23), we see from (2.7) that the integral is dominated by

$$Ck^{1-pq(n,p)} \int_{|t-r|}^{t+r} (\alpha+2k)^{(n-1-(n-1)p)/2} d\alpha.$$

Therefore by (4.24) we obtain (4.20). Thus we prove Lemma 4.3.

Next we shall estimate I_1.

LEMMA 4.4. *Let $I_1(r, t)$ be given by (4.15), where q is such a number as stated in the openning of this section. Suppose $n \geqq 4$ is even and $p > p_0(n)$. Then there is a constant C_2, depending only on n, p and q, such that*

(4.27) $\quad I_1(r, t) \leqq C_2 k^{1-pq}(t+r+2k)^{-(n-1)/2}(t-r+2k)^{-q(n, p)}$
for $r \leqq t+k$, *where* $q(n, p)$ *is as in the preceding lemma.*

PROOF. First suppose (4.21) holds. We see from (4.11) that the ρ-integral in (4.15) is dominated by

$$8^{n-3} r^{n-3}(t-\tau)^{n-3} \int_0^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho.$$

Besides,

$$\int_0^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho = t-\tau. \tag{4.28}$$

Hence (4.22) holds with $I=I_1$. Thus we obtain (4.27), as before.

Next suppose (4.25) holds. Then by (4.10) and (4.28) we get (4.26) with $I=I_1$. Hence we obtain (4.27), as before. The proof is complete.

Finally consider I_2.

LEMMA 4.5. *Let* $t>r$ *and* $I_2(r, t)$ *be given by* (4.16). *Suppose* n, p, q *and* $q(n, p)$ *are as in the preceding lemma. Then there is a constant* C_3, *depending only on* n, p *and* q, *such that, for* $r \leqq t+k$,

$$I_2(r, t) \leqq C_3 k^{1-pq}(t+r+2k)^{-(n-1)/2} \times (t-r+2k)^{-q(n, p)} \quad \text{if } p<2n/(n-1), \tag{4.29$_1$}$$

$$I_2(r, t) \leqq C_3 k^{1-pq}(t+r+2k)^{-(n-1)/2} \times (t-r+2k)^{-(n-1)/2}\left(1+\log\frac{t-r+2k}{2k}\right) \quad \text{if } p=2n/(n-1) \tag{4.29$_2$}$$

and

$$I_2(r, t) \leqq C_3 k^{1-pq-q(n, p)+((n-1)/2)} \times (t+r+2k)^{-(n-1)/2}(t-r+2k)^{-(n-1)/2} \quad \text{if } p>2n/(n-1). \tag{4.29$_3$}$$

REMARK. When $p \geqq 2n/(n-1)$, it follows from (1.11) that $q(n, p) \geqq (n-1)/2$ hence (4.29)$_1$ implies (4.29)$_2$ and (4.29)$_3$.

PROOF OF LEMMA 4.5. First we shall show

$$I_2(r, t) \leqq C k^{1-pq-q(n, p)}(t+r+2k)^{-(n-1)/2} \tag{4.30}$$

for $0<t-r \leqq k$. Since

$$I_2(r, t) \leqq I(r, t) = J(x, t)/(2^{3-n} w_{n-1}),$$

if $r \leqq t+k \leqq 3k$, we see from (4.3) and (4.4) that

$$I_2(r, t) \leqq Ck^{2-((n-1))p/2)-pq}.$$

Hence by (4.24) we obtain (4.30). Next suppose $t \geqq 2k$ and $0 < t-r \leqq k$. Then (4.30) is equivalent to

$$(4.31) \qquad I_2(r, t) \leqq Ck^{1-pq-q(n,p)} r^{(1-n)/2},$$

since $r \geqq k$ for such r, t. It follows from (4.10), (4.3) and (4.24) that

$$I_2(r, t) \leqq 2^{n-3}(2k)^{-1-q(n,p)} k^{-pq} r^{(1-n)/2} \times \int_0^k (t-\tau)^{-1} d\tau \int_0^{\tau+k} d\lambda \int_0^{t-\tau} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho.$$

Hence by (4.28) we obtain (4.31).

In what follows we assume

$$(4.32) \qquad t - r \geqq k.$$

First we shall prove

$$(4.33) \qquad I_2(r, t) \leqq Ck^{1-pq}(t+r+2k)^{(3/2)-n} \times \int_0^{t-r} (\alpha+2k)^{(n-1)(2-p)/2} (t-r-\alpha)^{-1/2} d\alpha.$$

It follows from (4.9) and (4.16) that

$$I_2(r, t) \leqq 4^{n-3} \frac{1}{r} \int_0^{t-r} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{n-2} b(\lambda, \tau) d\lambda \times \int_{|\lambda-r|}^{\lambda+r} \frac{\rho}{\sqrt{(t-\tau)^2-\rho^2}} d\rho.$$

Moreover, since

$$(t-\tau)^2-\rho^2 \geqq (t-\tau)^2-(\lambda+r)^2 = (t-\tau-\lambda-r)(t-\tau+\lambda+r)$$

and

$$\int_{|\lambda-r|}^{\lambda+r} \rho d\rho = 2r\lambda,$$

we have

$$I_2(r, t) \leqq 4^{n-3} 2 \int_0^{t-r} (t-\tau)^{2-n} d\tau \int_0^{t-r-\tau} \lambda^{n-1} b(\lambda, \tau) \times (t-r-\tau-\lambda)^{-1/2} (t+r-\tau+\lambda)^{-1/2} d\lambda.$$

