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Positive radial solutions of semilinear elliptic equations of order 2m in annular domains

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Abstract. We study the existence of positive radial solutions of $(-1)^{m}\Delta^{m}u=g(|x|)f(u)$ in an annulus with Dirichlet boundary conditions. In particular L^{∞} a priori bounds are obtained.

1. Introduction

In this paper we investigate the existence of positive radial solutions of the semilinear elliptic equation

$$(1.1)\qquad (-1)^{m}\Delta^{m}u=g(|x|)f(u)\qquad \text{in } \Omega(a, b)$$

$$(1.2)\qquad u=\frac{\partial u}{\partial\nu}=\ldots=\left(\frac{\partial}{\partial\nu}\right)^{m-1}u=0\qquad \text{on } \partial\Omega(a, b)$$

where $0<a<b<+\infty$, $\Omega(a, b)$ denotes the annulus $\{x\in \boldsymbol{R}^{n}; a<|x|<b\}$ $(n\geq 2)$, $\frac{\partial}{\partial\nu}$ is the outward normal derivative and m is a positive integer.

When $m=1$ problem (1.1), (1.2) has been intensively studied in recent years (see e.g. [1]-[3], [6], [10], [12], [15]). In most papers, the shooting method was used to establish the existence of positive radial solutions. In contrast the result of [1] was obtained by a variational method and the use of a priori estimates, while in [15] an expansion fixed point theorem was applied. The case $m=2$ was treated in [8] using a priori estimates and well-known properties of compact mappings taking a cone in a Banach space into itself (see [9]). However the technique used in [8] to obtain the a priori estimates does not extend to apply to higher order equations.

Our main result is the following L^{∞} bound for positive radial solutions of problem (1.1), (1.2).

THEOREM 1.1. *Let f and g satisfy the following hypotheses:*
(H_1) *$f:[0, +\infty)\rightarrow \boldsymbol{R}$ is a continuous function,*
(H_2) *$g:[a, b]\rightarrow[0, +\infty)$ is a continuous function such that $g\not\equiv 0$ in $[a, b]$,*
(H_3) $\lim_{u\rightarrow+\infty} f(u)/u=+\infty$.

Then there exists $M>0$ *such that*

$$\|u\|_\infty \leq M$$

for all positive radial solutions $u \in C^{2m}(\overline{\Omega(a, b)})$ *of* (1.1), (1.2).

Under some additional assumptions on the function f, we can use theorem 1.1 to establish the existence of a positive radial solution of problem (1.1), (1.2).

THEOREM 1.2. *Let* f *and* g *satisfy* (H_1)-(H_3). *Assume moreover*
(H_4) $f(u) \geq 0$ *for* $u>0$,
(H_5) $\lim_{u \to 0} f(u)/u=0$.

Then problem (1.1), (1.2) *possesses at least one positive radial solution* $u \in C^{2m}(\overline{\Omega(a, b)})$.

Since we are interested in positive radial solutions, the problem under consideration reduces to the one-dimensional boundary value problem

$$(1.3) \qquad (-1)^m \Delta^m u(t)=g(t) f(u(t)), \quad t \in (a, b)$$
$$(1.4) \qquad u^{(j)}(a)=u^{(j)}(b)=0, \quad j=0, \ldots, m-1$$

where Δ denotes the polar form of the Laplacian, i.e.:

$$\Delta=t^{1-n} \frac{d}{dt}\left(t^{n-1} \frac{d}{dt}\right).$$

In this paper our new key ingredient is the Green's function of the linear problem corresponding to (1.3), (1.4).

In our proofs we shall make an intensive use of the one dimensional maximum principle and the related Hopf boundary lemma [14], which we recall:

THEOREM A ([14] p. 2). *Suppose* $u \in C^2((a, b)) \cap C([a, b])$ *satisfies the differential inequality*

$$u''+g(x) u' \geq 0 \text{ for } a<x<b$$

with g *a bounded function. If* $u \leq M$ *in* (a, b) *and if the maximum* M *of* u *is attained at an interior point of* (a, b), *then* $u \equiv M$.

THEOREM B ([14] p. 4). *Suppose* $u \in C^2((a, b)) \cap C^1([a, b])$ *is a non-constant function which satisfies the differential inequality* $u''+g(x) u' \geq 0$ *in* (a, b) *and suppose* g *is bounded on every closed subinterval of* (a, b). *If the maximum of* u *occurs at* $x=a$ *and* g *is bounded below at* $x=a$, *then* $u'(a)<0$. *If the maximum occurs at* $x=b$ *and* g *is bounded above at* $x=$

b, then $u'(b)>0$.

REMARK 1.1. Theorems 1.1 and 1.2 can be easily extended to handle more general nonlinearities of the type $f(|x|, u)$.

Our paper is organized as follows. In Section 2 we give a maximum principle for higher order equations and we describe the special shape of nontrivial solutions of (1.3), (1.4) when $f\geq 0$ and $g\geq 0$. We also recall some simple inequalities of the Green's function of the linear problem corresponding to (1.3), (1.4). In Section 3 we prove Theorem 1.1. Finally Theorem 1.2 is proved in Section 4.

2. Preliminaries

The homogeneous Dirichlet problem

$$\begin{cases} \Delta^m v=0 \quad \text{in } (a, b) \\ v^{(j)}(a)=v^{(j)}(b)=0, \quad j=0, \ldots, m-1 \end{cases}$$

has only the trivial solution. Then it is well-known (see e.g. [13] p. 29) that the operator $(-1)^m\Delta^m$ with Dirichlet boundary conditions has one and only one Green's function $G_m(t, s)$.

THEOREM 2.1. $G_m(t, s)>0$ *for* $a<t,\ s<b$.

PROOF. Since $(-1)^m\Delta^m$ is a disconjugate operator on $[a, b]$, this follows readily from a theorem obtained in [7] (Theorem 11 p. 108).

THEOREM 2.2. *Let* $u\in C^{2m}([a, b])$ *be such that*

$$\begin{cases} (-1)^m\Delta^m u\geq 0 \text{ in } (a, b) \\ u^{(j)}(a)=u^{(j)}(b)=0, \quad j=0, \ldots, m-1. \end{cases}$$

Assume that $u\not\equiv 0$. *Then*:

(i) $u>0$ *on* (a, b).

(ii) $u^{(m)}(a)>0$ *and* $(-1)^m u^{(m)}(b)>0$.

(iiia) *Assume* $m=1$. *Then there exist* $d_1, d_2\in(a, b)$ *such that* $d_1\leq d_2$, $u'>0$ *on* $[a, d_1)$, $u'<0$ *on* $(d_2, b]$ *and* $u'\equiv 0$ *on* $[d_1, d_2]$.

(iiib) *Assume* $m\geq 2$. *Then there exists* $c\in(a, b)$ *such that* $u'>0$ *on* (a, c) *and* $u'<0$ *on* (c, b).

PROOF. Theorem 2.1 gives (i). Then (ii) is a simple consequence of a proposition obtained in [7] (Proposition 13 p. 109). We now prove (iiia). (ii) when $m=1$ gives $u'(a)>0$ and $u'(b)<0$. Let d_1 (resp. d_2) be the first (resp. the last) zero of u' on (a, b). If $d_1<d_2$ Theorems A and B imply

that u is constant on $[d_1, d_2]$. The proof of (iiib) requires some lemmas.

LEMMA 2.1. *Let $m \geq 2$ and $1 \leq j \leq m-1$. Then $\Delta^j u$ is neither nonnegative nor nonpositive in $[a, b]$.*

PROOF. Suppose first that $j=1$. If $\Delta u \geq 0$ on $[a, b]$, Theorem A implies $u \leq 0$ on $[a, b]$, a contradiction with (i). If $\Delta u \leq 0$ on $[a, b]$, (i) and Theorem B imply that $u'(a)>0$ and $u'(b)<0$, again a contradiction. Now if $2 \leq j \leq m-1$ (and necessarily $m \geq 3$), suppose for instance that $(-1)^j \Delta^j u \geq 0$ in $[a, b]$. Define $w=-\Delta u$. Then we have

$$(-1)^{j-1} \Delta^{j-1} w \geq 0 \text{ in } [a, b]$$

and

$$w(a)=w'(a)= \dots =w^{(j-2)}(a)=0, \; w(b)=w'(b)= \dots =w^{(j-2)}(b)=0.$$

Since by Theorem 2.1 (with $m=j-1$) the Green's function of $(-1)^{j-1}\Delta^{j-1}$ for the Dirichlet problem in $[a, b]$ is positive we get $w=-\Delta u \geq 0$ in $[a, b]$, which is impossible by what we have just seen. Clearly, the case $(-1)^j \Delta^j u \leq 0$ in $[a, b]$ can be handled in the same way. The proof of the lemma is complete.