Noting that $4(t-\tau)\geqq t+r+2k$ for $\tau\leqq t-r$ with (4.25) and that $6(t-\tau)\geqq t+r+2k$ for $\tau\leqq(t-r)/2$ with (4.32), we therefore obtain

$$(4.34)\qquad I_2(r, t)\leqq C(t+r+2k)^{2-n}I_3(r, t)+4^{n-3}2I_4(r, t),$$

where

$$(4.35)\qquad I_3(r, t)=\int_0^{t-\tau}d\tau\int_0^{t-r-\tau}\lambda^{n-1}b(\lambda, \tau)\times(t-r-\tau-\lambda)^{-1/2}(t+r-\tau+\lambda)^{-1/2}d\lambda$$

and $I_4(r, t)=0$ for $4r\geqq t+r+2k$,

$$(4.36)\qquad I_4(r, t)=\int_{(t-r)/2}^{t-r}d\tau\int_0^{t-r-\tau}\lambda b(\lambda, \tau)\times(t-r-\tau-\lambda)^{-1/2}(t+r-\tau+\lambda)^{-1/2}d\lambda \qquad \text{for } 4r\leqq t+r+2k.$$

because $\lambda^{n-2}\leqq(t-\tau)^{n-2}$ for $0\leqq\lambda\leqq t-\tau$.

First consider I_3. Changing variables by (4.18), we see from (4.19) and (4.35) that

$$(4.37)\qquad I_3(r, t)\leqq\int_0^{t-r}(\alpha+2k)^{(n-1)(2-p)/2}\times(t-r-\alpha)^{-1/2}d\alpha\int_{-k}^{\alpha}(\beta+2k)^{-pq}(t+r-\beta)^{-1/2}d\beta.$$

Furthermore we find that

$$(4.38)\qquad \int_{-k}^{\alpha}(\beta+2k)^{-pq}(t+r-\beta)^{-1/2}d\beta \leqq Ck^{1-pq}(t+r+2k)^{-1/2} \text{ for } 0\leqq\alpha\leqq t-r.$$

In fact, dividing the integral as

$$\int_{-k}^{\alpha/2}d\beta+\int_{\alpha/2}^{\alpha}d\beta,$$

we see that the left hand side of (4.38) is dominated by

$$\left(t+r-\frac{\alpha}{2}\right)^{-1/2}\int_{-k}^{\infty}(\beta+2k)^{-pq}d\beta+\left(\frac{\alpha}{2}+2k\right)^{-pq}2\frac{\alpha}{2}\left(t+r-\frac{\alpha}{2}\right)^{-1/2}$$

Hence (4.38) follows from (2.7) and (4.32).

Next consider I_4. Similarly to (4.37) we have, from (4.36) and (4.38),

(4.39) $$I_4(r,t)\leqq Ck^{1-pq}(t+r+2k)^{(3/2)-n}\times\int_0^{t-r}(a+2k)^{(n-1)(2-p)/2}(t-r-a)^{-1/2}da$$
$$\text{for } 4r\leqq t+r+2k,$$

since (4.21) implies

$$(a+2k)^{2-n}\leqq 4^{n-2}(t+r+2k)^{2-n}\quad\text{for } a\geqq(t-r)/2.$$

Consequently we obtain (4.33) by virtue of (4.34), (4.37), (4.38) and (4.39).

We are now in a position to prove (4.29). Dividing the integral in (4.33) as $\int_{(t-r)/2}^{t-r}da+\int_0^{(t-r)/2}da$, by (4.24) we have

$$\begin{aligned}I_2(r,t)\leqq{}& Ck^{1-pq}(t+r+2k)^{(1-n)/2}\\&\times\int_{(t-r)/2}^{t-r}(a+2k)^{-q(n,p)-(1/2)}(t-r-a)^{-1/2}da\\&+Ck^{1-pq}(t+r+2k)^{(3/2)-n}(t-r)^{-1/2}\int_0^{t-r}(a+2k)^{(n-1)(2-p)/2}da.\end{aligned}$$

Hence by (4.32) we get

$$\begin{aligned}I_2(r,t)\leqq{}& Ck^{1-pq}(t+r+2k)^{(1-n)/2}(t-r+2k)^{-q(n,p)}\\&+Ck^{1-pq}(t+r+2k)^{(1-n)/2}(t-r+2k)^{(1-n)/2}\\&\times\int_0^{t-r}(a+2k)^{(n-1)(2-p)/2}da.\end{aligned}$$

Therefore by (4.24) and (4.30) we obtain (4.29), noting that condition $(n-1)p>2n$ is equivalent to $(n-1)(2-p)<-2$. Thus we prove Lemma 4.5.

Proof of Lemma 2.1. If n is odd, then (BE) follows from (2.1), (2.2), (4.13) with $q=q(n,p)$ and (4.20), where $C=A_n2^{3-n}\omega_{n-1}C_1$, because (1.11) implies

(4.40) $$\frac{n-1}{2}p+1-\frac{n-1}{2}-q(n,p)=2.$$

From now on we assume n is even. We first deal with the case where the norm in (BE) is given by (2.3). If $p_0(n)<p<2n/(n-1)$, from (4.17), (4.27) and $(4.29)_1$ we obtain (BE) as above, taking $q=q(n,p)$.