LEMMA 2.2. *$\Delta^{m-1} u$ does not vanish throughout any subinterval of $[a, b]$.*

PROOF. Since $u>0$ on (a, b), the lemma is proved when $m=1$. Now assume $m \geq 2$. Suppose that there exist $\alpha, \beta \in [a, b]$ such that $a \leq \alpha < \beta \leq b$ and $w=(-1)^{m-1}\Delta^{m-1}u \equiv 0$ on $[\alpha, \beta]$. By Lemma 2.1 we have $\alpha > a$ or $\beta < b$. Let $t \in [a, \alpha) \cup (\beta, b]$. If $w(t)>0$ and $t \in [a, \alpha)$ (resp. $t \in (\beta, b]$) Theorems A and B imply that $w'(\alpha)<0$ (resp. $w'(\beta)>0$), a contradiction. Thus $w \leq 0$ on $[a, b]$ and this is impossible by Lemma 2.1.

LEMMA 2.3. *Assume $m \geq 2$. Then there exist $r, s \in (a, b)$ such that $r < s$, $\Delta u > 0$ on $(a, r) \cup (s, b)$ and $\Delta u < 0$ on (r, s).*

PROOF. Suppose first $m=2$. By (ii) $\Delta u(a)>0$ and $\Delta u(b)>0$. By Lemma 2.1 there exists $x \in (a, b)$ such that $\Delta u(x)<0$. Define r (resp. s) to be the first (resp. the last) zero of Δu on (a, b). Then Theorem A implies that $\Delta u<0$ on (r, s). Now assume $m \geq 3$. It follows from Lemma 2.2 that Δu does not vanish throughout any subinterval of $[a, b]$. Therefore we may apply Proposition 13 of [7] (p. 109) and conclude that Δu has at most two zeros on (a, b). Using Taylor's formula and (ii) we can show that there exists $\eta>0$ such that $\Delta u>0$ on $(a, a+\eta) \cup (b-\eta, b)$. Then the result follows with the aid of Lemma 2.1.

Now we can prove (iiib). Lemma 2.3, (i), Theorem A and Theorem B imply that $u'>0$ on $(a, r]$ and $u'<0$ on $[s, b)$. Let t_0 (resp. t_1) be the first (resp. the last) zero of u' in (a, b). Then $r<t_0\leq t_1<s$. Suppose that $t_0<t_1$. Since by Lemma 2.3 u is not constant on $[t_0, t_1]$, Theorems A and B imply that either $u'(t_0)>0$ or $u'(t_1)<0$, a contradiction. Thus $t_0=t_1=c$ and (iiib) is proved. The proof of the theorem is complete.

Now we recall some simple inequalities obtained in [4] for the Green's function of the linear problem corresponding to (1.3), (1.4). Below Δ^* denotes the adjoint of Δ.

Let $v, v^*, w, w^*\in C^{2m}([a, b])$ be defined by the following relations:

$$(2.1)\qquad \begin{cases} \Delta^m v=\Delta^{*m}v^*=0 \text{ in } (a, b) \\ v^{(j)}(a)=v^{*(j)}(b)=0,\ j=0, \dots, m-1 \\ v^{(j)}(b)=v^{*(j)}(a)=0,\ j=0, \dots, m-2 \ (\text{if } m\geq 2) \\ v^{(m-1)}(b)=(-1)^{m-1},\ v^{*(m-1)}(a)=1 \end{cases}$$

and

$$(2.2)\qquad \begin{cases} \Delta^m w=\Delta^{*m}w^*=0 \text{ in } (a, b) \\ w^{(j)}(a)=w^{*(j)}(b)=0,\ j=0, \dots, m-2 \ (\text{if } m\geq 2) \\ w^{*(j)}(a)=w^{(j)}(b)=0,\ j=0, \dots, m-1 \\ w^{(m-1)}(a)=1,\ w^{*(m-1)}(b)=(-1)^{m-1}. \end{cases}$$

The functions defined in (2.1), (2.2) are positive on (a, b) because of the disconjugacy of the operators Δ^m and Δ^{*m}. Applying Corollary 3.2 of [4] and Theorem 2.1 we get

THEOREM 2.3. *On the upper triangle* $a\leq t\leq s\leq b$,

$$0\leq G_m(t, s)\leq\frac{1}{v^{(m)}(a)}v(t)v^*(s)$$

and on the lower triangle $a\leq s\leq t\leq b$,

$$0\leq G_m(t, s)\leq\frac{1}{|w^{(m)}(b)|}w(t)w^*(s).$$

We easily deduce the following corollary.

COROLLARY 2.1. *There exists* $C>0$ *such that*

$$0\leq G_m(t, s)\leq C(s-a)^m(b-s)^m \textit{ for } a\leq t,\ s\leq b.$$

3. Proof of Theorem 1.1

We shall prove that there exists $M>0$ such that

(3.1) $\quad \|u\|_\infty \leq M$

for all positive solutions $u \in C^{2m}([a, b])$ of (1.3), (1.4).

Define

$$\rho(t)=(t-a)^m(b-t)^m \text{ for } a \leq t \leq b.$$

Let $\varphi \in C^{2m}([a, b])$ be the solution of the boundary value problem

$$\begin{cases} (-1)^m \Delta^m \varphi = g\rho \text{ in } (a, b) \\ \varphi^{(j)}(a)=\varphi^{(j)}(b)=0, \ j=0, \dots, m-1. \end{cases}$$

By Theorem 2.2 $\varphi>0$ in (a, b) and there exist $c_1>0$ and $c_2>0$ such that

(3.2) $\quad c_1\rho \leq \varphi \leq c_2\rho$ on $[a, b]$.

By (H_3), there exist $\lambda > c_1^{-1}$ and a positive constant c_3 such that

(3.3) $\quad f(u) \geq \lambda u - c_3$ for $u \geq 0$.

Now let $u \in C^{2m}([a, b])$ be a positive solution of (1.3), (1.4). If we multiply equation (1.3) by $t^{n-1}\varphi$ and integrate by parts $2m$ times we obtain

(3.4) $\quad \int_a^b t^{n-1}\varphi g f(u) dt = \int_a^b t^{n-1}\rho g u dt.$

From (3.2), (3.3) and (3.4) we deduce that

$$\int_a^b t^{n-1}\rho g u dt \geq \lambda \int_a^b t^{n-1}\varphi g u dt - c_4 \geq \lambda c_1 \int_a^b t^{n-1}\rho g u dt - c_4$$

for some positive constant c_4, hence

(3.5) $\quad \int_a^b t^{n-1}\rho g u dt \leq \frac{c_4}{\lambda c_1 - 1}.$

It easily follows that there is a positive constant c_5 such that

(3.6) $\quad \int_a^b t^{n-1}\rho g|f(u)| dt \leq c_5$

Using Corollary 2.1 and (3.6) we get

$$u(t)=\int_a^b G_m(t, s)g(s)f(u(s))ds \leq Ca^{1-n}c_5 \text{ for } t \in [a, b]$$

and (3.1) is proved.

4. Proof of Theorem 1.2

We shall prove that problem (1.3), (1.4) has at least one positive solution $u\in C^{2m}([a, b])$. The proof makes use of a fixed point theorem originally due to Krasnosel'skii [11] and Benjamin [5]. Here we use the following modified version.

PROPOSITION 4.1 ([9] p. 56). *Let C be a cone in a Banach space X and $\Phi: C\to C$ a compact map such that $\Phi(0)=0$. Assume that there exist numbers $0<r<R$ such that*

(i) *$u\neq\theta\Phi(u)$ for $\theta\in[0, 1]$ and $u\in C$ such that $\|u\|=r$,*

(ii) *there exists a compact map $F: \overline{B_R}\times[0, +\infty)\to C$ (where $B_\rho=\{u\in C; \|u\|<\rho\}$) such that $F(u, 0)=\Phi(u)$ for $\|u\|=R$, $F(u, x)\neq u$ for $\|u\|=R$ and $0\leq x<\infty$ and $F(u, x)=u$ has no solution $u\in\overline{B_R}$ for $x\geq x_0$. Then if $U=\{u\in C; r<\|u\|<R\}$, one has:*

$$i_C(\Phi, B_R)=0,\ i_C(\Phi, B_r)=1,\ i_C(\Phi, U)=-1,$$

where $i_C(\Phi, W)$ denotes the fixed point index of Φ on W. In particular Φ has a fixed point in U.