Next suppose $p>2n/(n-1)$. Then, since $t-r+2k\geqq k$ and (2.6) implies $(n-1)/2<q(n,p)$, it follows from (4.17), (4.27) and $(4.29)_3$ that

$$|L(|u|^p)(x, t)| \leqq C\|u\|^p k^{1+((n-1)p/2)-q(n, p)+((n-1)/2)} \times (t+r+2k)^{-(n-1)/2}(t-r+2k)^{-(n-1)/2}.$$

Hence by (4. 40) we obtain (BE), taking $q=(n-1)/2$.

Finally suppose

$$p=2n/(n-1),$$

so that $q(n, p)=(n-1)/2$. We shall then prove (BE) with a weaker norm of u instead of $\|u\|$ on the right hand side. Set

$$\|u\|_q=\sup_{\substack{x\in\mathbf{R}^n\\ t\geqq 0}}\left[\left(\frac{t+r+2k}{k}\right)^{(n-1)/2}\left(\frac{t-r+2k}{k}\right)^q|u(x, t)|\right],$$

where q is a number satisfying (2. 4), say, $q=1/p_0(n)$. Then it follows from (4. 17), (4. 27) and $(4.29)_2$ that

(4. 41) $$|L(|u|^p)(x, t)| \leqq C_4(\|u\|_q)^p k^{1+((n-1)p/2)}(t+r+2k)^{-(n-1)/2} \times(t-r+2k)^{-(n-1)/2}\left(1+\log\frac{t-r+2k}{2k}\right)$$

for $r\leqq t+k$, where $q=1/p_0(n)$ and C_4 is a constant depending only on n, p and q. Moreover, since

$$1+\log\frac{t-r+2k}{2k}\leqq\frac{1}{\log 2}\log\left(1+\frac{t-r+2k}{k}\right),$$

by (2. 1), (2. 3) and (4. 40) we get

$$\|L(|u|^p)\|\leqq\frac{C_4}{\log 2}k^2(\|u\|_q)^p.$$

Besides, since $p_0(n)>2/(n-1)$, we have

$$s^{1/p_0(n)}\leqq C_5 s^{(n-1)/2}/\log(1+s) \text{ for } s\geqq 1$$

with some constnat C_5 hence $\|u\|_q\leqq C_5\|u\|$. Thus we obtain (BE) for $p=2n/(n-1)$.

We next deal with the case where the norm is given by (2. 5). If

$$p_0(n)<p<\frac{2}{n-1}\left(\bar{q}+\frac{n+1}{2}\right),$$

then, since $\bar{q}<(n-1)/2$, we have $p<2n/(n-1)$. Hence the present case is a part of the preceding one.

From now on we assume

$$p\geqq\frac{2}{n-1}\left(\bar{q}+\frac{n+1}{2}\right).$$

Then (2.6) imlies

$$\bar{q}\leqq q(n,p).$$

Therefore, if $p<2n/(n-1)$, it follows from (4.17), (4.27) and $(4.29)_1$ with $q=\bar{q}$ that

$$(4.42)\quad |L(|u|^p)(x,t)|\left(\frac{t+r+2k}{k}\right)^{(n-1)/2}\left(\frac{t-r+2k}{k}\right)^{\bar{q}}$$
$$\leqq C\|u\|^p(t-r+2k)^{\bar{q}-q(n,p)}k^{((n-1)p/2)+1-((n-1)/2)-\bar{q}}$$
$$\leqq C\|u\|^p k^{((n-1)p/2)+1-((n-1)/2)-q(n,p)},$$

because $t-r+2k\geqq k$. By (4.40) we thus obtain (BE).

Next suppose

$$p>2n/(n-1).$$

Then, using $(4.29)_3$ instead of $(4.29)_1$, one can dominate analogously the left hand side of (4.42) by

$$C\|u\|^p(t-r+2k)^{\bar{q}-((n-1)/2)}k^{((n-1)p/2)+1-q(n,p)-\bar{q}}$$
$$\leqq C\|u\|^p k^{((n-1)p/2)+1-((n-1)/2)-q(n,p)},$$

because $\bar{q}<(n-1)/2$. Hence we get (BE) as above.

Finally suppose $p=2n/(n-1)$. Then (4.41) is still valid with q replaced by $\bar{q}$. Therefore by (2.1), (2.5) and (4.40) we obtain (BE), because now $\|u\|=\|u\|_{\bar{q}}$ and $s^{\bar{q}-((n-1)/2)}(1+\log s)$ is bounded for $s\geqq 1$. Thus we prove Lemma 2.1.

§ 5. Proof of Theorem 1

In this section, we shall construct the solution to the integral equation (1.2) in $X_j (j=1.2)$ by employing Lemma 2.1 and the classical iteration method by Picard;

$$(5.1)\quad \begin{cases} u_0=v \\ u_l=v+L(F(u_{l-1})) \text{ for } l\in\mathbf{N}. \end{cases}$$

Throughout the present section we fix j as $j=1$ or $j=2$, unless otherwise stated. We divide the proof, which claims that the sequence $\{u_l\}_{l\in\mathbf{N}}$ defined by (5.1) converges in X_j $(j=1,2)$ as $l\to\infty$, into three parts according as the order of derivatives. In each part, we need the following inequalities for all functions v_1, v_2 with the properties stated above (2.1);

$$(5.2)\quad |v_1(x,t)|\leqq\|v_1\|$$

and

(5.3) $\||v_1|^\theta|v_2|^{1-\theta}\| \leqq \|v_1\|^\theta\|v_2\|^{1-\theta}$ for $0\leqq\theta\leqq 1$.

One can readily derive these inequalities from (2.1).

We note that, by the definition of $p_0(n)$, (1.9),

(5.4) $$1<p_0(n)=\frac{n+1+\sqrt{(n+1)^2+8(n-1)}}{2(n-1)}\leqq 2 \text{ for } n\geqq 4$$

and the equality holds only for $n=4$. When $n=2$ or 3, since $p_0(n)>2$, it is natural to assume $F(0)=F'(0)=F''(0)=0$. However, there exists a typical example $F(s)=s^2$ for $n\geqq 5$, which satisfies the hypothesis $(H)_2$ but does not satisfy $F''(0)=0$. Thus some derices will be required in the proof of existence of C^2-solutions.

PART 1. *Convergence of* $\{u_l\}_{l\in\mathbf{N}}$ *for* $v\in X_j$ $(j=1, 2)$

We choose a function $v\in X_j$ such that

(5.5) $\|v\|^{p-1}\leqq(2^p(A+1)AC_k)^{-1}$ for $p>p_0(n)$
and $\|v\|\leqq 2^{-1}$,

where A is the constant in the hypothesis $(H)_j$ and $C_k=Ck^2$ is the one in Lemma 2.1. Then one can show by induction with respect to l that

(5.6) $\|u_l\|\leqq 2\|v\|$ for $l\in\mathbf{N}$.

In fact, assume that $\|u_{l-1}\|\leqq 2\|v\|$. Then it follows from $(H)_j$ that

$$|L(F(u_{l-1}))(x, t)|\leqq AL(|u_{l-1}|^p)(x, t).$$

Recall that L is a positive linear operator by its definition (1.3). By virtue of (BE) in Lemma 2.1, we get

(5.7) $\|L(F(u_{l-1}))\|\leqq AC_k\|u_{l-1}\|^p$.

Hence (5.1) and (5.5) yield

$$\begin{aligned}\|u_l\| &\leqq \|v\|+AC_k\|u_{l-1}\|^p\\ &\leqq \|v\|+AC_k2^p\|v\|^{p-1}\|v\|\\ &\leqq \|v\|+(A+1)^{-1}\|v\|\\ &\leqq 2\|v\|.\end{aligned}$$

Therefore we obtain (5.6). Besides, (5.6) and (5.5) implies

(5.8) $\|u_l\|\leqq 1$ for $l\in\mathbf{N}$.

Next, we shall show that

(5.9) $\|u_{l+1}-u_l\|\leqq 2^{-1}\|u_l-u_{l-1}\|$ for $l\in\mathbf{N}$.

It follows from (5.1) that

$$u_{l+1}-u_l=L(F(u_l)-F(u_{l-1})).$$

Moreover, $(H)_j$ implies

$$\begin{aligned}|F(u_l)-F(u_{l-1})| &\leqq |F'(\xi_l)||u_l-u_{l-1}| \\ &\leqq A|\xi_l|^{p-1}|u_l-u_{l-1}| \\ &\leqq Aw_l^{p-1}|u_l-u_{l-1}|,\end{aligned}$$

where

$$(5.10)\qquad \xi_l=u_{l-1}+\lambda(u_l-u_{l-1})$$

with some $0<\lambda<1$. Here and in what follows, we set

$$(5.11)\qquad \begin{cases} w_l(x,t)=\max\{|u_l(x,t)|,\ |u_{l-1}(x,t)|\} \\ \|w_l\|=\max\{\|u_l\|,\ \|u_{l-1}\|\}.\end{cases}$$

Hence we get

$$\|u_{l+1}-u_l\|\leqq A\|L(w_l^{p-1}|u_l-u_{l-1}|)\|.$$

Here we write

$$w_l^{p-1}|u_l-u_{l-1}|=(w_l^{(p-1)/p}|u_l-u_{l-1}|^{1/p})^p.$$

Then it follows from (BE) and (5.3) that

$$(5.12)\qquad \begin{aligned}\|u_{l+1}-u_l\| &\leqq AC_k\|w_l^{(p-1)/p}|u_l-u_{l-1}|^{1/p}\|^p \\ &\leqq AC_k\|w_l\|^{p-1}\|u_l-u_{l-1}\|.\end{aligned}$$

Therefore (5.11), (5.6) and (5.5) yield (5.9), which implies

$$(5.13)\qquad \|u_{l+1}-u_l\|\leqq C_1\cdot 2^{-l} \text{ for } l\in\mathbf{N},$$

where $C_1=\|u_1-v\|$.

PART 2. *Convergence of* $\{D_iu_l\}_{l\in\mathbf{N}}$ *for* $v\in X_j$ $(j=1, 2)$.

We first claim that, for each $i=1, \cdots, n$,

$$(5.14)\qquad \|D_iu_l\|\leqq C_2 \text{ for } l\in\mathbf{N},$$

where $C_2=2\|D_iv\|$. Indeed, it follows from (5.1) that

$$\|D_iu_l\|\leqq\|D_iv\|+\|L(|F'(u_{l-1})D_iu_{l-1}|)\|.$$

Note that L and $D_i=\partial/\partial_{xi}$ commute by the definition (1.3). Thus, similarly to the proof of (5.9), we get

$$\|D_i u_l\| \leqq \|D_i v\| + AC_k \|u_{l-1}\|^{p-1} \|D_i u_{l-1}\|$$
$$\leqq \|D_i v\| + 2^{-1} \|D_i u_{l-1}\|.$$

Therefore we obtain (5.14) by induction with respect to l.