Now let X denote the Banach space $C([a, b])$ endowed with the sup norm. Define the cone

$$C=\{u\in C([a, b]); u\geq 0\}.$$

For $(u, x)\in C\times[0, +\infty)$ we define

$$F(u, x)(t)=\int_a^b G_m(t, s)g(s)f(u(s)+x)ds \text{ for } t\in[a, b]$$

and

$$\Phi(u)=F(u, 0).$$

We shall show that the hypotheses of Proposition 4.1 are satisfied. By Theorem 2.1, (H_2) and (H_4) F maps $C\times[0, +\infty)$ into C. Since G_m is continuous, it is well-known that F is compact. (H_1), (H_4) and (H_5) imply that $f(0)=0$, hence $\Phi(0)=0$.

Let $\alpha\in(0, c_2^{-1})$, where c_2 is the constant in (3.2). By (H_5) we can choose $r>0$ such that $f(s)\leq\alpha s$ for $0\leq s\leq r$. Suppose that there exist $\theta\in[0, 1]$ and $u\in C$ with $\|u\|_\infty=r$ such that $u=\theta\,\Phi(u)$. Then $(-1)^m\Delta^m u=\theta gf(u)$ and u satisfies (1.4). By Theorem 2.2 (i) $u>0$ on (a, b). With the notations of Section 3 we have

$$\int_a^b t^{n-1}\rho gudt=\int_a^b t^{n-1}u(-1)^m\Delta^m\varphi dt=\int_a^b t^{n-1}\varphi(-1)^m\Delta^m udt$$
$$=\theta\int_a^b t^{n-1}\varphi gf(u)dt\leq\alpha c_2\theta\int_a^b t^{n-1}\rho gudt$$

$$< \int_a^b t^{n-1} \rho g u dt$$

and we reach a contradiction because the integrals are nonzero. Thus condition (i) of Proposition 4.1 is satisfied.

By (H_3), there exist $\lambda > c_1^{-1}$ where c_1 is the constant in (3.2) and $x_0 > 0$ such that

$$f(s+x) \geq \lambda(s+x) \geq \lambda s \text{ for } s \geq 0 \text{ and } x \geq x_0 > 0. \tag{4.1}$$

We shall show that

$$F(u, x) \neq u \text{ for all } u \in C \text{ and } x \geq x_0. \tag{4.2}$$

Indeed, suppose that there exist $u \in C$ and $x \geq x_0$ such that $F(u, x) = u$. Then $(-1)^m \Delta^m u(t) = g(t) f(u(t) + x)$ for $t \in [a, b]$ and u satisfies (1.4). If $u \equiv 0$ then $f(x) = 0$, a contradiction to (4.1). Thus $u \not\equiv 0$. Therefore $u > 0$ by Theorem 2.2 (i). Now with the notations of the proof of (3.1) we have

$$\begin{aligned}
\int_a^b t^{n-1} \rho(t) g(t) u(t) dt &= \int_a^b t^{n-1} \varphi(t) g(t) f(u(t)+x) dt \\
&\geq \lambda \int_a^b t^{n-1} \varphi(t) g(t) u(t) dt \\
&\geq \lambda c_1 \int_a^b t^{n-1} \rho(t) g(t) u(t) dt \\
&> \int_a^b t^{n-1} \rho(t) g(t) u(t) dt
\end{aligned}$$

and this yields a contradiction because the integrals are nonzero. Thus (4.2) holds and the third condition of (ii) is satisfied.

Now we note that the constant in (3.1) can be chosen independently of the parameter $x \in [0, x_0]$ for each fixed $x_0 \in (0, +\infty)$ if we consider positive solutions of (1.3), (1.4) for the family of nonlinearities $f_x(t) = f(t+x)$, $t \geq 0$. Thus we can find a constant $R > r$ such that

$$F(u, x) \neq u \text{ for all } x \in [0, x_0] \text{ and } u \in C \text{ with } \|u\|_\infty = R. \tag{4.3}$$

Therefore (4.2) and (4.3) prove the second condition of (ii).

Thus we may apply Proposition 4.1 to conclude that Φ has a nontrivial fixed point $u \in C$. By Theorem 2.1, (H_2), (H_4) and the properties of the Green's function any nontrivial fixed point of Φ in C yields a positive solution of (1.3), (1.4) in $C^{2m}([a, b])$. The proof of the theorem is complete.

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