Now, we shall estimate $\|D_i(u_{l+1} - u_l)\|$. In the sequel we will derive (5.17) and (5.22) below. It follows from $(H)_j$ that

$$|D_i(F(u_l) - F(u_{l-1}))|$$

(5.15) $$\leqq |F'(u_l) D_i(u_l - u_{l-1})| + |(F'(u_l) - F'(u_{l-1})) D_i u_{l-1}|$$
$$\leqq A|u_l|^{p-1} |D_i(u_l - u_{l-1})| + |(F'(u_l) - F'(u_{l-1})) D_i u_{l-1}|.$$

Dealing with the first term as in the proof of (5.9), we get

(5.16) $$\|L(A|u_l|^{p-1} |D_i(u_l - u_{l-1})|)\| \leqq AC_k \|u_l\|^{p-1} \|D_i(u_l - u_{l-1})\|$$
$$\leqq 2^{-1} \|D_i(u_l - u_{l-1})\|.$$

For the second term, we have to divide its estimate into the following two cases.

First, suppose $(H)_1$ holds and $v \in X_1$. Then we see that there is a positive constant B_1 depending only on F such that

$$|F'(u_l) - F'(u_{l-1})| \leqq B_1 |u_l - u_{l-1}|^\delta.$$

Hence

$$|F'(u_l) - F'(u_{-1})) D_i u_{l-1}| \leqq B_1 |u_l - u_{l-1}|^{p-1} |D_i u_l|$$

with $p_0(n) < p = 1 + \delta < 2$.

Thus, analogously to the proof of (5.12), it follows from (5.13) and (5.14) that

$$\|L(|(F'(u_l) - F'(u_{-1})) D_i u_{l-1}|)\| \leqq B_1 C_k \|u_l - u_{l-1}\|^{p-1} \|D_i u_l\|$$
$$\leqq B_1 C_k (C_1 \cdot 2^{-(l-1)})^{p-1} C_2$$
$$= C_3 \cdot 2^{-(p-1)l},$$

where $C_3 = 2^{p-1} B_1 C_k C_2 C_1^{p-1}$.

Therefore from (5.1), (5.15) and (5.16) we have

(5.17) $$\|D_i(u_{l+1} - u_l)\|$$
$$\leqq 2^{-l} \|D_i(u_1 - u_0)\| + \sum_{\nu=0}^{l-1} C_3 \cdot 2^{-(p-1)(l-\nu)-\nu}$$
$$\leqq C_4 l 2^{-(p-1)l} \text{ for } l \in \mathbf{N}$$

if $(H)_1$ holds and $v \in X_1$, where $C_4 = \max\{\|D_i(u_1 - v)\|, C_3\}$.

From now on we suppose that $(H)_2$ holds and $v\in X_2$. Then, since $F\in C^2(\mathbf{R})$, one can write the second term on the right hand side of (5.15) as

$$(5.18)\qquad |(F'(u_l)-F'(u_{l-1}))D_iu_{l-1}|=|F''(\xi_l)(u_l-u_{l-1})D_iu_{l-1}|,$$

where ξ_l has the form (5.10) with another λ. In view of (5.4), we have to deal with the right hand side separately according as $p_0(n)<p\leqq 2$ or $2<p$.

Let $p_0(n)<p\leqq 2$. Then it follows from (5.2) and (5.8) that

$$(5.19)\qquad |F''(\xi_l)|\leqq C_5 \text{ for } l\in\mathbf{N},$$

where $C_5=\sup\{|F''(s)| : |s|\leqq 1\}$. Moreover, writing as

$$|D_iu_{l-1}|=|D_iu_{l-1}|^{p-1}|D_iu_{l-1}|^{2-p},$$

we have from (5.14)

$$|D_iu_{l-1}|\leqq C_2^{2-p}|D_iu_l|^{p-1}.$$

Hence

$$|F''(\xi_l)(u_l-u_{l-1})D_iu_{l-1}|\leqq C_5C_2^{2-p}|u_l-u_{l-1}||D_iu_{l-1}|^{p-1}.$$

Thus, analogously to the case where $v\in X_1$, it follows from (5.13) and (5.14) that

$$\begin{aligned}(5.20)\qquad &\|L(|F''(\xi_l)(u_l-u_{l-1})D_iu_{l-1}|)\|\\ &\leqq C_5C_2^{2-p}C_k\|u_l-u_{l-1}\|\cdot\|D_iu_{l-1}\|^{p-1}\\ &\leqq C_5C_2^{2-p}C_kC_1\cdot 2^{-(l-1)}C_2^{p-1}.\end{aligned}$$

Next, let $p>2$. Then, according to $(H)_2$ and (5.11), the second term on the right hand side of (5.15) is dominated by

$$\begin{aligned}&Aw_l^{p-2}|u_l-u_{l-1}||D_iu_{l-1}|\\ &\quad=A(w_l^{(p-2)/2}|u_l-u_{l-1}|^{1/p}|D_iu_{l-1}|^{1/p})^p.\end{aligned}$$

To estimate this, we here use the following inequality which follows from (5.3).

$$(5.21)\qquad \||v_1|^\alpha|v_2|^\beta|v_3|^\gamma\|\leqq\|v_1\|^\alpha\|v_2\|^\beta\|v_3\|^\gamma \quad \text{for } 0\leqq\alpha,\ \beta,\ \gamma\leqq 1 \text{ and } \alpha+\beta+\gamma=1.$$

Thus, by virtue of (BE), (5.13) and (5.14), we get

$$\|L(Aw_l^{p-2}|u_l-u_{l-1}||D_iu_{l-1}|)\|$$

$$\leqq AC_k\|w_l\|^{p-2}\|u_l-u_{l-1}\|\cdot\|D_iu_{l-1}\|$$
$$\leqq AC_k\|u_l-u_{l-1}\|\cdot\|D_iu_{l-1}\|$$
$$\leqq AC_kC_1\cdot 2^{-(l-1)}C_2,$$

because (5.11) and (5.8) imply $\|w_l\|\leqq 1$. Hence it follows from (5.1), (5.15), (5.16) and (5.18)-(5.20) that

$$\|D_i(u_{l+1}-u_l)\|\leqq 2^{-1}\|D_i(u_l-u_{l-1})\|+C_6\cdot 2^{-l}$$

for $l\in\mathbf{N}$, where $C_6=2C_kC_2C_1\max\{A, C_5\}$. This implies the desired estimate;

(5.22) $\quad \|D_i(u_{l+1}-u_l)\|\leqq C_7l2^{-l}$ for $l\in\mathbf{N}$

if $(H)_2$ holds and $v\in X_2$, where $C_7=\max\{\|D_i(u_1-v)\|, C_6\}$. By virtue of (5.13) and (5.17), one can conclude that $\{u_l\}_{l\in\mathbf{N}}$ converges in X_1 as $l\to\infty$ under the condition on the size of $\|v\|$, (5.5). Therefore it remains to show the convergence in X_2.

PART 3. *Convergence of* $\{D_iD_ju_l\}_{l\in\mathbf{N}}$ *for* $v\in X_2$

Assume $(H)_2$ holds and $v\in X_2$.

We first claim that, for each $i, j=1, \cdots, n$, $\|D_iD_ju_l\|$ is bounded for $l\in\mathbf{N}$. It follows from $(H)_2$ that

$$|D_iD_jF(u_{l-1})|\leqq|F'(u_{l-1})D_iD_ju_{l-1}|+|F''(u_{l-1})D_iu_{l-1}D_ju_{l-1}|.$$

First, let $p_0(n)<p\leqq 2$. Then we get

$$|D_iD_jF(u_{l-1})|\leqq A|u_{l-1}|^{p-1}|D_iD_ju_{l-1}|+C_5C_2^{2-p}|D_iu_{l-1}||D_ju_{l-1}|^{p-1},$$

where C_5 is the constant in (5.19), because (5.14) yields

$$|D_ju_{l-1}|=|D_ju_{l-1}|^{p-1}|D_ju_{l-1}|^{2-p}$$
$$\leqq C_2^{2-p}|D_ju_{l-1}|^{p-1}.$$

Thus, analogously to Part 2, we see that

$$\|D_iD_ju_l\|\leqq\|D_iD_jv\|+\|L(|D_iD_jF(u_{l-1})|)\|$$
$$\leqq\|D_iD_jv\|+AC_k\|u_{l-1}\|^{p-1}\|D_iD_ju_{l-1}\|$$
$$+C_5C_2^{2-p}C_k\|D_iu_{l-1}\|\cdot\|D_ju_{l-1}\|^{p-1}$$
$$\leqq\|D_iD_jv\|+2^{-1}\|D_iD_ju_{l-1}\|+C_kC_5C_2^2.$$

Next, let $p>2$. Then, by virtue of $(H)_2$, (BE) and (5.21), we have

$$\|D_iD_ju_l\|\leqq\|D_iD_jv\|+AC_k\|u_{l-1}\|^{p-1}\|D_iD_ju_{l-1}\|$$
$$+AC_k\|u_{l-1}\|^{p-2}\|D_iu_{l-1}\|\cdot\|D_ju_{l-1}|$$

Besides, (5.8) implies $\|u_{l-1}\|^{p-2}\leqq 1$. Hence (5.6), (5.5) and (5.14) yield

$$\|D_iD_ju_l\|\leqq\|D_iD_jv\|+2^{-1}\|D_iD_ju_{l-1}\|+AC_kC_2^2.$$

Therefore we obtain, for $p>p_0(n)$,

$$\|D_iD_ju_l\|\leqq 2^{-1}\|D_iD_ju_{l-1}\|+C_8,$$

where $C_8=\|D_iD_jv\|+C_kC_2^2\max\{A, C_5\}$. This implies the desired conclusion

$$(5.23)\qquad \begin{aligned}\|D_iD_ju_l\|&\leqq 2^{-l}\|D_iD_ju_0\|+\sum_{\nu=0}^{l-1}C_8\cdot 2^{-\nu}\\ &\leqq C_9 \text{ for } l\in\mathbf{N},\end{aligned}$$

where $C_9=\|D_iD_jv\|+2C_8$.

Now, we shall estimate $\|D_iD_j(u_{l+1}-u_l)\|$. To this end, we write

$$\begin{aligned}&L[D_iD_j(F(u_l)-F(u_{l-1}))]\\ &=L[(F'(u_l)-F'(u_{l-1}))D_iD_ju_l]\\ &\quad+L[F'(u_{l-1})D_iD_j(u_l-u_{l-1})]\\ &\quad+L[F''(u_{l-1})D_i(u_l-u_{l-1})D_ju_{l-1}]\\ &\quad+L[F''(u_{l-1})D_iu_lD_j(u_l-u_{l-1})]\\ &\quad+L[(F''(u_l)-F''(u_{l-1}))D_iu_lD_ju_l]\\ &\equiv L_1+L_2+L_3+L_4+L_5.\end{aligned}$$

First of all, we shall estimate L_1. Note that

$$L_1=L[F''(\xi_l)(u_l-u_{l-1})D_iD_ju_l],$$

where ξ_l has the form (5.10) with another λ. Then it follows analogously to Part 2, with (5.14) replaced by (5.23), that

$$\begin{aligned}\|L_1\|&\leqq\begin{cases}C_kC_3C_9^{2-p}\|u_l-u_{l-1}\|\cdot\|D_iD_ju_l\|^{p-1} & \text{if } p_0(n)<p\leqq 2,\\ C_kA\|\xi_l\|^{p-2}\|u_l-u_{l-1}\|\cdot\|D_iD_ju_l\| & \text{if } 2<p\end{cases}\\ &\leqq\begin{cases}C_kC_9C_5C_1\cdot 2^{-(l-1)} & \text{if } p_0(n)<p\leqq 2,\\ C_kAC_1\cdot 2^{-(l-1)}C_9 & \text{if } 2<p.\end{cases}\end{aligned}$$

Therefore we obtain

$$(5.24)\qquad \|L_1\|\leqq C_{10}\cdot 2^{-l} \text{ for } l\in\mathbf{N},$$

where $C_{10}=2C_kC_9C_1\max\{A, C_5\}$. Next, we note that L_3 and L_4 have the same form as L_1. Thus, as above, we have from (5.14) and (5.22)

$$\|L_3\|+\|L_4\|\leqq 2\begin{cases}C_kC_5C_2^{2-p}C_7\cdot(l-1)2^{1-l}C_2^{p-1} & \text{if } p_0(n)<p\leqq 2\\ C_kAC_2C_7\cdot(l-1)2^{1-l} & \text{if } 2<p.\end{cases}$$

Hence we get

(5. 25) $\|L_3\|+\|L_4\|\leqq C_{11}(l-1)2^{-l}$ for $l\in\mathbf{N}$,

where $C_{11}=2C_kC_7C_2\max\{A, C_5\}$. Moreover, the procedure in the proof of (5.9) yields

(5. 26) $$\|L_2\|\leqq AC_k\|u_l\|^{p-1}\|D_iD_j(u_l-u_{l-1})\| \leqq 2^{-1}\|D_iD_j(u_l-u_{l-1})\| \text{ for } l\in\mathbf{N}.$$

As last step, we shall estimate L_5. In view of $(H)_2$, we see that there is a positive constant B_2 depending only on F such that

$$|F''(u_l)-F''(u_{l-1})|\leqq B_2|u_l-u_{l-1}|^{\delta} \text{ for } 0<\delta<1.$$

Hence we have

$$|L_5|\leqq B_2L(|u_l-u_{l-1}|^{\delta}|D_iu_l||D_ju_l|).$$

Let $p_0(n)<p\leqq 2$. Then it follows from (5.2), (5.13) and (5.14) that

$$|u_l-u_{l-1}|^{\delta}|D_iu_l||D_ju_l|\leqq(C_1\cdot 2^{-(l-1)})^{\delta}|D_iu_l||D_ju_l|^{p-1}C_2^{2-p}.$$

Thus, analogously to Part 2, we get

$$\|L_5\|\leqq B_2C_k(C_1\cdot 2^{-(l-1)})^{\delta}C_2^{2-p}\|D_iu_l\|\cdot\|D_ju_l\|^{p-1} \leqq 2^{\delta}B_2C_kC_2^2C_1^{\delta}\cdot 2^{-\delta l}.$$

Next, let $2<p\leqq 2+\delta$. Then it follows from (5.13) and $\delta\geqq p-2>0$ that

$$|u_l-u_{l-1}|^{\delta}|D_iu_l||D_ju_l| \leqq(C_1\cdot 2^{-(l-1)})^{\delta-(p-2)}|u_l-u_{l-1}|^{p-2}|D_iu_l||D_ju_l|.$$

Hence, as above, we have

$$\|L_5\|\leqq B_2C_k(C_1\cdot 2^{-(l-1)})^{\delta-(p-2)}\|u_l-u_{l-1}\|^{p-2}\|D_iu_l\|\cdot\|D_ju_l\| \leqq 2^{\delta}B_2C_kC_2^2C_1^{\delta}\cdot 2^{-\delta l}.$$

Therefore we obtain

(5. 27) $\|L_5\|\leqq 2^{\delta}B_2C_kC_2^2C_1^{\delta}\cdot 2^{-\delta l}$ for $l\in\mathbf{N}$.

In the end, it follows from (5.1) and (5.24)-(5.27) that

$$\|D_iD_j(u_{l+1}-u_l)\|\leqq 2^{-1}\|D_iD_j(u_l-u_{l-1})\|+C_{12}\cdot l2^{-\delta l},$$

where $C_{12}=\max\{C_{10}, C_{11}, 2^{\delta}B_2C_kC_2^2C_1^{\delta}\}$. This implies

(5. 28) $\|D_iD_j(u_{l+1}-u_l)\|$

$$\leqq 2^{-l}\|D_iD_j(u_1-u_0)\|+\sum_{\nu=0}^{l-1}C_{12}(l-\nu)2^{-\delta(l-\nu)-\nu}$$
$$\leqq(\|D_iD_j(u_1-v)\|+C_{12}\sum_{\nu=0}^{l-1}(l-\nu))\cdot 2^{-\delta l} \text{ for } l\in\mathbf{N}.$$

Thus, by (5.13), (5.22) anb (5.28), we can conclude that $\{u_l\}_{l\in\mathbf{N}}$ converges in X_2 as $l\to\infty$ under the condition on the size of $\|v\|$, (5.5).

Now, the uniqueness of the solution to (1.2) follows from a standard argument. Thus we prove Theorem 1.

Appendix

In this appendix we show that the decay rate of $L(|u|^p)(x, t)$ in the basic estimate (BE) is best possible when n is even and $p>2n/(n-1)$. More precisely we have

PROPOSITION A. *Let n be even. Suppose the function N in* (2.1) *is of the form $N(s)=s^q$. Assume that* (*BE*) *holds for all such functions u as stated in Lemma* 2. 1. *Then $q\leqq(n-1)/2$.*

PROOF. It suffices to show that one can construct a function u such that $\|u\|=1$ and $\|L(|u|^p)\|=\infty$ provided $q>(n-1)/2$.

Set

(A. 1) $$u(y, \tau)=k^{((n-1)/2)+q}\varphi(\tau)\psi(y)(b(|y|, \tau))^{1/p}.$$

Here $b(\lambda, \tau)$ is the function defined by (4.3) and $\varphi\in C_0^\infty(\mathbf{R}^1)$, $\psi\in C_0^\infty(\mathbf{R}^n)$ are cutoff functions such that $0\leqq\varphi\leqq 1$, $0\leqq\psi\leqq 1$, $\varphi(\tau)=1$ for $3k\leqq\tau\leqq 4k$, supp $\varphi\subset(2k, 5k)$, $\psi(y)=1$ for $k\leqq|y|\leqq 2k$ and supp $\psi\subset\{0<|y|<3k\}$. Then one can see easily that $u\in C_0^\infty(\mathbf{R}^n\times(0,\infty))$, supp $u(y,\tau)\subset\{(y,\tau): |y|<\tau+k\}$ and $\|u\|=1$. Thus it suffices to verify that $\|L(|u|^p)\|=\infty$ if $q>(n-1)/2$.

In what follows we suppose

$$r<t-9k.$$

Then it follows from (1.1), (1.3) and (A.1) that

$$L(|u|^p)(x, t)=A_n\int_0^t\frac{d\tau}{(t-\tau)^{n-2}}\int_{|y-x|\leqq t-\tau}\frac{|u(y,\tau)|^p}{\sqrt{(t-\tau)^2-|y-x|^2}}dy$$
$$\geqq A_nk^{((n-1)p/2)+pq}\int_{3k}^{4k}\frac{d\tau}{(t-\tau)^{n-2}}\int_{k\leqq|y|\leqq 2k}\frac{b(|y|,\tau)}{\sqrt{(t-\tau)^2-|y-x|^2}}dy,$$

because, if $3k\leqq\tau\leqq 4k$, $k\leqq|y|\leqq 2k$ and $r<t-9k$, then $\varphi(\tau)=1$, $\psi(y)=1$ and $|y-x|\leqq t-\tau-3k$. In addition, since $\tau+|y|+2k\leqq 8k$ and $\tau-|y|+2k\leqq 6k$, (4.3) yields

$$b(|y|, \tau) \geqq (8k)^{-(n-1)p/2}(6k)^{-pq}.$$

Moreover

$$(t-\tau)^{n-2}\sqrt{(t-\tau)^2-|y-x|^2} \leqq t^{n-1} \text{ for } \tau \geqq 0.$$

Therefore we have

$$L(|u|^p)(x, t) \geqq C_k t^{1-n}$$
$$\geqq C_k t^{(1-n)/2}(t+r+2k)^{(1-n)/2},$$

where $C_k = 8^{-(n-1)p/2}6^{-pq}A_n\omega_n k^{n+1}$. Consequently, it follows from (2.1) with $N(s)=s^q$ that

$$\text{(A. 2)} \quad \|L(|u|^p)\|$$
$$\geqq \sup_{r<t-9k} |L(|u|^p)(x, t)| \left(\frac{t+r+2k}{k}\right)^{(n-1)/2} \left(\frac{t-r+2k}{k}\right)^q$$
$$\geqq C'_k t^{(1-n)/2}(t-r+2k)^q$$

for $r<t-9k$, where $C'_k = C_k k^{-((n-1)/2)-q}$.

Now let δ be an arbitrary positive number with $\delta<1$. Suppose

$$r<(1-\delta)t \text{ and } t>9k/\delta,$$

so that $r<t-9k$. Then (A. 2) implies

$$\|L(|u|^p)\| \geqq C'_k \delta^q t^{q-((n-1)/2)}.$$

Thus we find that $\|L(|u|^p)\|=\infty$ if $q>(n-1)/2$. The proof is complete.

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